Minimum output entropy of a non-Gaussian quantum channel

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We introduce a model of a non-Gaussian quantum channel that stems from the composition of two physically relevant processes occurring in open quantum systems, namely, amplitude damping and dephasing. For it we find input states approaching zero output entropy while respecting the input energy constraint. These states fully exploit the infinite dimensionality of the Hilbert space. Upon truncation of the latter, the minimum output entropy remains finite, and optimal input states for such a case are conjectured thanks to numerical evidence.

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I. INTRODUCTION

Recently the subject of quantum channels has catalyzed the attention for its usefulness in foundational issues as well as in technological applications (for a recent review, see Ref. [1]). Formally a quantum channel is a completely positive and trace preserving map acting on the set of states (density operators) living in a Hilbert space. Since any physical process involves a state change, it can be regarded as a quantum channel mapping the initial (input) state to the final (output) state. As such it can be characterized in terms of its information transmission capability. This implies the use of entropic functionals among which the minimum output entropy plays a dominant role. In fact it is related to the minimum amount of noise inherent to the channel since it quantifies the minimum uncertainty occurring at the output of a channel when inputting pure states. More precisely, the output entropy measures the entanglement of the input pure state with the environment. Being this latter is not accessible, such entanglement induces loss of quantum coherence and thus injection of noise at the channel output. Clearly, low values of entanglement, i.e., of output entropy, correspond to low communication noise. As a consequence, the study of output entropy yields useful insight about channel capacities. In particular, an upper bound on the classical capacity can be derived from a lower bound on the output entropy of multiple channel uses [2].

When studying quantum channels a dichotomy between discrete and continuous channels usually appears. The former act on states living in finite-dimensional Hilbert spaces. In contrast the latter act on states living in infinite-dimensional Hilbert spaces. This is reflected in the possibility of using discrete or continuous variables to encode classical information. Among continuous quantum channels, attention has been almost exclusively devoted to Gaussian quantum channels, that is, channels mapping Gaussian input states into Gaussian output ones [3]. The reason is that they are easily implementable at the experimental level and moreover they are handy at the theoretical level. For these channels the minimum output entropy was largely investigated [4] and then showed that their classical capacity is actually achieved through states minimizing the output entropy [5].

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Here, we go beyond the restriction of Gaussianity of continuous quantum channels and propose a model of a non-Gaussian quantum channel that stems from the composition of two physically relevant processes that occur in open quantum systems, namely, amplitude damping and dephasing. We then single out a class of input states approaching zero output entropy while respecting the input energy constraint. Among them we analytically find the most economical in terms of a maximal number of quanta resources, namely, those having the smallest maximal number of quanta for a fixed small but nonzero value of output entropy. They consist of the superposition of two-number states one farthest away from the other. In truncated Hilbert spaces, we find that besides superposition of two-number states, the so-called binomial states [6] can be optimal depending on the value of channel parameters. We support this latter result by numerical investigations.

The paper is organized as follows. In Sec. II we introduce the model, and then we show the existence of input states achieving zero output entropy in Sec. III. Within such states we prove the optimality of the superposition of two-number states one farthest away from the other to get small but nonzero values of output entropy. Then focusing the attention on truncated Hilbert spaces, in Sec. IV we conjecture about the optimality of binomial states besides superposition of two-number states, and we give numerical evidence for this idea. Section V is for concluding remarks.

II. THE MODEL

Let us start by considering the Hilbert space $L^2(\mathbb{R})$ associated with a single bosonic mode with ladder operator $a,a^\dagger$.

In the framework of dynamical maps, a typical example of the Gaussian process is provided by the amplitude damping effect described by the master equation [7],

$$\frac{d}{dt}\varrho = 2a\varrho a^\dagger - a^\dagger a\varrho - \varrho a^\dagger a =: \mathcal{L}_{\mathrm{AD}}(\varrho)$$

for the density operator ϱ. In contrast, a typical example of the non-Gaussian process is provided by the purely dephasing effect described by the master equation [7],

$$\frac{d}{dt}\varrho = 2a^\dagger a\varrho a^\dagger a - (a^\dagger a)^2\varrho - \varrho(a^\dagger a)^2 =: \mathcal{L}_{\mathrm{PD}}(\varrho).$$

In order to interpolate between these two regimes we are going to consider the following dynamics:

$$\frac{d}{dt}\varrho = (1-\epsilon)\mathcal{L}_{\mathrm{AD}}(\varrho) + \epsilon\mathcal{L}_{\mathrm{PD}}(\varrho), \tag{1}$$

with $\epsilon \in [0,1]$. It is easy to see that

$$\mathcal{L}_{\mathrm{AD}}(\mathcal{L}_{\mathrm{PD}}(\varrho)) = \mathcal{L}_{\mathrm{PD}}(\mathcal{L}_{\mathrm{AD}}(\varrho)).$$

