FUNCTIONS ON THE SPHERE WITH CRITICAL POINTS IN PAIRS AND ORTHOGONAL GEODESIC CHORDS

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ABSTRACT. Using an estimate on the number of critical points for a *Morse-even* function on the sphere $\mathbb{S}^m$, $m \geqslant 1$, we prove a multiplicity result for orthogonal geodesic chords in Riemannian manifolds with boundary that are diffeomorphic to Euclidean balls. This yields also a multiplicity result for brake orbits in a potential well.

1. INTRODUCTION

The topology of spheres does not allow good estimates on the number of critical points of smooth functions. In fact, given the fact that any C^1-function on a compact manifold admits at least two critical points, Reeb's theorem characterizes spheres as the only compact manifold that admits functions with *exactly* 2 critical points (see for instance [14, Theorem 4.1, p. 25]). However, functions having some type of symmetry tend to have more than two critical points. For instance, if $f : \mathbb{S}^m \to \mathbb{R}$ is even, i.e., $f(x) = f(-x)$ for all x, then f defines a function $\widetilde{f}$ on the projective space $\mathbb{R}P^m$, that must have at least $m + 1$ distinct critical points, by Lusternik–Schnirelman (or Morse) theory. Thus, the original function f must have at least $m + 1$ distinct pairs of antipodal critical points.

1.1. **Morse-even functions.** Motivated by an example in classical Riemannian geometry, that will be described below, in this paper we prove that the same estimate on the number of critical points holds in a slightly more general situation, when resorting to the Lusternik–Schnirelman category of the projective space is not possible. More precisely, we will consider functions whose critical points come in pairs with the same Morse index.

Definition. Let $\mathcal{M}^m$ be a compact m-dimensional manifold. A Morse function $f : \mathcal{M} \to \mathbb{R}$ is said to be *Morse-even* if for every $k = 0, \ldots, m$, the set of critical points of f having Morse index equal to k is an even number.

The starting observation for Morse-even functions on spheres is the following:

Theorem A. *Let $\mathcal{M}$ be a smooth manifold that is homeomorphic to an m-sphere, and let $f : \mathcal{M} \to \mathbb{R}$ be a Morse-even function. Then, f has critical points of arbitrary Morse index in $\{0, 1, \ldots, m\}$. In particular, f admits at least $(m + 1)$ distinct pairs of critical points.*

Proof. The Poincaré polynomial (with integer coefficients) of the m-sphere is given by $P_m(\lambda) = 1 + \lambda^m$. By Morse theory, there exists a polynomial $Q(\lambda) = a_0 + a_1\lambda + \ldots + a_{m-1}\lambda^{m-1}$ with nonnegative integer coefficients such that:
(1.1)
$$1 + \lambda^m + (1+\lambda)Q(\lambda) = (1+a_0) + (a_0+a_1)\lambda + \ldots + (a_{m-2}+a_{m-1})\lambda^{m-1} + (1+a_{m-1})\lambda^m$$

is the Morse polynomial $M_f(\lambda)$ of f. Recall that $M_f(\lambda) = \sum_{k=0}^{m} \kappa_k \lambda^k$, where κ_k is the number of critical points of f whose Morse index is equal to k. Since f is Morse-even, then each κ_k is an even number. Using (1.1), one shows by an elementary argument that all the a_i are non-zero and odd, $i = 0, \ldots, m-1$. Thus, all the κ_k are positive, $k = 0, \ldots, m$. □

Remark 1. The above result has a natural generalization to Morse-even functions defined on a connected, orientable compact manifold $\mathcal{M}$ whose Betti numbers $\beta_k(\mathcal{M})$ are even for $k = 1, \ldots, m - 1$. If $f : \mathcal{M} \to \mathbb{R}$ is a Morse-even function on such a manifold, then for all $k = 1, \ldots, m-1$, f admits a number strictly larger than $\frac{1}{2}\beta_k(M)$ of pairs of critical points having Morse index equal to k (this statement holds trivially also for $k = 0$ and $k = m$). A proof of this is obtained readily from the Morse relations, given by the equality:

$$\sum_{k=0}^{m} \beta_k x^k + (1 + x) \sum_{k=1}^{m-1} a_k x^k = \sum_{k=0}^{m} \kappa_k x^k,$$

i.e.:

$$\kappa_0 = 1 + a_0, \quad \kappa_k = \beta_k + a_k + a_{k-1}, \text{ for } k = 1, \ldots, m-1, \quad \kappa_m = 1 + a_m,$$

for some integer coefficients $a_k \geqslant 0$. The parity of the coefficients implies $a_0, a_m \geqslant 1$, and $a_k + a_{k-1} \geqslant 2$ for $k = 1, \ldots, m - 1$.

