TOMOGRAPHIC PORTRAIT OF QUANTUM CHANNELS

G. G. AMOSOV

Steklov Mathematical Institute of Russian Academy of Sciences,
ul. Gubkina 8, Moscow 119991, Russia
(e-mail: gramos@mi.ras.ru)

S. MANCINI

School of Science and Technology,
University of Camerino, I-62032 Camerino, Italy
and INFN Sezione di Perugia, I-06123 Perugia, Italy
(e-mail: stefano.mancini@unicam.it)

and

V. I. MANKO

Lebedev Physical Institute of Russian Academy of Sciences,
Leninsky pr. 53, Moscow 119991, Russia
and Moscow Institute of Physics and Technology,
Institutski per. 9, Dolgoprudny 141700, Russia
(e-mail: manko@lebedev.ru)

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We formulate the notion of quantum channels in the framework of quantum tomography and address there the issue of whether such maps can be regarded as classical stochastic maps. In particular, kernels of maps acting on probability representation of quantum states are derived for qubit and bosonic systems. In the latter case it results that a single mode Gaussian quantum channel corresponds to non-Gaussian classical channels.

Keywords: quantum channels, quantum tomography, quantizer and de-quantizer formalism.

1. Introduction

Following the general approach of [1], given a Hilbert space H and a set of operators $\hat{U}(\mathbf{x})$ acting on it, labelled by an n-dimensional real vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, we construct a complex-valued function associate to an operator $\hat{A}$ on H as

$$f_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}\big(\hat{U}(\mathbf{x})\hat{A}\big), \tag{1}$$

and call it hereafter *symbol of operator* $\hat{A}$. Suppose now that there exists a set of operators $\hat{D}(\mathbf{x})$ on H such that we can write

$$\hat{A} = \int \hat{D}(\mathbf{x}) f_{\hat{A}}(\mathbf{x}) d\mathbf{x}. \tag{2}$$

The requirement that the composition of maps (1) and (2) leads to the identity operator results in

$$\int \mathrm{Tr}\big(\hat{U}(\mathbf{x})\hat{D}(\mathbf{y})\big) f_{\hat{A}}(\mathbf{y}) d\mathbf{y} = f_{\hat{A}}(\mathbf{x}). \tag{3}$$

The sets $\hat{D}(\mathbf{x})$ and $\hat{U}(\mathbf{x})$ are said to be quantizer and de-quantizer respectively[1]. If one defines the map for which the symbol of identity operator $\hat{\mathrm{I}}$ is equal to the unit function, then operators $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$ satisfy the conditions

$$\mathrm{Tr}\big(\hat{U}(\mathbf{x})\big) = 1, \qquad \int \hat{D}(\mathbf{x}) d\mathbf{x} = \hat{\mathrm{I}}. \tag{4}$$

In this framework the symbol $\omega_{\hat{\rho}}(\mathbf{x})$ of a quantum state (i.e. an operator $\hat{\rho}$ on H such that $\hat{\rho} > 0$ and $\mathrm{Tr}\hat{\rho} = 1$) is said to be a quantum tomogram. We hereafter denote by $\mathfrak{T}(H)$ the set of all tomograms obtainable on H. Taking into account (2) we get

$$\mathrm{Tr}(\hat{\rho}) = \int \mathrm{Tr}\big(\hat{D}(\mathbf{x})\big) f_{\hat{\rho}}(\mathbf{x}) d\mathbf{x} = 1. \tag{5}$$

The alternative demand to (4) is

$$\mathrm{Tr}\big(\hat{D}(\mathbf{x})\big) = 1, \qquad \int \hat{U}(\mathbf{x}) d\mathbf{x} = \hat{\mathrm{I}}. \tag{6}$$

In this case the symbol of a quantum state $\hat{\rho}$ satisfies the relation

$$\int \omega_{\hat{\rho}}(\mathbf{x}) d\mathbf{x} = 1. \tag{7}$$

It should be noted that in general $\omega_{\hat{\rho}}(\mathbf{x}) \ngeq 0$. Hence $\omega_{\hat{\rho}}(\mathbf{x})$ is not always a probability distribution. Nevertheless, it is so for important cases such as spin [4], optical [5] and symplectic [6] tomographies. In such contexts the quantizer $\hat{D}(\mathbf{x})$ and de-quantizer $\hat{U}(\mathbf{x})$ give rise to a dual structure [7, 8]. It also should be noted that the symbol (1) becomes a characteristic function of the quantum state $\hat{\rho}$ whenever Weyl operators are used in place of $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$ [9]. Moreover $f_{\hat{A}}(\mathbf{x})$ can be a generalized function [10].

