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ON PSEUDO-HYPERKÄHLER PREPOTENTIALS

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Abstract. An explicit surjection from a set of (locally defined) unconstrained holomorphic functions on a certain submanifold of $\mathrm{Sp}_1(\mathbb{C})\times\mathbb{C}^{4n}$ onto the set $\mathrm{HK}_{p,q}$ of local isometry classes of real analytic pseudo-hyperkähler metrics of signature $(4p, 4q)$ in dimension $4n$ is constructed. The holomorphic functions, called prepotentials, are analogues of Kähler potentials for Kähler metrics and provide a complete parameterisation of $\mathrm{HK}_{p,q}$. In particular, there exists a bijection between $\mathrm{HK}_{p,q}$ and the set of equivalence classes of prepotentials. This affords the explicit construction of pseudo-hyperkähler metrics from specified prepotentials. The construction generalises one due to Galperin, Ivanov, Ogievetsky and Sokatchev. Their work is given a coordinate-free formulation and complete, self-contained proofs are provided. An appendix provides a vital tool for this construction: a reformulation of real analytic G-structures in terms of holomorphic frame fields on complex manifolds.

1. Introduction

This paper is about a parametrisation of local isometry classes of real analytic pseudo-hyperkähler metrics on $4n$-dimensional manifolds. This parametrisation is surjective onto the space of local isometry classes and it allows the explicit construction of metrics. The parameter space consists of unconstrained holomorphic functions on a certain submanifold of $\mathrm{Sp}_1(\mathbb{C})\times\mathbb{C}^{4n}$.

A pseudo-Riemannian manifold (M, g) is determined by the holonomy subbundle $P \subset O_g(M)$ of its orthonormal frame bundle $\pi\colon O_g(M) \to M$. In

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turn, P is determined, up to local equivalence, by its fundamental vector fields (E_A, e_a), the infinitesimal transformations E_A of its structure group and the horizontal vector fields e_a given by the Levi-Civita connection form on P. Two pseudo-Riemannian manifolds (M, g) and (M', g') are *locally isometric* if and only if their respective vector fields (E_A, e_a) and (E'_A, e'_a) are related by a local diffeomorphism.

In the case of a pseudo-hyperkähler manifold (M, g), the associated holonomy bundle $P \subset O_g(M)$ is locally identifiable with the trivial bundle $\pi\colon P|_{\mathcal{V}} \simeq \mathrm{Sp}_{p,q}\times\mathcal{V} \to \mathcal{V}$, for some open subset $\mathcal{V} \subset \mathbb{R}^{4n}$. We shall regard the holonomy bundle as a subbundle of a larger bundle of orthonormal frames with structure group $\mathrm{Sp}_1{\cdot}\mathrm{Sp}_{p,q}$. This larger bundle has a double covering identifiable with $\mathrm{Sp}_1\times\mathrm{Sp}_{p,q}\times\mathcal{V}$, a real submanifold of the complex Lie group $\mathcal{P} = (\mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(\mathbb{C})) \ltimes \mathbb{C}^{4n}$.

Using the above local identifications, the vector fields (E_A, e_a) associated with (M, g) can be identified with corresponding vector fields on the larger space $\mathrm{Sp}_1\times\mathrm{Sp}_{p,q}\times\mathcal{V} \subset \mathcal{P}$. If g is real analytic, these vector fields admit holomorphic extension to an open subset $\mathcal{U} \subset \mathcal{P}$. Including the basis vector fields (H_0, H_{++}, H_{--}) of $\mathfrak{sp}_1(\mathbb{C}) \subset \mathrm{Lie}(\mathcal{P}) = \mathfrak{sp}_1(\mathbb{C}) + \mathfrak{sp}_n(\mathbb{C}) + \mathbb{C}^{4n}$, we obtain a set $\mathcal{A} = (H_0, H_{\pm\pm}, E_A, e_a)$ of holomorphic vector fields on $\mathcal{U}$, which is naturally associated with the pseudo-hyperkähler metric $g|_{\mathcal{V}}$. This mapping from real analytic pseudo-hyperkähler metrics to sets of holomorphic vector fields admits an explicit inversion. Introducing the notion of an *hk-pair* $(\mathcal{A}, M)$, consisting of a set $\mathcal{A} = (H_0, H_{\pm\pm}, E_A, e_a)$ of holomorphic vector fields on an open subset $\mathcal{U} \subset \mathcal{P}$, satisfying certain Lie bracket relations, and a real submanifold $M \subset \mathcal{P}$, satisfying appropriate transversality conditions with respect to the fields $H_0, H_{\pm\pm}$ and E_A, we shall show that every hk-pair $(\mathcal{A}, M)$ determines a pseudo-hyperkähler metric g on the manifold M. Further, the real submanifold $M \cdot \mathrm{Sp}_{p,q} \subset \mathcal{U}$ is identifiable with the (trivial) holonomy bundle $\pi\colon P = M \times \mathrm{Sp}_{p,q} \to M$ of (M, g).

The correspondence between hk-pairs and pseudo-hyperkähler metrics is crucial in order to obtain a complete parametrisation of the local isometry classes of real analytic pseudo-hyperkähler metrics. We shall prove that:

A) There exists a bijection between the local isometry classes of real analytic pseudo-hyperkähler metrics and local equivalence classes of hk-pairs.

B) Every local equivalence class of hk-pairs contains a distinguished subclass of *canonical hk-pairs*, each completely determined by a single unconstrained holomorphic function $\mathcal{L}$, the *prepotential*, defined on a complex submanifold of $\mathcal{P} = (\mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(\mathbb{C})) \ltimes \mathbb{C}^{4n}$.

Combining A and B:

For every real analytic pseudo-hyperkähler manifold (M, g)*, the restriction of the metric* $g|_{\mathcal{V}}$ *to a sufficiently small open subset* $\mathcal{V} \subset M$ *can be associated with an unconstrained prepotential* $\mathcal{L}$*. Conversely, an arbitrary holomorphic function* $\mathcal{L}$ *on a certain complex submanifold of* $\mathcal{P}$ *determines a pseudo-hyperkähler metric* g*, unique up to isometry. We call* $\mathcal{L}$ *a prepotential of the pseudo-hyperkähler metric* g*.*

One of the striking developments in theoretical physics, which animated much mathematical interest in supersymmetry, was the appearance of special geometries in various surprising contexts. For instance, requiring the harmonic map equations on a four-dimensional Lorentzian manifold to be supersymmetric automatically provided the target manifold with a Kähler structure [15]. Soon, it was found [2] that extended versions of supersymmetry yielded hyperkähler targets. The search for a supersymmetric action functional gave rise to the harmonic space method [6], which yielded a construction of a supersymmetric Lagrangian $\mathcal{L}$ in an extended space called *harmonic superspace*. The construction established a correspondence between the functions $\mathcal{L}$ and hyperkähler metrics. In an interesting collateral development [8, 9, 10], these authors extracted the latter correspondence from the original context of supersymmetric field theories, presenting a construction of hyperkähler metrics, parametrised by a prepotential $\mathcal{L}$, much as the Kähler potential parametrises Kähler metrics. A streamlined presentation of the correspondence was given in [3]. This was amenable to a generalisation to supersymmetric hyperkähler spaces [4, 5]. Further, a discussion of the prepotential in the framework of quaternionic Kähler metrics has been given in [7].

The correspondence between (pseudo-)hyperkähler metrics and free prepotentials provides an efficient parameterisation of *all* local isometry classes of real analytic pseudo-hyperkähler metrics and is important from both field theoretical and differential geometric points of view. The purpose of this paper is to give an appropriate mathematical presentation, in coordinate-free language, with complete and self-contained proofs; thus opening the way to further developments and applications. An analogous correspondence between Yang-Mills connections on (generalised) hyperkähler manifolds and free prepotentials has been discussed in [1].

Structure of the paper. Section 2 contains our notation and certain basic facts. In Sect. 3 and 4 we discuss hk-pairs, show how the associated pseudo-hyperkähler metrics may be determined and we state the two main theorems, which establish the surjective correspondence between prepotentials and equivalence classes of hk-pairs. In Sect. 5 we obtain some technical results on differential equations on harmonic spaces required for the proofs of our main theorems in Sect. 6 and 7. A summary of our construction, a five-step recipe to obtain a pseudo-hyperkähler metric from a prepotential, is given in Sect. 8. The appendix discusses real and complex G-structures, reformulating them in terms of holomorphic frame fields on complex manifolds. The latter provide a useful tool for the investigation of local properties of manifolds with real analytic G-structures. In Sect. A3 we discuss the particular case of G-structures corresponding to real analytic pseudo-hyperkähler manifolds. Finally, in Sect. A4, we prove the bijection between local isometry classes of real analytic pseudo-hyperkähler metrics and local equivalence classes of hk-pairs.

The discussion in the appendix has been kept general enough with a view to being directly applicable to other geometries. In particular, we intend to use it to obtain a new parametrisation of the local isometry classes of quaternionic Kähler metrics.

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2. Basic notions

2.1. A basis for $\mathfrak{p} = (\mathfrak{sp}_1(\mathbb{C}) + \mathfrak{sp}_n(\mathbb{C})) + \mathbb{C}^{4n}$

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sp}_1(\mathbb{C}) + \mathfrak{sp}_n(\mathbb{C})$. Since the vector space $V = \mathbb{C}^2 \otimes \mathbb{C}^{2n} \simeq \mathbb{C}^{4n}$ is a $\mathfrak{g}$-module, we may extend $\mathfrak{g}$ to the Lie algebra $\mathfrak{p} = \mathfrak{g} + V$ with additional brackets,

$$[v, v'] = 0\,, \quad [A, v] = A \cdot v\,, \qquad v, v' \in V, \quad A \in \mathfrak{g}\,,$$

where $A\cdot\colon V \to V,\ v \mapsto A\cdot v$, denotes the standard action of A on V. As our standard basis for $\mathfrak{sp}_1(\mathbb{C})$, we use the triple

$$H^o_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H^o_{++} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H^o_{--} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The standard bases of $\mathbb{C}^2$ and $\mathbb{C}^{2n}$ are denoted respectively by (h^o_+, h^o_-) and $(e^o_a), a = 1, \dots, 2n$. In terms of these, the basis elements of $V = \mathbb{C}^{4n}$ are given by $e^o_{\pm a} := h^o_\pm \otimes e^o_a$. In this basis V manifestly decomposes as $V = V_+ + V_-$, where $V_\pm := \operatorname{span}_{\mathbb{C}}\{e^o_{\pm 1}, \dots, e^o_{\pm 2n}\}$ are eigenspaces of H^o_0, since $h^o_\pm$ are its eigenvectors with eigenvalues ± 1. As a basis of $\mathfrak{sp}_n(\mathbb{C})$, we consider the N-tuple of $2n\times 2n$ matrices, $(E^o_1,\ldots,E^o_N)$, $N=(2n+1)n$, corresponding, by the classical identification $\mathfrak{sp}_n(\mathbb{C})\simeq \vee^2\mathbb{C}^{2n}$, to the tensors $E^o_{ab}:=e^o_a\vee e^o_b$. The nonzero Lie brackets of the basis elements of $\mathfrak{p}$ are given by

$$\begin{aligned} [H^o_{++},H^o_{--}] &= H^o_0\,, \quad [H^o_0,H^o_{\pm\pm}]=\pm 2H^o_{\pm\pm}\,, \quad [E^o_A,E^o_B]=c_{AB}^C E^o_C\,, \\ [H^o_{\pm\pm},e^o_{\mp a}] &= e^o_{\pm a}\,, \quad [E^o_A,e^o_{\pm a}]=(E^o_A)^b_a e^o_{\pm b}\,, \quad [H^o_0,e^o_{\pm a}]=\pm e^o_{\pm a}\,, \end{aligned} \tag{2.1}$$

where $(E^o_A)^b_a$ denote the entries of the matrix E^o_A and c_{AB}^C the structure constants of $\mathfrak{sp}_n(\mathbb{C})$ with respect to the basis (E^o_A).

The above basis manifestly displays $\mathfrak{p}=\mathfrak{g}+V$ as a Lie algebra with a five-fold gradation,

$$\mathfrak{p}=\mathfrak{p}_{-2}\oplus\mathfrak{p}_{-1}\oplus\mathfrak{p}_0\oplus\mathfrak{p}_{+1}\oplus\mathfrak{p}_{+2}\ ,$$

with $[\mathfrak{p}_i,\mathfrak{p}_j]\subset\mathfrak{p}_{i+j}$ and $\mathfrak{p}_{i+j}=0$ if $|i+j|>2$. Here the one-dimensional submodules $\mathfrak{p}_{\pm 2}$ are generated by $H^o_{\pm\pm}$, and $\mathfrak{p}_{\pm 1}=V_\pm$. The element $H^o_0\in\mathfrak{p}_0$ is the grading element, which defines the gradation of $\mathfrak{p}$ by virtue of the space $\mathfrak{p}_j$ being the eigenspace of $\mathrm{ad}_{H^o_0}$ with eigenvalue j. We say that an element $x\in\mathfrak{p}_j$ has *charge* j and we write x, as in the basis above, with an appropriate number of $+$ or $-$ signs in the subscript.

2.2. Coordinate systems and left-invariant vector fields

The connected subgroups of $\mathrm{GL}_{4n}(\mathbb{C})\ltimes\mathbb{C}^{4n}$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{p}$, respectively, are denoted by

$$G=\mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(\mathbb{C}) \quad\text{and}\quad \mathcal{P}=G\ltimes\mathbb{C}^{4n}.$$

We parametrise $\mathrm{GL}_{2n}(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$ using the entries of their respective elements, $B=(B^a_b)\in\mathrm{GL}_{2n}(\mathbb{C})$ and

$$U=\begin{pmatrix} u^1_+ & u^1_- \\ u^2_+ & u^2_- \end{pmatrix}\in\mathrm{GL}_2(\mathbb{C})\,.$$

The elements of the subgroup $\mathrm{Sp}_n(\mathbb{C})\subset\mathrm{GL}_{2n}(\mathbb{C})$ are characterised by the constraints $B^c_a\omega_{cd}B^d_b=\omega_{ab}$, where ω is the $2n\times 2n$ matrix of the symplectic form. Similarly, the elements of $\mathrm{Sp}_1(\mathbb{C})=\mathrm{SL}_2(\mathbb{C})\subset\mathrm{GL}_2(\mathbb{C})$ satisfy

$$\det(u^i_\pm)=u^1_+u^2_- - u^2_+u^1_- = 1\,, \tag{2.2}$$

which is tantamount to $\varepsilon^{AB}\varepsilon_{ij}u^i_Bu^j_C=\delta^A_C$, where the 2×2 skewsymmetric matrices $(\varepsilon_{ij})_{i,j\in\{1,2\}}$, $(\varepsilon_{AB})_{A,B\in\{+,-\}}$ and their respective inverses $(\varepsilon^{ij})=(\varepsilon_{\ell m})^{-1}$, $(\varepsilon^{AB})=(\varepsilon_{CD})^{-1}$, have nonzero elements $\varepsilon_{12}=-\varepsilon^{12}=1$ and $\varepsilon_{+-}=-\varepsilon^{+-}=1$.

Then, the inverse matrix $U^{-1} \in \mathrm{Sp}_1(\mathbb{C})$ takes the form

$$U^{-1} = \left(-u_i^{\pm}\right) = -(\varepsilon^{AB}\varepsilon_{j\ell}u_B^{\ell}) = \begin{pmatrix} u_-^2 & -u_-^1 \\ -u_+^2 & u_+^1 \end{pmatrix}. \tag{2.3}$$

Here, we adopt the convention $\alpha_i := \varepsilon_{i\ell}\alpha^{\ell}$ and $\beta^j := \varepsilon^{j\ell}\beta_{\ell}$ for raising and lowering $\mathrm{Sp}_1(\mathbb{C})$-indices.

We denote by (z^{1a}, z^{2b}), $a, b = 1, \dots 2n$, the elements of $V = \mathbb{C}^{4n} = \mathbb{C}^2 \otimes \mathbb{C}^{2n}$ and call the standard system of coordinates on $(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \ltimes \mathbb{C}^{4n}$ the *central coordinate system*,

$$((u_{\pm}^i), (B_b^a), (z^{ia})) \colon \ (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \ltimes \mathbb{C}^{4n} \longrightarrow \mathbb{C}^4 \times \mathbb{C}^{4n^2} \times \mathbb{C}^{4n}.$$

The basis elements of $\mathfrak{p}$ in (2.1) are restrictions to $\mathcal{P}$ of left-invariant vector fields of the Lie group $(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \ltimes \mathbb{C}^{4n}$, which contains $\mathcal{P}$ as a proper Lie subgroup. In central coordinates we have

$$\begin{aligned} H_0^o &= u_+^i \frac{\partial}{\partial u_+^i} - u_-^i \frac{\partial}{\partial u_-^i}, \qquad H_{++}^o = u_+^i \frac{\partial}{\partial u_-^i}, \qquad H_{--}^o = u_-^i \frac{\partial}{\partial u_+^i}, \\ E_A^o &= B_c^a (E_A^o)_b^c \frac{\partial}{\partial B_b^a}, \qquad e_{\pm a}^o = B_a^b u_{\pm}^j \frac{\partial}{\partial z^{jb}}. \end{aligned} \tag{2.4}$$

A useful alternative coordinate system is the *analytic coordinate system*,

$$((u_{\pm}^i), (B_b^a), (z^{\pm a})) \colon \ (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \ltimes \mathbb{C}^{4n} \longrightarrow \mathbb{C}^4 \times \mathbb{C}^{4n^2} \times \mathbb{C}^{4n},$$

determined by the coordinate transformation

$$(u_{\pm}^i) \mapsto (u_{\pm}^i), \quad (B_b^a) \mapsto (B_b^a), \quad (z^{ia}) \mapsto (z^{\pm a})\big|_{(U,B,z)} := U^{-1} \cdot (z^{ia})\big|_{(U,B,z)},$$

where the last mapping is equivalent to

$$\left.\begin{pmatrix} z^{1a} \\ z^{2a} \end{pmatrix}\right|_{(U=(u_{\pm}^i), B, z=z^{ia})} = \begin{pmatrix} u_+^1 & u_-^1 \\ u_+^2 & u_-^2 \end{pmatrix} \left.\begin{pmatrix} z^{+a} \\ z^{-a} \end{pmatrix}\right|_{(U=(u_{\pm}^i), B, z=z^{ia})}.$$

In these coordinates, the vector fields $e_{\pm a}^o$ have the simple expression $e_{\pm a}^o = B_a^b \frac{\partial}{\partial z^{\pm b}}$. In terms of the elements of U^{-1} given in (2.3), we may also write $z^{\pm a} = -u_i^{\pm} z^{ia}$.

2.3. U(1)-charge

Let F be a $\mathbb{C}^N$-valued holomorphic function, defined on (an open subset of) $\mathrm{Sp}_1(\mathbb{C})$ or $\mathrm{Sp}_1(\mathbb{C}) \ltimes \mathbb{C}^{4n}$. Our main results depend crucially on certain properties of such functions and we discuss these in Section 5. In analogy with the terminology for the elements of $\mathfrak{p}$, we say that F has **charge** k if it is a solution of the differential equation[1]

$$H_0^o \cdot F = kF\,, \quad k \in \mathbb{Z}\,. \tag{2.5}$$

We shall write such functions with $|k|$ plus or minus signs in the subscript or, when it is less cumbersome, as $F_{(\pm|k|)}$. We also adopt the sign convention that a plus or minus sign in the *superscript* denotes, respectively, a negative or positive charge, i.e. the opposite charge to that denoted by the same sign in the subscript. So, for instance, the coordinates $z^{\pm a}$ defined above satisfy the condition $H_0^o \cdot z^{\pm a} = \mp z^{\pm a}$.

2.4. Real structures on $\mathcal{P} = (\mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C})) \ltimes \mathbb{C}^{4n}$

Consider non-negative integers p, q with $p + q = n$, and organise the matrix $I_{2p,2q}$ of the flat metric of signature $(2p, 2q)$ and the complex structures $\mathbb{J}$ and J of $\mathbb{C}^{2n}$ and $\mathbb{C}^2$, respectively, as follows:

$$I_{2p,2q} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, \quad (\mathbb{J}_a^b) = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}, \quad (J_i^j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\eta := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Using these matrices we may define a holomorphic map

$$\begin{aligned} \psi\colon\ (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \ltimes \mathbb{C}^{4n} &\to (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \ltimes \mathbb{C}^{4n} \\ (U, B, (z^{ia})) &\mapsto (\psi(U), \psi(B), (\psi^{jb}(z)))\,, \end{aligned}$$

where

$$\psi(U) = (U^T)^{-1}, \quad \psi(B) = (I_{2p,2q} B^T I_{2p,2q})^{-1}, \quad \psi^{jb}(z) = -J_i^j\ \mathbb{J}_a^b\ z^{ia}.$$

Under ψ the left-invariant vector fields of $\mathcal{P}$ transform as:

$$\psi_*(H_0^o) = -H_0^o\,, \quad \psi_*(H_{\pm\pm}^o) = -H_{\mp\mp}^o\,, \quad \psi_*(e_{\pm a}^o) := \pm \widehat{\mathbb{J}}_a^b e_{\mp b}^o\,, \tag{2.6}$$

where the $\mathrm{GL}_{2n}(\mathbb{C})$-valued function $\widehat{\mathbb{J}}\colon \mathcal{P} \to \mathrm{GL}_{2n}(\mathbb{C})$ is defined by

$$\widehat{\mathbb{J}}|_{(U,B,z)} := -I_{2p,2q} \cdot \left(B \cdot \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot B^T \right)^{-1}. \tag{2.7}$$

The map ψ determines, by conjugation, an anti-holomorphic map,

$$\tau(U, B, z) := \overline{\psi(U, B, z)} = \left((\overline{U^T})^{-1}\,, \, (I_{2p,2q} \overline{B^T} I_{2p,2q})^{-1}\,, \, \overline{\psi(z)} \right). \tag{2.8}$$

The map ψ and complex conjugation clearly commute. The push-forwards of the complex vector fields H_0^o, $H_{\pm\pm}^o$, $e_{\pm a}^o$ under the anti-involution τ are

$$\tau_*(H_0^o) = -H_0^o\,, \quad \tau_*(H_{\pm\pm}^o) = -H_{\mp\mp}^o\,, \quad \tau_*(e_{\pm a}^o) = \pm \widehat{\mathbb{J}}_a^b e_{\mp b}^o\,. \tag{2.9}$$

[1] We use the notation $X \cdot f := X(f)$ to denote the directional derivative of a function f on a manifold M along a vector field X.

Now, $\tau(\mathcal{P}) \subset \mathcal{P}$ and the τ-fixed point set is $\mathcal{P}^\tau = (\mathrm{Sp}_1 \times \mathrm{Sp}_{p,q}) \ltimes (\mathbb{C}^{4n})^\tau$, where

$$(\mathbb{C}^{4n})^\tau := \{\, (z^{1a}, \overline{z^{2a}}, \eta^a_b z^{2b}, -\eta^a_b \overline{z^{1b}}),\ a = 1, \ldots, n\,\}.$$

We call $\tau|_{\mathcal{P}}$ the **real structure of signature $(4p, 4q)$ on $\mathcal{P}$** . For simplicity, we shall use τ instead of $\tau|_{\mathcal{P}}$ and we similarly denote each of the three component parts, the anti-involutions on $\mathrm{Sp}_1(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ and $\mathbb{C}^{4n}$ given by (2.8). Which anti-involution is meant will always be clear from the context.

The space $(\mathbb{C}^{4n})^\tau$ is endowed with an $(\mathrm{Sp}_1 \times \mathrm{Sp}_{p,q})$-invariant quaternionic structure $\mathfrak{J}\colon \mathbb{H}^n \to \mathbb{H}^n$, $\mathfrak{J}^2 = -1$,

$$\mathfrak{J}(z^{1a}, \overline{z^{2b}}, \eta^c_a z^{2a}, -\eta^d_a \overline{z^{1a}}) := (z^{2b}, -\overline{z^{1a}}, \eta^c_a z^{1a}, \eta^d_a \overline{z^{2a}}) ,$$

and is naturally identifiable with $\mathbb{H}^n = \{(z^{1a}, \overline{z^{2b}}),\ z^{ja} \in \mathbb{C}\}$, n-dimensional quaternion space.

2.5. $\mathfrak{sp}_n(\mathbb{C})$-equivariance

Let $\rho\colon \mathfrak{sp}_n(\mathbb{C}) \to \mathfrak{gl}(S)$ be a linear representation of $\mathfrak{sp}_n(\mathbb{C})$ on a complex vector space S and $f\colon \mathcal{U} \subset \mathcal{P} \to S$ a holomorphic map. We say that f is **$\mathfrak{sp}_n(\mathbb{C})$-equivariant** if it satisfies the differential equation

$$E^o_A \cdot f = \rho(E^o_A)(f) \ . \tag{2.10}$$

For instance, functions (f^a) , (h_b) , (g^a_b) and (ℓ^a_{bc}), taking values in the spaces $\widetilde{V} := \mathbb{C}^{2n}$, $\widetilde{V}^*$, $\widetilde{V} \otimes \widetilde{V}^*$ and $\widetilde{V} \otimes \widetilde{V}^* \otimes \widetilde{V}^*$, respectively, are $\mathfrak{sp}_n(\mathbb{C})$-equivariant if they satisfy the differential equations

$$\begin{aligned} E^o_A \cdot f^a &= -(E^o_A)^a_b f^b \ , \quad E^o_A \cdot h_a = (E^o_A)^b_a h_b \ , \\ E^o_A \cdot g^a_b &= -(E^o_A)^a_c g^c_b + (E^o_A)^c_b g^a_c \ , \\ E^o_A \cdot \ell^a_{bc} &= -(E^o_A)^a_d \ell^d_{bc} + (E^o_A)^d_b \ell^a_{dc} + (E^o_A)^d_c \ell^a_{bd} \ . \end{aligned}$$

3. HK-FRAMES

Let p, q be non-negative integers with $p+q = n$ and τ the real structure (2.8) of signature $(4p, 4q)$ on $\mathcal{P} = G \ltimes V$, $V = \mathbb{C}^{4n}$, $G = \mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C})$. Following [9, 10] we introduce:

Definition 3.1. Let $\mathcal{V} \subset V = \mathbb{C}^{4n}$ be a connected, simply connected neighbourhood of 0, invariant under the involution τ. The **harmonic space** of $\mathcal{V}$ is the set $\mathcal{H}|_{\mathcal{V}} := \mathrm{Sp}_1(\mathbb{C}) \times \{I_{2n}\} \times \mathcal{V}$. When $\mathcal{V} = \mathbb{C}^{4n}$, we write simply $\mathcal{H}$. Further, an open subset $\mathcal{U} \subset \mathcal{P}$ is called **appropriate** if it is a τ-invariant simply connected neighbourhood of $e = (I_2, I_{2n}, 0)$, such that $\mathcal{U} \cap \mathcal{H} = \mathcal{H}|_{\mathcal{V}}$ for some open subset $\mathcal{V} \subset \mathbb{C}^{4n}$.