Therefore we can write the formal solution of (1) as

$$\varrho(t) = e^{(1-\epsilon)t\mathcal{L}_{\mathrm{AD}}} e^{\epsilon t \mathcal{L}_{\mathrm{PD}}} \varrho(0). \tag{2}$$

Actually this map can be regarded as a quantum channel $\Phi_{\epsilon,t}$ (depending on the parameters ϵ and t) mapping,

$$\varrho(0) \mapsto \varrho(t) = \Phi_{\epsilon,t}(\varrho(0)) = \sum_{j,k=0}^{\infty} E_{jk}\varrho(0)E_{jk}^{\dagger}, \tag{3}$$

where E_{jk} are the Kraus operators [1]. In view of (2),

$$E_{jk} = A_j P_k,$$

where A_j are the amplitude damping Kraus operators [8],

$$A_j = \sum_{l=j}^{\infty} \sqrt{\binom{l}{j}} [1-f(\epsilon,t)]^{(l-j)/2} [f(\epsilon,t)]^{j/2} |l-j\rangle\langle l|, \tag{4}$$

with $f(\epsilon,t) := 1 - e^{-2(1-\epsilon)t}$ and P_k are the phase damping Kraus operators [8],

$$P_k = \sum_{l=0}^{\infty} \sqrt{\frac{(2l^2\epsilon t)^k}{k!}} e^{-l^2\epsilon t} |l\rangle\langle l|. \tag{5}$$

In Eqs. (4) and (5) the Fock basis $\{|l\rangle\}_{l\in\mathbb{N}_0}$ representation is used. Expanding $\varrho(0)$ in the same basis as $\varrho(0) = \sum_{m,n=0}^{\infty} C_{m,n}(0)|m\rangle\langle n|$ and considering the channel in (3), we obtain

$$\varrho(t) = \sum_{m,n=0}^{\infty} C_{m,n}(t)|m\rangle\langle n|, \tag{6}$$

with

$$C_{m,n}(t) = e^{-Y_{m,n}(\epsilon)t} \sum_{l=0}^{\infty} C_{m+l,n+l}(0) \times \left[\binom{m+l}{l}\binom{n+l}{l}\right]^{1/2} f^l, \tag{7}$$

in which $Y_{m,n}(\epsilon) := (1-\epsilon)(m+n) + \epsilon(m-n)^2$. Equation (7) is also the solution of the following recursive relation:

$$\dot{C}_{m,n}(t) = 2(1-\epsilon)\sqrt{(m+1)(n+1)}C_{m+1,n+1}(t) - Y_{m,n}(\epsilon)C_{m,n}(t), \tag{8}$$

which is obtainable from the master equation (1).

When using quantum channels acting on the set of states living in an infinite-dimensional Hilbert space, the cost associated with each input (e.g., power cost) might diverge for large inputs (not to mention practical limits in the range acceptable to a receiver), likewise the classical case (see, e.g., Ref. [9]). It is therefore conventional to constrain an average cost to be less than or equal to some value. Here we will constrain the average input energy as

$$\mathrm{Tr}(\varrho(0)a^{\dagger}a) = N. \tag{9}$$

III. MINIMIZING OUTPUT ENTROPY

The output entropy of the quantum channel $\Phi_{\epsilon,t}$ in Eq. (3) is the von Neumann entropy of the output state, namely,

$$S[\Phi_{\epsilon,t}(\rho)] := -\mathrm{Tr}\{\Phi_{\epsilon,t}(\rho)\log_2[\Phi_{\epsilon,t}(\rho)]\}. \tag{10}$$

In order to quantify the noise inherent to the quantum channel $\Phi_{\epsilon,t}$ we look for its minimal output entropy and call the state with minimum output entropy the optimal input state.

By the following theorem we introduce a class of states with limiting zero output entropy.