A quantum channel Φ is a linear, completely positive trace-preserving map on the set of all states $\mathfrak{S}(H)$ that can be represented as [11]

$$\Phi(\hat{\rho}) = \sum_i \hat{A}_i \hat{\rho} \hat{A}_i^{\dagger}, \qquad \sum_i \hat{A}_i^{\dagger} \hat{A}_i = \hat{\mathrm{I}}, \tag{8}$$

[1]Eq. (3) can be regarded as the completeness relation for generalized tomographies [2, 3].

$\hat{A}_i$ being operators on H.

Any quantum channel Φ generates a map $\breve{\Phi}$ on the set $\mathfrak{T}(H)$ by the formula

$$\breve{\Phi}(\omega_{\hat{\rho}})(\mathbf{x}) = \omega_{\Phi(\hat{\rho})}(\mathbf{x}), \qquad \hat{\rho} \in \mathfrak{S}(H). \tag{9}$$

Here we address the problem of representing (9) in the form

$$\breve{\Phi}(\omega)(\mathbf{x}) = \int \mathcal{K}(\mathbf{x}; \mathbf{x}')\, \omega(\mathbf{x}')d\mathbf{x}', \qquad \omega \in \mathfrak{T}(H), \tag{10}$$

to compare quantum channels with classical stochastic maps. The situation is considered both for finite- and infinite-dimensional Hilbert spaces H. In particular, it is shown that for the bosonic Gaussian quantum channel the kernel (10) gives rise to classical stochastic maps, but having a non-Gaussian form.

By referring to (8) we can write the map (10) with the kernel given by

$$\mathcal{K}(\mathbf{x}; \mathbf{x}') := \sum_i \operatorname{Tr}\big(\hat{U}(\mathbf{x})\hat{A}_i\hat{D}(\mathbf{x}')\hat{A}_i^\dagger\big). \tag{11}$$

If (6) is satisfied and

$$\int \mathcal{K}(\mathbf{x}; \mathbf{x}')\, d\mathbf{x}' = 1, \tag{12}$$

then the map defined by (10) has the property

$$\int \breve{\Phi}(\omega)(\mathbf{x})d\mathbf{x} = \int \omega(\mathbf{x})d\mathbf{x}$$

which is equivalent to preserving the trace for Φ. Nevertheless (12) does not take place in general because the set $\mathfrak{T}(H)$ cannot coincide with the set of all probability distributions [10]. Moreover, $\mathcal{K}(\mathbf{x}; \mathbf{x}') \not\geq 0$. Thus $\mathcal{K}(\mathbf{x}; \mathbf{x}')$ is not in general a conditional probability. Analogously the unitality of a channel Φ, i.e.

$$\Phi\left(\frac{1}{\dim H}\hat{I}\right) = \frac{1}{\dim H}\hat{I},$$

is not equivalent to the claim

$$\int \mathcal{K}(\mathbf{x}; \mathbf{x}')\, d\mathbf{x} = 1. \tag{13}$$

Taking into account that Φ is completely positive iff

$$\sum_{j,k}\langle\xi_j|\Phi\big(|\eta_j\rangle\langle\eta_k|\big)|\xi_k\rangle \geq 0, \qquad \forall|\xi_j\rangle, |\eta_k\rangle \in H, \tag{14}$$

we obtain the necessary and sufficient condition on $\mathcal{K}$ to determine a quantum channel in tomographic representation. That is

$$\sum_{j,k}\int\int \mathcal{K}(\mathbf{x}; \mathbf{x}')\langle\xi_j|\hat{D}(\mathbf{x})|\xi_k\rangle\langle\eta_k|\hat{U}(\mathbf{x}')|\eta_j\rangle d\mathbf{x}d\mathbf{x}' \geq 0, \qquad \forall|\xi_j\rangle, |\eta_k\rangle \in H. \tag{15}$$

2. Qubit channels

The qubit (spin-$\frac{1}{2}$) tomogram is given by [4, 12]

$$w_{\hat{\rho}}(\mathbf{x}) = w(\mathbf{x}) = \mathrm{Tr}\big(\hat{\rho}\,\hat{U}(\mathbf{x})\big), \tag{16}$$

where $\mathbf{x} := (m, \alpha, \beta)$. Here $m = \pm\frac{1}{2}$ are the two possible outcomes of the spin measurement performed along the direction $(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\beta)$ determined by the Euler angles α, β.