Let $\mathfrak{X}^{\mathrm{hol}}(\mathcal{U})$ be the space of holomorphic vector fields on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$.

Definition 3.2. A collection $\mathcal{A} = (H_0, H_{\pm\pm}, E_A, e_{\pm a})$ of holomorphic vector fields in $\mathfrak{X}^{\mathrm{hol}}(\mathcal{U})$, $\mathbb{C}$-linearly independent at all points, is an **hk-frame** if:

a) $\mathcal{U}$ carries a holomorphic right action $\rho\colon \mathcal{U}\times G \to \mathcal{U}$ of $G = \mathrm{Sp}_1(\mathbb{C})\times \mathrm{Sp}_n(\mathbb{C})$ such that the associated group homomorphism $\widehat{\rho}\colon G \to \mathrm{Diff}(\mathcal{U})$ satisfies

$$\widehat{\rho}_*(H^o_0) = H_0\,, \quad \widehat{\rho}_*(H^o_{\pm\pm}) = H_{\pm\pm}\,, \quad \widehat{\rho}_*(E^o_A) = E_A\,.$$

Since the map $\widehat{\rho}_*\colon \mathfrak{g} \simeq T_eG \to \mathfrak{X}^{\mathrm{hol}}(\mathcal{U})$ is an injective Lie algebra homomorphism, the vector fields $(H_0, H_{\pm\pm}, E_A)$ satisfy the Lie bracket relations (cf. (2.1))

$$[H_0, H_{\pm\pm}] \;=\; \pm 2H_{\pm\pm}\;, \quad [H_{++}, H_{--}] = \;H_0\;, \quad [E_A, E_B] = c^C_{AB}E_C\;. \tag{3.1}$$

b) The Lie brackets of the other fields of $\mathcal{A}$ are given by

$$\begin{aligned}
&[H_0, e_{\pm a}] = \pm e_{\pm a}\;, && [H_{\pm\pm}, e_{\pm a}] = 0\;, && [H_{\pm\pm}, e_{\mp a}] = e_{\pm a}\;,\\
&[E_A, e_{\pm a}] = (E^o_A)^b_a e_{\pm b}\;, && [e_{\pm a}, e_{\pm b}] = 0\;, && [e_{+a}, e_{-b}] = R^A_{ab}E_A\;,
\end{aligned} \tag{3.2}$$

where $R^A_{ab}\colon \mathcal{U} \to \mathbb{C}$ are holomorphic functions.

c) The orbit space $M = \mathcal{U}/G$ is a manifold and $\pi\colon \mathcal{U} \to M = \mathcal{U}/G$ is a principal G-bundle over M.

A pair of hk-frames $\mathcal{A}\,, \mathcal{A}'$ defined on appropriate open subsets $\mathcal{U}\,, \mathcal{U}' \subset \mathcal{P}$, respectively, are **locally equivalent** if there exists a G-equivariant biholomorphism $\varphi\colon \mathcal{U} \to \mathcal{U}'$ which maps the fields of $\mathcal{A}$ into the corresponding fields of $\mathcal{A}'$. We write: $\mathcal{A}' = \varphi_*(\mathcal{A})$.

A particularly important class of hk-frames is given by:

Definition 3.3. A **canonical hk-frame** is an hk-frame $\mathcal{A} = (H_0, H_{\pm\pm}, E_A, e_{\pm a})$ on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$, in which the vector fields take the form[2]

$$\begin{aligned}
H_0 &= H^o_0\\
H_{\pm\pm} &= (H^o_{\pm\pm} + \delta H_{\pm\pm})\;, \quad \delta H_{\pm\pm} = v^{-b}_{\pm\pm}e^o_{-b} + v^{+b}_{\pm\pm}e^o_{+b} + A^B_{++}E^o_B\\
E_A &= E^o_A\\
e_{+a} &= e^o_{+a}\\
e_{-a} &= (e^o_{-a} + \delta e_{-a})\;, \quad \delta e_{-a} = v^{+b}_{-a}e^o_{+b} + A^B_{-a}E^o_B\;,
\end{aligned}$$

with components v^{+b}_{++} identically vanishing on the submanifold $\{z^{+a} = 0\} \subset \mathcal{U}$.

[2] Here, the components $v^{\pm b}_{\pm\pm}$ and v^{+b}_{-a} are complex functions on $\mathcal{U}$ and have charges in accordance with the notation of Sect. 2.3: a plus (minus) sign in the superscript denotes a negative (positive) charge and vice versa for subscripts. So, for instance, the components v^{-b}_{++}, which satisfy $H^o_0{\cdot}v^{-b}_{++} = 3v^{-b}_{++}$ in virtue of eq. (3.1), have charge $+3$.

The $\mathbb{C}^{2n}$-valued function $(v^{-b}_{++})|_{\mathcal{H}\cap\mathcal{U}} : \mathcal{H}\cap\mathcal{U} \to \mathbb{C}^{2n}$, appearing as the coefficient of e^o_{-b} in $H_{++}|_{\mathcal{H}\cap\mathcal{U}}$, is called the **v-potential** of $\mathcal{A}$. We shall see that it effectively parametrises the equivalence classes of hk-frames.

Remark. Canonical hk-frames are called *analytic frames* in the harmonic space literature [10].

There is another important class of hk-frames:

Definition 3.4. A **central hk-frame** is an hk-frame $\mathcal{A}$ on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$, in which the vector fields take the form

$$\begin{gathered} H_0 = H^o_0 \ , \quad H_{\pm\pm} = H^o_{\pm\pm} \ , \quad E_A = E^o_A \ , \\ e_{\pm a} = e^o_{\pm a} + v^{+b}_{\pm a} e^o_{+b} + v^{-b}_{\pm a} e^o_{-b} + A^B_{\pm a} E^o_B \ , \end{gathered}$$

where the components $v^{+b}_{\pm a}$, $v^{-b}_{\pm a}$, $A^B_{\pm a}$ are holomorphic functions. The collection $\mathcal{A}^o = (H^o_0, H^o_{\pm\pm}, E^o_A, e^o_{\pm a})$ of left-invariant vector fields on $\mathcal{P}$, forming the standard basis of $\mathfrak{p}$, is called the **flat hk-frame**.

We shall see that if appropriate reality conditions are satisfied, every hk-frame $\mathcal{A}$, defined on an appropriate open set $\mathcal{U} \subset \mathcal{P}$, determines a real analytic pseudo-Riemannian metric g on $M = \mathcal{U}/G$. In this case, the functions R^A_{ab}, appearing in (3.2), are components of the curvature tensor of (M, g). In particular, the flat hk-frame corresponds to a flat pseudo-Riemannian metric.

A local biholomorphism $\varphi : \mathcal{U} \to \mathcal{U}'$ between two appropriate open subsets of $\mathcal{P}$, with components in central coordinates $\varphi = (\varphi^i_\pm, \varphi^a_b, \varphi^{ia})$ such that $\varphi^i_\pm(U, B, z) = u^i_\pm$ and mapping a central hk-frame $\mathcal{A}$ into a canonical hk-frame $\mathcal{A}' = \varphi_*(\mathcal{A})$, is called a **bridge** between $\mathcal{A}$ and $\mathcal{A}'$. From the definitions of central and canonical hk-frames, this means that φ is a biholomorphism satisfying

$$\begin{aligned} &\varphi^i_\pm(U, B, z) = u^i_\pm \\ &\varphi_*(E^o_A) = E^o_A \ , \quad \varphi_*(H^o_0) = H^o_0 \ , \quad \varphi_*(e_{+a}) = e^o_{+a} \ , \end{aligned} \tag{3.3}$$

with the property that the vector fields $H'_{\pm\pm} := \varphi_*(H^o_{\pm\pm})$, $e'_{-a} := \varphi_*(e_{-a})$ have the form prescribed in Def. 3.3.

Canonically associated with an hk-frame $\mathcal{A} = (H_0, H_{\pm\pm}, E_A, e_{\pm a})$, there exists an **absolute hk-parallelism**, a $\mathbb{C}$-linear map $\alpha^{\mathcal{A}} : \mathfrak{p} \to \mathfrak{X}^{\text{hol}}(\mathcal{U})$ from the (abstract) Lie algebra $\mathfrak{p}$ to the holomorphic vector fields on $\mathcal{U} \subset \mathcal{P}$, defined by

$$\alpha^{\mathcal{A}}(H^o_0) = H_0 \ , \ \ \alpha^{\mathcal{A}}(H^o_{\pm\pm}) = H_{\pm\pm} \ , \ \ \alpha^{\mathcal{A}}(E^o_A) = E_A \ , \ \ \alpha^{\mathcal{A}}(e^o_{\pm a}) = e_{\pm a} \ . \tag{3.4}$$

This map satisfies the following conditions:

a) $\alpha^{\mathcal{A}}$ is a holomorphic absolute parallelism, i.e. it gives a linear isomorphism between $\mathfrak{p}$ and $T^{10}_w\mathcal{U}$ for every $w \in \mathcal{U}$.

b) The restriction $\alpha^{\mathcal{A}}|_{\mathfrak{g}} \colon \mathfrak{g} \to \mathrm{span}_{\mathbb{C}}\{H_0, H_{\pm\pm}, E_A\}$ coincides with the map $\widehat{\rho}_* \colon \mathfrak{g} \simeq T_eG \to \mathfrak{X}^{\mathrm{hol}}(\mathcal{U})$ in Def. 3.2 corresponding to the right action $\rho \colon \mathcal{U}\times G \to \mathcal{U}$.

c) $\alpha^{\mathcal{A}}([X, v]) = [\alpha^{\mathcal{A}}(X), \alpha^{\mathcal{A}}(v)]$ for all $X \in \mathfrak{g}$ and $v \in V$.

d) $[\alpha^{\mathcal{A}}(v), \alpha^{\mathcal{A}}(v')]_w \in \alpha^{\mathcal{A}}(\mathfrak{sp}_n(\mathbb{C}))$ for all $v, v' \in V$, $w \in \mathcal{U}$.

Conversely, for a given right action ρ of G on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$ and a $\mathbb{C}$-linear map $\alpha^{\mathcal{A}}$ satisfying a) - d), the vector fields defined by (3.4) constitute an hk-frame.

4. The main theorems

4.1. **Canonical hk-pairs**

For a real $\mathcal{C}^\infty$-manifold M, we denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M. Given an hk-frame $\mathcal{A}$ on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$, we call the $\mathbb{R}$-linear map

$$\alpha^{\mathcal{A}}_{(\mathbb{R})} \colon \mathfrak{p} \to \mathfrak{X}(\mathcal{U}) , \qquad X \mapsto \alpha^{\mathcal{A}}_{(\mathbb{R})}(X) := 2\,\mathrm{Re}(\alpha^{\mathcal{A}}(X)) , \tag{4.1}$$

the **real absolute hk-parallelism** associated with $\mathcal{A}$. Notice that

$$\alpha^{\mathcal{A}}_{(\mathbb{R})}(iX) = J_o \alpha^{\mathcal{A}}_{(\mathbb{R})}(X)$$

for all $X \in \mathfrak{g}$, where J_o is the *real* $(1,1)$-tensor field corresponding to the standard complex structure of $\mathcal{P}$, and that $\alpha^{\mathcal{A}}(X) = (\alpha^{\mathcal{A}}_{(\mathbb{R})}(X))^{10}$.

Our classification of (local) isometry classes of pseudo-hyperkähler metrics is based on the following:

Definition 4.1. An **hk-pair** of signature $(4p, 4q)$ is a pair $(\mathcal{A}, M)$, consisting of an hk-frame $\mathcal{A}$ on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$ and a totally real $4n$-dimensional submanifold $M \subset \mathcal{U}$ passing through $e = (I_2, I_{2n}, 0)$ and satisfying the following conditions:

i) M is transversal to the $\mathfrak{sp}_{p,q}$-orbits, i.e. $T_xM \cap \alpha^{\mathcal{A}}_{(\mathbb{R})}(\mathfrak{sp}_{p,q})|_x = \{0\}$ for all x

ii) $T_xM \subset \alpha^{\mathcal{A}}_{(\mathbb{R})}(V^\tau + \mathfrak{sp}_{p,q})|_x$ for all $x \in M$.

Two hk-pairs $(\mathcal{A}, M), (\mathcal{A}', M')$ are called **locally equivalent** if there exists a G-equivariant biholomorphism $\varphi \colon \mathcal{U} \to \mathcal{U}'$ mapping $\mathcal{A}$ into $\mathcal{A}'$ and M into M'.

The **flat hk-pair** $(\mathcal{A}^o, M^o)$ of signature $(4p, 4q)$ consists of the flat hk-frame $\mathcal{A}^o$ on $\mathcal{P}$ together with the real submanifold

$$M^o = \{I_2\}\times\{I_{2n}\}\times(\mathbb{C}^{4n})^\tau \subset \mathcal{P} . \tag{4.2}$$

Definition 4.2. An hk-pair $(\mathcal{A}, M)$ on $\mathcal{U} \subset \mathcal{P}$ is called **canonical** if

a) the hk-frame $\mathcal{A}$ is canonical (Def. (3.3)),
b) there exists a bridge $\varphi\colon \mathcal{U}' \to \mathcal{U}$ that maps a central hk-frame $\widehat{\mathcal{A}}$ on $\mathcal{U}'$ to $\mathcal{A} = \varphi_*(\widehat{\mathcal{A}})$ and
c) there exists a real submanifold $\widehat{M} \subset \{I_2\}\times\{I_{2n}\}\times\mathbb{C}^{4n}$ passing through $e = (I_2, I_{2n}, 0)$ such that the bridge φ determines a local equivalence between $(\widehat{\mathcal{A}}, \widehat{M})$ and $(\mathcal{A}, M)$.

4.2. Correspondence between hk-pairs and pseudo-hyperkähler metrics

Pseudo-hyperkähler metrics and hk-pairs are related as follows. Consider an hk-pair $(\mathcal{A}, M)$ of signature $(4p, 4q)$, with associated holomorphic action $\rho\colon \mathcal{U}\times G \to G$, and let

$$\rho^\tau = \rho|_{\mathcal{U}\times G^\tau}\colon \mathcal{U}\times G^\tau \to \mathcal{U}\ , \qquad G^\tau = \mathrm{Sp}_1 \times \mathrm{Sp}_{p,q}\ ,$$

be the induced right action of G^τ. The infinitesimal transformations of ρ^τ are, by construction, the real vector fields in $\mathfrak{g}^{\mathcal{A},\tau} := \alpha^{\mathcal{A}}_{(\mathbb{R})}(\mathfrak{g}^\tau)$. In accordance with Def. 4.1, the union of $\mathrm{Sp}_{p,q}$-orbits

$$\mathcal{U}^{(\mathrm{Sp}_{p,q})} = M{\cdot}\mathrm{Sp}_{p,q} := \bigcup_{x\in M} x\cdot \mathrm{Sp}_{p,q} \tag{4.3}$$

is a manifold, $\mathrm{Sp}_{p,q}$-equivariantly diffeomorphic to $M{\times}\mathrm{Sp}_{p,q}$, and the $4n$-vectors,

$$e^\tau_I|_x = \alpha^{\mathcal{A}}_{(\mathbb{R})}(e^{o\tau}_I)|_x, \quad I = 1, \dots, 4n, \quad x \in M\ , \tag{4.4}$$

with I labelling the ordered index pairs $(+1, \dots, +2n, -1, \dots, -2n)$, belong to the vector space

$$T_x\mathcal{U}^{(\mathrm{Sp}_{p,q})} = T_xM + \alpha^{\mathcal{A}}_{(\mathbb{R})}(\mathfrak{sp}_{p,q}). \tag{4.5}$$

Here $e^{o\tau}_I$ denote a choice of basis vectors for the $4n$-dimensional real τ-invariant subspace $V^\tau \subset V = \mathbb{C}^{4n}$. By (4.5) and $\mathrm{Sp}_{p,q}$-equivariance, the restrictions to $\mathcal{U}^{(\mathrm{Sp}_{p,q})}$ of the vector fields $e^\tau_I = \alpha^{\mathcal{A}}_{(\mathbb{R})}(e^{o\tau}_I)$ are tangent to $\mathcal{U}^{(\mathrm{Sp}_{p,q})}$ at all its points.

Now, we choose a section $\sigma\colon M \to \mathcal{U}^{(\mathrm{Sp}_{p,q})}$ of the trivial bundle $\pi\colon \mathcal{U}^{(\mathrm{Sp}_{p,q})} \simeq M \times \mathrm{Sp}_{p,q} \to M$ and we consider the vector fields on M

$$e^{(\sigma)}_I = \left\{ e^{(\sigma)}_I|_x := \pi_*(e^\tau_I|_{\sigma(x)})\,, \quad x \in M \right\}\ .$$

There clearly exists a unique real analytic, pseudo-Riemannian metric g of signature $(4p, 4q)$, for which the $(e^{(\sigma)}_I|_x)$ are vielbeins, i.e.

$$g(e^{(\sigma)}_I, e^{(\sigma)}_J) = (I_4 \otimes \eta)_{IJ}\ . \tag{4.6}$$

The properties of the absolute hk-parallelism $\alpha^{\mathcal{A}}$ imply that for any other section $\sigma'\colon M \to \mathcal{U}^{(\mathrm{Sp}_{p,q})}$, the corresponding frames $\big(e_I^{(\sigma')}|_x := \pi_*(e_I^\tau|_{\sigma'(x)})\big)$ are also vielbeins for this metric, which thus does not depend on the choice of σ and is uniquely associated with the hk-pair $(\mathcal{A}, M)$ (see Lemma A18).

Moreover, the following proposition holds by construction:

Proposition 4.3. *Let (M, g) and (M', g') be real analytic pseudo-hyperkähler manifolds of signature $(4p, 4q)$ corresponding, in the above-described fashion, to hk-pairs $(\mathcal{A}, M)$ and $(\mathcal{A}', M')$, respectively. Then (M, g) and (M', g') are locally isometric if and only if the hk-pairs $(\mathcal{A}, M)$ and $(\mathcal{A}', M')$ are locally equivalent.*

Further (Theorem A19):
In each local isometry class of (germs of) real analytic pseudo-hyperkähler manifolds of signature $(4p, 4q)$, there is a pseudo-hyperkähler manifold (M, g) which is determined by an hk-pair $(\mathcal{A}, M)$ of signature $(4p, 4q)$ in the above-described fashion.

It follows immediately that:

Theorem 4.4. *There is a natural one to one correspondence between the local isometry classes of (germs of) real analytic pseudo-hyperkähler manifolds and the local equivalence classes of (germs of) hk-pairs.*

4.3. **Prepotentials of pseudo-hyperkähler metrics**

According to the above results, the classification of local isometry classes of real analytic pseudo-hyperkähler metrics corresponds to the classification of local equivalence classes of hk-pairs. The latter is achieved by means of the following two fundamental results.

Theorem 4.5. *Every local equivalence class of (germs of) hk-pairs contains a canonical hk-pair. Moreover, if hk-pairs $(\mathcal{A}, M)$ and $(\mathcal{A}', M')$ are both canonical and have identical v-potentials, then $\mathcal{A} = \mathcal{A}'$ and $(\mathcal{A}, M)$ and $(\mathcal{A}', M')$ are locally equivalent.*

Theorem 4.6. *There exists a one-to-one correspondence between canonical hk-pairs and holomorphic functions on harmonic space $\mathcal{L}_{(+4)}\colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}$ satisfying the system of first order equations,*

$$e^o_{+a}\cdot \mathcal{L}_{(+4)} = 0\ , \quad H^o_0\cdot \mathcal{L}_{(+4)} = 4\mathcal{L}_{(+4)}\ , \quad \mathcal{L}_{(+4)}|_{\mathrm{Sp}_1(\mathbb{C})\times\{I_{2n}\}\times\{0\}} = 0\ . \tag{4.7}$$

More precisely, given such an $\mathcal{L}_{(+4)}$, there exists a canonical hk-pair $(\mathcal{A}, M)$ on an appropriate subset $\mathcal{U} \subset \mathcal{P}$ with $\mathcal{U} \cap \mathcal{H} = \mathcal{H}|_{\mathcal{V}}$, whose v-potential is equal to

$$v^{-a}_{++}|_{\mathcal{U}\cap\mathcal{H}} = \omega^{ab}\big(e^o_{-b}\cdot \mathcal{L}_{(+4)}\big). \tag{4.8}$$

Here (ω^{ab}) *is the inverse matrix of* (ω_{ab})*. Conversely, given a canonical hk-pair* $(\mathcal{A}, M)$*, with v-potential* $v^{-a}_{++}|_{\mathcal{U}\cap\mathcal{H}}$*, there exists a unique holomorphic function* $\mathcal{L}_{(+4)}$ *satisfying* (4.7) *and* (4.8)*.*

The holomorphic function $\mathcal{L}_{(+4)}$ is the **prepotential**[3] of the canonical hk-pair $(\mathcal{A}, M)$. The space of prepotentials parametrises the local equivalence classes of real analytic pseudo-hyperkähler manifolds. Given an unconstrained prepotential $\mathcal{L}_{(+4)}$ satisfying (4.7), all the vector fields of the associated canonical hk-pair $(\mathcal{A}, M)$ may be obtained explicitly by solving a system of partial differential equations on harmonic space $\mathcal{H}|_{\mathcal{V}}$. The corresponding pseudo-hyperkähler manifold can then be determined according to the procedure of Sect. 4.2. Since the equivalence classes of (germs of) hk-pairs are in one to one correspondence with the (germs of) real analytic pseudo-hyperkähler metrics (Theorem 4.4) and each of them contains a canonical hk-pair (Theorem 4.5), the parametrisation of pseudo-hyperkähler metrics advertised in the Introduction is established.

In the next section we discuss some technical properties of holomorphic functions on $\mathrm{Sp}_1(\mathbb{C})$, which are essential in our discussion. We then prove Theorem 4.5 in Sect. 6 and Theorem 4.6 in Sect. 7. In Sect. 8, we describe a five-step recipe for the explicit construction of a pseudo-hyperkähler metric from its prepotential.

5. Holomorphic functions on $\mathrm{Sp}_1(\mathbb{C})$

Consider the standard coordinates of $\mathrm{GL}_2(\mathbb{C})$,

$$(u^1_+, u^2_+, u^1_-, u^2_-)\colon \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathbb{C}^4,$$

which associate with every matrix $U = \begin{pmatrix} u^1_+ & u^1_- \\ u^2_+ & u^2_- \end{pmatrix}$ the values of its entries, and the class of meromorphic functions $h\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ of the form

$$h = \sum_{p,q,r,s\in\mathbb{Z}} c_{pqrs}(u^1_+)^p(u^2_+)^q(u^1_-)^r(u^2_-)^s. \tag{5.1}$$

Two such maps h, h' are called $\mathbf{Sp_1(\mathbb{C})}$**-equivalent** if $h|_{\mathrm{Sp}_1(\mathbb{C})} = h'|_{\mathrm{Sp}_1(\mathbb{C})}$. Since the elements $U = (u^i_\pm) \in \mathrm{Sp}_1(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ are constrained by

$$\det U = u^1_+ u^2_- - u^2_+ u^1_- = 1\,, \tag{5.2}$$

any one coordinate from $\{u^1_+, u^2_+, u^1_-, u^2_-\}$ is $\mathrm{Sp}_1(\mathbb{C})$-equivalent to a rational function of the other three. It follows that every function (5.1) is $\mathrm{Sp}_1(\mathbb{C})$-equivalent to four others, obtained by expressing each of the four coordinates in terms of the others in accordance with (5.2). The meromorphic functions

[3]Our sign convention for the prepotential differs from the customary one (e.g. [10, 3]).

obtained in this way are said to be in **reduced form**. Clearly, each $\mathrm{Sp}_1(\mathbb{C})$-equivalence class of meromorphic functions (5.1) contains at most four distinct functions in reduced form.

The meromorphic functions of the form (5.1) are related to the holomorphic functions of $\mathrm{Sp}_1(\mathbb{C})$. Indeed, we have the following:

Lemma 5.1. *Any holomorphic function* $g\colon \mathrm{Sp}_1(\mathbb{C}) \to \mathbb{C}$ *is a restriction of some meromorphic function* $h\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ *of the form* (5.1), $g = h|_{\mathrm{Sp}_1(\mathbb{C})}$. *Further, if the function h thus associated with g is in reduced form, its coefficients* c_{pqrs} *are uniquely determined by the expansion of* $g|_{\mathrm{Sp}_1}$ *in generalised spherical functions.*

Proof. Since Sp_1 is a three dimensional, totally real submanifold of the three dimensional complex manifold $\mathrm{Sp}_1(\mathbb{C})$, the holomorphic map $g\colon \mathrm{Sp}_1(\mathbb{C}) \to \mathbb{C}$ is uniquely determined by its restriction $g|_{\mathrm{Sp}_1}$, which is of class $\mathcal{C}^\infty$ and hence in $L^2(\mathrm{Sp}_1)$. This implies that $g|_{\mathrm{Sp}_1}$ admits a unique series expansion in terms of generalised spherical functions (see e.g. [11] p. 94). We recall that these are the functions

$$T^\lambda_{mn}\colon \mathrm{Sp}_1 \to \mathbb{C}\,, \quad \text{with } \lambda = \frac{\ell}{2}\,,\ \ell \in \mathbb{N}\,,\ m, n = -\ell, -\ell+1, \ldots, \ell-1, \ell\,,$$

which associate with every $U = (u^i_\pm) \in \mathrm{Sp}_1$ the (m,n)-element of the matrix $T^\lambda(U)$ representing the action of U on the (unique, up to an isomorphism) irreducible Sp_1-module of highest weight λ. Since every such irreducible Sp_1-module is a symmetric power of the standard module $\mathbb{C}^2$, the entries of $T^\lambda(U)$ are polynomials in the entries of U. A generalised spherical function T^λ_{mn} is therefore a polynomial in these variables and has an expression of the form

$$T^\lambda_{mn} = \sum T^\lambda_{mn|pqrs}(u^1_+)^p(u^2_+)^q(u^1_-)^r(u^2_-)^s\big|_{\mathrm{Sp}_1}\,, \quad T^\lambda_{mn|pqrs} \in \mathbb{C}\,. \tag{5.3}$$

Since the restrictions $u^i_\pm|_{\mathrm{Sp}_1}$'s are constrained by (5.2), the coefficients in the expansion (5.3) are in general not uniquely determined by the spherical function T^λ_{mn}. However, replacing one of the functions $u^i_\pm|_{\mathrm{Sp}_1}$ by a rational expression of the others, one can always reduce to an expression for T^λ_{mn} as a Laurent series of the other three functions. Summing up, a spherical function $T^\lambda_{mn}\colon \mathrm{Sp}_1 \to \mathbb{C}$ admits at most four specific expansions (5.3), each of them equal to the restriction $\mathcal{T}^\lambda_{mn}|_{\mathrm{Sp}_1}$ of a meromorphic function $\mathcal{T}^\lambda_{mn}\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ as in (5.1) and in reduced form. We therefore have that $g|_{\mathrm{Sp}_1}$ can be expanded in a series of the form

$$g|_{\mathrm{Sp}_1} = \sum_{\lambda,m,n} c^{mn}_\lambda \mathcal{T}^\lambda_{mn}|_{\mathrm{Sp}_1} = \Big(\sum c_{pqrs}(u^1_+)^p(u^2_+)^q(u^1_-)^r(u^2_-)^s\Big)\Big|_{\mathrm{Sp}_1}\,, \tag{5.4}$$

where the coefficients $c_{pqrs} = \sum c_\lambda^{mn} T^\lambda_{mn|pqrs}$ are completely determined by the coefficients c_λ^{mn} of the expansion of $g|_{\mathrm{Sp}_1}$ in generalised spherical functions if the $T^\lambda_{mn|pqrs}$ are coefficients of a reduced form of the maps $\mathcal{T}^\lambda_{mn}\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$.