Theorem 1. Any state ρ belonging to the set,

$$\mathcal{C} := \Big\{(1-\delta)|0\rangle\langle 0| + \delta|\xi_{\mathcal{G}}\rangle\langle\xi_{\mathcal{G}}| + \tau\sqrt{\delta(1-\delta)}(|0\rangle\langle\xi_{\mathcal{G}}| + |\xi_{\mathcal{G}}\rangle\langle 0|) \Big| \delta = \frac{N}{\langle\xi_{\mathcal{G}}|a^{\dagger}a|\xi_{\mathcal{G}}\rangle} \leqslant 1,\ -1 \leqslant \tau \leqslant 1\Big\},$$

with $|\xi_{\mathcal{G}}\rangle$ normalized and such that $\langle 0|\xi_{\mathcal{G}}\rangle = 0$ gives

$$\lim_{\delta\to 0} S[\Phi_{\epsilon,t}(\rho)] = 0.$$

Proof. First it is easy to see that the energy constraint $\mathrm{Tr}(\rho a^{\dagger}a) = N$ is satisfied by states $\rho \in \mathcal{C}$. Since vacuum is invariant under the action of the channel, the output state corresponding to input $\rho \in \mathcal{C}$ is as follows:

$$\Phi_{\epsilon,t}(\rho) = (1-\delta)|0\rangle\langle 0| + \delta\Phi_{\epsilon,t}(|\xi_{\mathcal{G}}\rangle\langle\xi_{\mathcal{G}}|) + \tau\sqrt{\delta(1-\delta)}\Phi_{\epsilon,t}(|0\rangle\langle\xi_{\mathcal{G}}| + |\xi_{\mathcal{G}}\rangle\langle 0|).$$

For δ approaching zero, which means finite N and $\langle\xi_{\mathcal{G}}|a^{\dagger}a|\xi_{\mathcal{G}}\rangle$ going to infinity, the output state results in the vacuum, which has zero entropy. ■

Although the content of Theorem 1 may appear not surprising, it raises a less obvious issue. In fact, in order to achieve the limit of zero output entropy, the average energy of $|\xi_{\mathcal{G}}\rangle$ must go to infinity ($\langle\xi_{\mathcal{G}}|a^{\dagger}a|\xi_{\mathcal{G}}\rangle \to \infty$). Expanding $|\xi_{\mathcal{G}}\rangle$ on the Fock basis, it becomes clear that this happens when the maximal number of energy quanta goes to infinity. However the maximal number of energy quanta should be considered as a resource itself, hence it would be relevant to find a state ρ that requires the minimum such resource to achieve a fixed small value of output entropy.

The next theorem formalizes this result.

Theorem 2. For sufficiently small but nonzero values of output entropy, say $0 < S[\Phi_{\epsilon,t}(\rho)] \leqslant \mathcal{E}$, the following states:

$$|\kappa_\alpha\rangle = \sqrt{1-\frac{N}{K}}|0\rangle + \sqrt{\frac{N}{K}}e^{i\alpha}|K\rangle, \quad K \in \mathbb{N}, \quad \alpha \in \mathbb{R} \tag{11}$$

are the most economical in terms of the maximal number of energy quanta resources among the states of set $\mathcal{C}$. An estimate of K is given by

$$K \geqslant \left\lceil \frac{N}{\mathcal{E}^2}\left(\frac{2}{\ln 2} + \sqrt{\frac{\pi}{2}eNf(1-f)}\right)^2 \right\rceil.$$

Proof. Let $N/\mathcal{K}$ be the value of δ corresponding to the (nonzero but small) value $\mathcal{E}$ of output entropy. Setting $K := \lceil\mathcal{K}\rceil$ it will be $\delta = \frac{N}{K} + O(\frac{1}{K^2})$. For small enough $\mathcal{E}$ it will be $\delta \ll 1$ and $S[\Phi_{\epsilon,t}(\rho)]$ monotonically increasing vs δ, i.e., monotonically decreasing vs K. Thus taking $\delta \approx \frac{N}{K}$ guarantees $S[\Phi_{\epsilon,t}(\rho)] \leqslant \mathcal{E}$. In turn this means that the average energy of state $|\xi_\wp\rangle = \sum_0^\infty c_l|l\rangle$ (expanded in the Fock basis) is

$$\sum_{l=1}^{\infty} l|c_l|^2 = K.$$

Looking for states with minimum numbers of energy quanta and satisfying such a constraint yields

$$\sum_{l=1}^{K} l|c_l|^2 = K.$$

On the other hand the normalization of $|\xi_\wp\rangle$ demands $\sum_{l=1}^K |c_l|^2 = 1$, thus leading to $c_l = e^{i\alpha}\delta_{l,K}$ ($\alpha \in \mathbb{R}$). Therefore $|\xi_\wp\rangle = e^{i\alpha}|K\rangle$ and the state with output entropy smaller than (or equal to) $\mathcal{E}$ has the following form:

$$\rho = \left(1 - \frac{N}{K}\right)|0\rangle\langle 0| + \frac{N}{K}|K\rangle\langle K| + \tau\sqrt{\frac{N}{K}\left(1-\frac{N}{K}\right)}(|0\rangle\langle K|e^{-i\alpha} + e^{i\alpha}|K\rangle\langle 0|). \quad (12)$$