The operators $\hat{U}(\mathbf{x})$ read

$$\hat{U}(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m\begin{pmatrix} \cos\beta & -e^{i\alpha}\sin\beta \\ -e^{-i\alpha}\sin\beta & -\cos\beta \end{pmatrix}. \tag{17}$$

The tomograms satisfy the normalization conditions

$$\sum_{m=-1/2}^{1/2} w(m, \alpha, \beta) = 1, \qquad \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} w(m, \alpha, \beta)\,\sin\beta\, d\beta\, d\alpha = 1. \tag{18}$$

Eq. (16) can be inverted by expressing the density operator in terms of tomograms as

$$\hat{\rho} = \int \hat{D}(\mathbf{x})\, w(\mathbf{x}) d\mathbf{x}, \tag{19}$$

where

$$\int d\mathbf{x} := \sum_{m=-1/2}^{1/2} \frac{1}{2\pi}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta, \tag{20}$$

and

$$\hat{D}(\mathbf{x}) := 3\hat{U}(\mathbf{x}) - \hat{\mathrm{I}}. \tag{21}$$

A channel $\Phi : \mathfrak{S}(\mathbb{C}^2) \to \mathfrak{S}(\mathbb{C}^2)$ defines the linear map $\breve{\Phi}$ on the set $\mathfrak{T}(\mathbb{C}^2)$ of spin-$\frac{1}{2}$ tomograms by the formula

$$\breve{\Phi}(w_{\hat{\rho}})(m, \alpha, \beta) = w_{\Phi(\hat{\rho})}(m, \alpha, \beta). \tag{22}$$

The matrix (17) can be represented as follows,

$$\hat{U}(\mathbf{x}) = \frac{1}{2}\hat{\mathrm{I}} - m\cos\alpha\sin\beta\,\hat{\sigma}_x - m\sin\alpha\sin\beta\,\hat{\sigma}_y + m\cos\beta\,\hat{\sigma}_z, \tag{23}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ are the standard Pauli operators. Thus, to determine $\breve{\Phi}$ one should check the action of a conjugate map Φ^*, that is $\mathrm{Tr}\big(\hat{\rho}\Phi^*(\hat{\sigma})\big) = \mathrm{Tr}\big(\Phi(\hat{\rho})\hat{\sigma}\big)$, on (23).

2.1. Unital qubit channel

All unital qubit channels $\Phi : \mathfrak{S}(\mathbb{C}^2) \to \mathfrak{S}(\mathbb{C}^2)$ are mixture of unitary channels, i.e. there are unitary operators $\hat{U}_j : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$\Phi(\hat{\rho}) = \sum_j \pi_j \hat{U}_j \hat{\rho} \hat{U}_j^*, \tag{24}$$

$\pi_j \geq 0$, $\sum_j \pi_j = 1$. Moreover, picking up unitaries $\hat{U}, \hat{V} : \mathbb{C}^2 \to \mathbb{C}^2$ we can obtain the representation (24) for the channel $\Psi(\hat{\rho}) = \hat{U}\Phi(\hat{V}\hat{\rho}\hat{V}^*)\hat{U}^*$ with $\hat{U}_j \in \mathrm{SU}(2)$.

Let us write $\mathbf{x} = (m, \overline{n})$, where $\overline{n} := (\cos\alpha \sin\beta, \sin\alpha \sin\beta, \cos\beta)$. It follows from (23) and (24) that

$$\breve{\Phi}(w)(m, \overline{n}) = \sum_j \pi_j w(m, \hat{V}_j \overline{n}), \tag{25}$$

where $\hat{V}_j \in \mathrm{O}(3)$.