Consider now the function $h = \sum c_{pqrs}(u^1_+)^p(u^2_+)^q(u^1_-)^r(u^2_-)^s$ on $\mathrm{GL}_2(\mathbb{C})$. Being meromorphic, it is holomorphic on a dense open subset $\mathcal{U} \subset \mathrm{Sp}_1(\mathbb{C})$. Since $h|_{\mathrm{Sp}_1} = g|_{\mathrm{Sp}_1}$, it follows that $h|_{\mathcal{U}} = g|_{\mathcal{U}}$, so that, by continuity, $h|_{\mathrm{Sp}_1(\mathbb{C})} = g$, proving the first claim of the lemma. The second claim follows from the construction of h and the fact that h is in reduced form if and only if all meromorphic functions $\mathcal{T}^\lambda_{mn}$ appearing in (5.4) are taken in reduced form. □

We now solve certain equations for holomorphic functions $f\colon \mathrm{Sp}_1(\mathbb{C}) \to \mathbb{C}$.

Lemma 5.2. *i) Every solution of*

$$H^o_{++}\cdot f = 0 \tag{5.5}$$

is a restriction $f = h|_{\mathrm{Sp}_1(\mathbb{C})}$ of a holomorphic map $h\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ of the form

$$h = \sum_{n,m\geq 0} c_{nm}(u^1_+)^n(u^2_+)^m. \tag{5.6}$$

ii) Every solution of

$$H^o_0\cdot f = kf\ ,\ \ k \in \mathbb{Z}\,, \tag{5.7}$$

is a restriction $f = h|_{\mathrm{Sp}_1(\mathbb{C})}$ of a meromorphic map $h\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ of the form

$$h = \sum_{\substack{n,m,p,q\,\in\,\mathbb{Z}\\ n+m-p-q\,=\,k}} c_{nmpq}(u^1_+)^n(u^2_+)^m(u^1_-)^p(u^2_-)^q. \tag{5.8}$$

Proof. A holomorphic function f on $\mathrm{Sp}_1(\mathbb{C})$ is of the form $f = h|_{\mathrm{Sp}_1(\mathbb{C})}$, for some meromorphic h in reduced form (Lemma 5.1). The coordinate expression for H^o_{++} (2.4) and holomorphicity imply that f satisfies $H^o_{++}\cdot f = 0$ if and only if h has the form (5.6), proving i). A similar argument proves ii). □

Simultaneous solutions of (5.5) and (5.7) may now be constructed. More generally:

Lemma 5.3. *Let $g\colon \mathrm{Sp}_1(\mathbb{C}) \to \mathbb{C}$ be holomorphic. The system of equations for a holomorphic function $f\colon \mathrm{Sp}_1(\mathbb{C}) \to \mathbb{C}$,*

$$\begin{aligned} H^o_0\cdot f &= kf\,, \quad k \in \mathbb{Z}\,,\\ H^o_{++}\cdot f &= g\,, \end{aligned} \tag{5.9}$$

admits solutions if and only if g satisfies the equation,

$$H^o_0\cdot g = (k+2)g\,. \tag{5.10}$$

If (5.10) *holds, the set of solutions to* (5.9) *is as follows.*

a) For $k < 0$ there exists exactly one holomorphic solution.

b) For $k \geq 0$, the solutions are precisely all the functions $f = h|_{\mathrm{Sp}_1(\mathbb{C})}$ determined by holomorphic maps $h\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ of the form

$$h = h_o + \sum_{\substack{m,n\geq 0\\ m+n=k}} c_{mn}(u^1_+)^m(u^2_+)^n, \tag{5.11}$$

where $f_o = h_o|_{\mathrm{Sp}_1(\mathbb{C})}$ is some solution of (5.9).

Proof. If a solution f to (5.9) exists, then

$$H^o_0{\cdot}g \;=\; [H^o_0, H^o_{++}]\cdot f + H^o_{++}\cdot(H^o_0{\cdot}f) \;=\; 2H^o_{++}{\cdot}f + kH^o_{++}{\cdot}f \;=\; (k+2)g \;,$$

so (5.10) is a necessary condition for the integrability of the system (5.9). On the other hand, by Lemma 5.2 ii) we have that

$$g = \widetilde{g}|_{\mathrm{Sp}_1(\mathbb{C})} \;, \quad \widetilde{g} \;=\; \sum_{p+q-r-s\,=\,k+2} c_{pqrs}(u^1_+)^p(u^2_+)^q(u^1_-)^r(u^2_-)^s.$$

Now, by integration of $H^o_{++}{\cdot}h_o = \widetilde{g}$ we obtain the series,

$$h_o = \sum_{p+q-r-s\,=\,k+2} c_{pqrs}\;(u^1_+)^{p-1}(u^2_+)^{q-1}(u^1_-)^r(u^2_-)^s\left(\frac{u^2_+u^1_-}{r+1} + \frac{u^1_+u^2_-}{s+1}\right).$$

This converges uniformly to a holomorphic solution of (5.11) on relatively compact neighbourhoods of the points of $\mathrm{Sp}_1(\mathbb{C})\backslash\mathcal{Y}$, where $\mathcal{Y} := \{(u^i_\pm) \in \mathrm{Sp}_1(\mathbb{C}) \mid u^1_+u^2_+ = 0\}$. Moreover, since there is no element of $\mathrm{Sp}_1(\mathbb{C})$, on which u^1_+ and u^2_+ are both zero, for any $U = (u^i_\pm) \in \mathcal{Y}$ we may replace the meromorphic functions $\widetilde{g}$ and h_o by equivalent functions $\widetilde{g}'$, h'_o in reduced form, both independent of either u^1_+ or u^2_+ and hence with no singularity at the chosen $U \in \mathcal{Y}$. This means that the functions $\widetilde{g}|_{\mathrm{Sp}_1(\mathbb{C})}$ and $h_o|_{\mathrm{Sp}_1(\mathbb{C})}$ are holomorphic on $\mathrm{Sp}_1(\mathbb{C}) \setminus \mathcal{Y}$ and extendable to all points of $\mathcal{Y}$, i.e. $\mathcal{Y}$ is a set of removable singularities for them. Thus, $f_o = h_o|_{\mathrm{Sp}_1(\mathbb{C})}$ is a solution of (5.11) which is holomorphic everywhere on $\mathrm{Sp}_1(\mathbb{C})$. To complete the proof it suffices to observe that if both $f_o\,, f\colon \mathrm{Sp}_1(\mathbb{C}) \to \mathbb{C}$ satisfy (5.11), their difference $\delta f = f - f_o$ satisfies (5.5) and (5.7). Therefore it is equal to $\delta f = \delta h|_{\mathrm{Sp}_1(\mathbb{C})}$ for some $\delta h\colon \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ of the form $\delta h = \sum_{\substack{m,n\geq 0\\ m+n=k}} c_{mn}(u^1_+)^m(u^2_+)^n$. From this, (a) and (b) follow immediately. □

We now consider an initial value problem for an important generalisation of the system (5.9) to harmonic space $\mathcal{H}|_{\mathcal{V}} = \mathrm{Sp}_1(\mathbb{C}){\times}\{I_{2n}\}{\times}\mathcal{V}$, $\mathcal{V} \subset \mathbb{C}^{4n}$. In what follows, we represent the elements $(U, I_{2n}, z) \in \mathcal{H}|_{\mathcal{V}}$ simply as (U, z).

Lemma 5.4. *Let $\mathcal{V} \subset \mathbb{C}^{4n}$ be a simply connected open neighbourhood of $0 \in \mathbb{C}^{4n}$. The system of differential equations on $\mathcal{H}|_{\mathcal{V}} := \mathrm{Sp}_1(\mathbb{C}) \times \mathcal{V}$ for* *holomorphic maps* $k = (k^{ic}) \colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}^{4n}$, $i = 1, 2$, $a, b, c = 1, \ldots, 2n$,

$$H_0^o \cdot k = 0 \ , \quad H_{++}^o \cdot k|_{(U,z)} = F(U, k(U, z)), \tag{5.12}$$

where $F = (F^{ia}) \colon \mathrm{Sp}_1(\mathbb{C}) \times \mathbb{C}^{4n} \to \mathbb{C}^{4n}$ *is holomorphic and satisfies the integrability condition*

$$H_0^o {\cdot} F = 2F \,, \tag{5.13}$$

admits

A) local *solutions around any point* $(u^i_{o\pm}, z^{ia}_o) \in \mathcal{H}|_{\mathcal{V}}$ *with arbitrary initial conditions* $k^{jd}(u^i_{o\pm}, z^{ia}_o) = c^{jd}$ *and*

B) *a* global *solution* k *on* $\mathcal{H}|_{\mathcal{V}}$ *for any choice of initial values* $k|_{(I_2, z)} = \widehat{k}(z)$, $z \in \mathcal{V}$, *having the property*

$$k(U, z) = k(\psi(U), z) \ , \qquad \psi(U) := (U^T)^{-1}. \tag{5.14}$$

Remark 5.5. The latter property is merely a tool in our proof of the existence of global solutions. It is by no means true that every global solution on $\mathcal{H}|_{\mathcal{V}}$ has this property.

Proof of A). The existence of a solution k of (5.12) is equivalent to the existence of a certain special submanifold $S^{(k)}$ corresponding to the graph of k in the cartesian product $\mathcal{N} := \mathcal{H}|_{\mathcal{V}} \times \mathbb{C}^{4n}$. Denote the coordinates of $\mathcal{N}$ by $(u^i_\pm, z^{ia}, w^{jd})$ and the standard projections onto its factors by $\pi_1 \colon \mathcal{N} \to \mathcal{H}|_{\mathcal{V}}$ and $\pi_2 \colon \mathcal{N} \to \mathbb{C}^{4n}$. Let $\widehat{F} \colon \mathcal{N} \to \mathbb{C}^{4n}$ be the map $\widehat{F}(u^i_\pm, z^{ia}, w^{jd}) := F(u^i_\pm, w^{jd})$ and $\widehat{H}_0$ and $\widehat{H}_{++}$ the vector fields on $\mathcal{N}$ uniquely determined at $x \in \mathcal{N}$ by the conditions,

$$\pi_{1*}(\widehat{H}_0|_x) = H_0^o|_{\pi_1(x)} \ , \qquad \pi_{1*}(\widehat{H}_{++}|_x) = H_{++}^o|_{\pi_1(x)} \ ,$$

$$\pi_{2*}(\widehat{H}_0|_x) = 0 \ , \qquad \pi_{2*}(\widehat{H}_{++}|_x) = \widehat{F}^{ia}(x) \frac{\partial}{\partial w^{ia}}\Big|_{\pi_2(x)} .$$

Further, let $\mathcal{D} \subset T\mathcal{N}$ be the complex distribution generated by $\widehat{H}_0$ and $\widehat{H}_{++}$. We immediately see that a map $k \colon \mathcal{W} \subset \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}^{4n}$ is a solution of (5.12) if and only if the vector fields of $\mathcal{D}$ are everywhere tangent to the graph $S^{(k)} \subset \mathcal{N}$ given by

$$S^{(k)} := \{(u^i_\pm, z^{ia}, w^{jd}) \mid (u^i_\pm, z^{ia}) \in \mathcal{W} \ , \ w^{jd} = k^{jd}(u^i_\pm, z^{ia})\}.$$

We have:

$$[\widehat{H}_0, \widehat{H}_{++}] = 2\widehat{H}_{++} + (\widehat{H}_0 {\cdot} \widehat{F}^{ia} - 2\widehat{F}^{ia}) \frac{\partial}{\partial w^{ia}} \ .$$

From (5.13) $\widehat{H}_0 {\cdot} \widehat{F}^{ia} - 2\widehat{F}^{ia} = 0$, so $\mathcal{D}$ is involutive. Let $x_o = (u^i_{o\pm}, z^{ia}_o, c^{jd}) \in \mathcal{N}$ and choose a disk $\Delta_\varepsilon \subset \mathbb{C}$ of radius ε and centre 0, a neighbourhood $\mathcal{V}' \subset \mathcal{V}$ of

(z_o^{ia}) and a holomorphic map (possibly constant) $\widehat{k}\colon \Delta_\varepsilon \times \mathcal{V}' \to \mathbb{C}^n$ such that $\widehat{k}^{jd}(0, z_o^{ia}) = c^{jd}$. If ε is sufficiently small, the set

$$\mathcal{T} := \left\{ \left(u^i_\pm = \exp(\zeta H^o_{--}) \cdot u^i_{o\pm}\ ,\ z^{ia}\ ,\ w^{jb} = \widehat{k}^{jd}(\zeta, z^{ia}) \right) \text{ with } \zeta \in \Delta_\varepsilon\ , z^{ia} \in \mathcal{V}' \right\}$$

is a $(4n{+}1)$-dimensional $\mathcal{D}$-transversal complex submanifold of $\mathcal{N}$. By the complex Frobenius Theorem, there exists a family of two-dimensional integral leaves of $\mathcal{D}$, each passing through a distinct point of $\mathcal{T}$, which combine to form a complex manifold of dimension $4n{+}3$ with the property that the vector fields in $\mathcal{D}$ are everywhere tangent to it. This submanifold is the graph $S^{(k)}$ of a map k in a neighbourhood of $(u^i_{o\pm}, z_o^{ia})$ such that $k^{jd}(u^i_{o\pm}, z_o^{ia}) = c^{jd}$. This is one of the required local solutions.

Proof of B). We now turn to the existence of global solutions. We recall that the standard transitive action of $\mathrm{Sp}_1(\mathbb{C})$ on $\mathbb{C}P^1$ yields a natural identification $\mathbb{C}P^1 \simeq \mathrm{Sp}_1(\mathbb{C})/B$, where B is the Borel subgroup formed by upper triangular matrices in $\mathrm{Sp}_1(\mathbb{C})$,

$$B := \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix}\ ,\ \lambda \in \mathbb{C}^*\ , \mu \in \mathbb{C} \right\} \simeq \mathbb{C}^* \times \mathbb{C}.$$

The affine subspaces of $\mathbb{C}P^1$

$$\mathbb{C}_{(0)} = \{[1 : \zeta]\,;\, \zeta \in \mathbb{C}\} \quad \text{and} \quad \mathbb{C}_{(\infty)} = \{[\zeta : 1]\,;\, \zeta \in \mathbb{C}\},$$

can be identified with the cosets in $\mathrm{Sp}_1(\mathbb{C})/B$ given by the points of

$$\widetilde{\mathbb{C}}_{(0)} := \left\{ \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}\ ,\ \zeta \in \mathbb{C} \right\} = \exp(\mathbb{C} H^o_{--})$$

and

$$\widetilde{\mathbb{C}}_{(\infty)} := \left\{ \begin{pmatrix} \zeta & -1 \\ 1 & 0 \end{pmatrix}\ ,\ \zeta \in \mathbb{C} \right\} = J_o \cdot \exp(\mathbb{C} H^o_{--})\,, \quad J_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. This means that $\mathcal{H}|_\mathcal{V}$ is the union of the two patches,

$$\mathcal{H}|_\mathcal{V} = \mathrm{Sp}_1(\mathbb{C}) \times \mathcal{V} = (\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}) \cdot B \cup (\widetilde{\mathbb{C}}_{(\infty)} \times \mathcal{V}) \cdot B\,, \tag{5.15}$$

with their intersection (a tube over an annulus) having two equivalent descriptions,

$$\begin{aligned}(\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}) \cdot B \cap (\widetilde{\mathbb{C}}_{(\infty)} \times \mathcal{V}) \cdot B &= (\exp(\mathbb{C}^* H^o_{--}) \times \mathcal{V}) \cdot B \\ &= (J_o \cdot \exp(\mathbb{C}^* H^o_{--}) \times \mathcal{V}) \cdot B\,. \end{aligned} \tag{5.16}$$

Now, every B-orbit $x \cdot B$, $x \in \mathcal{H}|_\mathcal{V}$, is biholomorphic to $B \simeq \mathbb{C}^* \times \mathbb{C}$ and the non-trivial elements of its fundamental group $\pi_1(x \cdot B)$ are given by the H^o_0-orbits in $x \cdot B$. A local solution of (5.12) is constant along any open subset of an H^o_0-orbit. Using this and the existence of local solutions around every point, we see that if there exists a solution k on a given simply connected open subset $\mathcal{S} \subset \mathcal{H}|_{\mathcal{V}}$, it can always be extended to a solution defined on the union of B-orbits $\mathcal{S}{\cdot}B := \bigcup_{y\in\mathcal{S}} y{\cdot}B$.

Now consider a simply connected subset $\mathcal{Z} \subset \mathcal{H}|_{\mathcal{V}}$ transversal to the B-orbits. It may be covered by a collection of open sets $\mathcal{W}_x \subset \mathcal{H}|_{\mathcal{V}}$, $x \in \mathcal{Z}$, each admitting, by part A, a local solution with arbitrary initial data on $\mathcal{W}_x \cap \mathcal{Z}$. The initial conditions can be chosen so that the solutions agree on non-empty intersections $\mathcal{W}_x \cap \mathcal{W}_{x'}$, $x' \neq x$. By the simple connectedness of $\mathcal{Z}$ these solutions combine to give a solution on a neighbourhood of $\mathcal{Z}$ for any choice of initial data $\widetilde{k}$ on $\mathcal{Z}$. Such a solution uniquely extends to $\mathcal{Z}{\cdot}B$.

Since $\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}$ is simply connected and transversal to B-orbits it follows that for any choice of data on $\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}$

$$\widetilde{k}\colon \widetilde{\mathbb{C}}_{(0)} \times \mathcal{V} = \exp(\mathbb{C}H^o_{--}) \times \mathcal{V} \longrightarrow \mathbb{C}^{4n}$$

there is a unique solution k on the collection of B-orbits $(\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}){\cdot}B$ with $k|_{\widetilde{\mathbb{C}}_{(0)}\times\mathcal{V}} = \widetilde{k}$. We now prove that the solution k satisfies $k(U,z) = k(\psi(U),z)$ and $k(I_2,z) = \widehat{k}(z)$, provided that $\widetilde{k}$ is chosen appropriately.

Let $K\colon B \times \mathcal{V} \to \mathbb{C}^{4n}$ be the unique holomorphic function satisfying (5.12) at the points of the B-orbits $(I_2,z){\cdot}B$, $z \in \mathcal{V}$, with initial data $K(I_2,z) = \widehat{k}(z)$. Then set $\widetilde{k}$ to be the unique holomorphic function at the points $(\exp(\zeta H_{--}),z)$ such that $\widetilde{k}(I_2,z) = \widehat{k}(z)$ and

$$H^o_{--}{\cdot}\widetilde{k}(\exp(\zeta H_{--}),z) = -F(\psi(\exp(\zeta H^o_{--})), K(\psi(\exp(\zeta H^o_{--})),z))\,.$$

Now consider the modified differential problem on maps $h\colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}^{4n}$

$$\begin{aligned} H^o_0{\cdot}h|_{(U,z)} &= 0\\ H^o_{++}{\cdot}h|_{(U,z)} &= F(U\,,\, h(U,z))\\ H^o_{--}{\cdot}h|_{(U,z)} &= -F(\psi(U)\,,\, h(\psi(U),z))\,. \end{aligned} \tag{5.17}$$

Note that (5.17) is simply (5.12) with the addition of a third equation, which is non-local; the right hand side depends on the value of h at the shifted point $(\psi(U),z)$. However, every solution h of (5.12) satisfying $h(U,z) = h(\psi(U),z)$, necessarily satisfies the third equation of (5.17) also. Indeed, since $\psi_*(H^o_0) = -H^o_0$, $\psi_*(H^o_{\pm\pm}) = -H^o_{\mp\mp}$ (see (2.6)), we have:

$$\begin{aligned} H^o_{--}{\cdot}h\big|_{(U,z)} &= H^o_{--}{\cdot}h(\psi(\cdot),\cdot))\big|_{(U,z)} = (\psi_*(H^o_{--}){\cdot}h)\big|_{(\psi(U),z)}\\ &= -(H^o_{++}{\cdot}h)\big|_{(\psi(U),z)} = -F(\psi(U)\,,\, h(\psi(U),z))\ . \end{aligned}$$

The solution k of (5.12), which we constructed on $(\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}){\cdot}B$ with initial data $k|_{\widetilde{\mathbb{C}}_{(0)}\times\mathcal{V}} = \widetilde{k}$, clearly solves the first two equations of (5.17) and by the above choice of $\widetilde{k}$, it also satisfies the third equation at the points $(\widehat{U},\widehat{z}) \in \widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}$. Further, on the points $(U^{(\lambda)}, z) = (\widehat{U},\widehat{z})\cdot \exp(\lambda H^o_{++})$, $\lambda \in \mathbb{C}$, of their H^o_{++}-orbits we have

$$\begin{aligned}
H^o_{--}\cdot k|_{(U^{(\lambda)},z)} &= H^o_{--}\cdot k|_{(\widehat{U},\widehat{z})} + \int_0^\lambda H^o_{++}\cdot H^o_{--}\cdot k|_{(U^{(\mu)},\widehat{z})}\, d\mu \\
&= H^o_{--}\cdot k|_{(\widehat{U},\widehat{z})} + \int_0^\lambda \left(H^o_0 + H^o_{--}\cdot H^o_{++}\right)\cdot k|_{(U^{(\mu)},\widehat{z})}\, d\mu \\
&= -F(\psi(\widehat{U}), k(\psi(\widehat{U}),\widehat{z})) + \int_0^\lambda H^o_{--}\cdot F(\cdot, k(\cdot,\cdot))|_{(U^{(\mu)},\widehat{z})}\, d\mu \\
&= -F(\psi(\widehat{U}), k(\psi(\widehat{U}),\widehat{z})) - \int_0^\lambda H^o_{++}\cdot F(\psi(\cdot), k(\psi(\cdot),\cdot))|_{(\psi(U^{(\mu)}),\widehat{z})} d\mu \\
&= -F(\psi(U^{(\lambda)}), k(\psi(U^{(\lambda)}), z))\ .
\end{aligned}$$

Thus k solves the third equation in (5.17) at points of $(\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V})\cdot \exp(\mathbb{C}H^o_{++})$ as well. A similar argument shows that it solves the third equation also at the points of the H^o_0-orbits in

$$((\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V})\cdot \exp(\mathbb{C}H^o_{++}))\cdot \exp(\mathbb{C}H^o_0) = (\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V}) \cdot B\,.$$

So, k solves (5.17) at *all* points of its domain.

Now, the new map $k'(U,z) := k(\psi(U),z)$ satisfies

$$\begin{aligned}
&H^o_0\cdot k'|_{(U,z)} = -(H^o_0\cdot k)|_{(\psi(U),z)} = 0 \\
&H^o_{++}\cdot k'|_{(U,z)} = \psi_*(H^o_{++})\cdot k|_{\psi(U,z)} = -H^o_{--}\cdot k|_{(\psi(U),z)} = F(U, k(U,z)) \\
&H^o_{--}\cdot k'|_{(U,z)} = \psi_*(H^o_{--})\cdot k|_{(\psi(U),z)} = -H^o_{++}\cdot k|_{(\psi(U),z)} = -F(\psi(U)\,,\, k(\psi(U),z)).
\end{aligned}$$

So, k and k' are both solutions of the system

$$\begin{aligned}
H^o_0\cdot h|_{(U,z)} &= 0 \\
H^o_{++}\cdot h|_{(U,z)} &= F(U\,,\, k(U,z)) \\
H^o_{--}\cdot h|_{(U,z)} &= -F(\psi(U)\,,\, k(\psi(U),z))
\end{aligned} \tag{5.18}$$

with identical initial data $k'|_{\{I_2\}\times\mathcal{V}} = \widehat{k} = k|_{\{I_2\}\times\mathcal{V}}$. We thus have $k' \equiv k$ by the uniqueness of local solutions of (5.18) and the connectedness of the domain $(\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V})\cdot B$. This concludes the proof that $k(U,z) = k(\psi(U),z)$.