The condition on the parity of the Betti numbers is satisfied by a large class of manifolds. For instance, when $m = 2$, the condition β_1 even is satisfied by every compact oriented surface Σ. Namely, in this case $\beta_1(\Sigma) = 2 \operatorname{gen}(\Sigma)$ (here $\operatorname{gen}(\Sigma)$ denotes the genus of Σ). It is not hard to classify the homeomorphism classes of simply connected manifolds satisfying the condition in low dimensions. When $m = 3$, if $\mathcal{M}$ is simply connected then $\beta_1(\mathcal{M}) = 0$ (and thus also $\beta_2(\mathcal{M}) = 0$). By the Poincaré conjecture, the unique simply connected compact manifold that satisfies the assumptions is the 3-sphere. Many interesting cases of even Betti numbers are found in dimension 4. In this case, for a simply connected manifold it suffices to require that β_2 is even. Recall that in dimension 4, the second Betti number β_2 is additive by connected sums, i.e., given 4-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, then $\beta_2(\mathcal{M}_1 \# \mathcal{M}_2) = \beta_2(\mathcal{M}_1) + \beta_2(\mathcal{M}_2)$. Recall also that $\beta_2(\mathbb{CP}^2) = 1$ and $\beta_2(\mathbb{S}^2 \times \mathbb{S}^2) = 2$. Thus, connected sums of any number of copies of $\mathbb{S}^2 \times \mathbb{S}^2$ and any even number of copies of $\mathbb{CP}^2$ have even β_2.

1.2. **Orthogonal geodesic chords.** Our motivation for the result of Theorem A comes from a classical problem in Riemannian geometry, which consists in finding lower estimates on the number of geodesics departing and arriving orthogonally to the boundary of a compact Riemannian manifold. These objects are called *orthogonal geodesic chords* (OGC). It is interesting to observe that there are manifolds diffeomorphic to Euclidean balls that have no OGC's, see for instance [3]. Orthogonal geodesics chords for metrics in a ball have a special interest in the case where the boundary $\mathbb{S}^m$ of B^{m+1} is strictly concave. Namely, in this situation, a multiplicity result for OCG's yields an analogous multiplicity result for brake orbits of natural Hamiltonians or Lagrangian in a potential well (see for instance [5]).

In order to apply Theorem A to obtain information on the number of OGC's, let us consider the following situation. Given a compact Riemannian manifold (M, g) with boundary ∂M, let exp denote the corresponding exponential map. Let $\vec{\nu}$ be the unit normal field along ∂M pointing inwards. The metric g will be said to be *regular* (with respect to ∂M) if there exists a (necessarily smooth) function $s_g : \partial M \to]0, +\infty[$ such that, for any $p \in \partial M$, the geodesic $[0, s_g(p)] \ni t \mapsto \exp_p(t \cdot \vec{\nu}_p)$ meets transversally ∂M at $t = s_g(p)$. In this situation, we will call s_g the *crossing time function* of the metric g.

It is easy to see that the set of regular metrics on a given manifold with boundary M, that will be denoted by $\operatorname{Reg}(M)$, is open in the C^1-topology. We also define *non-focal* a metric on M for which there are no ∂M-focal points along ∂M. Also in this case, it is not hard to show that non-focal metrics form an open subset in the C^2-topology, see Proposition 3.2.

Let us denote by $\operatorname{Reg}_*(M)$ *the set of non-focal regular metrics on* M. Our main interest is in the case when M is diffeomorphic to the unit ball B^{m+1} in the Euclidean space $\mathbb{R}^{m+1}$. In this case, the set of regular and non-focal metrics is an open subset of all Riemannian metrics containing, for instance, the set of radially symmetric metrics, see Corollary 3.3.

As an application of Theorem A, we prove the following:

Theorem B. *For a* generic[1] *set of metrics* $\mathfrak{g}$ *in* $\mathrm{Reg}_*(B^{m+1})$*, there are at least* $(m+1)$ *distinct orthogonal geodesic chords in* $(B^{m+1}, \mathfrak{g})$.

More precisely, the result of Theorem B holds for all metrics in $\mathrm{Reg}_*(B^{m+1})$ for which every OGC is nondegenerate in an appropriate sense, see the discussion after Corollary 2.2 and Section 3.2 for details.

When M is convex and homeomorphic to the $m+1$-dimensional disk the multiplicity problem for OGC's is studied in [3, 12]. If M is concave there is an existence result and a multiplicity result of two OGC's (see [7] and the references therein). To the authors' knowledge, Theorem B is the first result about multiplicity of OGC's without convexity or concavity assumption.

For the Morse-theoretical aspects in the proof of Theorem B, one of the key ingredients will be an index theorem for orthogonal geodesic chords (see Corollary 2.4 and Corollary 2.5), in the formulation given in [15]. This result, together with a stability result for focal points proved in [13], is used to prove that the crossing time function $s_\mathfrak{g}$ is even-Morse, providing the desired link between even-Morse functions and orthogonal geodesic chords.

1.3. **Brake orbits of Lagrangian systems.** The result of Theorem B can be applied to prove a new multiplicity result for brake orbits, as illustrated below. We will present here a Lagrangian formulation of the brake orbits problem. An equivalent formulation can be given for periodic solutions of Hamiltonian systems, via Legendre transform.