Given a unital qubit channel $\Phi : \mathfrak{S}(\mathbb{C}^2) \to \mathfrak{S}(\mathbb{C}^2)$, there exist unitary operators $\hat{U}, \hat{V} : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$\Psi(\hat{\rho}) = \hat{U}\Phi(\hat{V}\hat{\rho}\hat{V}^*)\hat{U}^* = \pi_0\hat{\rho} + \pi_x\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \pi_y\hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \pi_z\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z, \qquad \hat{\rho} \in \mathfrak{S}(\mathbb{C}^2), \tag{26}$$

where $\{\pi_0, \pi_x, \pi_y, \pi_z\}$ is a probability distribution. Thus, it suffices to study only channels Ψ of the form (26). Denote by $\breve{\Sigma}_a$ the unitary quantum channel implemented by the Pauli matrix $\hat{\sigma}_a$, i.e.

$$\breve{\Sigma}_a(\hat{\rho}) = \hat{\sigma}_a\hat{\rho}\hat{\sigma}_a, \qquad \hat{\rho} \in \mathfrak{S}(\mathbb{C}^2),$$

with $a \in \{x, y, z\}$.

PROPOSITION 1. *The linear maps $\breve{\Sigma}_x$, $\breve{\Sigma}_y$ and $\breve{\Sigma}_z$ act on the set $\mathfrak{T}(\mathbb{C}^2)$ of qubit tomograms as follows*

$$\begin{aligned}
\breve{\Sigma}_x &: w(m, \alpha, \beta) \to w\left(m, \alpha - \frac{\pi}{2}, \beta + \frac{\pi}{2}\right),\\
\breve{\Sigma}_y &: w(m, \alpha, \beta) \to w\left(m, \alpha + \frac{\pi}{2}, \beta + \frac{\pi}{2}\right),\\
\breve{\Sigma}_z &: w(m, \alpha, \beta) \to w\left(m, \alpha, \beta - \frac{\pi}{2}\right).
\end{aligned} \tag{27}$$

Proof: It is

$$\breve{\Sigma}_a(w_{\hat{\rho}})(\mathbf{x}) = \mathrm{Tr}(\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a\hat{U}(\mathbf{x})) = \mathrm{Tr}(\hat{\rho}\hat{\sigma}_a\hat{U}(\mathbf{x})\hat{\sigma}_a),$$

$a \in \{x, y, z\}$. Taking into account (23) we get

$$\begin{aligned}
\hat{\sigma}_x\hat{U}(m, \alpha, \beta)\hat{\sigma}_x &= \hat{U}\left(m, \alpha - \frac{\pi}{2}, \beta + \frac{\pi}{2}\right),\\
\hat{\sigma}_y\hat{U}(m, \alpha, \beta)\hat{\sigma}_y &= \hat{U}\left(m, \alpha + \frac{\pi}{2}, \beta + \frac{\pi}{2}\right),\\
\hat{\sigma}_z\hat{U}(m, \alpha, \beta)\hat{\sigma}_z &= \hat{U}\left(m, \alpha, \beta - \frac{\pi}{2}\right).
\end{aligned}$$

□

COROLLARY 1. *The linear map $\breve{\Phi}$ on the set $\mathfrak{T}(\mathbb{C}^2)$ of qubit tomograms is* *associated with a unital quantum channel iff it is (up to unitary equivalence) a convex linear combination of the identity map and the three maps (27).*

Proof: It immediately follows from the representation of unital channel in the form (26). □

PROPOSITION 2. *The maps $\breve{\Sigma}_x$, $\breve{\Sigma}_y$ and $\breve{\Sigma}_z$ can be represented in the form of integral operators*

$$\breve{\Sigma}_a(w)(\mathbf{x}) = \int K_a(\mathbf{x};\mathbf{x}')w(\mathbf{x}')\,d\mathbf{x}', \qquad a \in \{x, y, z\},$$

with the kernels defined by the formulae:

$$K_x(\mathbf{x};\mathbf{x}') = \frac{1}{2}\delta_{mm'}(1+3\cos\alpha\sin\beta\cos\alpha'\sin\beta' - 3\sin\alpha\sin\beta\sin\alpha'\sin\beta' - 3\cos\beta\cos\beta'),$$

$$K_y(\mathbf{x};\mathbf{x}') = \frac{1}{2}\delta_{mm'}(1-3\cos\alpha\sin\beta\cos\alpha'\sin\beta' + 3\sin\alpha\sin\beta\sin\alpha'\sin\beta' - 3\cos\beta\cos\beta'),$$

$$K_z(\mathbf{x};\mathbf{x}') = \frac{1}{2}\delta_{mm'}(1-3\cos\alpha\sin\beta\cos\alpha'\sin\beta' - 3\sin\alpha\sin\beta\sin\alpha'\sin\beta' + 3\cos\beta\cos\beta').$$