We now show that the solution k extends holomorphically to a solution defined on all of $\mathcal{H}|_{\mathcal{V}}$. Since $k(U,z) = k(\psi(U),z)$, we have

$$k|_{\left(\left(\begin{smallmatrix}\zeta & \zeta-1\\ 1 & 1\end{smallmatrix}\right),z\right)} = k|_{\left(\left(\begin{smallmatrix}1 & -1\\ 1-\zeta & \zeta\end{smallmatrix}\right),z\right)} \qquad \text{for } \zeta \in \mathbb{C}^*\ .$$

Since k is holomorphic on $(\widetilde{\mathbb{C}}_{(0)}\times\mathcal{V})\cdot B$ and $\left(\left(\begin{smallmatrix}1 & -1\\ 1-\zeta & \zeta\end{smallmatrix}\right), z\right) = \left(\left(\begin{smallmatrix}1 & 0\\ 1-\zeta & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right), z\right)$ belongs to $(\widetilde{\mathbb{C}}_{(0)} \times \mathcal{V})\cdot B$ for any $\zeta \in \mathbb{C}$ (including $\zeta = 0$), it follows that for every $z \in \mathcal{V}$ the map on $\mathbb{C}^*$

$$\zeta \longmapsto k|_{\left(\left(\begin{smallmatrix} \zeta & \zeta-1 \\ 1 & 1 \end{smallmatrix}\right), z\right)}$$

admits a holomorphic extension to $\zeta = 0$. Now, the B-orbits of the points $\left(\begin{smallmatrix} \zeta & \zeta-1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \zeta & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generate the entire set

$$(\widetilde{\mathbb{C}}_{(\infty)} \times \mathcal{V})\cdot B = \bigcup_{\zeta \in \mathbb{C}, z \in \mathcal{V}} \left(\left(\begin{smallmatrix} \zeta & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), z\right) \cdot B \ .$$

So we may take solutions of (5.12) along B-orbits having $k\big|_{\left(\left(\begin{smallmatrix} \zeta & \zeta-1 \\ 1 & 1 \end{smallmatrix}\right), z\right)}$ as initial values and combine them into a holomorphic extension of k to $(\widetilde{\mathbb{C}}_{(\infty)} \times \mathcal{V})\cdot B$. In virtue of (5.15) and (5.16), k extends to $\mathcal{H}|_{\mathcal{V}}$ and, by continuity, it satisfies (5.14) everywhere. □

Remark 5.6. Given a global solution $k = (k^{ic})\colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}^{4n}$ of (5.12), we set

$$k^{\pm c}(U, z) := -u_i^{\pm} k^{ic}(U, z) \ , \qquad U = (u^i_{\pm}) \in \mathrm{Sp}_1(\mathbb{C}) \ , \ z \in \mathcal{V}.$$

For any $k^{\pm c}$, the corresponding k^{ia} are recovered using the inverse formula $k^{ia} = u^i_+ k^{+a} + u^i_- k^{-a}$. The lemma says that there exists a global solution to (5.12), (5.14) for any choice of initial values $\widehat{k}^{\pm a} = k^{\pm a}|_{\mathcal{V} \times \{I_2\}}\colon \mathcal{V} \times \{I_2\} \to \mathbb{C}^{2n}$.

6. The existence of canonical hk-pairs

Proof of Theorem 4.5

To begin, we need the following:

Lemma 6.1. *In every local equivalence class of hk-pairs of signature $(4p, 4q)$ there exists an hk-pair $(\mathcal{A}, M)$ with $\mathcal{A}$ central (Def. 3.4) and $M \subset \{I_2\} \times \{I_{2n}\} \times \mathbb{C}^{4n}$.*

Proof. As shown in Sect. 4.2, every hk-pair $(\mathcal{A}, M)$ is associated with a pseudohyperkähler metric g on M. We may then use a local system of coordinates to identify M with an open subset $M' \subset \mathbb{R}^{4n}$, g with a pseudohyperkähler metric g' on M' and the hk-frame $\mathcal{A}$ with the hk-frame $\mathcal{A}'$ of holomorphic extensions of vertical and horizontal vector fields of the covering of the holonomy bundle of (M', g') with structure group $\mathrm{Sp}_1 \times \mathrm{Sp}_n$. This means that $(\mathcal{A}, M)$ is locally equivalent to $(\mathcal{A}', M')$ and the explicit construction of the holomorphic extensions that give the vector fields in $\mathcal{A}'$ (see Lemma A16) shows that that $\mathcal{A}'$ is a central hk-frame and $M' \subset \{I_2\}\times\{I_{2n}\}\times\mathbb{C}^{4n}$. □

It follows from this lemma that in order to prove that every equivalence class of hk-pairs includes a canonical one, it suffices to show the following: Given an hk-pair $(\mathcal{A}, M)$, with $\mathcal{A} = (H^o_0, H^o_{\pm\pm}, E^o_A, e_{\pm a})$ central and M contained in $\{I_2\}\times\{I_{2n}\}\times\mathbb{C}^{4n}$ and passing through $e=(I_2,I_{2n},0)$, there exists a local biholomorphism $\varphi\colon\mathcal{U}\to\mathcal{U}'$, with $\varphi(e)=e$, between two appropriate open sets mapping $\mathcal{A}$ into a canonical hk-frame $\varphi_*(\mathcal{A})$ having central coordinate components $\varphi=(\varphi^a_b,\varphi^i_\pm,\varphi^{ia})$ with $\varphi^i_\pm(U,B,z)=u^i_\pm$. Indeed, if we are able to prove this, we immediately have that $(\varphi_*(\mathcal{A}),\varphi(M))$ is a canonical hk-pair in the local equivalence class of $(\mathcal{A},M)$, as desired.

Let $(\mathcal{A},M)$ be an hk-pair on an appropriate open subset $\mathcal{U}\subset\mathcal{P}$ with $\mathcal{A}=(H^o_0,H^o_{\pm\pm},E^o_A,e_{\pm a})$ central and $M\subset\{I_2\}\times\{I_{2n}\}\times\mathbb{C}^{4n}$. With no loss of generality, we may assume that the restriction of e_{-a} to $\mathcal{V}=\mathcal{U}\cap\{I_2\}\times\{I_{2n}\}\times\mathbb{C}^{4n}$ has the form $e_{-a}|_\mathcal{V}=\frac{\partial}{\partial z^{-a}}\big|_\mathcal{V}+A^B_{-a}E^o_B|_\mathcal{V}$. Indeed, this form can always be attained by applying a biholomorphism of the form $(U,B,z)\mapsto(U,B,\psi(z))$ to $(\mathcal{A},M)$, for some appropriate local transformation ψ of $\mathbb{C}^{4n}$. Such appropriate transformation ψ surely exists because the images of the vector fields e_{-a} on $\mathbb{C}^{4n}$ under the standard projection $\pi\colon\mathcal{P}\to\mathbb{C}^{4n}$ are commuting vector fields.

Let us now show the existence of a local biholomorphism φ, with $\varphi^i_\pm=u^i_\pm$, that maps $\mathcal{A}$ to a canonical hk-frame. We denote the components in central coordinates of the required biholomorphism as $\varphi=(\varphi^A)=(\varphi^i_\pm=u^i_\pm\,,\,\varphi^a_b\,,\,\varphi^{ia})$ and those in analytic coordinates as $(\varphi^i_\pm=u^i_\pm\,,\,\varphi^a_b\,,\,\varphi^{\pm a}=-\,u^\pm_i\varphi^{ia})$. The images of the vector fields in $\mathcal{A}$ under such a biholomorphism φ have the form:

$$\begin{aligned}
\varphi_*(H^o_0)&=H^o_0+H^o_0\cdot\varphi^a_b\frac{\partial}{\partial B^a_b}+H^o_0\cdot\varphi^{ia}\frac{\partial}{\partial z^{ia}}\\
&=H^o_0+H^o_0\cdot\varphi^a_b\frac{\partial}{\partial B^a_b}+(H^o_0\cdot\varphi^{+a}-\varphi^{+a})\frac{\partial}{\partial z^{+a}}+(H^o_0\cdot\varphi^{-a}+\varphi^{-a})\frac{\partial}{\partial z^{-a}}\\
\varphi_*(H^o_{\pm\pm})&=H^o_{\pm\pm}+H^o_{\pm\pm}\cdot\varphi^a_b\frac{\partial}{\partial B^a_b}+H^o_{\pm\pm}\cdot\varphi^{ia}\frac{\partial}{\partial z^{ia}}\\
\varphi_*(E^o_B)&=E^o_B\cdot\varphi^{ia}\frac{\partial}{\partial z^{ia}}+E^o_B\cdot\varphi^a_b\frac{\partial}{\partial B^a_b}\\
\varphi_*(e_{\pm a})&=e_{\pm a}\cdot\varphi^c_b\frac{\partial}{\partial B^c_b}+e_{\pm a}\cdot\varphi^{ib}\frac{\partial}{\partial z^{ib}}\ .
\end{aligned}$$

Hence the pushed-forward hk-frame $\varphi_*(\mathcal{A})$ is canonical if and only if φ satisfies the following four conditions.

i) $\varphi_*(E^o_B)-E^o_B=0$, which means that φ^{ia} does not depend on B^a_b and φ^a_b has the form

$$\varphi^a_b\left((u^i_\pm),(B^e_f),(z^{ja})\right)=\varphi^a_c\left((u^i_\pm),I_2,(z^{ja})\right)B^c_b \tag{6.1}$$

ii) $\varphi_*(H^o_0)-H^o_0=0$ and $\varphi_*(e_{+a})-e^o_{+a}=0$, which are equivalent to

$$H^o_0\cdot\varphi^a_b=0\,,\qquad H^o_0\cdot\varphi^{\pm a}=\mp\varphi^{\pm a}, \tag{6.2}$$

$$e_{+a}\cdot\varphi^c_b=0\,,\qquad e_{+a}\cdot\varphi^{-b}=0\,,\qquad e_{+a}\cdot\varphi^{+b}=\varphi^b_a \tag{6.3}$$

iii) the components v^{+b}_{++} of the vector field

$$H_{++} = \varphi_*(H^o_{++}) = H^o_{++} + v^{\pm b}_{++} e^o_{\pm b} + A^B_{++} E^o_B \ ,$$

which are given by $v^{+b}_{++} = -u^+_i (H^o_{++} \cdot \varphi^{ib}) = H^o_{++} \cdot \varphi^{+b} - \varphi^{-b}$, are such that $v^{+b}_{++}|_{\{z^{+a}=0\}} = 0$

iv) the components v^{-b}_{-a} of of the vector field

$$\delta e_{-a} = \varphi_*(e_{-a}) - e^o_{-a} = v^{\pm b}_{-a} e^o_{\pm b} + A^B_{-a} E^o_B \ ,$$

which are given by $v^{-b}_{-a} = e_{-a} \cdot \varphi^{-b} - \varphi^a_b$, are identically equal to 0.

It remains to prove that there exists a $\varphi = (\varphi^i_\pm = u^i_\pm \, , \, \varphi^a_b \, , \, \varphi^{ia})$ satisfying (i)-(iv), with $\varphi(I_2, I_{2n}, 0) = (I_2, I_{2n}, 0)$. First we define $\widehat{\mathcal{V}} := \mathcal{V} \cap \{z^{+a} = 0\}$, with $\mathcal{V} := \mathcal{U} \cap \{I_2\} \times \{I_{2n}\} \times \mathbb{C}^{4n}$, and consider a holomorphic map

$$(g^{-b})\colon \mathcal{H}|_{\widehat{\mathcal{V}}} \cdot \mathrm{Sp}_n(\mathbb{C}) := \mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C}) \times \widehat{\mathcal{V}} \to \mathbb{C}^{2n}$$

independent of B, with charge $+1$, such that $g^{-b}(I_2, I_{2n}, 0) = 0$ and $e_{-a} \cdot g^{-b}|_{(I_2, I_{2n}, 0)} = \delta^b_a$. Second, we set

$$g^b_a \colon \mathcal{H}|_{\widehat{\mathcal{V}}} \cdot \mathrm{Sp}_n(\mathbb{C}) \to \mathbb{C} \ , \qquad g^b_a := e_{-a} \cdot g^{-b} \ . \tag{6.4}$$

Third, using Lemma 5.3, we determine functions $g^{+a}\colon \mathcal{H}|_{\widehat{\mathcal{V}}} \cdot \mathrm{Sp}_n(\mathbb{C}) \to \mathbb{C}$ satisfying

$$H^o_0 \cdot g^{+b} = -g^{+b}, \qquad H^o_{++} \cdot g^{+b} = g^{-b}, \tag{6.5}$$

with initial data chosen to be independent of B and with $g^{+a}(I_2, I_{2n}, 0) = 0$.

We now extend the functions $g^{\pm a}, g^c_d \colon \mathcal{H}|_{\widehat{\mathcal{V}}} \cdot \mathrm{Sp}_n(\mathbb{C}) \to \mathbb{C}$ to holomorphic functions $\varphi^{\pm c}, \varphi^c_d$ on an appropriate open set $\mathcal{U} = \mathcal{H}|_{\mathcal{V}} \cdot \mathrm{Sp}_n(\mathbb{C})$ as follows. First, we consider the points $y \in \mathcal{P}$ of the form

$$y(x, t^1, \ldots, t^{2n}) := \Phi^{e_{+1}}_{t^1} \circ \ldots \circ \Phi^{e_{+2n}}_{t^{2n}}(x) \ , \qquad x \in \mathcal{P}|_{\widehat{\mathcal{V}}} \ , \ t^j \in \Delta_\varepsilon(0) \subset \mathbb{C} \ , \tag{6.6}$$

where $\Phi^{e_{+b}}_s$ is the holomorphic flow of the vector field e_{+b} parametrised by s. Second, we set

$$\begin{aligned}
&\varphi^{-b}(y(x, t^1, \ldots, t^{2n})) := g^{-b}(x) \ , \\
&\varphi^{+b}(y(x, t^1, \ldots, t^{2n})) := g^b_a(x) t^a + g^{+b}(x), \\
&\varphi^b_c(y(x, t^1, \ldots, t^{2n})) := g^b_c(x).
\end{aligned}$$

By construction, the map $\varphi = (\varphi^i_\pm = u^i_\pm, \varphi^a_b, \varphi^{\pm a})$ is such that $\varphi(I_2, I_{2n}, 0) = (I_2, I_{2n}, 0)$ and it satisfies (i). Since $g^a_b, g^{\pm a}$ satisfy (6.4) and (6.5), the map φ is a solution of (6.2), (6.3), hence it satisfies (ii). Moreover, (6.5) implies $(H^o_{++} \cdot \varphi^{+b} - \varphi^{-b})|_{\{z^{+a}=0\}} = 0$, so that (iii) holds. Finally, from (6.4) we see that also (iv) is satisfied, meaning that φ is a bridge.

To conclude the proof of Theorem 4.5, it remains to show that canonical hk-pairs with the same v-potential have the same hk-frame and are locally equivalent. Let $(\mathcal{A}, M)$, $(\mathcal{A}', M')$ be canonical hk-pairs on an appropriate open set $\mathcal{U} \subset \mathcal{P}$, with bridges φ, φ' and identical v-potentials,

$$v_{++}^{-a}|_{\mathcal{H}|_{\mathcal{V}}} = v'^{-a}_{++}|_{\mathcal{H}|_{\mathcal{V}}} \ , \qquad \mathcal{H}|_{\mathcal{V}} := \mathcal{U} \cap \mathcal{H} \ .$$

We claim that all components of H_{++} (and, similarly, of H'_{++}) are completely determined by the v-potentials $v_{++}^{-a}|_{\mathcal{H}|_{\mathcal{V}}}$ $(= v'^{-a}_{++}|_{\mathcal{H}|_{\mathcal{V}}})$. For this, we first observe that $[H_0^o, H_{++}] = 0$ implies that the components v_{++}^{-a}, v_{++}^{+a}, $A_{++}^B (E_B^o)_a^b$ of H_{++} have charges $k = 3, 1$ and 2, respectively. Further, the relation $[E_A^o, H_{++}] = 0$ implies that these components are $\mathfrak{sp}_n(\mathbb{C})$-equivariant and hence are uniquely determined by their restrictions to $\mathcal{H}|_{\mathcal{V}}$. It therefore suffices to check that $v_{++}^{+a}|_{\mathcal{H}|_{\mathcal{V}}}$, $A_{++}^B (E_B^o)_a^b|_{\mathcal{H}|_{\mathcal{V}}}$ are uniquely determined by the v-potential $v_{++}^{-a}|_{\mathcal{H}|_{\mathcal{V}}}$.

We now recall that $e_{+a}^o = [H_{++}, e_{-a}]$ and $[H_{++}, e_{+a}^o] = 0$. Expanding all vector fields in terms of the flat hk-frame, we get

$$\begin{aligned}
e_{+a}^o = [H_{++}, e_{-a}] &= [H_{++}^o + v_{++}^{\pm b} e_{\pm b}^o + A_{++}^A E_A^o \, , \, e_{-a}^o + v_{-a}^{+b} e_{+b}^o + A_{-a}^B E_B^o] \\
&= \delta_a^b e_{+b}^o + H_{++}^o \cdot v_{-a}^{+b} e_{+b}^o + H_{++}^o \cdot A_{-a}^B E_B^o - e_{-a}^o \cdot v_{++}^{+b} e_{+b}^o - e_{-a}^o \cdot v_{++}^{-b} e_{-b}^o \\
&\quad + v_{++}^{+c} e_{+c}^o \cdot v_{-a}^{+b} e_{+b}^o + v_{++}^{-c} e_{-c}^o \cdot v_{-a}^{+b} e_{+b}^o - v_{-a}^{+c} e_{+c}^o \cdot v_{++}^{+b} e_{+b}^o - v_{-a}^{+c} e_{+c}^o \cdot v_{++}^{-b} e_{-b}^o \\
&\quad + v_{++}^{+c} e_{+c}^o \cdot A_{-a}^B E_B^o + v_{++}^{-c} e_{-c}^o \cdot A_{-a}^B E_B^o + A_{++}^B (E_B^o)_a^b e_{-b}^o - e_{-b}^o \cdot A_{++}^B E_B^o \\
&\quad + A_{++}^A (E_A^o)_a^c v_{-c}^{+b} e_{+b}^o - v_{-a}^{+c} e_{+c}^o \cdot A_{++}^B E_B^o + A_{++}^A (E_A^o)_a^c A_{-c}^B E_B^o \\
&= \left(-e_{-a}^o \cdot v_{++}^{-b} - v_{-a}^{+c} e_{+c}^o \cdot v_{++}^{-b} + A_{++}^A (E_A^o)_a^b \right) e_{-b} \mod \{ e_{+b}^o, E_B^o \} \ ,
\end{aligned}$$

$$\begin{aligned}
0 = [H_{++}, e_{+a}^o] &= [H_{++}^o + v_{++}^{\pm b} e_{\pm b}^o + A_{++}^A E_A^o \, , \, e_{+a}^o] \\
&= \left(A_{++}^A (E_A^o)_a^b - e_{+a}^o \cdot v_{++}^{+b} \right) e_{+b}^o - e_{+a}^o \cdot v_{++}^{-b} e_{-b}^o - e_{+a}^o \cdot A_{++}^A E_A^o \ .
\end{aligned}$$

It follows that

$$\begin{aligned}
e_{+a}^o \cdot v_{++}^{-b} &= 0 \ , \\
A_{++}^A (E_A^o)_a^b &= e_{-a}^o \cdot v_{++}^{-b} + v_{-a}^{+c} \, e_{+c}^o \cdot v_{++}^{-b} = e_{-a}^o \cdot v_{++}^{-b} \ , \\
e_{+a}^o \cdot v_{++}^{+b} &= A_{++}^A (E_A^o)_a^b = e_{-a}^o \cdot v_{++}^{-b} \ .
\end{aligned} \tag{6.7}$$

Since $v_{++}^{+a}|_{\{z^{+a}=0\}} = 0$, these equations show that v_{++}^{+a} and $A_{++}^B (E_B^o)_a^b$ are uniquely determined by the (first derivatives of the) functions v_{++}^{-b}, as claimed. So H_{++} (and H'_{++}) is completely determined by the v-potential, as claimed. Since the two v-potentials are equal, it follows also that $H_{++} = H'_{++}$.

Now, applying the inverse of the bridge φ to both hk-frames $\mathcal{A}$ and $\mathcal{A}'$ we obtain the new hk-frames

$$\begin{aligned}
\varphi_*^{-1}(\mathcal{A}) &= (H_0^o, H_{++}^o, H_{--}^o, E_A^o, \widehat{e}_{\pm a} := \varphi_*^{-1}(e_{\pm a})) \ , \\
\varphi_*^{-1}(\mathcal{A}') &= (H_0^o, H_{++}^o, \widehat{H}'_{--} := \varphi_*^{-1}(H'_{--}), E_A^o, \widehat{e}'_{\pm a} := \varphi_*^{-1}(e'_{\pm a})) \ ,
\end{aligned}$$

where we used the facts that $H_{++} = H'_{++}$ and that φ is a bridge from a central hk-frame to the canonical hk-frame $\mathcal{A}$. Now, if we can prove that $\widehat{H}'_{--} = H^o_{--}$, it would immediately follow that $H'_{--} = \varphi_*(\widehat{H}'_{--})$ coincides with H_{--} and that $e'_{-a} = [H'_{--}, e^o_{+a}] = [H_{--}, e^o_{+a}] = e_{-a}$, meaning that $\mathcal{A} = \mathcal{A}'$.

Since $\varphi^i_{\pm} = u^i_{\pm}$, the vector field $\widehat{H}'_{--}$ has the form

$$\widehat{H}'_{--} = H^o_{--} + v^{+a}_{--} e^o_{+a} + v^{-a}_{--} e^o_{-a} + A^B_{--} E_B \ .$$

On the other hand,

$$[H^o_0, \widehat{H}'_{--}] = \varphi_*^{-1}([H^o_0, H'_{--}]) = \varphi_*^{-1}(-2H'_{--}) = -2\widehat{H}'_{--} \ .$$

Thus the components v^{+a}_{--} , v^{-a}_{--} and A^B_{--} have charges -3, -1 and -2, respectively. Further,

$$\begin{aligned} H^o_0 &= \varphi_*^{-1}([H'_{++}, H'_{--}]) = [H^o_{++}, \widehat{H}'_{--}] \\ &= [H^o_{++}\ ,\ H^o_{--} + v^{+a}_{--} e^o_{+a} + v^{-a}_{--} e^o_{-a} + A^B_{--} E_B] \end{aligned}$$

implies that

$$H^o_{++} \cdot v^{+a}_{--} + v^{-a}_{--} = 0\,, \qquad H^o_{++} \cdot v^{-a}_{--} = 0\,, \qquad H^o_{++} \cdot A^B_{--} = 0\,.$$

Since the functions v^{+a}_{--} , v^{-a}_{--} and A^B_{--} are negatively charged, they vanish by Lemma 5.3. Thus $\widehat{H}'_{--} = H^o_{--}$ and $\mathcal{A} = \mathcal{A}'$, as required.

We now observe that by definition of hk-pairs, the $\mathrm{Sp}_{p,q}$-orbits of the points of M and M' (namely, the submanifolds $\mathfrak{U}^{(\mathrm{Sp}_{p,q})}$, $\mathfrak{U}'^{(\mathrm{Sp}_{p,q})} \subset \mathfrak{U}$ defined in (4.3)) determine two integral submanifolds of the distribution generated by the vector fields in $\alpha^{\mathcal{A}}_{(\mathbb{R})}(V^\tau + \mathfrak{sp}_{p,q})$. Since $e = (I_2, I_{2n}, 0)$ belongs to both of them, $\mathfrak{U}^{(\mathrm{Sp}_{p,q})} = \mathfrak{U}'^{(\mathrm{Sp}_{p,q})}$ and M' can be identified with a section of the (trivial) $\mathrm{Sp}_{p,q}$-bundle $\pi : \mathfrak{U}^{(\mathrm{Sp}_{p,q})} \simeq M \times \mathrm{Sp}_{p,q} \to M$. So, if we construct pseudo-hyperkähler metrics g and g' on M and M', respectively, as in Sect.4.2, we see that the projection $\pi|_{M'} : M' \to M$ maps the vielbeins of (M', g') onto vielbeins of (M, g) and is therefore an isometry between (M', g') and (M, g). Proposition 4.3 implies that $(\mathcal{A}, M)$ and $(\mathcal{A} = \mathcal{A}', M')$ are locally equivalent. This concludes the proof of Theorem 4.5.

7. Parameterisation of canonical hk-pairs

Proof of Theorem 4.6

The proof is divided into two steps. We first need to prove that for every prepotential $\mathcal{L}_{(+4)}$ there exists a canonical hk-pair whose v-potential is related to $\mathcal{L}_{(+4)}$ by (4.8). We then need to prove the converse statement: every canonical hk-pair has a uniquely associated prepotential satisfying (4.8).

Step 1: Existence of a canonical hk-pair for every prepotential

Consider a charge $k{=}4$ holomorphic map $\mathcal{L}_{(+4)}\colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}$ on an harmonic space $\mathcal{H}|_{\mathcal{V}}$ satisfying (4.7). It follows from (6.7) that if there exists an hk-pair $(\mathcal{A}, M)$ with v-potential satisfying (4.8), then the components of the vector field

$$H_{++} = H^o_{++} + v^{+c}_{++} e^o_{+c} + v^{-c}_{++} e^o_{-c} + A^C_{++} E^o_C \tag{7.1}$$

necessarily have the following form at any point $x{\cdot}B \in \mathcal{U} := \mathcal{H}|_{\mathcal{V}}{\cdot}\mathrm{Sp}_n(\mathbb{C})$, $x \in \mathcal{H}$, $B \in \mathrm{Sp}_n(\mathbb{C})$,

$$\begin{aligned} v^{-a}_{++}|_{x\cdot B} &= (B^{-1})^a_c\, \omega^{cd} \frac{\partial \mathcal{L}_{(+4)}}{\partial z^{-d}}\Big|_x \\ A^B_{++}(E^o_B)^a_b|_{x\cdot B} &= (B^{-1})^a_c\, B^d_b\, \omega^{ce} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-d}\partial z^{-e}}\Big|_x \\ v^{+a}_{++}|_{x\cdot B} &= (B^{-1})^a_b\, \widetilde{v}^{+b}_{++}|_x \ . \end{aligned} \tag{7.2}$$

Here $(\omega^{ab}) = (\omega_{ab})^{-1}$ and the functions $\widetilde{v}^{+a}_{++}\colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}^{2n}$ are solutions to the differential problem

$$\frac{\partial \widetilde{v}^{+a}_{++}}{\partial z^{+b}} = \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-b}\partial z^{-c}}\ , \qquad \widetilde{v}^{+a}_{++}|_{\{z^{+a}=0\}} \equiv 0\ .$$

Since $e^o_{+a}{\cdot}\mathcal{L}_{(+4)} = \frac{\partial \mathcal{L}_{(+4)}}{\partial z^{+a}} = 0$, this has a unique solution, linear in z^{+b},

$$\widetilde{v}^{+a}_{++} = \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-b}\partial z^{-c}} z^{+b}\ . \tag{7.3}$$

Now, as an ansatz we take H_{++} with components thus determined by $\mathcal{L}_{(+4)}$ and we search for a local biholomorphism φ, with $\varphi(I_2, I_{2n}, 0) = (I_2, I_{2n}, 0)$, whose components in central coordinates satisfy a system of differential equations which corresponds to a special subset of the conditions that characterise a bridge. In the next three lemmata, we shall show that: a) these equations admit at least one global solution φ on $\mathcal{H}|_{\mathcal{V}}$ (Lemma 7.1), b) in the class of global solutions there exists one satisfying a special set of initial conditions (Lemma 7.3), c) using such a special solution we may construct an hk-pair $(\mathcal{A}, M)$ having $\mathcal{L}_{(+4)}$ as prepotential and the map φ as a bridge (Lemma 7.4). These lemmata will conclude the proof of Step 1.