Let $(\mathfrak{M}^{m+1}, \mathfrak{g})$ be a Riemannian manifold (without boundary), representing the configuration space of some dynamical systems, and let $V : \mathfrak{M} \to \mathbb{R}$ be a smooth function, representing the potential energy of some conservative force acting on the system. One looks for periodic solutions $x : [0, T] \to \mathfrak{M}$ of the Lagrangian systems:

$$\text{(1.2)} \qquad \frac{D}{dt}\dot{x} = -\nabla V,$$

where $\frac{D}{dt}$ denotes the covariant derivative of the Levi–Civita connection of $\mathfrak{g}$ for vector fields along x, and ∇V is the gradient of V. Solutions of (1.2) satisfy the conservation of energy law $\frac{1}{2}\mathfrak{g}(\dot{x}, \dot{x}) + V(x) = E$, where E is a real constant called the *energy* of the solution x. It is a classical problem to give estimate of periodic solutions of (1.2) having a fixed value of the energy E. This problem has been, and still is, the main topic of a large amount of literature, also for autonomous Hamiltonian systems, see for instance [8, 9, 11, 16] an the references therein. We will give here a very short account of a geometric approach to periodic solutions of (1.2)

By the classical Maupertuis principle, solutions of (1.2) having energy E are, up to a parameterization, geodesics in the conformal metric:

$$\text{(1.3)} \qquad g_E = \big(E - V(p)\big) \cdot \mathfrak{g},$$

defined in the closed E-sublevel $M_E = V^{-1}\big(\,]-\infty, E]\,\big)$ of V. Observe that, in fact, g_E degenerates on the boundary $\partial M_E = V^{-1}(E)$. Among all periodic solutions of (1.2), historical importance is given to a special class called *brake orbits*; these are "pendulum-like" solutions, that oscillate with constant frequency along a trajectory that joins two endpoints lying in $V^{-1}(E)$. Thus, brake orbits correspond to g_E-geodesics in M_E with endpoints in ∂M_E, or, more precisely, to g_E-geodesics $\gamma : \,]0, T[\, \to V^{-1}\big(\,]-\infty, E[\,\big)$, with $\lim\limits_{t \to 0^-} \gamma(t)$ and $\lim\limits_{t \to T^-} \gamma(t)$ in ∂M_E.

For such degenerate situation, it has been proved in [5] that, if E is a regular value of the function V (which implies in particular that ∂M_E is a smooth hypersurface of $\mathfrak{M}$), then

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[1]Here generic is in meant in the topological sense. A subset of a topological set is generic if it contains a residual set (countable intersection of open dense subsets).

g_E defines a distance-to-the-boundary function $\text{dist}_E : M_E \to [0, +\infty[$ which is smooth in the interior of M_E and extends continuously to 0 on the boundary ∂M_E. Moreover, if $\delta > 0$ is small enough, then any OGC in the Riemannian manifold $M = \text{dist}_E^{-1}\big([\delta, +\infty[\big)$ endowed with the metric g_E (which is now non-singular) can be extended uniquely to a g_E-geodesic γ in M_E with endpoints in ∂M_E, as above. In conclusion, any result on multiplicity of OGC's can be reformulated to a multiplicity result for brake orbits at level a fixed regular energy level of a conservative dynamical system.

A very famous conjecture due to Seifert, see [17], asserts that, given a Lagrangian system as in (1.2), if the sublevel $V^{-1}\big(]-\infty, E]]\big)$ is homeomorphic to an $(m + 1)$-ball B^{m+1}, then there are at least $m + 1$ distinct brake orbits. This estimate is known to be sharp, i.e., there are examples of Lagrangian systems having energy levels homeomorphic to an $(m + 1)$-ball and admitting exactly $m + 1$ distinct brake orbits. A proof of Seifert's conjecture in its full generality is still open, but the question has been solved affirmatively in some cases. When V is even and convex, multiplicity results are obtained in [9, 10, 18, 19, 20]. In particular in [9] there is the proof of the Seifert conjecture for euclidean metrics and even and convex potentials.

When the E-sublevel $V^{-1}\big(]-\infty, E]$ has the topology of the annulus, the multiplicity of brake orbits is studied in depth in [4] and [6].

Theorem B yields the following contribution to Seifert's conjecture:

Theorem C. *Let* E *be a regular value of* V, *such that* $V^{-1}\big(]-\infty, E]\big)$ *is homemorphic to an* $(m + 1)$*-ball* B^{m+1}. *Assume that there exists* $\delta > 0$ *sufficiently small such that the Riemannian manifold* $M = \text{dist}_E^{-1}\big([\delta, +\infty[\big)$ *endowed with the metric* g_E *satisfies the assumptions of Theorem B. Then there are at least* $m + 1$ *distinct brake orbits of energy* E *for the Lagrangian system* (1.2). $\square$

In particular, from Proposition 3.2 below we obtain the following:

Corollary D. *Seifert conjecture is generically true in a* C^2*-open set of potentials* V *that contains the ones that are* rotationally symmetric *at level* E. $\square$

By a potential V rotationally symmetric at level E we mean that there is a continuous action of the rotation group $SO(m + 1)$ on $V^{-1}\big(]-\infty, E]\big)$ that makes the sublevel equivariantly diffeomorphic to the Euclidean ball B^{m+1} with the canonical $SO(m + 1)$-action, and such that V is constant along the orbits of this action.

2. VARIATIONAL THEORY FOR OGC'S

Let us show how obtain multiple orthogonal geodesic chords in a compact Riemannian manifold with boundary (M, g), using a function on ∂M whose critical points are OGC's. Next Proposition shows that, for a regular metric g, such function is precisely the crossing time function s_g. In order to prove this, let us introduce some notations.