Proof: Let us define the inner product by the formula

$$(f, g) := \int_0^{2\pi}\int_0^{\pi} \overline{f(\alpha,\beta)}g(\alpha,\beta)\,\sin\beta d\beta\,d\alpha. \tag{28}$$

Then, the functions

$$\begin{aligned} f_0(\alpha,\beta) &= 1, & f_1(\alpha,\beta) &= \cos\alpha\sin\beta, \\ f_2(\alpha,\beta) &= \sin\alpha\sin\beta, & f_3(\alpha,\beta) &= \cos\beta, \end{aligned} \tag{29}$$

become orthogonal with respect to (28). Moreover,

$$\|f_0\|^2 = 2, \quad \|f_1\|^2 = \|f_2\|^2 = \|f_3\|^2 = \frac{2}{3}.$$

To fullfil the transformation from Proposition 1 one can construct the kernels using this set of orthogonal functions. □

REMARK 1. The kernels determined in Proposition 3 are not positive definite. Thus, the maps $\breve{\Sigma}_x$, $\breve{\Sigma}_y$ and $\breve{\Sigma}_z$ are not classical channels.

2.2. Nonunital qubit channels

Given a qubit channel $\Phi : \mathfrak{S}(\mathbb{C}^2) \to \mathfrak{S}(\mathbb{C}^2)$ there exist unitaries $\hat{U}, \hat{V} : \mathbb{C}^2 \to \mathbb{C}^2$, and a set of real numbers $(t_x, t_y, t_z, \lambda_x, \lambda_y, \lambda_z)$ such that

$$\Psi(\hat{\rho}) = \hat{U}\Phi(\hat{V}\hat{\rho}\hat{V}^*)\hat{U}^* = \frac{1}{2}\big(\hat{\mathrm{I}} + (t_x + \lambda_x a_x)\hat{\sigma}_x + (t_y + \lambda_y a_y)\hat{\sigma}_y + (t_z + \lambda_z a_z)\hat{\sigma}_z\big), \tag{30}$$

where

$$\hat{\rho} = \frac{1}{2}(\hat{\mathrm{I}} + a_x\hat{\sigma}_x + a_y\hat{\sigma}_y + a_z\hat{\sigma}_z).$$

The image of the Bloch sphere of pure states under a map of the form (30) is the ellipsoid

$$\left(\frac{x_1 - t_1}{\lambda_1}\right)^2 + \left(\frac{x_2 - t_2}{\lambda_2}\right)^2 + \left(\frac{x_3 - t_3}{\lambda_3}\right)^2 = 1.$$

The conditions on the parameters $(t_x, t_y, t_z, \lambda_x, \lambda_y, \lambda_z)$ for which Ψ is a channel are quite complicated and derived in [13].

The extreme points of the set (30) for nonunital case correspond (up to unitary equivalence) to

$$t_x = t_y = 0, \qquad \lambda_z = \lambda_x\lambda_y, \qquad t_z^2 = (1 - \lambda_x^2)(1 - \lambda_y^2). \tag{31}$$

For the conjugate map we obtain

$$\begin{aligned}\Psi^*(\hat{\rho}) &= \hat{U}\Phi(\hat{V}\hat{\rho}\hat{V}^*)\hat{U}^*\\ &= \frac{1}{2}\big((1 + t_xa_x + t_ya_y + t_za_z)\hat{\mathrm{I}} + \lambda_xa_x\hat{\sigma}_x + \lambda_ya_y\hat{\sigma}_y + \lambda_za_z\hat{\sigma}_z\big).\end{aligned} \tag{32}$$