Lemma 7.1. *On an appropriate open set $\mathcal{U} = \mathcal{H}|_{\mathcal{V}} \cdot \mathrm{Sp}_n(\mathbb{C}) \subset \mathcal{P}$, the differential equations*

$$\varphi_*(E^o_A) = E^o_A\ , \quad \varphi_*(H^o_0) = H^o_0\ , \quad \varphi_*(H^o_{++}) = H_{++}\ , \tag{7.4}$$

admit at least one global solution $\varphi\colon \mathcal{U} \to \mathcal{U}$ with $\varphi^i_\pm = u^i_\pm$.

Proof. In virtue of (2.4), the differential equations take the form

$$(E^o_B \cdot \varphi^A)\frac{\partial}{\partial x^A} = \varphi^a_c (E^o_B)^c_b \frac{\partial}{\partial B^a_b}\,, \tag{7.5}$$

$$(H^o_0 \cdot \varphi^A)\frac{\partial}{\partial x^A} = \varphi^i_+ \frac{\partial}{\partial u^i_+} \;-\; \varphi^i_- \frac{\partial}{\partial u^i_-}\,, \tag{7.6}$$

$$(H^o_{++} \cdot \varphi^A)\frac{\partial}{\partial x^A} = \varphi^i_+ \frac{\partial}{\partial u^i_-} \;+\; v^{\pm b}_{++}\big|_\varphi\, \varphi^a_b \varphi^j_\pm \frac{\partial}{\partial z^{ja}} + A^B_{++}\big|_\varphi\, \varphi^a_c (E^o_B)^c_b \frac{\partial}{\partial B^a_b}\,. \tag{7.7}$$

To prove the existence of solutions with $\varphi^i_\pm = u^i_\pm$, we first note that solutions of (7.5) are maps such that a) the components φ^{ia} do not depend on B, i.e. $\varphi^{ia} = \varphi^{ia}((u^i_\pm),(z^{ja}))$, and b) the components φ^a_b satisfy eq. (6.1), with $\varphi^a_b((u^i_\pm),(z^{jc}))$ denoting the restriction $\varphi^a_b|_{\mathcal{H}}$. Thus the problem reduces to looking for holomorphic functions φ^a_c , φ^{ia} on $\mathcal{H}|_{\mathcal{V}}$ satisfying (7.6) and (7.7), with $\varphi^i_\pm = u^i_\pm$. These equations say that φ^a_c , φ^{ia} have charge 0 and using (7.2)-(7.3) we obtain

$$\begin{aligned} H^o_{++} \cdot \varphi^{ia} &= v^{+b}_{++}|_\varphi \varphi^a_b\, u^i_+ + v^{-b}_{++}|_\varphi \varphi^a_b\, u^i_- \\ &= u^i_+ \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-b}\partial z^{-c}}\Big|_{(u^i_\pm,\varphi^{ja})} u^+_j \varphi^{jb} + u^i_- \omega^{ad} \frac{\partial \mathcal{L}_{(+4)}}{\partial z^{-d}}\Big|_{(u^i_\pm,\varphi^{ja})} \end{aligned} \tag{7.8}$$

$$H^o_{++} \cdot \varphi^a_b = \varphi^a_c\, A^B_{++}|_\varphi\, (E^o_B)^c_b = \varphi^c_b\, \omega^{ad} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-c}\partial z^{-d}}\Big|_{(u^i_\pm,\varphi^{ja})}. \tag{7.9}$$

Now, writing $\varphi^a_b = (e^\psi)^a_b$ with $\psi\colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$, equation (7.9) takes the form

$$H^o_{++} \cdot \psi^a_b = \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-c}\partial z^{-b}}\Big|_{(u^i_\pm,(\varphi^{ia}))}. \tag{7.10}$$

Equation (7.8) decouples. Its right hand side has charge $k = 2$, so by Lemma 5.4, it admits a global solution φ^{ia} on $\mathcal{H}|_{\mathcal{V}}$. Inserting this in (7.10), we obtain a inhomogeneous linear equation for ψ^a_b, which admits a global solution by Lemma 5.3. □

Remark 7.2. Since $H_{++}{\cdot}u^+_i = -u^-_i$, $H_{++}{\cdot}u^-_i = 0$, writing $\varphi^{\pm a} := -u^\pm_j \varphi^{ja}$, equation (7.8) allows the convenient reformulation

$$H^o_{++} \cdot \varphi^{-a} = \omega^{ab} \frac{\partial \mathcal{L}_{(+4)}}{\partial z^{-b}}\Big|_{(u^i_\pm,\varphi^{-c})}\,, \tag{7.11}$$

$$H^o_{++} \cdot \varphi^{+a} = \varphi^{+b} \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-b}\partial z^{-c}}\Big|_{(u^i_\pm,(\varphi^{-d}))} + \varphi^{-a}. \tag{7.12}$$

Note that the first equation is a nonlinear differential equation in φ^{-a} only, while the second is linear and inhomogeneous in the remaining variable φ^{+a}.

Lemma 7.3. *There exists a global solution* $\varphi = (\varphi^i_\pm = u^i_\pm\,,\ \varphi^a_b\,,\ \varphi^{ia})$ *to the system* (7.4) *on an appropriate open set* $\mathcal{U} = \mathcal{H}|_{\mathcal{V}}{\cdot}\mathrm{Sp}_n(\mathbb{C})$ *satisfying the additional condition*

$$\varphi^{+a}|_{\{I_2\}\times\mathcal{V}} = (-H_{--}\cdot\varphi^{-a} + c^{+a})|_{\{I_2\}\times\mathcal{V}}\,, \quad \varphi^a_b|_{\{I_2\}\times\mathcal{V}} = \delta^a_b\,, \tag{7.13}$$

where $\varphi^{\pm a} := -u^\pm_j\varphi^{ja}$ *and* $c^{+a} := H_{--}\cdot\varphi^{-a}|_{(I_2,0)}$.

Proof. Let $\widetilde{\varphi} = (\widetilde{\varphi}^i_\pm = u^i_\pm\,,\ \widetilde{\varphi}^a_b\,,\ \widetilde{\varphi}^{ia})$ be a global solution to (7.4) on an appropriate open set $\mathcal{U} = \mathcal{H}|_{\mathcal{V}}\cdot\mathrm{Sp}_n(\mathbb{C})$. As shown in the proof of Lemma 7.1, $\widetilde{\varphi}^{ia}$ and $\widetilde{\varphi}^a_b$ are solutions to (7.11), (7.12) and (7.10). We now consider the *linear* system for functions $\varphi'^{\pm a}$

$$H^o_{++}\cdot\varphi'^{-a} = 0\,,$$
$$H^o_{++}\cdot\varphi'^{+a} = \varphi'^{+b}\omega^{ac}\frac{\partial^2\mathcal{L}_{(+4)}}{\partial z^{-b}\partial z^{-c}}\Big|_{(u^i_\pm,(\widetilde{\varphi}^{-d}))} + \widetilde{\varphi}^{-a}$$

Writing $\varphi'^{ia} := u^i_+\varphi'^{+a} + u^i_-\varphi'^{-a}$, we obtain a system for functions φ'^{ia} satisfying the hypotheses of Lemma 5.4. Therefore (see Rem. 5.6) there exists a global solution φ'^{ia} to this system satisfying the initial conditions

$$\varphi'^{+a}|_{\{I_2\}\times\mathcal{V}} = \left(-H^o_{--}\cdot\widetilde{\varphi}^{-a} + c^{+a} - \widetilde{\varphi}^{+a}\right)|_{\{I_2\}\times\mathcal{V}}\,.$$

Inserting

$$\varphi^{ia} := u^i_+(\widetilde{\varphi}^{+a} + \varphi'^{+a}) + u^i_-\widetilde{\varphi}^{-a}$$

in (7.10), we choose a global solution ψ^a_b satisfying $\psi^a_b|_{\{I_2\}\times\mathcal{V}} = 0$; it exists by Lemma 5.3. A direct check shows that $\varphi = (\varphi^i_\pm = u^i_\pm, \varphi^a_b = (e^\psi)^a_b, \varphi^{ia})$ is a global solution to (7.11), (7.12) and (7.10) and thus to (7.4). It is defined on an appropriate open set and satisfies (7.13). □

Lemma 7.4. *Let*

i) H_{++} *be the vector field defined by* (7.1)-(7.3),

ii) $\varphi\colon \mathcal{U}\subset\mathcal{P}\to\mathcal{U}\subset\mathcal{P}$, *with* $\varphi^i_\pm = u^i_\pm$, *be a global solution to the differential equations* (7.4) *on an appropriate open set* $\mathcal{U} = \mathcal{H}|_{\mathcal{V}}{\cdot}\mathrm{Sp}_n(\mathbb{C})$ *satisfying the condition* (7.13) *and*

iii) $H_{--} := \varphi_*(H^o_{--})\,,\quad e_{+a} := e^o_{+a}\,,\quad e_{-a} := [H_{--}, e^o_{+a}]$.

Then $\mathcal{A} := (H^o_0, H_{\pm\pm}, E^o_A, e_{\pm a})$ *is a canonical hk-frame with v-potential satisfying* (4.8) *and there exists a* $4n$*-dimensional real submanifold* $M\subset\mathcal{U}$ *so that* $(\mathcal{A}, M)$ *is a canonical hk-pair with bridge* φ.

Proof. We first prove that $\mathcal{A}$ is an hk-frame. Consider the vector fields $\widehat{e}_{\pm a} := \varphi_*^{-1}(e_{\pm a})$. By construction $\widehat{\mathcal{A}} = \left(H^o_0, H^o_{\pm\pm}, E^o_A, \widehat{e}_{\pm a}\right)$ is such that $\varphi_*(\widehat{\mathcal{A}}) = \mathcal{A}$. So the proof that $\mathcal{A}$ is an hk-frame reduces to showing that $\widehat{\mathcal{A}}$ is an hk-frame.

The vector fields $\widehat{e}_{\pm a}$ may be expressed in terms of the flat hk-frame,

$$\widehat{e}_{\pm a} = \widehat{v}^{+b}_{\pm a} e^o_{+b} + \widehat{v}^{-b}_{\pm a} e^o_{-b} + \widehat{v}^{++}_{\pm a} H^o_{++} + \widehat{v}^{0}_{\pm a} H^o_0 + \widehat{v}^{--}_{\pm a} H^o_{--} + \widehat{A}^A_{\pm a} E^o_A,$$

where the components $\widehat{v}^{\pm b}_{\pm a}$, $\widehat{v}^{\pm\pm}_{\pm a}$, $\widehat{v}^{0}_{\pm a}$, $\widehat{A}^A_{\pm a}$ are holomorphic functions. Since $\varphi_*(E^o_A) = E^o_A$ and $\varphi_*(\widehat{e}_{+a}) = e_{+a} = e^o_{+a}$, we have that $du^i_\pm(\varphi_*(\widehat{e}_{+a})) = 0$. Inserting the above expression for $\widehat{e}_{+a}$ we have

$$\begin{aligned} 0 &= du^i_\pm\big(\widehat{v}^{+b}_{+a}\varphi_*(e^o_{+b}) + \widehat{v}^{-b}_{+a}\varphi_*(e^o_{-b}) + \widehat{v}^{++}_{+a} H^o_{++} + \widehat{v}^{0}_{+a} H^o_0 + \widehat{v}^{--}_{+a} H^o_{--}\big) \\ &= du^i_\pm\big(\widehat{v}^{++}_{+a} H^o_{++} + \widehat{v}^{0}_{+a} H^o_0 + \widehat{v}^{--}_{+a} H^o_{--}\big) \ , \quad \text{since } \varphi^i_\pm = u^i_\pm \ . \end{aligned}$$

Since H^o_0 , H^o_{++} and H^o_{--} are linearly independent at each point, it follows that

$$\widehat{v}^{++}_{+a} = \widehat{v}^{0}_{+a} = \widehat{v}^{--}_{+a} = 0 \ . \tag{7.14}$$

Further, since $\widehat{e}_{-a} = \varphi_*^{-1}([H_{--}, e^o_{+a}]) = [H^o_{--}, \widehat{e}_{+a}]$, we find that $\widehat{e}_{-a}$ has no component along H^o_0 and $H^o_{\pm\pm}$, i.e.

$$\widehat{v}^{++}_{-a} = \widehat{v}^{0}_{-a} = \widehat{v}^{--}_{-a} = 0 \ . \tag{7.15}$$

We now check that the Lie brackets between $\widehat{e}_{\pm a}$ and the other fields in $\widehat{\mathcal{A}}$ have the required form. By direct computation:

$$\begin{aligned} [E^o_A, \widehat{e}_{+a}] &= \varphi_*^{-1}([E^o_A, e^o_{+a}]) = (E^o_A)^b_a\, \varphi_*^{-1}(e^o_{+a}) = (E^o_A)^b_a\, \widehat{e}_{+b} \\ [E^o_A, \widehat{e}_{-a}] &= \varphi_*^{-1}([E^o_A, [H_{--}, e^o_{+a}]]) = (E^o_A)^b_a\, \varphi_*^{-1}([H_{--}, e^o_{+b}]) = (E^o_A)^b_a\, \widehat{e}_{-b} \\ [H^o_0, \widehat{e}_{+a}] &= \varphi_*^{-1}([H^o_0, e^o_{+a}]) = \widehat{e}_{+a} \\ [H^o_0, \widehat{e}_{-a}] &= \varphi_*^{-1}([H^o_0, [H_{--}, e^o_{+a}]]) \\ &= -2\varphi_*^{-1}([H_{--}, e^o_{+a}]) + \varphi_*^{-1}([H_{--}, e^o_{+a}]) = -\widehat{e}_{-a} \end{aligned}$$

$$[H^o_{++}, \widehat{e}_{+a}] = \varphi_*^{-1}([H_{++}, e^o_{+a}]) = -\varphi_*^{-1}((e^o_{+a}\cdot v^{\pm b}_{++})\, e^o_{\pm b} + (e^o_{+a}\cdot A^B_{++})\, E^o_B) = 0\,.$$

The last equality follows from (7.2) and $e^o_{+a}\cdot\mathcal{L}_{(+4)} = 0$. Further,

$$[H^o_{++}, \widehat{e}_{-a}] = \varphi_*^{-1}([H_{++}, [H_{--}, e^o_{+a}]]) = [H^o_{++}, [H^o_{--}, \widehat{e}_{+a}]] = [H^o_0, \widehat{e}_{+a}] = \widehat{e}_{+a}$$

and by construction $[H^o_{--}, \widehat{e}_{+a}] = \widehat{e}_{-a}$. It remains to verify that $X_{---a} := [H^o_{--}, \widehat{e}_{-a}] = 0$ and that $\widehat{Y}_{ab} := [\widehat{e}_{+a}, \widehat{e}_{-b}]$ has terms only in the directions of the E^o_A. Expanding X_{---a} in the vector fields of the flat hk-frame $\mathcal{A}^o$,

$$X_{---a} = X^{\pm b}_{---a} e^o_{\pm b} + X^0_{---a} H^o_0 + X^{\pm\pm}_{---a} H^o_{\pm\pm} + X^A_{---a} E^o_A \ ,$$

we see that since $[H^o_0, X_{---a}] = -2[H^o_{--}, \widehat{e}_{-a}] - [H^o_{--}, \widehat{e}_{-a}] = -3X_{---a}$, each component of X_{---a} has a negative charge. Further, from the expansion in the flat hk-frame of the equality

$$[H^o_{++}, X_{---a}] = [H^o_0, \widehat{e}_{-a}] + [H^o_{--}, \widehat{e}_{+a}] = -\widehat{e}_{-a} + \widehat{e}_{-a} = 0 \ , \tag{7.16}$$

we find that $H^o_{++}\cdot X_{---a}^{-b} = H^o_{++}\cdot X_{---a}^{--} = H^o_{++}\cdot X_{---a}^{A} = 0$. It follows from Lemma 5.3 a) that $X_{---a}^{-b} = X_{---a}^{--} = X_{---a}^{A} = 0$. Expanding once again (7.16) in the flat hk-frame and using the vanishing of these components, we get that $H^o_{++}\cdot X_{---a}^{+b} = H^o_{++}\cdot X_{---a}^{0} = 0$. Lemma 5.3 a) then implies $X_{---a}^{+b} = X_{---a}^{0} = 0$ and we get that the remaining component in the expansion of (7.16) gives $X_{---a}^{++} = 0$. It follows that $X_{---a} = [H^o_{--}, \widehat{e}_{-a}] = 0$, as required.

Now, in the image $\varphi(\mathcal{H}|_{\mathcal{V}}) \subset \mathcal{P}$ we have:

$$H_{--} = \varphi_*(H^o_{--}) = H^o_{--} + v_{--}^{+a} e^o_{+a} + v_{--}^{-a} e^o_{-a} + A_{--}^{A} E^o_A \tag{7.17}$$

$$e_{-b} \;=\; \varphi_*(\widehat{e}_{-b}) \;= [H_{--}, e^o_{+b}] = e^o_{-b} + v_{-b}^{-c} e^o_{-c} + v_{-b}^{+c} e^o_{+c} + A_{-b}^{A} E^o_A\,. \tag{7.18}$$

The components of these vector fields are:

$$\begin{aligned} v_{--}^{\pm a}|_{(U,B,z)} &= -(u_i^{\pm}(e^{-\psi})^a_c H^o_{--}\cdot \varphi^{ic})|_{\Phi(U,B,z)} \\ A_{--}^{B}(E^o_B)^a_b|_{(U,B,z)} &= ((e^{-\psi})^a_c H^o_{--}\cdot \varphi^c_b)|_{\Phi(U,B,z)} \\ v_{-b}^{+c}|_{(U,B,z)} &= (A_{--}^{B}(E^o_B)^c_b - e^o_{+b}\cdot v_{--}^{+c})|_{(U,B,z)} \\ v_{-b}^{-c}|_{(U,B,z)} &= -e^o_{+b}\cdot v_{--}^{-c}|_{(U,B,z)} \\ A_{-b}^{B}(E_B)^c_a|_{(U,B,z)} &= (-e^o_{+b}\cdot (A_{--}^{B}(E_B)^c_a))|_{(U,B,z)}\,, \end{aligned}$$

where we denote the inverse map of φ by $\Phi = \varphi^{-1}$ and write $(\varphi^a_b) = (e^{\psi})^a_b$. From this and (7.5) we see that v_{-b}^{-c} is entirely determined by the map φ as follows:

$$\begin{aligned} v_{-b}^{-c}|_{(U,B,z)} &= e^o_{+b}|_{(U,B,z)}\cdot (u_i^{\pm}(e^{-\psi})^a_c H^o_{--}\cdot \varphi^{ic}|_{\Phi(U,B,z)}) \\ &= -\frac{\partial((e^{-\psi})^a_c(H^o_{--}\cdot\varphi^{-c} + \varphi^{+c}))}{\partial Y^M}\Big|_{\Phi(U,B,z)} \frac{\partial \Phi^M}{\partial z^{+b}}\Big|_{(U,B,z)} \\ &= \Big((e^{-\psi})^a_c \frac{\partial \psi^c_f}{\partial B^d_e}(H^o_{--}\cdot\varphi^{-f} + \varphi^{+f})\Big)\Big|_{\Phi(U,B,z)} \frac{\partial \Phi^d_e}{\partial z^{+b}}\Big|_{(U,B,z)} \\ &\quad + \Big((e^{-\psi})^a_c \frac{\partial \psi^c_f}{\partial z^{\pm d}}\big(H^o_{--}\cdot\varphi^{-f} + \varphi^{+f}\big)\Big)\Big|_{\Phi(U,B,z)} \frac{\partial \Phi^{\pm d}}{\partial z^{+b}}\Big|_{(U,B,z)} \\ &\quad - \Big((e^{-\psi})^a_c \frac{\partial\big(H^o_{--}\cdot\varphi^{-c} + \varphi^{+c}\big)}{\partial z^{\pm d}}\Big)\Big|_{\Phi(U,B,z)} \frac{\partial \Phi^{\pm d}}{\partial z^{+b}}\Big|_{(U,B,z)} \end{aligned} \tag{7.19}$$

where $(Y^M) = (u^i_{\pm}, B^a_b, z^{\pm})$. To proceed we need the following technical lemma.

Lemma 7.5. *If φ satisfies* (7.13), *the components v_{-b}^{-c} of the vector field e_{-b} are identically equal to* 0.

Proof. Expanding the relation $e_{+b} = [H_{++}, e_{-b}]$ in the flat basis, where H_{++} is the vector field in (7.1)-(7.3), and using the $\mathfrak{sp}_n(\mathbb{C})$-equivariance for the components of e_{-b} implied by $[E^o_A, e_{-b}] = (E^o_A)^d_b e_{-d}$, we have:

$$\begin{aligned}
e^o_{+b} &= [H_{++}\ ,\ e_{-b}] \\
&= \big[H_{++}\ ,\ (\delta^c_b + v^{-c}_{-b})e^o_{-b} + v^{+c}_{+b}\ e^o_{+c} + A^B_{-b}\ E^o_B\big] \\
&= (\delta^c_b + v^{-c}_{-b})\,\big[H_{++}\ ,\ e^o_{-c}\big] + H_{++}{\cdot}v^{\pm c}_{-b}\ e^o_{\pm c} + H_{++}{\cdot}A^B_{-b}\,E^o_B \\
&= (\delta^c_b + v^{-c}_{-b})\,\big[H^o_{++} + v^{\pm d}_{++}e^o_{\pm d} + A^C_{++}E^o_C\ ,\ e^o_{-c}\big] \quad \bmod \langle e^o_{+c}, E^o_B\rangle \\
&= \big(H_{++}\cdot v^{-c}_{-b} + (\delta^d_b + v^{-d}_{-b})(A^C_{++}(E^o_C)^c_d - e^o_{-d}\cdot v^{-c}_{++})\big)\,e^o_{-c} \quad \bmod \langle e^o_{+c}, E^o_B\rangle .
\end{aligned}$$

Now, from (7.2), we have that

$$A^C_{++}(E^o_C)^c_d = e^o_{-d}\cdot v^{-c}_{++}\,,$$

so the remaining components in the e^o_{-c}-directions imply that $(H_{++}{\cdot}v^{-c}_{-b})\circ\varphi = H^o_{++}{\cdot}(v^{-c}_{-b}\circ\varphi) = 0$. Since v^{-c}_{-b} has charge zero, we have $(H^o_0{\cdot}v^{-c}_{-b})\circ\varphi = H^o_0{\cdot}(v^{-c}_{-b}\circ\varphi) = 0$ and the conditions for the applicability of Lemma 5.3 hold for $v^{-c}_{-b}\circ\varphi$. We deduce that v^{-c}_{-b} is constant along orbits of H^o_0, H_{++} and H_{--}, the images under the map φ of the orbits of $\mathrm{Sp}_1(\mathbb{C})$ in harmonic space. Thus, $v^{-c}_{-b} = 0$ everywhere on $\mathcal{U}$ if and only if $v^{-c}_{-b}|_{\varphi(\{I_2\}\times\{I_{2n}\}\times\mathcal{V})} = 0$. From (7.19), this follows if and only if, for any $z\in\mathcal{V}$,

$$\begin{aligned}
0 = \Big(\frac{\partial\psi^c_f}{\partial B^d_e}(H^o_{--}{\cdot}\varphi^{-f} + \varphi^{+f})\Big)\Big|_{(I_2,I_{2n},z)}\frac{\partial\Phi^d_e}{\partial z^{+b}}\Big|_{\varphi(I_2,I_{2n},z)} \\
+ \Big(\frac{\partial\psi^c_f}{\partial z^{\pm d}}(H^o_{--}{\cdot}\varphi^{-f} + \varphi^{+f})\Big)\Big|_{(I_2,I_{2n},z)}\frac{\partial\Phi^{\pm d}}{\partial z^{+b}}\Big|_{\varphi(I_2,I_{2n},z)} \\
- \Big(\frac{\partial(H^o_{--}{\cdot}\varphi^{-c} + \varphi^{+c})}{\partial z^{\pm d}}\Big)\Big|_{(I_2,I_{2n},z)}\frac{\partial\Phi^{\pm d}}{\partial z^{+b}}\Big|_{\varphi(I_2,I_{2n},z)}
\end{aligned}$$

This holds since the initial data satisfy (7.13). □

Since the functions v^{-c}_{-b} are identically vanishing, the vector fields e_{-b} have the form $e_{-b} = e^o_{-b} + v^{+c}_{-b}e^o_{+c} + A^B_{-b}E^o_B$. It follows that

$$Y_{ab} := \varphi_*(\widehat{Y}_{ab}) = [e^o_{+a}, e_{-b}] = T^{+c}_{ab}e^o_{+c} + R^B_{ab}E^o_B\,,$$

with $T^{+c}_{ab} = e^o_{+a}{\cdot}v^{+c}_{-b} - A^B_{-b}(E^o_B)^c_a$ and $R^B_{ab} = e^o_{+a}{\cdot}A^B_{-b}$. From the equations $[H^o_0, [e_{+a}, e_{-b}]] = 0$ and $[H^o_{++}, [e_{+a}, e_{-b}]] = 0$, we see that T^{+c}_{ab} has charge -1 and that $H_{++}{\cdot}T^{+c}_{ab} = 0$. By Lemma 5.3a), applied to the functions $\widetilde{T}^{+c}_{ab} := \varphi^*(T^{+c}_{ab})$, it follows that $T^{+c}_{ab} = 0$ and that $\widehat{Y}_{ab} = [\widehat{e}_{+a}, \widehat{e}_{-b}] = \varphi^{-1}_*([e_{+a}, e_{-b}])$ has terms only in the directions of the E^o_A, as required. This concludes the proof that $\mathcal{A}$ is an hk-frame.

We now observe that the above construction, together with (7.18) and Lemma 7.5, shows that $\mathcal{A}$ is in fact canonical and that φ is a bridge from the central hk-frame $\widehat{\mathcal{A}} = \varphi^{-1}_*(\mathcal{A})$ to $\mathcal{A}$. It therefore remains to show that there exists a $4n$-dimensional real submanifold $M \subset \mathcal{U}$ such that $(\mathcal{A}, M)$ is a canonical hk-pair.