For $p \in \partial M$, let $S_p : T_p(\partial M) \to T_p(\partial M)$ denote the shape operator of ∂M at p in the normal direction $\vec{v}_p$. Let $\exp^\perp : U \subset T(\partial M)^\perp \to M$ denote the normal exponential map of g along ∂M; for $p \in \partial M$, let $\gamma_p : [0, s_g(p)] \to M$ denote the geodesic $t \mapsto \exp_p(t \cdot \vec{v}_p)$. Recall that a point $q \in M$ is a singular value of $\exp^\perp$ exactly when q is *focal* to ∂M. If $q \in M$ is a singular value of $\exp^\perp$ and $v \in T_p(\partial M)^\perp$ is the corresponding critical point, so that $\gamma_p(t_*) = q$ for some $t_* \in]0, s_g(p)]$, then the kernel of $d \exp^\perp(v)$ consists of ∂M-*Jacobi fields* along the geodesic γ_p that vanish at t_*. Recall that a Jacobi field along γ_p is called a ∂M-Jacobi field if it satisfies the initial conditions:

$$(2.1) \qquad J(0) \in T_p(\partial M), \quad J'(0) + S_p\big(J(0)\big) \in T_p(\partial M)^\perp,$$

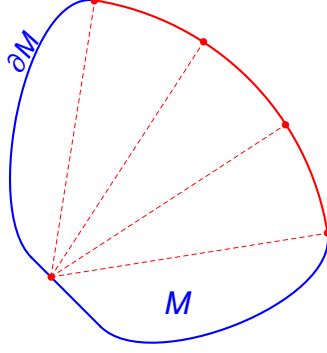where J' denotes the covariant derivative of J along γ_p.

FIGURE 1. The picture represents a situation in which critical points of the function s_g do not correspond to orthogonal geodesic chords (see Remark 2). Here, M is a compact subset of a Euclidean space. A portion of the boundary of M (in red) is a spherical hypersurface, whose center, which is a ∂M-focal point, lies on ∂M. Orthogonal geodesic chords starting from this spherical surfaces converge and meet at the center; thus the function s_g is constant in this spherical region of the boundary. However, these critical points do not correspond to OGC's in M.

For $p \in \partial M$, denote by $q_p = \gamma_p\big(s_g(p)\big) \in \partial M$, and let $\mathcal{E}_p \subset T_{q_p} M$ the image of the linear map $d \exp^\perp \big(s_g(p)\vec{v}_p\big)$. Equivalently:

$$\mathcal{E}_p = \Big\{ J\big(s_g(p)\big) : J \text{ is a } \partial M\text{-Jacobi field along } \gamma_p \Big\}.$$

When q_p is not ∂M-focal, then $\mathcal{E}_p = T_{q_p} M$.

Proposition 2.1. *Assume g regular. A point $p \in \partial M$ is critical for $s_g : \partial M \to \,]0, +\infty[$ if and only if the vector $\dot{\gamma}_p\big(s_g(p)\big) \in T_{q_p} M$ is orthogonal to the intersection[2] $\mathcal{E}_p \cap T_{q_p}(\partial M)$. In particular, if q_p is not ∂M-focal, then γ_p is an orthogonal geodesic chord in M.*

Proof. Since $s_g > 0$, the critical points of s_g coincide with those of s_g^2. For $p \in \partial M$, denote by $\sigma_p : [0,1] \to M$ the affinely reparameterized geodesic $\sigma_p(t) = \gamma_p\big(s_g(p) \cdot t\big)$, $t \in [0,1]$. Integration on $[0,1]$ gives:

$$(2.2) \qquad \int_0^1 g\,(\dot{\sigma}_p, \dot{\sigma}_p)\,dt = s_g(p)^2,$$

i.e., σ_g^2 can be thought of the geodesic action functional of g applied to the geodesic σ_p. Let $p \in \partial M$ be fixed, and let $\rho : \,]-\varepsilon, \varepsilon[\,\to \partial M$ be a C^1-curve satisfying $\rho(0) = p$ and $\dot{\rho}(0) = w \in T_p(\partial M)$. Then, p is critical for s_g^2 iff for any such ρ one has $\frac{d}{dt}\big|_{t=0} s_g^2\,(\rho(t)) = 0$. Using (2.2) one has:

$$(2.3) \qquad \tfrac{d}{dt}\big|_{t=0} s_g^2\,(\rho(t)) = 2 \int_0^1 g\,(\dot{\sigma}_p, J_w')\,dt,$$

where J_w is the Jacobi field along σ_p given by:

$$J_w(s) = \tfrac{d}{dt}\big|_{t=0} \sigma_{\rho(t)}(s), \quad s \in [0,1].$$

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[2] We observe that, under the assumptions of Proposition 2.1, q_p is ∂M-focal if and only if $\mathcal{E}_p \cap T_{q_p} \partial M \neq T_{q_p} \partial M$. This follows easily from the fact that $\mathcal{E}_p$ always contains $\dot{\gamma}_p\big((s_g(p))\big)$ which is transversal to $T_{q_p} \partial M$.

Keeping in mind that $J_w(0) = w \in T_p(\partial M)$ and that $\dot{\sigma}_p(0) \in T_p(\partial M)^\perp$, integration by parts in (2.3) gives:
$$\frac{d}{dt}\Big|_{t=0} s_g^2(\rho(t)) = 2g(\dot{\sigma}_p(1), J_w(1)).$$
It is easily seen that the map $T_p(\partial M) \ni w \mapsto J_w(1) \in \mathcal{E}_p \cap T_{q_p}(\partial M)$ is surjective, and from this observation the thesis follows readily. $\qquad\square$

Remark 2. It is not hard to give examples of regular metrics that do not satisfy the non-focal property, and in which critical points of s_g do not correspond to orthogonal geodesic chords. See Figure 1.