Substituting (23) into (32) we get

$$\begin{aligned}\Psi^*(\hat{U}(\mathbf{x})) = {}& \frac{1}{2}\big(1 - t_xm\cos\alpha\sin\beta - t_ym\sin\alpha\sin\beta + t_zm\cos\beta\big)\hat{\mathrm{I}}\\ &- \lambda_xm\cos\alpha\sin\beta\,\hat{\sigma}_x - \lambda_ym\sin\alpha\sin\beta\,\hat{\sigma}_y + \lambda_zm\cos\beta\,\hat{\sigma}_z.\end{aligned} \tag{33}$$

PROPOSITION 3. *The map* (22) *associated with the channel* (30) *can be represented in the form of integral operator*

$$\breve{\Psi}(w)(\mathbf{x}) = \int \mathcal{K}(\mathbf{x};\mathbf{x}')w(\mathbf{x}')\,d\mathbf{x}',$$

with the kernel

$$\begin{aligned}\mathcal{K}(\mathbf{x};\mathbf{x}') = {}& \frac{\delta_{mm'}}{2}(1 - m\cos\alpha\sin\beta t_x - m\sin\alpha\sin\beta t_y + m\cos\alpha t_z)\\ &+ \frac{3}{2}\delta_{mm'}\big(-\cos\alpha\sin\beta\cos\alpha'\sin\beta'\lambda_x - \sin\alpha\sin\beta\sin\alpha'\sin\beta'\lambda_y + \cos\beta\cos\beta'\lambda_z\big).\end{aligned} \tag{34}$$

Proof: Following the idea of proof in Proposition 3, take into account that the functions (29) are orthogonal. Then, by means of them we construct the kernel corresponding to the transformation (33). □

REMARK 2. Like for unital channels the kernel (34) is not positive definite and the map $\hat{\Psi}$ is not a classical channel determined by conditional probabilities.

3. One-mode bosonic channel

In this section we shall move to the framework of optical homodyne tomography of a single-mode radiation field (see e.g. [14, 15]). The optical tomogram $\omega_{\hat\rho}(x,\varphi)$ of a state $\hat\rho$ in $L^2(\mathbb{R})$ is given by the formula [5]

$$\omega(x,\varphi)=\omega_{\hat\rho}(x,\varphi)=\mathrm{Tr}\big(\hat\rho\,\delta\big(x-\cos\varphi\hat Q-\sin\varphi\hat P\big)\big), \tag{35}$$

where $\hat Q$, $\hat P$ are the canonical conjugate quadratures operators and $x\in\mathbb{R}$, $\varphi\in[0,2\pi]$. The characteristic function $F(q,p)$ relative to $\hat\rho$ is defined as

$$F(q,p)=F_{\hat\rho}(q,p)=\mathrm{Tr}\big(\hat\rho\,e^{i(q\hat Q+p\hat P)}\big). \tag{36}$$

The optical tomogram $\omega(x,\varphi)$ is connected with the characteristic function $F(q,p)$ as follows,

$$F(t\cos\varphi,t\sin\varphi)=\int\limits_{\mathbb{R}}e^{itx}\omega(x,\varphi)dx, \tag{37}$$

$$\omega(x,\varphi)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}e^{-ixt}F(t\cos\varphi,t\sin\varphi)dt. \tag{38}$$

Following up (9), consider a map $\breve\Phi$ on the set of optical tomograms given by the formula

$$\breve\Phi(\omega_{\hat\rho})(x,\varphi)=\omega_{\Phi(\hat\rho)}(x,\varphi). \tag{39}$$

Below we shall deal with quantum Gaussian channels, widely used in quantum information (see e.g. [16]).

3.1. Covariant channel

Let us take a one-mode covariant bosonic channel Φ transforming the characteristic function $F(q,p)$ by the formula [9]

$$F(q,p)\to F(kq,kp)e^{-\frac{\alpha(q^2+p^2)}{2}}, \tag{40}$$

where

$$k\geq 0,\qquad k\neq 1,\qquad \alpha\geq\frac{|k^2-1|}{2}.$$

PROPOSITION 4. *The map* (39) *associated with the bosonic channel* (40) *can be represented as an integral operator with a Gaussian kernel*

$$\breve\Phi(\omega)(x,\varphi)=\frac{1}{\sqrt{2\pi\alpha}}\int\limits_{\mathbb{R}}e^{-\frac{(x-kx')^2}{2\alpha}}\omega(x',\varphi)dx'. \tag{41}$$