Thanks to Lemmas 1 and 2 in the Appendix, we know that pure states give smaller output entropy, or in other words, for a fixed output entropy, they give smaller K. Hence, below we will assume $\tau = 1$, i.e.,

$$\rho = |\kappa_\alpha\rangle\langle\kappa_\alpha|, \quad |\kappa_\alpha\rangle := \sqrt{1-\frac{N}{K}}|0\rangle + \sqrt{\frac{N}{K}}e^{i\alpha}|K\rangle. \quad (13)$$

Furthermore, due to the covariance property of the channel under unitary transformations,

$$U \in \mathcal{U} := \left\{\sum_n e^{i\alpha n}|n\rangle\langle n| \,\middle|\, \alpha \in \mathbb{R}, n \in \mathbb{N}_0\right\},$$

all the states (13) have the same output entropy. Therefore, from now on, we can restrict the attention to $|\kappa_0\rangle$. Using Eq. (3), the corresponding output reads

$$\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|) = \left(1 - \frac{N}{K}(1-f^K)\right)|0\rangle\langle 0| + \sqrt{\frac{N}{K}\left(1-\frac{N}{K}\right)} \times (1-f)^K e^{-\epsilon K^2 t}(|0\rangle\langle K| + |K\rangle\langle 0|) + \frac{N}{K}\sum_{m=1}^{K}\binom{K}{m}(1-f)^m f^{K-m}|m\rangle\langle m|.$$

The matrix form of this output state is block diagonal, so the eigenvalues can be easily found as

$$\lambda_{0,K} = \frac{1}{2}[A + B \pm \sqrt{(A-B)^2 + 4C^2}],$$
$$\lambda_m = \frac{N}{K}\binom{K}{m}(1-f)^m f^{K-m}, \quad m = 1,\ldots,K-1,$$

with

$$A := 1 - \frac{N}{K}(1-f^K), \quad B := \frac{N}{K}(1-f)^K$$
$$C := \sqrt{\frac{N}{K}\left(1-\frac{N}{K}\right)}(1-f)^K e^{-\epsilon K^2 t}.$$

The input state (13) has output entropy,

$$\begin{aligned} S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)] &= -\lambda_0\log_2\lambda_0 - \lambda_K\log_2\lambda_K \\ &\quad - \frac{N}{K}[1 - f^K - (1-f)^K]\log_2\left(\frac{N}{K}\right) \\ &\quad + \frac{N}{K}\{f^K\log_2(f^K) + (1-f)^K\log_2[(1-f)^K]\} \\ &\quad + \frac{N}{K}\frac{1}{2}\log_2[2\pi eKf(1-f)] + O\left(\frac{1}{K}\right). \end{aligned} \quad (14)$$

Now we should find the K's such that $S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)] \leqslant \mathcal{E}$. To this end, we first find an upper bound for (14). Since projective measurements increase entropy [10], we have the inequality,

$$S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)] \leqslant H[p_{\epsilon,t}(n)],$$

where the right-hand side is the Shannon entropy of the probability mass function $p_{\epsilon,t}(n) := \langle n|\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)|n\rangle$. Explicitly the latter reads

$$p_{\epsilon,t}(n) = \begin{cases} 1 - \frac{N}{K}(1-f^K), & n = 0, \\ \frac{N}{K}\binom{K}{n}(1-f)^m f^{K-n}, & n = 1,\ldots,K. \end{cases}$$

As a consequence,

$$\begin{aligned} H[p_{\epsilon,t}(n)] &= -\left[1 - \frac{N}{K}(1-f^K)\right]\log_2\left[1-\frac{N}{K}(1-f^K)\right] \\ &\quad - \frac{N}{K}\log_2\left(\frac{N}{K}\right) + \frac{N}{K}f^K\log_2\left(\frac{N}{K}f^K\right) \\ &\quad + \frac{1}{2}\frac{N}{K}\log_2[2\pi eKf(1-f)] + O\left(\frac{1}{K}\right). \end{aligned}$$

Using the inequalities $-x\ln x < \sqrt{x(1-x)}$ and $\log_2 x < 0$ for $0 < x < 1$ we then get

$$\begin{aligned} H(p_{\epsilon,t}(n)) &\leqslant \frac{1}{\ln 2}\sqrt{\left(1-\frac{N}{K}(1-f^K)\right)\frac{N}{K}(1-f^K)} \\ &\quad + \frac{1}{\ln 2}\sqrt{\frac{N}{K}\left(1-\frac{N}{K}\right)} + \frac{1}{2}\frac{N}{K}\sqrt{2\pi eKf(1-f)} \\ &\leqslant \frac{2}{\ln 2}\sqrt{\frac{N}{K}} + \sqrt{\frac{N}{K}\frac{\pi}{2}eNf(1-f)}. \end{aligned}$$