From Proposition 2.1, we obtain immediately the following:

Corollary 2.2. *Let* g *be regular and non-focal. Then, a point* p *is critical for* s_g *if and only if the point* $q_p = \gamma_p(s_g(p))$ *is critical for* s_g. *In this case,* $s_g(p) = s_g(q_p)$, *and* $\gamma_{q_p} = \gamma_p$ *up to orientation.* $\qquad\square$

Proposition 2.1 gives a first order variational principle relating orthogonal geodesic chords in M to critical points of a smooth function on ∂M. For our purposes, we need a related second order variational principle, relating nondegeneracy and Morse index of OGC's in M and critical points of s_g. Let us recall a few facts from the variational theory of OGC's.

Assume that $\gamma_p : [0, s_g(p)] \to M$ is an OGC in M, i.e., that $\dot{\gamma}_p(s_g(p)) \in T_{q_p}(\partial M)^\perp$. The *index form* along γ_p is the symmetric bilinear form I_p defined on the vector space $\mathcal{V}_p$ of (piecewise smooth) vector fields V along γ_p satisfying $V(0) \in T_p(\partial M)$ and $V(s_g(p)) \in T_{q_p}(\partial M)$, defined[3] by:

$$(2.4) \quad I_p(V, W) = \int_0^{s_g(p)} g(V', W') + g(R(\gamma_p', V)\gamma_p', W)\, ds$$
$$- \Big[g(\mathcal{S}_p(V(0)), W(0)) + g(\mathcal{S}_{q_p}(V(s_g(p))), W(s_g(p))) \Big],$$

where R is the curvature tensor of g, chosen with the sign convention
$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

It is well known that I_p is the second variation of the geodesic action functional, defined in the set of paths with endpoints in ∂M, at the critical point γ_p. The OGC γ_p is said to be *nondegenerate* if I_p is a nondegenerate bilinear form on $\mathcal{V}_p$, i.e., if γ_p is a nondegenerate critical point of the geodesic action functional of M in the space of paths with endpoints in ∂M. The kernel of I_p is the space of ∂M-Jacobi fields J along γ_p that satisfy, in addition to (2.1), the following boundary condition at $t = s_g(p)$:

$$(2.5) \quad J(\sigma_g(p)) \in T_{q_p}(\partial M), \quad J'(s_g(p)) + \mathcal{S}_{q_p}(J(s_g(p))) \in T_{q_p}(\partial M)^\perp.$$

Thus, γ_p is nondegenerate if and only if there exists no non-trivial Jacobi field J along γ satisfying (2.1) and (2.5). The *Morse index* of γ_p is the index of the symmetric bilinear form I_p, which is the dimension of a maximal subspace of $\mathcal{V}_p$ on which I_p is negative definite. Recall that this is a (finite) nonnegative integer, that can be computed in terms of some focal invariants of ∂M, see [15] for details.

Proposition 2.3. *Assume* g *regular. Let* $p \in \partial M$ *be a critical point of* s_g, *and assume that* $q_p \in \partial M$ *is not* ∂M-*focal along* γ_p. *Then:*

(a) p *is a nondegenerate critical point of* s_g *if and only if* γ_p *is a nondegenerate OGC;*

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[3]Observe that we have a different sign convention from the standard literature in the second boundary term of I_p, because $\mathcal{S}_{q_p}$ has been defined as the shape operator in the normal direction $\vec{v}_{q_p}$, and $\vec{v}_{q_p} = -\dot{\gamma}_p(s_g(p))$. Similarly, the second condition in (2.5) has a sign different from the standard literature.

(b) *the Hessian of $\frac{1}{2}s_g^2$ at the critical point p is identified with the symmetric bilinear form:*

$$(2.6) \qquad \mathbb{J}_p \times \mathbb{J}_p \ni (J_1, J_2) \longmapsto g\Big(J_1'(s_g(p)) - \mathcal{S}_{q_p}\big(J_1(s_g(p))\big), J_2(s_g(p))\Big) \in \mathbb{R},$$

where $\mathbb{J}_p$ is the vector space of all ∂M-Jacobi fields J along γ_p that satisfy

$$J\big(s_g(p)\big) \in T_{q_p}(\partial M).$$

Its index is less than or equal to the Morse index of the OGC γ_p.

Proof. As in Proposition 2.1, both statements are obtained by identifying the points p of ∂M with the curve γ_p, as an element of the space of curves with endpoints in ∂M. Using this identification, the tangent space $T_p(\partial M)$ is identified with the space of variations of γ_p by geodesics γ_q that start orthogonally to ∂M and arrive onto ∂M. This space of variations is given by the vector space $\mathbb{J}_p$. The function $\frac{1}{2}s_g^2$ is the restriction to the set $\{\gamma_p : p \in \partial M\}$ of the geodesic action functional of M. We have proved that, under our assumptions, γ_p is a critical point of this restriction, but also a critical point of the *full* geodesic action functional. Hence, the second derivative of $\frac{1}{2}s_g^2$ at p is given by the restriction of the index form I_p to the space $\mathbb{J}_p$; this implies, in particular, that the Morse index of p is less than or equal to the Morse index of the OGC γ_p. Formula (2.6) is obtained easily using partial integration in (2.4), the Jacobi equation and (2.1). This proves (b).