Proof: Taking into account the relations (37), (38) and (40) we get

$$\breve\Phi(\omega)(x,\varphi)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}e^{-ixy}e^{-\frac{\alpha y^2}{2}}\int\limits_{\mathbb{R}}e^{ikyx'}\omega(x',\varphi)dx'dy.$$

Changing the order of integration we arrive at

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx'-x)y}e^{-\frac{\alpha y^2}{2}}dy = \frac{1}{\sqrt{2\pi\alpha}}e^{-\frac{(kx'-x)^2}{2\alpha}}. \qquad \square$$

REMARK 3. The kernel

$$\mathcal{K}(x,\varphi;x',\varphi') = \frac{1}{\sqrt{2\pi\alpha}}e^{-\frac{(x-kx')^2}{2\alpha}}\delta(\varphi'-\varphi)$$

resulting from (41) is positive definite and

$$\int \mathcal{K}(x,\varphi;x',\varphi')dxd\varphi = 1, \qquad \int \mathcal{K}(x,\varphi;x',\varphi')dx'd\varphi' = \frac{1}{k}.$$

Hence the map (41) results stochastic, but not bi-stochastic. As a matter of fact $\mathcal{K}(x,\varphi;x',\varphi')$ does not represent a conditional probability distribution.

3.2. Contravariant channel

Let us now take a one-mode contravariant bosonic channel Φ transforming the characteristic function $F(q,p)$ by the formula [9]

$$F(q,p) \to F(kq,-kp)e^{-\frac{\alpha(q^2+p^2)}{2}}, \tag{42}$$

where

$$k \geq 0, \qquad \alpha \geq \frac{k^2+1}{2}.$$

PROPOSITION 5. *The map* (39) *associated with the bosonic channel* (42) *can be represented as an integral operator with a Gaussian kernel*

$$\breve{\Phi}(\omega)(x,\varphi) = \frac{1}{\sqrt{2\pi\alpha}}\int_{\mathbb{R}} e^{-\frac{(x-kx')^2}{2\alpha}}\omega\left(x',\varphi-\frac{\pi}{2}\right)dx'. \tag{43}$$

Proof: Taking into account the relations (37), (38) and (42) we get

$$\breve{\Phi}(\omega)(x,\varphi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixy}e^{-\frac{\alpha y^2}{2}}\int_{\mathbb{R}} e^{ikyx'}\omega\left(x',\varphi-\frac{\pi}{2}\right)dx'dy.$$

Changing the order of integration we arrive at

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx'-x)y}e^{-\frac{\alpha y^2}{2}}dy = \frac{1}{\sqrt{2\pi\alpha}}e^{-\frac{(kx'-x)^2}{2\alpha}}. \qquad \square$$

REMARK 4. The kernel

$$\mathcal{K}(x,\varphi;x',\varphi') = \frac{1}{\sqrt{2\pi\alpha}}e^{-\frac{(x-kx')^2}{2\alpha}}\delta(\varphi'-\varphi+\pi/2)$$

resulting from (43) is positive definite and

$$\int \mathcal{K}(x,\varphi;x',\varphi')dxd\varphi = 1, \qquad \int \mathcal{K}(x,\varphi;x',\varphi')dx'd\varphi' = \frac{1}{k}.$$

Hence the map (43) results stochastic, but not bi-stochastic. As a matter of fact $\mathcal{K}(x,\varphi;x',\varphi')$ does not represent a conditional probability distribution.

3.3. The representation on the plane

Following [7] let us define the function $\Omega(x,y)$ on the plane $\mathbb{R}^2$ in polar coordinates by the formula

$$\Omega(r\cos\varphi, r\sin\varphi) = \Omega_{\hat\rho}(r\cos\varphi, r\sin\varphi) := \frac{1}{r}\omega(r,\varphi). \tag{44}$$

Then,

$$\Omega(x,y) \geq 0, \qquad \frac{1}{2\pi}\int_{\mathbb{R}^2} \Omega(x,y)dxdy = 1,$$

hence Ω results to be probability distribution function on $\mathbb{R}^2$. It follows from the definition (44) that the characteristic function can be reconstructed from (44) by the formula

$$F(t\cos\varphi, t\sin\varphi) = \int_0^{+\infty} re^{itr}\Omega(r\cos\varphi, r\sin\varphi)dr. \tag{45}$$