By imposing that the above r.h.s. becomes smaller than $\mathcal{E}$, it follows

$$K \geqslant \left\lceil \frac{N}{\mathcal{E}^2}\left(\frac{2}{\ln 2} + \sqrt{\frac{\pi}{2}eNf(1-f)}\right)^2 \right\rceil.$$

■

In Fig. 1 the output entropy of input states $|\kappa_0\rangle$ (red circle line), $\sigma = (1 - \frac{K}{N})|0\rangle\langle 0| + \frac{N}{K}|K\rangle\langle K|$ (green star line), and

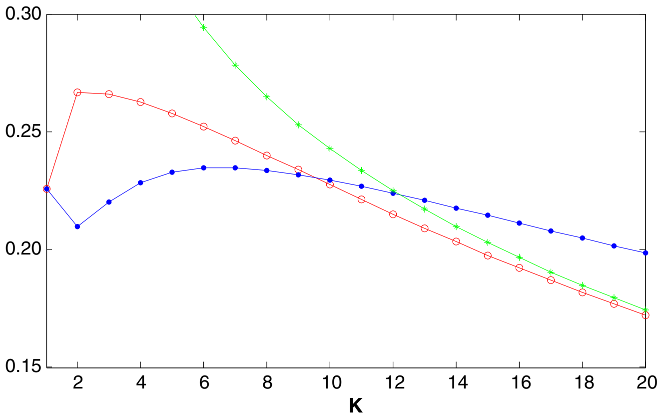

FIG. 1. Output entropy for input state $|\kappa_0\rangle$ (red circle line), $|\phi\rangle$ (dotted blue line), and σ (green star line) with $N = 0.6$, $t = 1.5$, and $\epsilon = 0.3$.

$|\phi\rangle = \sqrt{1-\frac{2N}{K+1}}|0\rangle + \sqrt{\frac{2N}{K(K+1)}}\sum_{l=1}^{K}|l\rangle$ (dotted blue line) vs K for $N = 0.6$, $t = 1.5$, and $\epsilon = 0.3$ is reported. This is an example showing how, by fixing a small but nonzero value of $\mathcal{E}$ (in this case $0 < \mathcal{E} < 0.2$), the output entropy of $|\kappa_0\rangle$ corresponds to the smallest value of K.

IV. SPACE TRUNCATION

In the previous section we showed that, although all states in $\mathcal{C}$ give limiting zero output entropy, the states (13) are optimal among them in terms of the maximal number of quanta resources. It means that, for a fixed value of a maximal number of quanta K, the states (13) minimize the output entropy among all states in $\mathcal{C}$.

On the other hand, once fixing the value of the maximal number of quanta K, we are allowed to exploit input states with the number of quanta between 0 and K (whereas always respecting the average energy constraint). This leads us to consider in this section the problem of minimizing the output entropy in the truncated Hilbert space of dimension $(K+1)$ spanned by the number state basis $\{|0\rangle,|1\rangle,\dots,|K\rangle\}$.

To accomplish the task, we have to go beyond the class of states $\mathcal{C}$. Hence, we introduce a class of states knows as binomial states [6],

$$|B\rangle_{M,\mu} := \sum_{n=0}^{M}\beta_n|n\rangle, \quad \beta_n := \left[\binom{M}{n}\mu^n(1-\mu)^{M-n}\right]^{1/2}, \tag{15}$$

with parameters $M \in \mathbb{N}$ and $\mu \in [0,1]$. The binomial state (15) reduces to the number state $|0\rangle$ for $\mu = 0$ and to the number state $|M\rangle$ for $\mu = 1$. In contrast, in the limits $\mu \to 0$, $M \to \infty$, and $\mu M = z \in \mathbb{R}$ the binomial state approaches the coherent state $|z\rangle$.