The assumption that q_p is not ∂M-focal along γ_p implies that, as J_2 runs in $\mathbb{J}_p$, the vector $J_2(b)$ is an arbitrary vector in $T_{q_p}(\partial M)$. Hence, $I_p(J_1, J_2) = 0$ for all J_2 if and only if J_2 satisfies (2.5), i.e., the kernel of the second derivative of s_g^2 at the critical point $p \in \partial M$ coincides with the kernel of the index form I_p. This proves statement (a). $\qquad \square$

Using the Morse index theorem for geodesics between two fixed submanifold, see for instance [15], one proves the following more precise result on the Morse index of critical points of s_g:

Corollary 2.4. *Under the assumption of Proposition 2.3, the Morse index of $\frac{1}{2}s_g^2$ at p is equal to the Morse index of the OGC γ_p minus the number of ∂M-focal points along γ_p, counted with multiplicity.*

Proof. It follows readily from Proposition 2.3 and the result of [15, Theorem 2.7]. More precisely, [15, Theorem 2.7] proves that, when q_p is not ∂M-focal along γ_p, then the space $\mathcal{V}_p$ (recall that this is the domain of the index form I_p) is the direct sum of:

- the space $\mathcal{V}_p^1$ of (piecewise smooth) vector fields V along γ_p satisfying $V(0) \in T_p(\partial M)$,
- the space $\mathbb{J}_p$ of (∂M)-Jacobi fields J along γ_p satisfying $J\big(s_g(p)\big) \in T_{q_p}(\partial M)$.

Such a direct sum decomposition is I_p-orthogonal. Hence, the index of I_p on $\mathcal{V}_p$, which is the Morse index of the OGC γ_p, is equal to the sum of the indices of the restriction of I_p to each one of the two spaces above. The index of the restriction of I_p to $\mathbb{J}_p$ is precisely the index of the bilinear form (2.6), i.e., the Morse index of $\frac{1}{2}s_g^2$ at p. The index of the restriction of I_p to the space $\mathcal{V}_p^1$, by the Morse index theorem for geodesics between a submanifold and a fixed point (see [15, Theorem 2.5]), is given by the number of ∂M-focal points along γ_p counted with multiplicity. $\qquad \square$

Corollary 2.5. *Let M be a compact manifold with connected boundary ∂M, and let g be a regular and non-focal Riemannian metric on M. If every OCG in M is nondegenerate, then s_g is an even-Morse function on ∂M.*

Proof. From Proposition 2.3, part (a), it follows that if every OCG is nondegenerate, then also every critical point of s_g is nondegenerate, i.e., s_g is a Morse function. Let us show
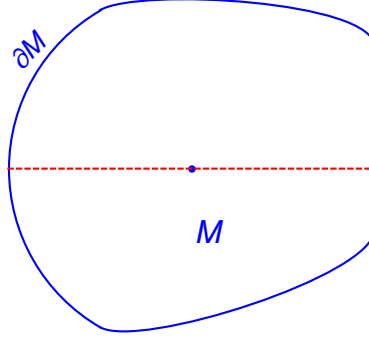
FIGURE 2. The picture illustrates an example of a non-degenerate OGC which has one ∂M-focal point, but its backward reparameterization has no ∂M-focal point. Here M is a compact subset of a Euclidean space. The boundary of M has a portion which is a spherical surface. An orthogonal geodesic chord departs from this surface, and it passes through the center of the sphere, which is a focal point. At the final endpoint, the geodesic meets orthogonally a portion of ∂M which is planar (i.e., flat). Thus, there are no focal points along the backward reparameterization of the geodesic. Note that this phenomenon only occurs when the metric fails to be non-focal, so that the picture shows also that the non-focal property is *not* generic. Namely, any sufficiently small perturbation of the metric preserves the existence of a non-degenerate OGC, which has different Morse indexes when run in the two directions, by the stability of the Morse index. Therefore, any perturbed metric fails to be non-focal.

that, given a critical point p of s_g, then q_p is a critical point with the same Morse index. From Corollary 2.4, the difference between the Morse indices of p and q_p equals the difference of the number of ∂M-focal points along γ_p and along γ_{q_p} (which incidentally is the same geodesic[4] as γ_p, with the opposite orientation, see Corollary 2.2). We claim that the number of ∂M-focal points along any geodesic $\gamma_r : [0, s_g(r)] \to M$ is constant for al $r \in \partial M$. This follows easily from the assumption that no γ_r has any ∂M-focal point in M, and an argument of stability of the number of focal points in Riemannian geometry, see [13] for details. Namely, as r runs in ∂M, the number of focal points (counted with multiplicity) along γ_r changes continuously, and by connectedness, if such number is not constant in ∂M, then there would be a ∂M-focal point in ∂M along some γ_r. This concludes the proof.   □

Remark 3. It is not hard to show that in all the results of this section, the assumptions cannot be relaxed. As we observed in Remark 2, Figure 1 shows an example of a metric which is not non-focal, and where the critical points of s_g do not correspond to OGC's (cf. Proposition 2.1 and Corollary 2.2). Figure 2 illustrates an example where the function s_g is a Morse function, but not even-Morse (cf. Corollary 2.5). Figure 3 gives an example of a regular and non-focal metric on a manifold of arbitrary dimension, whose boundary is not connected, and admitting only 2 OGC's.