Consider now the linear map on the set of functions (44)

$$\breve\Phi(\Omega_{\hat\rho})(x,y) = \Omega_{\Phi(\hat\rho)}(x,y). \tag{46}$$

PROPOSITION 6. *The map* (46) *associated with the bosonic channel* (40) *is the integral operator*

$$\breve\Phi(\Omega)(x,y) = \int_{\mathbb{R}^2} \mathcal{K}(x,y;x',y')\Omega(x',y')dx'dy',$$

with the kernel

$$\mathcal{K}(x,y;x',y') = \frac{1}{\sqrt{2\pi\alpha}}\exp\left(-\frac{(x-kx')^2+(y-ky')^2}{2\alpha}\right)\delta_{x,y}(x',y'), \tag{47}$$

where

$$\langle \delta_{x,y}, \psi\rangle := \frac{1}{\sqrt{x^2+y^2}}\int_0^{+\infty} r\psi\left(r\frac{x}{\sqrt{x^2+y^2}}, r\frac{y}{\sqrt{x^2+y^2}}\right)dr.$$

Proof: It is

$$\breve\Phi(\Omega)(\rho\cos\varphi, \rho\sin\varphi) = \frac{1}{2\pi\rho}\int_{\mathbb{R}} e^{-it\rho}e^{-\alpha\frac{t^2}{2}}\int_0^{+\infty} re^{iktr}\Omega(r\cos\varphi, r\sin\varphi)drdt.$$

Changing the order of integration we get

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{it(kr-\rho)}e^{-\alpha\frac{t^2}{2}}dt = \frac{1}{\sqrt{2\pi\alpha}}e^{-\frac{(\rho-kr)^2}{2\alpha}},$$

and

$$\breve{\Phi}(\Omega)(\rho\cos\varphi,\rho\sin\varphi) = \frac{1}{\sqrt{2\pi\alpha}\rho}\int_0^{+\infty} re^{-\frac{(\rho-kr)^2}{2\alpha}}\Omega(r\cos\varphi, r\sin\varphi)dr.$$

Substituting $x=\rho\cos\varphi,\ \ y=\rho\sin\varphi$ we obtain

$$\breve{\Phi}(\Omega)(x,y) = \frac{1}{\sqrt{2\pi\alpha(x^2+y^2)}}\int_0^{+\infty} re^{-\frac{(\sqrt{x^2+y^2}-kr)^2}{2\alpha}}\Omega\left(r\frac{x}{\sqrt{x^2+y^2}}, r\frac{y}{\sqrt{x^2+y^2}}\right)dr.$$

□

REMARK 5. It is worth remarking that the same conclusion of Proposition 7 can be drawn for contravariant channels simply changing (x, y) to $(y, -x)$ for Ω.

REMARK 6. The kernel (47) is positive definite and $\int \mathcal{K}(x, y; x', y')dxdy = 1$ and $\int \mathcal{K}(x, y; x', y')dx'dy' = 1/k$. Hence the map (46) results to be stochastic, but not bi-stochastic. As matter of fact $\mathcal{K}(x,\varphi; x',\varphi')$ does not represent a conditional probability distribution. Anyway, the one-mode bosonic channel (either covariant or contravariant) can be intended through the representation on the plane as a two-mode classical channel, i.e. acting on probability distribution functions on $\mathbb{R}\times\mathbb{R}$. This is in contrast to the map (39) where the argument is defined on $\mathbb{R}\times[0, 2\pi]$.

4. Conclusion

In conclusion, we have formulated the notion of quantum channel in the framework of quantum tomography, that is as a map acting on probability representation of quantum states (tomograms). Kernels for such maps were derived for qubit and bosonic systems. They show the existence of cases in which a quantum channel can be regarded as a classical stochastic map. In particular, this happens for the one-mode bosonic channel that corresponds to classical channels, though non-Gaussian.

The present study paves the way for finding further correspondences between quantum channels and classical stochastic maps. This could be helpful for characterizing the information transmission capabilities of quantum channels without the necessity of resorting to regularization procedures [17]. In fact it is known that (unlike quantum channels) classical channels admit single letter formula for capacity [18].

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