Let us consider the distribution $\mathcal{D} \subset T\mathcal{U}$, generated by

$$\alpha^{\widehat{\mathcal{A}}}_{(\mathbb{R})}(v)_x\ ,\ v \in V^\tau\ ,\ x \in \mathcal{U}\,, \quad \text{and} \quad \alpha^{\widehat{\mathcal{A}}}_{(\mathbb{R})}(E)\ ,\ E \in \mathfrak{sp}_n = \mathfrak{sp}_n(\mathbb{C})^\tau,$$

where $\alpha^{\widehat{\mathcal{A}}}_{(\mathbb{R})}$ is the real absolute parallelism (4.1) associated with $\widehat{\mathcal{A}}$. Its image $\mathcal{D}' = \pi_*(\mathcal{D}) \subset T\mathcal{V}$ under the natural projection $\pi\colon \mathcal{U} \to \mathcal{V} \simeq \{I_2\} \times \{I_{2n}\} \times \mathcal{V}$ is a totally real, $4n$-dimensional distribution and it is is involutive by virtue of the Lie brackets of vector fields in $\mathcal{A}$. By Frobenius' Theorem, $\mathcal{D}'$ admits integral submanifolds. Let $M' \subset \{I_2\} \times \{I_{2n}\} \times \mathcal{V}$ be an integral submanifold through $e = (I_2, I_2, 0)$. We now show that $M := \varphi(M')$ is totally real and satisfies the conditions of Def. 4.1.

M is totally real because it is the image under a biholomorphism of a totally real submanifold. Condition (i) of Def. 4.1 holds because φ is $\mathfrak{sp}_n(\mathbb{C})$-equivariant and M' is transversal to the $\mathfrak{sp}_n(\mathbb{C})$-orbits. Finally, from (7.14) and (7.15), the real absolute parallelism $\alpha^{\mathcal{A}}_{(\mathbb{R})}$ associated with $\mathcal{A} = \varphi_*(\widehat{\mathcal{A}})$ is such that for any $x = \varphi(y) \in M = \varphi(M')$ and $v \in V^\tau$

$$\begin{aligned}\alpha^{\mathcal{A}}_{(\mathbb{R})}(v)_x = \varphi_*(\alpha^{\widehat{\mathcal{A}}}_{(\mathbb{R})}(v)_y) &\in \varphi_*\big(T_y M' + \alpha^{\widehat{\mathcal{A}}}_{(\mathbb{R})}(\mathfrak{sp}_{p,q})|_y\big)\\ &= T_x M + \varphi_*\big(\alpha^{\widehat{\mathcal{A}}}_{(\mathbb{R})}(\mathfrak{sp}_{p,q})|_y\big)\\ &= T_x M + \alpha^{\mathcal{A}}_{(\mathbb{R})}(\mathfrak{sp}_{p,q})|_x\ .\end{aligned}$$

Hence condition (ii) of Def. 4.1 holds as well. This concludes the proof that $(\mathcal{A}, M)$ is a canonical hk-pair. □

Remark 7.6. The role of condition (7.13) in this proof is merely to simplify the proof of the existence of a submanifold on which the functions v^{-a}_{-b} vanish. Indeed, the argument at the end of Sect. 6 shows that for every pair of solutions φ, φ' of (7.4), the vector fields $H_{--} = \varphi_*(H^o_{--})$ and $H'_{--} = \varphi'_*(H^o_{--})$ necessarily coincide. This means that *any* solution φ of (7.4), not necessarily satisfying (7.13), can be used to construct the (unique) vector fields H_{--} and $e_{-a} = [H_{--}, e^o_{+a}]$ required to complete the vector fields $H^o_0, H_{++}, E^o_A, e^o_{+a}$ to a canonical hk-frame.

Step 2: Existence of a prepotential for any canonical hk-pair

Let $\mathcal{A} = (H^o_0, H_{\pm\pm}, E^o_A, e_{\pm a})$ be a canonical hk-frame defined on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$. It follows from (3.1) and (3.2) that the maps $(v^{-a}_{++}), (v^{+a}_{\pm\pm}), (v^{+b}_{-a})$ and $(A^{B}_{-a}(E^o)^b_c)$, considered as components taking values in $V = \mathbb{C}^{2n}$, $\mathfrak{gl}_{2n}(V)$ and $V \otimes V^* \otimes V^*$, respectively, are $\mathfrak{sp}_n(\mathbb{C})$-equivariant.

Further, we have

$$[H_{++}, e_{+a}] = [H^o_{++} + v^{\pm b}_{++} e^o_{\pm b} + A^B_{++} E^o_B, e^o_{+a}] = 0$$
$$[H_{++}, e_{-a}] = [H^o_{++} + v^{\pm b}_{++} e^o_{\pm b} + A^B_{++} E^o_B, e^o_{-a} + v^{+c}_{-a} e^o_{+c} + A^B_{-a} E^o_B] = e^o_{+a} \ .$$

Comparing components along e^o_{-b} on both sides of these equations, we see that $e^o_{+a} \cdot v^{-b}_{++} = 0$ and $e^o_{-a} \cdot v^{-b}_{++} = A^A_{++} (E^o_A)^b_a \in \mathfrak{sp}_n(\mathbb{C})$, or equivalently,

$$\omega_{cb}\, e^o_{-a} \cdot v^{-c}_{++} \ - \omega_{ca}\, e^o_{-b} \cdot v^{-c}_{++} = 0\,. \tag{7.20}$$

This means that on $\mathcal{H}|_{\mathcal{V}} = \mathcal{U} \cap \mathcal{H}$, we have

$$\frac{\partial(\omega_{cb} v^{-b}_{++})}{\partial z^{+a}} = 0\ , \qquad \frac{\partial(\omega_{cb} v^{-b}_{++})}{\partial z^{-a}} = \frac{\partial(\omega_{ab} v^{-b}_{++})}{\partial z^{-c}}\ , \tag{7.21}$$

so that there exists a holomorphic prepotential $\mathcal{L}_{(+4)}$ of charge $k = 4$ and independent of z^{+a}, such that

$$v^{-c}_{++} = \omega^{bc} \frac{\partial \mathcal{L}_{(+4)}}{\partial z^{-b}}\,, \qquad (\omega^{ab}) = (\omega_{ab})^{-1}\ . \tag{7.22}$$

This prepotential is determined up to an arbitrary function depending only on $u^i_{\pm}$, which is fixed by imposing the initial value $\mathcal{L}_{(+4)}|_{\mathrm{Sp}_1(\mathbb{C}) \times \{I_{2n}\} \times \{0\}} = 0$.

8. Construction of a pseudo-hyperkähler metric from its prepotential

In this section we summarise the correspondence between prepotentials and metrics, giving a recipe to construct a real analytic pseudo-hyperkähler metric from a specified prepotential $\mathcal{L}_{(+4)} \colon \mathcal{H}|_{\mathcal{V}} \to \mathbb{C}$.

Step 1. *Construct the vector field $H_{++}|_{\mathcal{H}}$.*

The vector field H_{++} of the canonical hk-pair, corresponding to $\mathcal{L}_{(+4)}$ is of the form $H_{++} = H^o_{++} + v^{-b}_{++} e^o_{-b} + v^{+b}_{++} e^o_{+b} + A^B_{++} E^o_B$. The components of its restriction $H_{++}|_{\mathcal{H}|_{\mathcal{V}}}$ are given by

$$v^{-b}_{++}\big|_{\mathcal{H}} = \omega^{bc} \frac{\partial \mathcal{L}_{(+4)}}{\partial z^{-c}}, \quad v^{+a}_{++} = \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-b} \partial z^{-c}} z^{+b}, \quad A^B_{++} (E^o_B)^a_b\big|_{\mathcal{H}} = \omega^{ac} \frac{\partial^2 \mathcal{L}_{(+4)}}{\partial z^{-b} \partial z^{-c}}\,.$$

The components at other points of $\mathcal{P}|_{\mathcal{V}} = \mathcal{H}|_{\mathcal{V}} \cdot \mathrm{Sp}_n(\mathbb{C})$ are determined using $\mathrm{Sp}_n(\mathbb{C})$-equivariance.

Step 2. *Construct a bridge φ.*

Determine holomorphic functions φ^a_b and $\varphi^{\pm a}$ on $\mathcal{H}|_{\mathcal{V}}$ by solving the system of equations

$$H^o_{++}\cdot\varphi^{-a} = \omega^{ab}\,\frac{\partial\mathcal{L}_{(+4)}}{\partial z^{-b}}\bigg|_{(u^i_\pm,\varphi^{-c})}$$

$$H^o_{++}\cdot\varphi^{+a} = \varphi^{+b}\omega^{ac}\,\frac{\partial^2\mathcal{L}_{(+4)}}{\partial z^{-b}\partial z^{-c}}\bigg|_{(u^i_\pm,(\varphi^{-d}))} + \varphi^{-a}$$

$$H^o_{++}\cdot\varphi^a_b = \omega^{ac}\varphi^d_b\,\frac{\partial^2\mathcal{L}_{(+4)}}{\partial z^{-c}\partial z^{-d}}\bigg|_{(u^i_\pm,(\varphi^{-a}))},$$

with $\varphi^a_b(I_2,0)=\delta^a_b$, $\varphi^{\pm a}(I_2,0)=0$. Then extend $\varphi^{ia}=-u^i_+\varphi^{+a}-u^i_-\varphi^{-a}$, as constant functions along $\mathrm{Sp}_n(\mathbb{C})$ orbits, to the appropriate open subset $\mathcal{U}=\mathcal{H}|_{\mathcal{V}}\cdot\mathrm{Sp}_n(\mathbb{C})\subset\mathcal{P}$ and extend the $\varphi^a_b|_{\mathcal{H}|_{\mathcal{V}}}$ to $\mathcal{U}$ using (6.1). Now set $\varphi=(\varphi^i_\pm=u^i_\pm,\varphi^a_b,\varphi^{ia})$.

Step 3. *Construct the hk-frame $\mathcal{A}$.*

Set: $H_0=H^o_0$, $H_{--}=\varphi_*(H^o_{--})$, $E_A=A^o_A$, $e_{+a}=e^o_{+a}$, $e_{-a}=[H_{--},e^o_{+a}]$.

Step 4. *Determine the manifold M for the hk-pair $(\mathcal{A},M)$.*

Consider the integrable distribution $\mathcal{D}'$ on $\mathcal{V}\subset\mathbb{C}^{4n}$ spanned by the real and imaginary parts of the vectors

$$\widehat{e}^{(U)}_a|_z := (\pi\circ\varphi^{-1})_*(e^o_{+a}|_{(U,z)}+\widehat{\mathbb{J}}^d_c e_{-d}|_{(U,z)})$$
$$\widehat{e}^{(U)}_{a+2n}|_z := (\pi\circ\varphi^{-1})_*(e_{-a}|_{(U,z)}-\widehat{\mathbb{J}}^d_c e^o_{+d})|_{(U,z)})\ ,$$

where $(\widehat{\mathbb{J}}^d_c):=-I_{2p,2q}\cdot\left(\begin{smallmatrix}0 & -I_n\\ I_n & 0\end{smallmatrix}\right)$, $U\in\mathrm{Sp}_1(\mathbb{C})$ and $\pi\colon\mathcal{H}|_{\mathcal{V}}\to\mathcal{V}$ is the standard projection. We need to find an integral submanifold M' of $\mathcal{D}'$ through $0\in\mathcal{V}$. This can be done, for instance, by choosing vector fields which locally generate $\mathcal{D}'$ and considering an orbit of 0 under the flows of these vector fields. Then, set $M=\varphi(M')$, where M' is considered as a submanifold of $\{I_2\}\times\{I_{2n}\}\times\mathcal{V}\simeq\mathcal{V}$.

Step 5. *Construct the pseudo-hyperkähler metric.*

Find the dual coframe field $\mathcal{A}^*=(H^0,H^{\pm\pm},E^A,e^{\pm a})$ of the hk-frame $\mathcal{A}=(H_0,H_{\pm\pm},E_A,e_{\pm a})$. A pseudo-hyperkähler metric on M in the isometry class associated with $\mathcal{L}_{(+4)}$ is:

$$g=\sum_{a=1}^{2n}\left(e^{+a}\vee e^{-a}\right)\big|_{TM\times TM}\ .$$

To conclude this section, we summarise the inverse construction of a prepotential from a given pseudo-hyperkähler metric g. In order to determine this (non-unique) prepotential, we first construct an hk-pair $(\widehat{\mathcal{A}}, M')$ associated with g, following the procedure in the proof of Lemma A16. Here $\widehat{\mathcal{A}}=(H^o_0, H^o_{\pm\pm}, E^o_A, e_{\pm a})$ is a central hk-frame on $\mathcal{U} = \mathcal{H}|_{\mathcal{V}} \cdot \mathrm{Sp}_n(\mathbb{C}) \subset \mathcal{P}$ and $M' \subset \{I_2\}\times\{I_{2n}\}\times\mathcal{V}$ with $e = (I_2, I_{2n}, 0) \in M'$. Applying a local biholomorphism of $\mathbb{C}^{4n}$ if required, we may choose $\widehat{\mathcal{A}}$ so that $e_{-a}|_{\mathcal{V}} = \frac{\partial}{\partial z^{-a}}\big|_{\mathcal{V}} + A^B_{-a}E^o_B|_{\mathcal{V}}$. Then, we construct a bridge $\varphi = (\varphi^i_{\pm} = u^i_{\pm}, \varphi^a_b, \varphi^{\pm a} = -u^{\pm}_i\varphi^{ia})$ with $\varphi(e) = e$ by solving the differential problem

$$
\begin{aligned}
&H^o_0 \cdot \varphi^{-a} = -\varphi^{-a}, \qquad && e_{-a} \cdot \varphi^{-a} = \varphi^a_b\,, \qquad && e_{+a} \cdot \varphi^{-b} = 0\,,\\
&H^o_0 \cdot \varphi^{+a} = \varphi^{+a}, && e_{+a} \cdot \varphi^{+b} = \varphi^a_b\,, && H^o_0 \cdot \varphi^a_b = 0\,,
\end{aligned}
$$

together with the conditions:

i) φ^{ia} is independent of B^a_b

ii) $\varphi^a_b\,(U, B, z) = \varphi^a_c\,(U, I_{2n}, z)\,B^c_b$ for any $(U, B, z) \in \mathcal{U}$

iii) $H^o_{++}\cdot\varphi^{+b}|_{\{z^{+a}=0\}} = \varphi^{-b}|_{\{z^{+a}=0\}}$.

Computing $H_{++} := \varphi_*(H^o_{++})$, the restriction to $\mathcal{H}|_{\mathcal{V}}$ of the components v^{-b}_{++} in the expansion $H_{++} = H^o_{++} + v^{\pm b}_{++}e^o_{\pm b} + A^B_{++}E^o_B$ gives the v-potential of the metric. Finally, the z^{+a}-independent potential $\mathcal{L}_{(+4)}$ for the exact 1-form $\alpha := \omega_{ab}v^{-b}_{++}dz^{-a}$ on $\mathcal{H}|_{\mathcal{V}}$, with $\mathcal{L}_{(+4)}|_{\mathrm{Sp}_1(\mathbb{C})\times\{I_{2n}\}\times\{0\}} = 0$, is the required prepotential.

Appendix A. G-structures and pseudo-hyperkähler manifolds

In this appendix we introduce real and complex $\mathfrak{g}$-structures, local reformulations of G-structures in terms of vector fields. They provide a useful tool for the investigation of local properties of manifolds with real analytic G-structures.

In Sect. A1 we show that there exists a natural one-to-one correspondence between local equivalence classes of (a) G-structures with connections and (b) complete $\mathfrak{g}$-structures. This correspondence allows the formulation of questions on local equivalences of G-structures in terms of local equivalence problems among sets of vector fields.

We then discuss (Sect. A2) *complexifications of real G-structures* and *real forms of complex G-structures*, with a view to expressing problems of equivalence among real analytic G-structures in terms of holomorphic vector fields. In Sect. A3 we discuss the particular case of G-structures corresponding to real analytic pseudo-hyperkähler metrics. As our main result in Sect. A4, we prove the bijection between local isometry classes of real analytic pseudo-hyperkähler metrics and local equivalence classes of hk-pairs, which was advertised in Sect. 4.2.

A1. G-structures and associated $\mathfrak{g}$-structures

We start with a slight generalisation of the classical notion of a G-structure.

Definition A1. Let G be a real Lie group admitting an almost exact linear representation $\rho\colon G \to \mathrm{GL}(W)$, $W = \mathbb{R}^n$, i.e. $\ker\rho$ is a discrete normal subgroup. A G**-structure** (P, ϑ) on an n-dimensional manifold M is a principal G-bundle $\pi\colon P \to M$ together with a **soldering form** $\vartheta\colon TP \to W$, a G-equivariant W-valued 1-form which is strictly horizontal, namely the vertical distribution T^vP of P is such that $T^v_uP = \ker\vartheta_u$ for any $u \in P$.

Remark A2. This definition can be thought of as a a minor generalisation of the classical notion of a G-structure as a G-reduction of the linear frame bundle $L(M)$ of M (see e.g. [12, 14]). Various examples motivate this generalisation. In particular, the Spin_n-bundle of a Riemannian manifold (M, g) is not a G-structure in the classical sense, but is indeed a Spin_n-structure in the sense of Def. A1. The relation between the two definitions may be understood as follows. Let (e^o_i) be a fixed basis of W and choose a point $u \in P$. Then find n vectors $\widehat{e}_i \in T_uP$ satisfying the equations $\vartheta_u(\widehat{e}_i) = e^o_i$. These vectors are determined up to elements in $\ker\vartheta = T^vP$, so that their projections $e_i := \pi_*(\widehat{e}_i) \in T_{\pi(u)}M$, are uniquely associated with the point $u \in P$. Thus, there exists a well-defined map

$$\widetilde{p}\colon P \to L(M)\ , \quad u \mapsto \widetilde{p}(u) := (e_i) \subset T_{\pi(u)}M\ .$$

In virtue of the G-equivariance of ϑ, we may check that $\widetilde{p}$ is G-equivariant, namely that $\widetilde{p}(u{\cdot}g) = \widetilde{p}(u){\cdot}\rho(g)$, for $u \in P$ and $g \in G$. This property, together with the assumption that $\rho\colon G \to \mathrm{GL}(W)$ is almost exact, implies that $P' := \widetilde{p}(P) \subset L(M)$ is a $\rho(G)$-reduction of $L(M)$, that $p\colon P \to P'$, $p(u) := \widetilde{p}(u)$, is a covering map and that $\vartheta = p^*(\vartheta')$, where ϑ' is the soldering form of P'.

Summing up, a *G-structure*, as defined in Def. A1, *always admits a covering map $p\colon P \to P'$ onto a $\rho(G)$-reduction P' of the linear frame bundle $L(M)$ such that ϑ is the pullback, $\vartheta = p^*(\vartheta')$, of the soldering form ϑ' of $P' \subset L(M)$.*

We recall that a connection on a principal G-bundle $\pi\colon P \to M = P/G$ is a G-equivariant $\mathfrak{g}$-valued 1-form $\omega\colon TP \to \mathfrak{g} = \mathrm{Lie}(G)$, for which the restriction $\omega|_{T^v_uP}$ to any vertical subspace T^v_uP coincides with the inverse of the canonical identification $\nu\colon \mathfrak{g} \to T^v_uP$ between $\mathfrak{g}$ and T^v_uP. A connection $\omega\colon TP \to \mathfrak{g}$ on a G-structure $(\pi\colon P \to M\ , \vartheta)$ yields a Cartan connection,

$$\kappa := \omega + \vartheta\ , \quad \kappa\colon TP \to \mathfrak{q} := \mathfrak{g} + W\ , \tag{A1}$$

namely a $\mathfrak{q}$-valued 1-form which a) has trivial kernel, b) extends the natural isomorphism $\nu^{-1}\colon T^v_u P \to \mathfrak{g}$ at every $u \in P$ and c) satisfies $(R^*_g \kappa)(v) = \mathrm{Ad}_{g^{-1}}(\kappa(v))$ for $g \in G$ and $v \in TP$, where $R_g\colon P \to P$ is the right action of g on P. The notion of a Cartan connection is related to the following:

Definition A3. An **absolute parallelism** on an n-dimensional manifold N is an $\mathbb{R}$-linear map from a fixed n-dimensional real vector space, say $\mathbb{R}^n$, into the space of smooth vector fields $\mathfrak{X}(N)$,

$$\beta\colon \mathbb{R}^n \to \mathfrak{X}(N)\,,$$

with the property that the induced map $\beta_x\colon \mathbb{R}^n \to T_x N\ ,\ \beta_x(v) := \beta(v)|_x$, is an isomorphism of vector spaces for each $x \in N$.

The existence of an absolute parallelism β on N is equivalent to the existence of a set $\{X_1, \ldots, X_n\}$ of n vector fields $X_i \in \mathfrak{X}(N)$, such that for every $x \in N$ the vectors $X_1|_x, \ldots, X_n|_x$ form a basis of $T_x N$. In fact, for a given β, such vector fields are images $X_i = \beta(e^o_i)$ of the elements of some basis (e^o_i) of $\mathbb{R}^n$.

It follows immediately that if κ is the $\mathfrak{q}$-valued one-form (A1), the $\mathbb{R}$-linear map

$$\alpha\colon \mathfrak{q} = \mathfrak{g} + W \to \mathfrak{X}(P)\,, \quad \alpha(X)|_u := \kappa_u^{-1}(X) \quad \text{for } u \in P\ ,\ X \in \mathfrak{q}\,, \qquad \text{(A2)}$$

is an absolute parallelism on P. So, if $(e^o_1, \ldots, e^o_n)$ and $(E^o_1, \ldots, E^o_N)$ are two fixed bases for W and $\mathfrak{g}$, respectively, the absolute parallelism α and consequently κ (which is the inverse of α in the sense of (A2)) are uniquely determined by the corresponding set of vector fields $\mathcal{A} = (e_i = \alpha(e^o_i), E_A = \alpha(E^o_A))$, which provides a field of linear frames for the tangent spaces of P. The absolute parallelism α, constructed from a connection on a G-structure (P, ϑ), is an example of special class of absolute parallelisms, which we call $\mathfrak{g}$-*structures*. Let $\mathfrak{g} \subset \mathfrak{gl}(W)$ be a *real* linear Lie algebra and $\mathfrak{q} := \mathfrak{g} + W$ the associated nonhomogeneous Lie algebra, with $[W, W] = 0$ and $[A\,,\, v] = A{\cdot}v$ for $A \in \mathfrak{g}$ and $v \in W$.

Definition A4. Let P be a manifold with $\dim P = \dim \mathfrak{q}$. A **$\mathfrak{g}$-structure** is an absolute parallelism $\alpha\colon \mathfrak{q} \to \mathfrak{X}(P)\,, \mathfrak{q} := \mathfrak{g} + W$, satisfying the following Lie bracket relations:

$$[\alpha(A), \alpha(B)] = \alpha([A, B])\,, \qquad A \in \mathfrak{g}\ ,\ B \in \mathfrak{q}\,. \qquad \text{(A3)}$$

Two $\mathfrak{g}$-structures α, α' on P are **equivalent** if there exists a diffeomorphism φ of P such that $\alpha' = \varphi_* \circ \alpha$. The **vertical** and **horizontal distributions** of a $\mathfrak{g}$-structure are the distributions $\mathfrak{V}$ and $\mathfrak{H}$ in TP generated by $\alpha(\mathfrak{g})$ and $\alpha(W)$, respectively.

The brackets not included in (A3), between a pair of horizontal vector fields $\alpha(v)$ and $\alpha(v')$, $v, v' \in W$, take the form

$$[\alpha(v), \alpha(v')] = T(v, v') + R(v, v') \;, \qquad v, v' \in W \;, \tag{A4}$$

where we denote by $T(v, v')$ and $R(v, v')$ the components of $[\alpha(v), \alpha(v')]$ along $\mathfrak{H}$ and $\mathfrak{V}$ respectively. The maps

$$\begin{aligned} T_u &\in \mathrm{Hom}(\Lambda^2 W, W)\,, \quad T_u(v, v') := \alpha_u^{-1}(T(v, v')|_u) \\ R_u &\in \mathrm{Hom}(\Lambda^2 W, \mathfrak{g})\,, \quad R_u(v, v') := \alpha_u^{-1}(R(v, v')|_u)\,, \end{aligned}$$

are respectively called **torsion** and **curvature** of the $\mathfrak{g}$-structure α at $u \in P$. These generalise the classical notions of torsion and curvature of a connection. As we shall see, if α satisfies certain additional conditions, there exists a right G-action on P, a W-valued 1-form ϑ and a connection 1-form ω on P such that (P, ϑ, ω) is a G-structure with a connection having the property that $\omega + \vartheta = \alpha^{-1}$, in the sense of (A2). Then, given bases (E^o_A) of $\mathfrak{g}$ and (e^o_i) of W, the components T^k_{ij} and R^A_{ij} of $T_u = T^k_{ij} e^o_k \otimes e^{oi} \otimes e^{oj}$ and $R_u = R^A_{ij} E^o_A \otimes e^{oi} \otimes e^{oj}$ are precisely the components of the torsion and of the curvature of the connection ω in the linear frame $(e_i) = p(u) \in P' \subset L(M)$ (see Remark A2).

Remark A5. The conditions (A3) are tantamount to the following:

a) The map $\alpha|_{\mathfrak{g}} : \mathfrak{g} \to \alpha(\mathfrak{g})$ is a faithful representation of $\mathfrak{g}$ in the Lie algebra of vector fields.
b) The adjoint representation of $\alpha(\mathfrak{g})$ in $\alpha(W)$ is equivalent to the linear representation of $\mathfrak{g} \subset \mathfrak{gl}(W)$ on W.

Definition A6. A $\mathfrak{g}$-structure α on P is called **complete** if

a) the Lie algebra $\alpha(\mathfrak{g})$ of vector fields of P defines a free right action of a corresponding connected Lie group G on P and
b) the orbit space $M = P/G$ is a smooth manifold and the projection $\pi \colon P \to M = P/G$ is a locally trivial fibration.

The absolute parallelism α on a G-structure (P, ϑ) with a connection ω given in (A2) is a complete $\mathfrak{g}$-structure. The following proposition shows that this correspondence is in fact invertible. Thus complete $\mathfrak{g}$-structures are in bijection with G-structures endowed with a connection.

Proposition A7. *Let α be a complete $\mathfrak{g}$-structure on a manifold P, which has a free right G-action $\rho \colon P \times G \to P$. Then there exists*

i) a W-valued 1-form $\vartheta \colon TP \to W$, such that (P, ϑ) is a G-structure and
ii) a connection ω on the G-structure (P, ϑ),

with the property that the Cartan connection $\kappa = \omega + \vartheta$ is the inverse of the map α in the sense of eq. (A2)*.*

Proof. By completeness, P is a principal G-bundle over $M = P/G$, where G is the connected group generated by the Lie algebra of vector fields $\alpha(\mathfrak{g}) \subset \mathfrak{X}(P)$. By Remark A5b), G has a linear representation on the vector space $\alpha(W) \simeq W$. This defines an almost exact representation of G in W.