Proposition 2.6. *Under the assumptions of Corollary 2.5, there are at least $(m+1)$ distinct orthogonal geodesic chords in (B^{m+1}, g).*

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[4]Observe that the fact that γ_{q_p} is the same as γ_p with the opposite orientation does not imply in principle that the number of ∂M-focal points along the two geodesics is the same, see Figure 2.
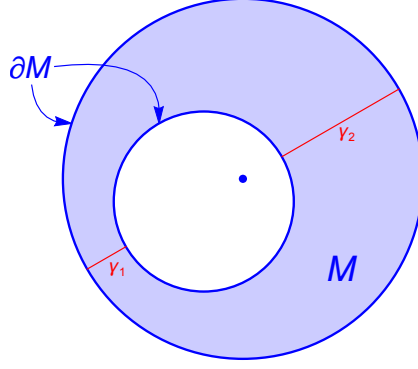
FIGURE 3. When the boundary of M is not connected, one cannot expect the existence of more than 2 orthogonal geodesic chords, regardless of the dimension of M. Consider for instance an annular region of $\mathbb{R}^{m+1}$ bounded by two non-concentrical spheres. In this situation, the metric is regular and, if the inner sphere contains the center of the outer sphere in its interior, then the metric is also non-focal. Thus, s_g is an even-Morse function on a non-connected manifold. However, there are only two orthogonal geodesic chords, and thus s_g admits only two pairs of critical points with the same Morse index. Observe that in Remark 1 on page 2 only connected manifolds were considered.

Proof. It follows immediately from Theorem A and Corollary 2.5. □

In next section we will discuss the assumptions of the results proved above, and we will prove the genericity of the nondegeneracy condition for OGC's.

3. PROOF OF THEOREM B. A DISCUSSION OF THE ASSUMPTIONS.

3.1. **On the genericity of metrics without degenerate OGC's.** Recall that a metric on a compact Riemannian manifold is called *bumpy* if every one of its closed geodesic (possibly iterated) is nondegenerate, i.e., it does not admit non-trivial periodic Jacobi fields. A classical result due to Anosov [1] asserts that bumpy metric are generic. Genericity of metrics all of whose geodesics satisfying more general boundary condition are nondegenerate is proved in [2].

Let us introduce the following terminology. Given a compact manifold M with boundary, a metric $g \in \text{Reg}_*(M)$ will be called *∂-bumpy* if every OGC in M is nondegenerate.

The following is what we need for the conclusion of Theorem B:

Proposition 3.1. *Let M be a compact manifold with boundary. The set:*

$$(3.1) \qquad \widetilde{\text{Reg}}(M) := \left\{ g \in \text{Reg}_*(M) : g \text{ is } \partial\text{-bumpy} \right\}$$

is C^k-generic in $\text{Reg}_*(M)$ for all $k = 2, \ldots, +\infty$.

Proof. It suffices to show that $\widetilde{\text{Reg}}(M)$ is C^k-generic in the set of all metrics on M. This is an application of the result in [2]. In order to obtain the desired conclusion, one must observe that the proof of [2, Theorem 5.10] yields the genericity of metrics having only nondegenerate geodesics satisfying arbitrary boundary conditions, provided that one can exclude *a priori* the existence of *strongly degenerate*[5] periodic geodesics in this class. This

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[5]The notion of strong degeneracy for a periodic geodesic has been introduced in [2], and it is stated in terms of the existence of a non-trivial periodic Jacobi field along an iterated periodic geodesic, satisfying some additional properties. A detailed discussion of this notion is irrelevant in the context of the present paper.

is always the case for orthogonal geodesic chords, because this class does not contain any periodic geodesic at all. $\square$

3.2. **∂M-focal points and non-focal metrics.** Unlike bumpy metrics, the set of non-focal Riemannian metrics on a compact manifold with boundary ∂M is not generic in general.

Example 1. For instance, one can construct examples of hypersurfaces Σ in $\mathbb{R}^n$, diffeomorphic to an $(n-1)$-disk, whose set of focal points contains another hypersurface $\mathcal{F}$. Now, consider the closure $\overline{\Omega}$ of a bounded open subset whose smooth boundary $\partial \Omega$ contains Σ, and such that $\partial \Omega$ intercepts $\mathcal{F}$ transversally. In this situation, transversality implies that any C^1-perturbation of the flat metric on $\overline{\Omega}$ will produce $\partial \Omega$-focal points along $\partial \Omega$. The situation is illustrated concretely in Figure 2. The picture provides an example of a metric which fails to be non-focal, and the same holds also for small perturbations of the metric.

However, the absence of ∂M-focal points on ∂M is still a quite general assumptions, that holds in a large variety of circumstances. Intuitively, concavity of the boundary (which is a property satisfied by Jacobi metric) and non-positive curvature have an effect of keeping focal points away from ∂M. More precisely, we can prove that when the sectional curvature of g and the eigenvalues of the shape operator of ∂M are not too large compared to the *size* of M, then there are no ∂M-focal points in M. In order to give basis to this assertion, let us introduce the following constants.