The energy constraint (9) yields the relation,

$$\mathrm{Tr}(|B\rangle_{M,\mu}\langle B|a^\dagger a) = M\mu = N.$$

Furthermore, inserting the coefficients β_n of (15) into (7) we get the explicit expression of the output density operator representation in the Fock basis,

$$\Phi_{\epsilon,t}(|B\rangle_{M,\mu}\langle B|) = \sum_{m,n=0}^{M} e^{-Y_{m,n}(\epsilon)t}\left(\frac{\mu}{1-\mu}\right)^{(m+n)/2} \sum_{l=0}^{M-\max\{m,n\}}\left[\binom{M}{m+l}\binom{m+l}{m}\binom{M}{n+l}\binom{n+l}{n}\right]^{1/2} \times(\mu f)^l(1-\mu)^{M-l}|m\rangle\langle n|.$$

Here we numerically evaluate the output entropy for binomial input states with average energy N. Once N is fixed we still have the freedom to vary μ and M in a way that $\mu M = N$. Since $\mu \leqslant 1$, for fixed N, we increase M from $\lceil N\rceil$ to K in order to find the minimum value of $S[\Phi_{\epsilon,t}(|B\rangle_{M,\mu}\langle B|)]$. From here on, when we refer to the binomial state $|B\rangle$, we mean the one which has minimum output entropy among other possible binomial states with average energy N.

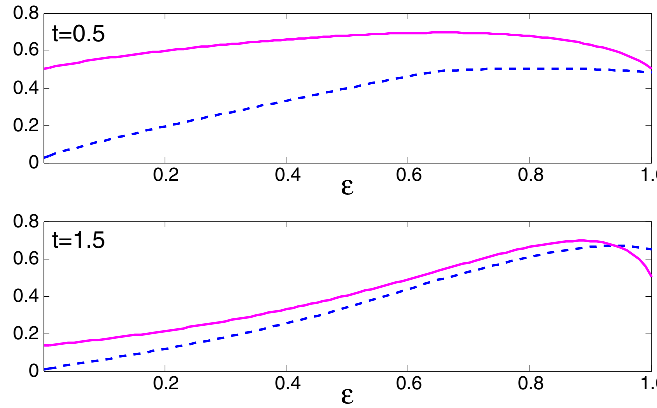

FIG. 2. Output entropy for input state $|B\rangle$ (dashed blue line) and for $|\kappa_0\rangle$ (magenta solid line) input states with $N = 0.6$ and $t = 0.5$ (top) and $t = 1.5$ (bottom).

Figure 2 shows the output entropy of state $|B\rangle$ in (15) (dashed blue line) and of state $|\kappa_0\rangle$ in (13) (magenta solid line) vs ϵ for $N = 0.6$ at $t = 0.5$ (top) and $t = 1.5$ (bottom). Here four-dimensional Hilbert space is considered. As can be argued from these figures, the output entropy of $|B\rangle$ remains smaller than the output entropy of $|\kappa_0\rangle$ (for any value of ϵ) until t reaches a threshold t_*. Then, for $t > t_*$ the state with less output entropy can be either $|B\rangle$ or $|\kappa\rangle$ depending on the value of ϵ (see also Fig. 3).

To have an estimation of t_*, we first point out that our numerical analysis shows that the output entropies of $|B\rangle$ and $|\kappa_0\rangle$ cross each other at large values of ϵ where the optimal value of M is 1. In such a case the output state of $|B\rangle$ lives in a two-dimensional subspace, and its output entropy turns out to be

$$S_B = -\sum_{j=1}^{2}\mu_j\log_2(\mu_j),$$
$$\mu_{1,2} := \frac{1\pm\sqrt{[1-2N(1-f)]^2+4N(1-N)e^{-2t}}}{2}.$$

Then solving the equation $S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)] = S_B$, we can find the value of t_*.

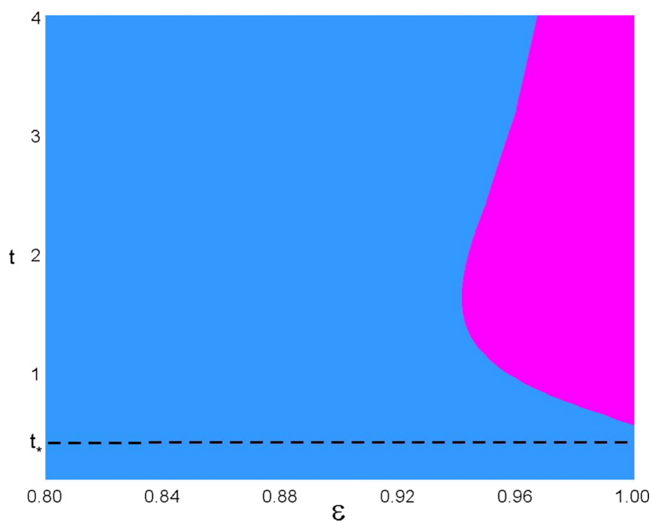

FIG. 3. Curve on the ϵ,t plane where $S[\Phi_{\epsilon,t}(|B\rangle\langle B|)] = S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)]$ for $N = 0.6$. On the left (respectively, on the right) of the curve it is $S[\Phi_{\epsilon,t}(|B\rangle\langle B|)] < S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)]$ (respectively, $S[\Phi_{\epsilon,t}(|B\rangle\langle B|)] > S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)]$). The horizontal dashed line represents the value of t_*.