Now, for $X \in T_uP$, $u \in P$, consider the natural projections $(X)^{\mathfrak{H}}$ and $(X)^{\mathfrak{V}}$ onto the horizontal and vertical subspaces $\mathfrak{H}_u, \mathfrak{V}_u \subset T_uP$. Since $\alpha(W)$ and $\alpha(\mathfrak{g})$ generate $\mathfrak{H}$ and $\mathfrak{V}$, respectively, there exist unique elements $v \in W$, $E \in \mathfrak{g}$ such that $\alpha(v)|_u = (X)^{\mathfrak{H}}$ and $\alpha(E)|_u = (X)^{\mathfrak{V}}$. Thus, the horizontal and vertical projections provide the mappings

$$\vartheta_u \colon \ T_uP \ni X \longmapsto v \in W$$
$$\omega_u \colon \ T_uP \ni X \longmapsto E \in \mathfrak{g}\,.$$

By construction, ϑ is G-equivariant, (P, ϑ) is a G-structure, ω is a connection on the G-bundle $\pi\colon P \to M = P/G$ and $\omega + \vartheta = \alpha^{-1}$. □

A2. Complex G-structures and their real forms

Given a complex manifold (N, J), its *holomorphic* and *anti-holomorphic tangent bundles*, $T^{10}N$ and $T^{01}N = \overline{T^{10}N}$ are the subbundles of $T^{\mathbb{C}}N$ given by the $+i$ and $-i$ eigenspaces, respectively, of the $\mathbb{C}$-linear map $J_x\colon T^{\mathbb{C}}_xN \to T^{\mathbb{C}}_xN$.

We recall that *holomorphic vector fields* of (N, J) coincide with complex vector fields of N of the form $X = Y - iJY$ for $Y \in \mathfrak{X}(N)$ satisfying $\mathcal{L}_Y J = 0$. This condition is equivalent to say that in any system of holomorphic complex coordinates $\xi = (\zeta^1, \dots, \zeta^m)\colon \mathfrak{U} \subset N \to \mathbb{C}^m$, $m = \dim_{\mathbb{C}} N$, the complex vector field $X = Y - iJY$ has the form $X = X^i \frac{\partial}{\partial \zeta^i}$ with $X^i = X^i(\zeta^1, \dots \zeta^m)$ holomorphic in the coordinates ζ^i.

A2.1. Complex G-structures and complex $\mathfrak{g}$-structures

In this section $G \subset \mathrm{GL}(V)$, $V = \mathbb{C}^n$, is a connected *complex* linear group with Lie algebra $\mathrm{Lie}(G) =: \mathfrak{g} \subset \mathfrak{gl}(V)$ and $\mathfrak{q} := \mathfrak{g} + V$ the associated non-homogeneous Lie algebra, with $[V, V] = 0$ and $[A, v] = A \cdot v$ for $A \in \mathfrak{g}$ and $v \in V$. We shall treat $\mathfrak{g}$ and V as $\mathbb{R}$-vector spaces endowed with the standard complex structures $J_o\colon \mathfrak{g} \to \mathfrak{g}$ and $J_o\colon V \to V$. Denote by (E^o_A) and (e^o_a) fixed choices of complex bases for $\mathfrak{g}$ and V, respectively. Further, let $\mathfrak{g}^{10}$ and $\mathfrak{g}^{01} = \overline{\mathfrak{g}^{10}}$, respectively, be the $+i$ and $-i$ eigenspaces of J_o in the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{10} + \mathfrak{g}^{01}$ of $\mathfrak{g}$. Recall that $\mathfrak{g}^{10}$ is naturally isomorphic to $\mathfrak{g}$ as complex Lie algebra and that each holomorphic element $X \in \mathfrak{g}^{10}$ has the form

$X = Y - iJ_oY$ for some $Y \in \mathfrak{g}$. We may generalise Def. A1 to the case of a complex Lie group G as follows:

Definition A 8. A **complex G-structure** (P, ϑ) on an n-dimensional (real) manifold M is a principal G-bundle $\pi\colon P \to M$ equipped with a **complex soldering form** $\vartheta\colon T^{\mathbb{C}}P \to V$, a V-valued $\mathbb{C}$-linear 1-form, which is

i) G-equivariant (i.e. $R_{g*}\vartheta = g^{-1} \cdot \vartheta$ for all $g \in G$) and
ii) strictly horizontal (i.e. $\ker \vartheta_u = T^{v\mathbb{C}}_{\ u}P$ for all $u \in P$, where $T^{v\mathbb{C}}_{\ u}P$ is the complexification of the vertical subspace $T^v_uP \subset T_uP$).

The main motivation for considering *complex* G-structures comes from the following relation to the *real* ones. Let $W = \mathbb{R}^n$ and let $H \subset \mathrm{GL}(W)$ be a real form of $G \subset \mathrm{GL}(V)$, with $V = W^{\mathbb{C}} = \mathbb{C}^n$. A (real) H-structure $(\widetilde{P}, \widetilde{\vartheta})$, in the classical sense with $\widetilde{P} \subset L(M)$, can be considered as a reduction of the $\mathrm{GL}(V)$-bundle $L^{\mathbb{C}}(M)$ of complex linear frames of $T^{\mathbb{C}}M$. Consider the unique $\mathrm{GL}(V)$-equivariant and strictly horizontal 1-form

$$\widehat{\vartheta}\colon T^{\mathbb{C}}(L^{\mathbb{C}}(M)) \longrightarrow W^{\mathbb{C}} = V \qquad \text{with} \ \ \widehat{\vartheta}|_{T\widetilde{P}} = \widetilde{\vartheta}$$

and the G-reduction $P = \widetilde{P} \cdot G \subset L^{\mathbb{C}}(M)$. The pair $(P, \vartheta = \widehat{\vartheta}|_{T^{\mathbb{C}}P})$ is a complex G-structure, which we call the **complexification** of the H-structure $(\widetilde{P}, \widetilde{\vartheta})$. We may therefore think of the class of complex G-structures as a natural generalisation of the principal bundles, which arise via the above complexification procedure from real G-structures of linear frames.

In the more general case, where $(\widetilde{P}, \widetilde{\vartheta})$ is a (possibly non-trivial) covering of an H-structure $(\widetilde{Q}, \widetilde{\vartheta})$ of linear frames $\widetilde{Q} \subset L(M)$, a **complexification** of $(\widetilde{P}, \widetilde{\vartheta})$ is a complex G-structure (P, ϑ), with $\widetilde{P} \subset P$, which is a covering of the complexification of $(\widetilde{Q}, \widetilde{\vartheta})$.

The complexification procedure of a real G-structure allows reversal. Let $\tau\colon V \to V$ be a $\mathbb{C}$-antilinear involution and consider the induced involutions on $\mathfrak{gl}(V)$ and $\mathrm{GL}(V)$:

$$\tau(A) := \tau \circ A \circ \tau\,, \quad \tau(g) := \tau \circ g \circ \tau\,, \ \text{for all } A \in \mathfrak{gl}(V)\ ,\ g \in \mathrm{GL}(V)\,.$$

When $\mathfrak{g} \subset \mathfrak{gl}(V)$ and $G \subset \mathrm{GL}(V)$ are preserved by τ we say that τ is **$\mathfrak{g}$-admissible** and we denote by V^τ, G^τ and $\mathfrak{g}^\tau$ the τ-fixed point sets in V, G and $\mathfrak{g}$, respectively. Note that in this case $\mathfrak{g}^\tau$ is a real form of $\mathfrak{g}$ (i.e. $(\mathfrak{g}^\tau)^{\mathbb{C}}$ is naturally isomorphic to $\mathfrak{g}$).

Definition A 9. Let $\tau\colon V \to V$ be a $\mathfrak{g}$-admissible $\mathbb{C}$-antilinear involution and (P, ϑ) a complex G-structure over M. A **real form** (P^τ, ϑ^τ) of (P, ϑ) is a G^τ-reduction $P^\tau \subset P$ with soldering form $\vartheta^\tau = \vartheta|_{TP^\tau}$ taking values in $V^\tau \simeq W = \mathbb{R}^n$.

Clearly, if (P^τ, ϑ^τ) is a real form of (P, ϑ), then (P, ϑ) is the complexification of (P^τ, ϑ^τ).

Let (P, ϑ) be a complex G-structure and $\omega \colon TP \to \mathfrak{g}$ a connection form on the G-bundle P. The $\mathbb{C}$-linear extension of the 1-form ω_u on the complexified tangent space $T^{\mathbb{C}}_u P$, $u \in P$, determines a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form ω on $T^{\mathbb{C}} P$. We call **complex Cartan connection** associated with ϑ and ω the map

$$\kappa \colon T^{\mathbb{C}} P \longrightarrow \mathfrak{g}^{\mathbb{C}} + V\,, \quad \kappa := \omega + \vartheta\,. \tag{A5}$$

We note that the restriction $\kappa_u = \kappa|_{T^{\mathbb{C}}_u P} \colon T^{\mathbb{C}}_u P \to \mathfrak{g}^{\mathbb{C}} + V$ is a $\mathbb{C}$-linear isomorphism for every $u \in P$. Now, since κ_u is $\mathbb{C}$-linear and $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ a real form of $\mathfrak{g}^{\mathbb{C}}$, the inverse isomorphism $\kappa_u^{-1} \colon \mathfrak{g}^{\mathbb{C}} + V \to T^{\mathbb{C}}_u P$ is uniquely determined by the map

$$\alpha_u := \kappa_u^{-1}|_{\mathfrak{g}+V} \colon \mathfrak{g} + V \longrightarrow T^{\mathbb{C}}_u P\ .$$

The family of linear maps α_u , $u \in P$, combine into the single map

$$\alpha \colon \mathfrak{g} + V \longrightarrow \mathfrak{X}^{\mathbb{C}}(P)\,, \qquad \alpha(X)|_u := \alpha_u(X)\ , \tag{A6}$$

which we call the **complex (absolute) parallelism** associated with κ. By definition, κ is completely determined by α. Further, α has, by construction, the following properties:

i) $\alpha(X) = \overline{\alpha(X)}$ for all $X \in \mathfrak{g}$ and $\alpha(J_o v) = i\alpha(v)$ for all $v \in V$.

ii) The vector fields $\alpha(X) \in \alpha(\mathfrak{g})$ generate the vertical distribution $\mathcal{D} := T^v P \subset TP$, on which the complex structure J_o on $\mathfrak{g}$ induces the family of complex structures $J = \{J_u\}$ defined by

$$J_u \colon \mathcal{D}_u \to \mathcal{D}_u\ , \quad J_u\ \alpha(X)|_u := \alpha(J_o X)|_u\ ; \quad X \in \mathfrak{g}\,,\ u \in P\,.$$

The pair $(\mathcal{D}, J)$ is a CR structure. We denote by $Y^{10} := \frac{1}{2}(Y - iJY) \in \mathcal{D}^{\mathbb{C}}$, for $Y \in \mathcal{D}$, the unique complex vector field satisfying $JY^{10} = iY^{10}$ and $Y = Y^{10} + \overline{Y^{10}}$.

iii) For $X \in \mathfrak{g}$ and $v \in V$,

$$[\alpha(X), \alpha(v)] = \alpha(X \cdot v)\ , \qquad [\alpha(J_o X), \alpha(v)] = i\alpha(X \cdot v)\ .$$

iv) The complex parallelism α uniquely determines the following pair of objects:

a) the CR structure $(\mathcal{D}, J)$ and

b) the collection of vector fields in $T^{\mathbb{C}} P$,

$$\mathcal{A}^{(\alpha)} = \left(e_i = \alpha(e^o_i)\ ,\ E_A = \tfrac{1}{2}\left(\alpha(E^o_A) - i\alpha(J_o E^o_A)\right)\right),$$

which, together with the vector fields $\overline{E_A}$, form a field of complex linear frames for $T^{\mathbb{C}} P$, with the fields E_A , $\overline{E_A}$ taking values in $\mathcal{D}^{\mathbb{C}}$.

Conversely, given a CR structure $(\mathcal{D}, J)$ and a collection of complex vector fields $\mathcal{A}^{(\alpha)}$, the complex parallelism α allows explicit determination. In fact, given $(\mathcal{D}, J)$ and the fields (e_i, E_A), the family of $\mathbb{C}$-antilinear involutions $\overline{(\cdot)}\colon \mathcal{D}^{\mathbb{C}}_u \to \mathcal{D}^{\mathbb{C}}_u$, $u \in P$, defined by

$$\overline{(A + iJA)} := A - iJ_uA\ , \quad \overline{(A - iJA)} := A + iJ_uA\ , \qquad \text{for } A \in \mathcal{D}^{\mathbb{C}}_u\ ,$$

affords the construction of α as the unique $\mathbb{C}$-linear map $\alpha\colon \mathfrak{g}+V \to \mathfrak{X}^{\mathbb{C}}(P)$ such that

$$\alpha(E^o_A)|_u = E_A|_u + \overline{E_A}|_u\ , \qquad \alpha(e^o_i)|_u = e_i|_u\ .$$

Note that this means that $E_A = \alpha(E^o_A)^{10}$.

Analogously to the absolute parallelisms of real G-structures with a connection, the map (A6) is an example of a special class of maps, called *complex* $\mathfrak{g}$*-structures.*

Definition A10. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$, $V = \mathbb{C}^n$, be a linear complex Lie algebra, $\mathfrak{p} = \mathfrak{g} + V$ the associated nonhomogeneous Lie algebra, with the standard complex structure $J_o\colon V \to V$, $J_o\colon \mathfrak{g} \to \mathfrak{g}$. Further, let P be a real manifold of dimension $\dim_{\mathbb{R}} \mathfrak{g} + \frac{1}{2}\dim_{\mathbb{R}} V$. A **complex $\mathfrak{g}$-structure** is an $\mathbb{R}$-linear map $\alpha\colon \mathfrak{p} \to \mathfrak{X}^{\mathbb{C}}(P)$ such that, for $X \in \mathfrak{g}$, $Y \in \mathfrak{p}$ and $v \in V$,

$$\alpha(X) = \overline{\alpha(X)}\,, \qquad \alpha(J_ov) = i\alpha(v)$$

and

$$\begin{aligned} [\alpha(X)\,,\, \alpha(Y)] &= \alpha([X,Y]) \\ [\alpha(J_oX), \alpha(v)] &= i\alpha(X \cdot v)\,, \end{aligned} \tag{A7}$$

and the fields in $\mathcal{A}^{(\alpha)} = (e_i := \alpha(e^o_i)\,,\, E_A := \alpha(E^o_A)^{10})$ are $\mathbb{C}$-linearly independent at each $u \in P$. The **CR structure of** α is the pair $(\mathcal{D}, J)$, consisting of the distribution $\mathcal{D}_u = \mathrm{span}_{\mathbb{R}}\{\ \alpha(X)_u\ ,\ X \in \mathfrak{g}\ \}$ and the family of complex structures $J_u\colon \mathcal{D}_u \to \mathcal{D}_u$ defined by

$$J_u\alpha(X)_u := \alpha(J_oX)_u\,.$$

Two complex $\mathfrak{g}$-structures α, α' on P are called **equivalent** if there exists a diffeomorphism φ of P such that $\alpha' = \varphi_* \circ \alpha$, where φ_* is extended to $T^{\mathbb{C}}P$ by $\mathbb{C}$-linearity.

Conditions (A7) may be reformulated as follows:

a) $\alpha|_{\mathfrak{g}}\colon \mathfrak{g} \to \alpha(\mathfrak{g})$ is an exact representation of (the real Lie algebra underlying) $\mathfrak{g}$ on the Lie algebra of real vector fields in TP.

b) The adjoint representation of $\alpha(\mathfrak{g})$ on $\alpha(V)$ is equivalent to the linear representation of $\mathfrak{g} \subset \mathfrak{gl}(V)$ on V.

The definitions of vertical and horizontal subbundles $\mathfrak{V}, \mathfrak{H} \subset T^{\mathbb{C}}P$, determined by a complex $\mathfrak{g}$-structure α, as well as the torsion and curvature of α, are analogous to those for a real $\mathfrak{g}$-structure. The vector fields of a complex $\mathfrak{g}$-structure in the vertical subbundle $\mathfrak{V} \subset T^{\mathbb{C}}P$ generate, as in the real case, a local action of the complex Lie group G, whose orbit space, locally identifiable with the base of the principal G-bundle $P \to M$, has real dimension $\dim P - \dim_{\mathbb{R}} \mathfrak{g} = \frac{1}{2}\dim_{\mathbb{R}} V = n$. However, there is a crucial difference between complex and real $\mathfrak{g}$-structures in the geometric interpretation of the horizontal subbundle $\mathfrak{H} \subset T^{\mathbb{C}}P$ generated by vector fields in $\alpha(V) \subset \mathfrak{X}^{\mathbb{C}}(P)$. For any $u \in P$, we have $\dim_{\mathbb{R}} \mathfrak{H}_u = \dim_{\mathbb{R}} V = 2n$. So the subbundle $\mathfrak{H} \subset T^{\mathbb{C}}P$ is isomorphic to $T^{\mathbb{C}}M$ and admits no natural interpretation as a real horizontal distribution in TP. The curvature and torsion of a complex $\mathfrak{g}$-structure, $R(v, v')$ and $T(v, v')$, thus have arguments v, v' in $V \simeq T^{\mathbb{C}}_u M$ rather than in $T_u M$.

A2.2. **Real forms of complex $\mathfrak{g}$-structures**

Let $\tau\colon V \to V$, $V = \mathbb{C}^n$, be a $\mathfrak{g}$-admissible $\mathbb{C}$-antilinear involution.

Definition A11. A torsionless complex $\mathfrak{g}$-structure $\alpha\colon \mathfrak{g} + V \to \mathfrak{X}^{\mathbb{C}}(P)$ is **τ-compatible** around $x_o \in P$ if there exists an n-dimensional submanifold $M \subset P$ containing x_o, such that, at every point $y \in M$, the following hold:

i) $T_y M$ is transversal to $\alpha(\mathfrak{g}^\tau)|_y \subset T_y P$

ii) $\alpha(v)_y = \overline{\alpha(v)_y}$ for all $v \in V^\tau$

iii) $\alpha(V^\tau)_y \subset T_y M + \alpha(\mathfrak{g}^\tau)|_y$.

Proposition A12. *Let $\alpha\colon \mathfrak{g} + V \to \mathfrak{X}^{\mathbb{C}}(P)$ be a complete, torsionless complex $\mathfrak{g}$-structure and let $\kappa = \omega + \vartheta\colon T^{\mathbb{C}}P \to \mathfrak{g}^{\mathbb{C}} + V$ be such that $\alpha(X)|_u = \kappa_u^{-1}(X)$ for $u \in P$, $X \in \mathfrak{g} + V$. The following conditions are equivalent:*

a) The complex $\mathfrak{g}$-structure α is τ-compatible around $x_o \in P$.

b) There exists a local diffeomorphism $\imath\colon \mathcal{U} \to \mathcal{U}' \subset P'$ between a neighbourhood $\mathcal{U} \subset P$ of x_o and an open subset $\mathcal{U}'$ of the complexification P' of a complete (real) G^τ-structure $(\widetilde{P}, \widetilde{\vartheta})$, such that $\imath_*(\omega) = \omega'$ is a torsionless connection on $\mathcal{U}' \subset P'$, which is the complexification of a connection on the real form $\widetilde{P} \cap \mathcal{U}'$.

Proof. We first check that (a) implies (b). Consider a submanifold $M \subset P$ containing x_o and satisfying conditions (i)-(iii) of Def. A11. By (i), M is transversal to the orbits of the action of the real Lie group G^τ determined by the flows of the fields in $\alpha(\mathfrak{g}^\tau)$, where $\mathfrak{g}^\tau = \mathrm{Lie}(G^\tau)$. By (ii), (iii) and the properties of the Lie brackets of the fields in $\alpha(\mathfrak{g}^\tau + V^\tau)$, the union of the

G^τ-orbits of the points of M, denoted by $\mathcal{U}^\tau = M \cdot G^\tau$, is (locally) a smooth submanifold of P passing through x_o and tangent to the fields of $\alpha(\mathfrak{g}^\tau + V^\tau)$. In particular, the restriction to $\mathcal{U}^\tau$ of the vector fields in $\alpha(\mathfrak{g}^\tau + V^\tau)$ determines a $\mathfrak{g}^\tau$-structure. By Prop. A7, $\mathcal{U}^\tau$ is locally diffeomorphic to an open set in a G^τ-structure $\widetilde{P}$, which we may assume, with no loss of generality, to be a Cartesian product $\widetilde{P} = M \times G^\tau$. By construction, the restriction of ω to the tangent space $T\mathcal{U}^\tau$ is mapped into a torsionless connection on $\widetilde{P}$. Imposing equivariance under local action of $G = (G^\tau)^{\mathbb{C}}$, the local diffeomorphism between $\mathcal{U}^\tau$ and $\widetilde{P}$ extends to a local diffeomorphism between an open set of the form $\mathcal{U} = \mathcal{U}^\tau \cdot G$ and an open set of the complexification $P' \simeq M \times G$ of $\widetilde{P}$. This proves (b). The proof that (b) implies (a) follows directly from the definitions. □

A3. **Complex G-structures, pseudo-hyperkähler metrics and hk-pairs**

A3.1. **Pseudo-hyperkähler metrics as $\mathrm{Sp}_{p,q}$-structures**

A **hypercomplex structure** on a $4n$-dimensional real vector space W is a triple (J_1, J_2, J_3) of endomorphisms of W satisfying the multiplication relations of the imaginary quaternions, $J_\alpha^2 = -\mathrm{Id}_W$, $J_\alpha J_\beta = J_\gamma$, for all cyclic permutations (α, β, γ) of $(1, 2, 3)$. An inner product g on W is called **hermitian** with respect to the hypercomplex structure (J_1, J_2, J_3) if every J_α is skew-symmetric with respect to g, i.e. $g(J_\alpha w, w') + g(w, J_\alpha w') = 0$ for all $w, w' \in W$.

Definition A13. A $4n$-dimensional pseudo-Riemannian manifold (M, g) of signature $(4p, 4q)$, with $p+q = n$, is called **pseudo-hyperkähler** if it is endowed with a triple (J_1, J_2, J_3) of global sections of End (TM) such that

i) $(J_1, J_2, J_3)_x$ is a hypercomplex structure on T_xM for every $x \in M$ and g_x is hermitian with respect to it, and
ii) $\nabla J_\alpha = 0$ for $\alpha = 1, 2, 3$, where ∇ is the Levi-Civita connection of the metric g.

Equivalently, a $4n$-dimensional pseudo-Riemannian manifold (M, g) of signature $(4p, 4q)$ is pseudo-hyperkähler if and only if its holonomy algebra $\mathfrak{hol}(M, g)$ is a subalgebra of $\mathfrak{sp}_{p,q}$. This is equivalent to requiring that the Levi-Civita connection on M preserves an $\mathrm{Sp}_{p,q}$-reduction $Q \subset \mathrm{O}_g(M)$ of the orthonormal frame bundle. When g is Riemannian, (M, g) is called **hyperkähler**.

Let (M, g) be a pseudo-hyperKähler manifold and ϑ the canonical soldering form of the orthonormal frame bundle $\mathrm{O}_g(M)$. The pair $(\pi\colon Q \to M\ ,\ \vartheta|_{TQ})$ is an $\mathrm{Sp}_{p,q}$-structure with a unique Levi-Civita (torsionless) connection $\omega\colon TQ \to \mathfrak{sp}_{p,q}$. The $\mathrm{Sp}_{p,q}$-structure with connection ω is uniquely associated with the manifold (M, g), modulo principal bundles equivalences.

Conversely (see Remark A2), every $\mathrm{Sp}_{p,q}$-structure $(\pi\colon Q{\to}M, \vartheta)$ with a torsionless connection ω determines a pseudo-hyperkähler metric on M. Q can be identified with an $\mathrm{Sp}_{p,q}$-reduction of $L(M)$. Further, every (local) section $\sigma\colon M \to Q$ determines a field of frames (e_i) on M, together with a pseudo-Riemannian metric g of signature $(4p, 4q)$, with respect to which the frames (e_i) are orthonormal. Since $Q \subset L(M)$ is an $\mathrm{Sp}_{p,q}$-bundle, it follows that g is independent of the choice of section σ in Q, it is pointwise hermitian with respect to a family of hypercomplex structures, $(J'_i|_x)_{i=1,2,3}$, $x \in M$, and that the restriction of the Levi-Civita connection of $\mathrm{O}_g(M)$ to Q is the torsionless connection ω. Thus g is pseudo-hyperkähler and Q is a holonomy reduction of $\mathrm{O}_g(M)$. We therefore have:

Proposition A14. *There exists a natural one-to-one correspondence between pseudo-hyperkähler metrics g of signature $(4p, 4q)$ on a manifold M, up to isometries, and $\mathrm{Sp}_{p,q}$-structures $(\pi\colon Q{\to}M\,,\ \vartheta)$ possessing a torsionless connection, up to principal bundle equivalences.*

An $\mathrm{Sp}_{p,q}$-structure $(\pi\colon Q{\to}M\,,\ \vartheta)$ with torsionless connection can be regarded locally as a real form of a complex $\mathrm{Sp}_n(\mathbb{C})$-structure. It can also be considered naturally as an $\mathrm{Sp}_{p,q}$-reduction of a real form of a complex $(\mathrm{Sp}_1(\mathbb{C}){\times}\mathrm{Sp}_n(\mathbb{C}))$-structure $(\pi\colon P{\to}M\,,\ \vartheta)$. The reason is the following: Since (Q, ϑ) is locally a bundle of orthonormal frames of a pseudo-hyperkähler manifold (M, g), it can also be considered as an $\mathrm{Sp}_{p,q}$-reduction of the $\mathrm{Sp}_1{\times}\mathrm{Sp}_{p,q}$-subbundle of $\mathrm{Spin}_{4p,4q}(M, g)$. Since this subbundle is a real form of its complexification, the bundle Q is in turn naturally identifiable with an $\mathrm{Sp}_{p,q}$-reduction of the complex $(\mathrm{Sp}_1(\mathbb{C}){\times}\mathrm{Sp}_n(\mathbb{C}))$-structure P.