Let g be a regular metric on M; the maximum of the function s_g on ∂M, denoted by $L(M, g)$ will be called the *maximal crossing time* of (M, g). Clearly, $L(M, g)$ is somewhat related to, however in general larger than, the diameter of (M, g). Moreover, denote by Λ the maximum, for $p \in \partial M$, of the largest eigenvalue of $\mathcal{S}_p$, and by K the maximum value of the sectional curvatures of all 2-planes tangent to M.

Proposition 3.2. *Let M be a compact manifold with smooth boundary.*

  (a) *The set of non-focal metrics on M is open with respect to the C^2-topology.*
  (b) *A sufficient condition for a metric g to be non-focal is that the following inequality holds:*

(3.2) $$K^+ \cdot L(M, g)^2 + \Lambda^+ \cdot L(M, g) - \tfrac{1}{4} < 0,$$

(where a^+ denotes the positive part of the real numner a.) Condition (3.2) *implies indeed that there is no focal point to ∂M in M.*

Proof. The set of focal points to a given compact hypersurface along a normal geodesic is stable by C^2-perturbations of all data. A formal proof of this fact can be found in reference [13]. From this stability, (a) follows easily.

For the proof of (b), let us show that, under the assumption (3.2), given any $p \in \partial M$, the index form I_p is positive definite on the space on the vector space $\mathcal{U}_p$ of (piecewise smooth) vector fields V along γ_p satisfying $V(0) \in T_p(\partial M)$, $V(s_g(p)) = 0$, and $g(V, \dot{\gamma}_p) \equiv 0$. Recall that $\gamma_p : [0, s_g(p)] \to M$ is a unit speed geodesic.

Let $V \in \mathcal{U}_p \setminus \{0\}$ be fixed; up to normalization, we can assume:

(3.3) $$\int_0^{s_g(p)} g(V, V) \, ds = 1.$$

Since $V(s_g(p)) = 0$, for all $t \in [0, s_g(p)]$ we have:

$$-g(V(t), V(t)) = 2 \int_t^{s_g(p)} g(V', V) \, ds,$$

which gives

(3.4) $$g(V(t), V(t)) \leqslant 2 \int_0^{s_g(p)} g(V', V')^{\frac{1}{2}} g(V, V)^{\frac{1}{2}} \, ds.$$

Integrating (3.4) on $[0, s_g(p)]$ and keeping in mind (3.3), Schwarz's inequality gives:

$$(3.5) \qquad 1 = \int_0^{s_g(p)} g(V, V) \, ds \leqslant 4 s_g(p)^2 \int_0^{s_g(p)} g(V', V') \, ds.$$

Setting $t = 0$ in (3.4) and using again Schwarz's inequality, we obtain:

$$(3.6) \qquad g\big(V(0), V(0)\big) \leqslant 4 s_g(p) \int_0^{s_g(p)} g(V', V') ds.$$

Let us now estimate $I_p(V, V)$, using formula (2.4); note that $g\big(\dot\gamma_p, V)\dot\gamma_p, V\big)$ is equal to $-K(\dot\gamma_p, V) g(V, V)$, where $K(\dot\gamma_p, V)$ is the sectional curvature of the 2-plane spanned by $\dot\gamma_p$ and V. Then, using the very definition of the constants Λ and K, we get:

$$(3.7) \quad I_p(V, V) \geqslant -\Lambda g\big(V(0), V(0)\big) + \int_0^{s_g(p)} \big[g(V', V') - K g(V, V)\big] \, ds$$

$$\overset{\text{by (3.6)}}{\geqslant} -4\Lambda^+ s_g(p) \int_0^{s_g(p)} g(V', V') \, ds + \int_0^{s_g(p)} \big[g(V', V') - K g(V, V)\big] \, ds$$

$$\overset{\text{by (3.5)}}{\geqslant} \Big[-4\Lambda^+ s_g(p) + 1 - 4K^+ s_g(p)^2 \Big] \cdot \int_0^{s_g(p)} g(V', V') \, ds.$$

Note that if $V \neq 0$, then $\int_0^{s_g(p)} g(V', V') \, ds > 0$, because $V\big(s_g(p)\big) = 0$, and therefore V cannot be parallel along γ. Thus, from (3.7) it follows that $I_p(V, V) > 0$ if:

$$(3.8) \qquad 1 - 4\Lambda^+ s_g(p) - 4K^+ s_g(p)^2 > 0.$$

Note that (3.2) coincides with (3.8) when $s_g(p) = L(M, g)$. On the other hand, if (3.2) is satisfied, it is easy to see that (3.8) holds for every $p \in \partial M$. $\qquad \square$

For instance, when M is a ball, from part (a) of Proposition 3.2 we deduce the following:

Corollary 3.3. *If M is diffeomorphic to a ball, then the set of regular and non-focal metrics on M is a C^2-open set that contains all the rotationally symmetric metrics.*

Proof. A rotationally symmetric metric in the ball is regular, because the geodesics starting orthogonally to the boundary are radial, and therefore they arrive transversally (orthogonally) to the boundary at the other endpoint. Moreover, for all these metrics the unique focal point to the boundary is the center of symmetry, which is far from the boundary. Hence, sufficiently small perturbations of rotationally symmetric metrics are non-focal. $\qquad \square$

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