To perform the similar calculation for any given N, we have numerically found that the optimal value of M is $\lceil N \rceil$. Therefore the output entropy of $|B\rangle_{M,\mu}$ with $M = \lceil N \rceil$ and $\mu = N/M$ should be found and equated to $S[\Phi_{\epsilon,t}(|\kappa_0\rangle\langle\kappa_0|)]$ in order to get t_*.

After having compared the behavior of the output entropy for inputs of the kind (13) and (15), we formulate the following conjecture.

Conjecture 1. In a truncated Hilbert space of dimension $K+1$, the minimal output entropy of the quantum channel (3) is achieved either by binomial states of Eq. (15) or by states $|\kappa_\alpha\rangle$ of Eq. (13), depending on the values of ϵ and t.

To support this conjecture we perform a uniform random search over all pure input states in the finite-dimensional Hilbert space. The restriction to search only among pure states is motivated by Lemmas 1 and 2 in the Appendix.

To generate random pure input states in $(K+1)$-dimensional Hilbert space, we employ the following parametrization:

$$|\psi\rangle = \sum_{n=0}^{K} \nu_n |n\rangle, \quad \nu_0 = \cos\theta_K,$$

$$\nu_{n>0} = e^{i\phi_n} \cos\theta_{K-n} \prod_{l=K-n+1}^{K} \sin\theta_l.$$

Then, according to Ref. [11], it is enough to generate $\phi_{n\geqslant 1} \in [0,2\pi)$ from a uniform distribution $p(\phi_{n\geqslant 1}) = \frac{1}{2\pi}$ and random independent variables ξ_n distributed uniformly in $[0,1]$ for $n = 1,\ldots,K$ defining

$$\theta_n := \arcsin\left(\xi_n^{1/(2n)}\right).$$

However, due to the energy constraint (9), we should consider states satisfying $\sum_{n=0}^{K} n|\nu_n|^2 = N$. This imposes a functional relation among θ_n's and so among ξ_n's, which can be written as $\xi_K = g(\xi_1,\xi_2,\ldots,\xi_{K-1};N)$. Therefore we should generate $K-1$ random variables with the following modified probability distribution function:

$$\tilde{p}(\xi_1,\ldots,\xi_{K-1}) = \mathcal{C}\int d\xi_K\, p(\xi_1,\ldots,\xi_K)\delta(\xi_K - g),$$

with $\mathcal{C}$ as a normalization factor and $p(\xi_1,\ldots,\xi_K)$ as the probability distribution function for the variables $\xi_1,\ldots,\xi_K$. Since these are chosen independently and with a standard uniform distribution in $[0,1]$, we conclude that we should generate $\xi_1,\ldots,\xi_{K-1}$ according to $\tilde{p}(\xi_1,\ldots,\xi_{K-1}) = p(\xi_1,\ldots,\xi_{K-1}) = 1$, and pick ξ_K as

$$\begin{aligned}\xi_K &= g(\xi_1,\ldots,\xi_{K-1};N)\\ &= \frac{N}{\left(1+\xi_{K-1}^{1/(K-1)}\{1+\xi_{K-2}^{1/(K-2)}[1+\cdots\xi_2^{1/2}(1+\xi_1)]\}\right)}.\end{aligned}$$

In our four-dimensional example with $N = 0.6$ and $t \in \{0.2,0.4,0.6,0.8,1,1.2,1.4\}$ the search over 10^5 states when generated as explained above confirms the statement of Conjecture 1 for all values of $\epsilon \in \{0,0.01,0.02,\ldots,0.99,1\}$. Further support comes from numerical investigations in five- and six-dimensional Hilbert spaces with 10^5 states generated for $\epsilon \in \{0,0.1,\ldots,0.9,1\}$ and $t \in \{0.2,0.6,1,1.4\}$.

V. CONCLUSION

We have opened an avenue for studying, from an information theoretical point of view, continuous quantum channels beyond the usual restriction of Gaussianity. Actually we have proposed a model for a non-Gaussian quantum channel that stems from a master equation accounting for two processes, amplitude damping and dephasing. Its physical relevance relies on the fact that amplitude damping and dephasing are applied in many concrete discussions to model the noise of quantum information processing with a single mode light field, vibration phonon mode, or excitonic wave, see, e.g., Ref. [12].