The latter has the following geometrical interpretation. Recall that an $\mathrm{Sp}_1{\cdot}\mathrm{Sp}_{p,q}$-reduction of the linear frame bundle $L(M)$ is uniquely associated with a (local) isomorphism $T^{\mathbb{C}}M \simeq H \otimes_M E$ between $T^{\mathbb{C}}M$ and the tensor product of two complex vector bundles $\pi^H\colon H{\to}M$ and $\pi^E\colon E{\to}M$, with fibres given by standard complex representations of $\mathrm{Sp}_1(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$, respectively (see e.g. [13]). This (local) identification allows us to consider complex frames for $T^{\mathbb{C}}_xM$ of the form $(h_i{\otimes}e_a)_{i=1,2;1\leq a\leq 2n}$, where (h_i) and (e_a) are complex frames for H_x and E_x, respectively, adapted to the standard symplectic forms of H_x and E_x. The collection of all such complex frames is an $\mathrm{Sp}_1(\mathbb{C}){\cdot}\mathrm{Sp}_n(\mathbb{C})$-reduction of the complex linear frame bundle $L^{\mathbb{C}}(M)$, whose double cover is the complex $(\mathrm{Sp}_1(\mathbb{C}){\times}\mathrm{Sp}_n(\mathbb{C}))$-structure P.

Now, by construction, the Levi-Civita connection of (M, g) uniquely corresponds to torsionless connections on (Q, ϑ) as well as on (P, ϑ). Since the latter connection is an $\mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C})$-equivariant extension of the former it follows that its curvature 2-form necessarily takes values only in $\mathfrak{sp}_n(\mathbb{C})$. We may now directly obtain the following:

Theorem A 15. *Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ and $\tau\colon V \to V$ be the anti-involutions of $\mathfrak{g} = \mathfrak{sp}_1(\mathbb{C}) + \mathfrak{sp}_n(\mathbb{C})$ and $V = \mathbb{C}^{4n}$ defined in Sect.2.4. There is a one-to-one correspondence between the following two sets of data, up to, respectively, local isometry and local equivalence:*

i) Pseudo-hyperkähler metrics of signature $(4p, 4q)$ over open subsets of $W = \mathbb{R}^{4n}$

ii) Torsionless complex $\mathfrak{g}$-structures $\alpha\colon \mathfrak{g} + V \to \mathfrak{X}^{\mathbb{C}}(\mathcal{U})$ on neighbourhoods $\mathcal{U}$ of the identity $e = (I_2, I_{2n}, 0)$ in $P = \mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(\mathbb{C})\ltimes W$, $W = V^\tau$, so that

a) the curvature $R_u \in \mathrm{Hom}(\Lambda^2 V, \mathfrak{g})$, $u \in \mathcal{U}$, takes values only in the $\mathfrak{sp}_n(\mathbb{C})$ part of $\mathfrak{g}$, the $\mathfrak{sp}_1(\mathbb{C})$ part being trivial and

b) there exists a submanifold $\widehat{\mathcal{U}} \subset \mathcal{U}$ containing $e \in \mathcal{U}$, tangent to the distribution defined by

$$\mathcal{D}_u = \alpha(\mathfrak{sp}_n(\mathbb{C}))_u + \mathrm{span}_{\mathbb{R}}\{\, \mathrm{Re}(X)_u\, ,\ X \in \alpha(V^\tau)\,\}\, , \tag{A8}$$

of dimension $\dim \widehat{\mathcal{U}} = \mathrm{rank}\,\mathcal{D}$, such that the map

$$\beta\colon \mathfrak{sp}_n(\mathbb{C}) + V \to \mathfrak{X}^{\mathbb{C}}(\widehat{\mathcal{U}})\ , \quad \beta(X) := \alpha(X)|_{\widehat{\mathcal{U}}}\ , \tag{A9}$$

is a complex $\mathfrak{sp}_n(\mathbb{C})$-structure, τ-compatible around e.

Proof. By Prop. A14, a pseudo-hyperkähler metric g is naturally associated, up to local equivalences, with a unique $\mathrm{Sp}_{p,q}$-structure with a torsionless connection. The latter is (locally) a reduction of a real form of an $\mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(\mathbb{C})$-structure with a torsionless connection. This real form corresponds to an associated τ-compatible torsionless complex $\mathfrak{g}$-structure α (see Prop. A12). This $\mathfrak{g}$-structure satisfies the conditions a) and b) by construction. Conversely, if a) and b) hold, then α is associated, up to local equivalences, with an $\mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(C)$-structure with a real form admitting an $\mathrm{Sp}_{p,q}$-reduction corresponding to a pseudo-hyperkähler metric g. □

Complex $\mathfrak{g}$-structures corresponding to pseudo-hyperkähler metrics, are said to be **reducible** to $\mathfrak{sp}_{p,q}$-structures.

A3.2. ($\mathfrak{sp}_1(\mathbb{C})$+$\mathfrak{sp}_n(\mathbb{C})$)-structures and hk-pairs

Consider a torsionless, complex $\mathfrak{sp}_1(\mathbb{C})$+$\mathfrak{sp}_n(\mathbb{C})$-structure

$$\alpha\colon (\mathfrak{sp}_1(\mathbb{C}) + \mathfrak{sp}_n(\mathbb{C})) + V \longrightarrow \mathfrak{X}^{\mathbb{C}}(\mathcal{U}')\, ,\ V = W^{\mathbb{C}} = \mathbb{C}^{4n},$$

on some open neighbourhood $\mathcal{U}'$ of $e = (I_2, I_{2n}, 0) \in P = \mathrm{Sp}_1(\mathbb{C})\times\mathrm{Sp}_n(\mathbb{C})\ltimes W$, $W = V^\tau$, which is reducible to an $\mathfrak{sp}_{p,q}$-structure. Let $\mathcal{A}_o = (H^o_0, H^o_{\pm\pm}, E^o_A, e^o_{\pm a})$ be the standard basis of $\mathfrak{p} = \mathfrak{sp}_1(\mathbb{C}) + \mathfrak{sp}_n(\mathbb{C}) + V$ (see Sect. 2). As discussed in Sect. A2.1, α is completely determined by the set of complex vector fields

$$\mathcal{A}^{(\alpha)} = \left(e_{\pm a} = \alpha(e^o_{\pm a}), H_0 = \alpha(H^o_0)^{10}, H_{\pm\pm} = \alpha(H^o_{\pm\pm})^{10}, E_A = \alpha(E^o_A)^{10}\right) ,$$

which we call the **frame associated with** α. The Lie brackets of the fields in $\mathcal{A}^{(\alpha)}$ are of the form (3.1)-(3.2) and it is therefore tempting to claim that $\mathcal{A}^{(\alpha)}$ is an hk-frame. Alas, this is not so (see Def. 3.2), since the vector fields in $\mathcal{A}^{(\alpha)}$ are not defined on an appropriate open neighbourhood $\mathcal{U} \subset \mathcal{P}$ of e, but rather on an open subset $\mathcal{U}'$ of P, a codimension n real submanifold of $\mathcal{P} = \mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C}) \ltimes V$. However, we have:

Lemma A16. *When the data are real analytic and $\mathcal{U}'$ is sufficiently small, there exists an appropriate neighbourhood $\mathcal{U} \subset \mathcal{P}$ of e, which contains $\mathcal{U}'$, on which the vector fields in $\mathcal{A}^{(\alpha)}$ admit unique holomorphic extensions.*
The set $\mathcal{A} = (H_0, H_{\pm\pm}, E_A, e_{\pm a})$ of such holomorphic extensions on $\mathcal{U}$ is a central hk-frame, uniquely associated with α up to local equivalence, such that

a) the intersection $((\{I_2\} \times \mathrm{Sp}_n(\mathbb{C})) \ltimes W) \cap \mathcal{U}$ is equal to a submanifold $\widehat{\mathcal{U}}$ of $\mathcal{U}'$ as in Theorem A15 b), i.e. tangent at all points the distribution $\mathcal{D}$ in (A8)
b) the pair $(\mathcal{A}, M)$, with $M := \mathcal{U} \cap \{I_2\}\times\{I_{2n}\}\times W$, is an hk-pair.

Remark A17. The distribution $\mathcal{D}$ can be also described as the restriction $\mathcal{D} = \widetilde{\mathcal{D}}|_{\mathcal{U}'}$ of the real distribution $\widetilde{\mathcal{D}} \subset T\mathcal{U}$, generated by an appropriate set of real and imaginary parts of vector fields in $\mathcal{A}$, namely by the real and imaginary parts of E_A, together with the vector fields $\mathrm{Re}(\alpha(w))$, $w \in V^\tau$.

Before proving the lemma, it is convenient to review the notion of *(local) holomorphic extensions* of real analytic complex vector fields.

Let (N, J) be a complex manifold, $\dim_{\mathbb{C}} N = m$ and $T^{\mathbb{C}}N = T^{10}N \oplus T^{01}N$ the decomposition in holomorphic and anti-holomorphic tangent bundles. For any chart of holomorphic complex coordinates $\xi = (\zeta^1, \dots, \zeta^m) \colon \mathcal{U} \subset N \to \mathbb{C}^m$ and $x \in \mathcal{U}$, we have:

$$T^{10}_x N = \mathrm{span}_{\mathbb{C}} \left\{ \left.\frac{\partial}{\partial \zeta^i}\right|_x \right\} , \quad T^{01}_x N = \mathrm{span}_{\mathbb{C}} \left\{ \left.\frac{\partial}{\partial \overline{\zeta^i}}\right|_x \right\} .$$

Given an open subset $\mathcal{V} \subset N$ and standard coordinates (z^i, w^j) of $\mathbb{C}^{2m}$, a **complexification** of $\mathcal{V}$ is a real analytic embedding $\imath \colon \mathcal{V} \subset N \to \mathbb{C}^{2m}$, satisfying

1) $\imath(\mathcal{V}) \subset \{ (z^i, w^i) : w^i - \overline{z^i} = 0 \} \subset \mathbb{C}^{2m}$ and

2) $\imath_*\left(T^{10}_x N|_{\mathcal{V}}\right) = \mathrm{span}_{\mathbb{C}}\left\{ \frac{\partial}{\partial z^i}\Big|_{\imath(z)} + \frac{\partial}{\partial \overline{w^i}}\Big|_{\imath(z)}\ ,\ 1 \leq i \leq m \right\}$ for all $x \in \mathcal{V}$.

Complexifications are easily constructed if holomorphic complex coordinates $\xi = (\zeta^i)$ on $\mathcal{V}$ exist. Namely, it suffices to consider the embedding

$$\imath\colon \mathcal{V} \subset N \longrightarrow \mathbb{C}^{2m}\ , \qquad \imath(x) = (\zeta^1(x), \dots, \zeta^m(x), \overline{\zeta^1}(x), \dots, \overline{\zeta^m}(x))\ .$$

Consider now a real analytic complex vector field on $\mathcal{V}$

$$X = X^i(\zeta^k, \bar\zeta^\ell)\frac{\partial}{\partial \zeta^i} + \mathcal{X}^j(\zeta^k, \bar\zeta^\ell)\frac{\partial}{\partial \overline{\zeta^j}}$$

and identify X with the field $\imath_*(X) \in T^{\mathbb{C}}\mathbb{C}^{2m}|_{\imath(\mathcal{V})}$ on $\imath(\mathcal{V})$. This vector field immediately extends to a holomorphic vector field X^{hol}, defined on an open neighbourhood $\mathcal{W} \subset \mathbb{C}^{2m}$ of $\imath(\mathcal{V})$ by setting

$$X^{hol}\big|_{(z^k, w^\ell)} := X^i(z^k, w^\ell)\frac{\partial}{\partial z^i} + \mathcal{X}^j(z^k, w^\ell)\frac{\partial}{\partial w^j}\ , \tag{A10}$$

i.e. replacing the (dependent) complex coordinates ζ^k and $\bar\zeta^\ell$ of $\mathcal{V}$ by the independent variables z^k and w^ℓ of $\mathbb{C}^{2m}$. The resulting holomorphic vector fields are called **(local) holomorphic extensions** of real analytic vector fields.

We now proceed to the missing proof:

Proof of Lemma A16. Consider the distribution $\mathcal{D} \subset T\mathcal{U}'$ and the submanifold $\widehat{\mathcal{U}} \subset \mathcal{U}'$, tangent to the distribution $\mathcal{D}$, described in Theorem A15. From (A9) and the hypotheses on the Lie brackets, it follows that $\mathcal{U}'$ is foliated by submanifolds of the form $\widehat{\mathcal{U}}{\cdot}g$ determined by images of $\widehat{\mathcal{U}}$ under the local action of the elements $g \in \mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C})$. By construction, all such submanifolds are integral leaves of $\mathcal{D}$, proving that the distribution $\mathcal{D}$ is indeed integrable. Consider the complex distribution $\mathcal{D}^{10}$ defined by $\mathcal{D}^{10}_x = \mathrm{span}_{\mathbb{C}}\{\ E_A|_x, e_{+\alpha}|_x\ \}$ for $x \in \mathcal{U}'$. Recall that $\mathcal{D}^{\mathbb{C}}_x = \mathcal{D}^{10}_x + \mathcal{D}^{01}_x$ where $\mathcal{D}^{01}_x := \overline{\mathcal{D}^{10}_x}$.

For $y \in \mathcal{U}'$, we denote by $\mathfrak{F}_y$ the integral leaf of $\mathcal{D}$ passing through y. Since the fields $(E_A + \overline{E_A})|_{\mathfrak{F}_y}$ generate a Lie algebra of real vector fields isomorphic to $\mathfrak{sp}_n(\mathbb{C})$, they determine a local right action of $\mathrm{Sp}_n(\mathbb{C})$ on $\mathfrak{F}_y$. Moreover, the complementary subbundles $\mathcal{D}^{10}|_{\mathfrak{F}_y}$, $\mathcal{D}^{01}|_{\mathfrak{F}_y}$ of $T^{\mathbb{C}}\mathfrak{F}_y$ are involutive and $\mathrm{Sp}_n(\mathbb{C})$-invariant. Therefore, there exists a unique $\mathrm{Sp}_n(\mathbb{C})$-invariant integrable complex structure J_y on $\mathfrak{F}_y$, which has the subbundles $\mathcal{D}^{10}|_{\mathfrak{F}_y}$, $\mathcal{D}^{01}|_{\mathfrak{F}_y}$ as associated holomorphic and anti-holomorphic distributions. By $\mathrm{Sp}_n(\mathbb{C})$-invariance, such a complex structure J_y naturally projects onto a complex structure $\widetilde{J}_y$ on the quotient $\widetilde{\mathfrak{F}}_y = \mathfrak{F}_y/\mathrm{Sp}_n(\mathbb{C})$. We may identify the pair $(\widetilde{\mathfrak{F}}_y, \widetilde{J}_y)$, without loss of generality, with an open neighbourhood $\widetilde{\mathfrak{F}}_y$ of 0 in $(\mathbb{C}^{2n})^\tau \simeq \mathbb{R}^{4n}$, endowed with an appropriate complex structure $\widetilde{J}_y$.

If $\mathcal{U}' \subset P$ is sufficiently small we may always assume that the following conditions are satisfied:

a) All integral leaves of $\mathcal{D}$ are transversal to the orbits of the local right action of $\mathrm{Sp}_1(\mathbb{C})$ generated by real vector fields in

$$\mathrm{span}_{\mathbb{R}}\{\mathrm{Re}(H_0), \mathrm{Im}(H_0), \mathrm{Re}(H_{\pm\pm}), \mathrm{Im}(H_{\pm\pm})\} \simeq \mathfrak{sp}_1(\mathbb{C}) .$$

b) The quotients $\widetilde{\mathfrak{F}}_y = \mathfrak{F}_y/\mathrm{Sp}_n(\mathbb{C}) \subset \mathbb{C}^{2n}$ are all diffeomorphic to a fixed suitable open subset $\widetilde{\mathfrak{F}}$ of $(\mathbb{C}^{4n})^\tau \simeq \mathbb{R}^{4n}$. Thus, $\widetilde{\mathfrak{F}} \subset \mathbb{R}^{4n}$ is equipped with a family integrable complex structures $\{\widetilde{J}_y \ , \ y \in \mathcal{U}' \ \}$, these being the push-forwards of the complex structures of the leaves $\widetilde{\mathfrak{F}}_y$, $y \in \mathcal{U}'$.

c) Any leaf $\mathfrak{F}_y$ admits a holomorphic trivialisation

$$\varphi_y \colon (\mathfrak{F}_y \, , \, J_y) \ \to \ (\mathrm{Sp}_n(\mathbb{C}) \times \widetilde{\mathfrak{F}} \, , \, J_o + \widetilde{J}_y) \ ,$$

where J_o is the standard complex structure of $\mathrm{Sp}_n(\mathbb{C})$.

d) For all $y \in \mathcal{U}'$, the complex manifold $(\widetilde{\mathfrak{F}}, \widetilde{J}_y) \subset (\mathbb{R}^{4n}, \widetilde{J}_y) \simeq \mathbb{C}^{2n}$ admits a complexification $\imath_y \colon \widetilde{\mathfrak{F}} \to V = \mathbb{C}^{4n}$, which together with the trivialisation φ_y, determines a real analytic $\mathrm{Sp}_n(\mathbb{C})$-equivariant embedding

$$\imath_y \colon \mathfrak{F}_y \ \to \ (\{I_2\} \times \mathrm{Sp}_n(\mathbb{C})) \ltimes V \, , \quad y \mapsto (I_2, I_{2n}, z^i(y)) \in \mathcal{P} \, .$$

Using this embedding, the vector fields $(E_A, e_{\pm a})|_{\mathfrak{F}_y}$ extend holomorphically to an open neighbourhood $\mathcal{U}_y \subset (\{I_2\}{\times}\mathrm{Sp}_n(\mathbb{C})){\ltimes}V$ of $\imath(\mathfrak{F}_y)$. We may choose the map $\imath_y$ so that $\imath_*(E_A) = E^o_A$.

e) Given a submanifold $M \subset \widehat{\mathcal{U}}$ satisfying the conditions of Def. A11, the maps $\imath_y$, $y \in M$, combine to determine an $\mathrm{Sp}_1(\mathbb{C}){\times}\mathrm{Sp}_n(\mathbb{C})$-equivariant real analytic embedding of $\mathcal{U}'$ into an appropriate open neighbourhood $\mathcal{U}$ of $e \in \mathcal{P}$,

$$\imath \colon \mathcal{U}' = \bigcup_{y \in M \cdot \mathrm{Sp}_1(\mathbb{C})} \mathfrak{F}_y \ \longrightarrow \ \mathcal{U} \subset \mathcal{P} = (\mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C})) \ltimes V \, . \tag{A11}$$

This embedding can be constructed to map the points $y \in M$ into points of $M^o = \{I_2\}{\times}\{I_{2n}\}{\times}(\mathbb{C}^{4n})^\tau$ and the complex vector fields H_0 , $H_{\pm\pm}$, E_A , $e_{\pm a}$ of $\mathcal{U}'$ into complex vector fields of $\imath(\mathcal{U}') \subset \mathcal{U}$, which extend holomorphically to all of $\mathcal{U}$. The equivariant embedding $\imath$ can also be constructed so that the holomorphic extensions of $\imath_*(H_0), \imath_*(H_{\pm\pm}), \imath_*(E_A)$ are $H^o_0, H^o_{\pm\pm}, E^o_A$, respectively.

By construction, the pair $(\mathcal{A}, M = M^o)$, formed by the collection $\mathcal{A}$ of the above holomorphic extensions of the vector fields in $\mathcal{A}^{(\alpha)}$ and the manifold $M^o = \imath(M)$ is a central hk-pair and it is uniquely determined by α up to local equivalences. □

By Theorem A15 and Lemma A16, we may associate an hk-pair $(\mathcal{A}, M)$ with every real analytic pseudo-hyperkähler manifold (M, g). In the next section, we show that, up to local equivalences, this correspondence is invertible, providing a bijection between local isometry classes of real analytic pseudo-hyperkähler manifolds and local equivalence classes of hk-pairs.

A4. **Inverse map between hk-pairs and pseudo-hyperkähler metrics**

Consider the pseudo-hyperkähler metric g determined by an hk-pair $(\mathcal{A}, M)$ and a section $\sigma \colon M \to \mathcal{U}^{(\mathrm{Sp}_{p,q})}$ (see Sect. 4.2). We now prove that g is uniquely associated with $(\mathcal{A}, M)$.

Lemma A18. *The metric* (4.6)*, constructed from an hk-pair* $(\mathcal{A}, M)$ *of signature* $(4p, 4q)$ *on an appropriate open subset* $\mathcal{U} \subset \mathcal{P}$*, is independent of the choice of section* $\sigma \colon M \to \mathcal{U}^{(\mathrm{Sp}_{p,q})}$ *and is a real analytic pseudo-hyperkähler metric of signature* $(4p, 4q)$*.*

Proof. Let $\alpha \colon \mathfrak{g} + V \to \mathfrak{X}^{\mathbb{C}}(\mathcal{U} \cap P)$ be the $\mathbb{R}$-linear map defined by

$$\alpha(X) = \mathrm{Re}(\alpha^{\mathcal{A}}(X))\big|_{\mathcal{U}\cap P}\,, \quad \alpha(v) = \alpha^{\mathcal{A}}(v)\big|_{\mathcal{U}\cap P}$$

for $X \in \mathfrak{g}$ and $v \in V = \mathbb{C}^{4n}$, where $\alpha^{\mathcal{A}}$ is the absolute hk-parallelism associated with $\mathcal{A}$ (see Sect. 3). By construction and the assumptions on M, the map α is a τ-compatible, complex $\mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_n(\mathbb{C})$-structure on $\mathcal{U} \cap P$. From the proof of Prop. A12, it follows that $M \cdot (\mathrm{Sp}_1(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})) \subset \mathcal{U}$ is an open subset of the complexification of an $\mathrm{Sp}_1 \times \mathrm{Sp}_{p,q}$-structure $\widetilde{P}$ over (an open subset of) M and that the set $\mathcal{U}^{(\mathrm{Sp}_{p,q})} = M \cdot \mathrm{Sp}_{p,q}$, defined in (4.3), is an $\mathrm{Sp}_{p,q}$-reduction of an $\mathrm{Sp}_1 \times \mathrm{Sp}_{p,q}$-invariant open subset of such $\mathrm{Sp}_1 \times \mathrm{Sp}_{p,q}$-structure.

The conditions on the curvature imply that $\widetilde{P}$ can be identified with a double covering of an $\mathrm{Sp}_1 \cdot \mathrm{Sp}_{p,q}$-reduction of $L(M)$, admitting a further reduction to an $\mathrm{Sp}_{p,q}$-bundle $Q \subset L(M)$. Thus the set $\mathcal{U}^{\mathrm{Sp}_{p,q}}$ is identifiable with an $\mathrm{Sp}_{p,q}$-invariant open subset of Q. As explained in Sect. A3.1, we therefore have that:

i) The bundles $\pi^Q \colon Q \to M$ and $\pi^{\widetilde{P}} \colon \widetilde{P} \to M$ are formed by linear frames which are orthonormal with respect to a pseudo-Riemannian metric g and pointwise hermitian with respect to a family of hypercomplex structures $(J_i|_x)_{i=1,2,3}\,, x \in M$.
ii) The fields $e^{\tau}_I = \alpha^{\mathcal{A}}_{(\mathbb{R})}(e^{o\tau}_I)$ are identifiable with (local) vector fields on Q, horizontal with respect to the Levi-Civita connection of g and $\vartheta(e^{\tau}_I) = e^{o\tau}_I$.

It follows that g is pseudo-hyperkähler and that the frame fields $e^{(\sigma)}_I = \pi_*(e^{\tau}_I)$ are orthonormal with respect to g regardless of the choice of the local section $\sigma \colon M \to \mathcal{U}^{(\mathrm{Sp}_{p,q})}$. □

We are finally in a position to prove the result quoted in Sect. 4.2.

Theorem A19. *Every real analytic pseudo-hyperkähler manifold of signature* $(4p, 4q)$ *is locally isometric to a pseudo-hyperkähler manifold* (M, g) *corresponding to an hk-pair of signature* $(4p, 4q)$*, with metric given by eq.* (4.6).

Proof. By the proof of Theorem A15, every pseudo-hyperkähler metric g of signature $(4p, 4q)$ on an open subset $M \subset \mathbb{R}^{4n}$ determines a bundle $\pi\colon P \to M$, the complexification of (a double covering of) an $\mathrm{Sp}_1{\cdot}\mathrm{Sp}_{p,q}$-bundle $(\widetilde{P}, \widetilde{\vartheta})$ of orthonormal frames of (M, g). The Levi-Civita connection and the soldering form of $\widetilde{P}$ determine a complex $\mathfrak{sp}_1(\mathbb{C}){\times}\mathfrak{sp}_n(\mathbb{C})$-structure on P, which is reducible to an $\mathfrak{sp}_{p,q}$-structure. This $\mathfrak{sp}_1(\mathbb{C}){\times}\mathfrak{sp}_n(\mathbb{C})$-structure is uniquely associated (see Sect. 4.3) with an hk-pair $(\mathcal{A}, M')$, where $\mathcal{A}$ is a central hk-frame on an appropriate open subset $\mathcal{U} \subset \mathcal{P}$ and $M' = \mathcal{U} \cap M^o$.

The claim is proved if we can show that the pseudo-hyperkähler metric g on $M \subset \mathbb{R}^{4n}$ coincides (modulo identifications) with the metric on $M' \simeq M$, associated with the hk-pair $(\mathcal{A}, M')$, i.e. the metric defined in eq. (4.6). For this, it suffices to observe that, by construction, the real submanifold $\mathcal{U}^{(\mathrm{Sp}_{p,q})} = M'{\cdot}\mathrm{Sp}_{p,q}$ of $\mathcal{U}$ considered in (4.3) coincides with the bundle of orthonormal frames $\widetilde{P}$ over $M(\simeq M')$, so that the vector fields $e^\tau_I|_{\mathcal{U}^{(\mathrm{Sp}_{p,q})}} = \alpha^{\mathcal{A}}_{(\mathbb{R})}(e^{o\tau}_I)|_{\mathcal{U}^{(\mathrm{Sp}_{p,q})}}$ are horizontal with respect to the Levi-Civita connection and satisfy the equation $\vartheta_u(e^\tau_a) = e^{o\tau}_a \in V^\tau$ for any $u \in \widetilde{P}$. Hence, the projections of the vectors $\vartheta_x(e^\tau_a)|_u$ onto the points

$$x = \pi(u) \in M' = \mathcal{U}^\tau/\mathrm{Sp}_1 \times \mathrm{Sp}_{p,q} \simeq \widetilde{P}/\mathrm{Sp}_1{\cdot}\mathrm{Sp}_{p,q} = M$$

constitute g-orthonormal frames and the metric (4.6) is necessarily equal to the metric g. □

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