Then, the first question that arises is how much the introduced channel deviates from Gaussianity. Arguably this depends on the parameter ϵ, however an exact quantification would be in order, maybe in a fashion similar to what has been performed for non-Gaussian states [13]. This could also shed light on the choice of optimal input states for communication tasks. Here we found input states approaching zero output entropy while respecting the input energy constraint. Among them we proved that the most economical ones in terms of space resources (those living in the smallest dimensional Hilbert space for a fixed small but nonzero value of output entropy) consist of the superposition of two number states one farthest away from the other. Then, in truncated Hilbert spaces, the optimal input states are conjectured to be binomial states besides the superposition of two-number states, depending on the values of the channel's parameters. This is corroborated by numerical results. The study performed in truncated Hilbert space is justified by the fact that in realistic physical situations it is hard to fully exploit the infinite dimensionality of the space $L^2(\mathbb{R})$.

As a further development one could address the issue of additivity of output entropy for two copies of the channel and then eventually of multiple copies. This would be motivated by the additivity of the classical capacity deriving from the additivity of the minimum output entropy [14].

Although challenging, the introduced map leaves concrete hopes for characterizing its (product states) classical capacity, which implies finding the optimal input ensemble of states maximizing the Holevo χ quantity [15].

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APPENDIX

Lemma 1. Given a self-adjoint operator $H:\mathbb{C}^{K+1}\to\mathbb{C}^{K+1}$, we can always decompose a density operator ρ on $\mathbb{C}^{K+1}$, satisfying a linear constraint $\mathrm{Tr}(\rho H)=N$, in terms of pure states $|\psi_k\rangle$ satisfying the same constraint, i.e., $\mathrm{Tr}(|\psi_k\rangle\langle\psi_k|\,H)=N$.

Proof. Consider the spectral decomposition of $H=\sum_j h_j|j\rangle\langle j|$. An arbitrary density operator represented in the H eigenvector's basis,

$$\rho=\sum_{i,j} r_{i,j}|i\rangle\langle j|,\quad r_{j,j}>0,\quad \sum_j r_{j,j}=1 \tag{A1}$$

satisfies the constraint if $\mathrm{Tr}(\rho H)=\sum_j h_j r_{j,j}=N$. Decomposing ρ in terms of Q pure states, we have

$$\rho=\sum_k p_k|\psi_k\rangle\langle\psi_k|,\quad p_k>0,\quad \sum_k p_k=1, \tag{A2}$$

Comparing Eqs. (A1) and (A2), we find that $\sum_k p_k|\langle\psi_k|j\rangle|^2=r_{j,j}$. If we take

$$|\langle\psi_k|j\rangle|^2=r_{j,j},\ \forall\, k, \tag{A3}$$

it will result in

$$\mathrm{Tr}(|\psi_k\rangle\langle\psi_k|\,H)=\sum_j h_j r_{j,j}=N,\ \forall\, k.$$

Hence to get a decomposition of ρ in terms of pure states satisfying the same constraint, it is enough to determine the $|\psi_k\rangle$'s from the condition (A3), that is,

$$|\psi_k\rangle=\sum_m\sqrt{r_{m,m}}e^{i\varsigma_{m,k}}|m\rangle.$$

Inserting this expression in (A2) and equating with (A1) we obtain $\frac{K(K+1)}{2}+1$ equations (including normalization of p_k's as well),

$$r_{m,n}=\sqrt{r_{m,m}r_{n,n}}\sum_k p_k e^{i(\varsigma_{m,k}-\varsigma_{n,k})},\qquad \sum_k p_k=1,$$

and have $Q(K+2)$ parameters (p_k's and $\varsigma_{m,k}$'s) to be found. As long as we choose $Q>\frac{K(K+1)+2}{2(K+2)}$, the existence of a solution is guaranteed. ■

Lemma 2. The minimum output entropy of a quantum channel Φ acting on states ρ on $\mathbb{C}^{K+1}$ satisfying the energy constraint (9) is achieved on pure states.

Proof. Assume that the minimum output entropy is achieved by the input state ρ satisfying the energy constraint. Decomposing it in terms of pure states that satisfy the same energy constraint $\rho=\sum_k p_k|\psi_k\rangle\langle\psi_k|$, and using the concavity of von Neumann entropy [10], we have

$$S[\Phi(\rho)]=S\left(\sum_k p_k\Phi(|\psi_k\rangle\langle\psi_k|)\right)\geqslant\sum_k p_k S[\Phi(|\psi_k\rangle\langle\psi_k|)].$$

In the decomposition, let us denote the pure state with minimum output entropy by $|\psi_*\rangle$. Therefore we have

$$S[\Phi(\rho)]\geqslant S[\Phi(|\psi_*\rangle\langle\psi_*|)],$$

that is, the optimal input state must be pure. ■

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