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Testing cellular automata interpretation of quantum mechanics in carbon nanotubes and superconductivity

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Abstract. Cellular Automata (CA) are represented at an effective level as intrinsic periodic phenomena, classical in the essence, reproducing the complete coherence (perfect recurrences) associated to pure quantum behaviours in condensed matter systems. By means of this approach it is possible to obtain a consistent, novel derivation of SuperConductivity (SC) essential phenomenology and of the peculiar quantum behaviour of electrons in graphene physics and Carbon Nanotubes (CNs), in which electrons cyclic dynamics simulate CA. In this way we will derive, from classical arguments, the essential electronic properties of these — or similar — graphene systems, such as energy bands and density of states. Similarly, in the second part of the paper, we will derive the fundamental phenomenology of SC by means of fundamental quantum dynamics and geometrical considerations, directly derived from the CA evolution law, rather than on empirical microscopical characteristics of the materials as in the standard approaches. This allows for a novel heuristic interpretation of the related gauge symmetry breaking and of the occurrence of high temperature superconductivity by means of simple considerations on the competition of quantum recurrence and thermal noise.

1. Introduction

As proven by G. 't Hoof in recent works, particular CA models have much in common with Quantum Mechanics (QM). This correspondence has originated a deterministic interpretation of fundamental aspects of QM [1, 2]. The idea considered in this paper stems from the model describing CA of generic number N of "ontic" sites on a circle and permutations among the neighbour sites and period T. The evolution law is $|t_l\rangle \to |t_{l+1} + \text{mod } T\rangle)$. Such a CA can be equivalently regarded as a "particle moving on a circle" of N lattice sites. For the continuous limit, 't Hooft has proven that "*there is a close relationship between a particle moving on a circle with period T and the Quantum Harmonic Oscillator (QHO) with the same period*" [1]. This is a fundamental result as the QHO is the essential ingredient of a second quantised field, and in turn of the whole Quantum Field Theory (QFT). This represents the central concept used in this paper to derive applications to condensed matter. We will name the continuum of ontic sites of the continuous CA model as "ontic" time t of period T. In this continuum limit the CA evolution law from a generic initial time t_i to a final time t_f is $|t_i\rangle \to |t_f + \text{mod } T\rangle$. To such a CA of period T is associated a fundamental energy $\hbar\omega = 2\pi\hbar/T$, according to the de Broglie phase harmony condition in the ontic time $\omega T = 2\pi$. According to 't Hooft, the CA periodic dynamics turn out to be described, at a statistical level, by a wave-function $\Phi(t)$, also named "CA physical state", whose generic solution is $\phi(t) = e^{-i\omega t}$, satisfying the Periodic Boundary

Conditions (PBCs) $\Phi(t) = \Phi(t+T)$. These PBCs encode the CA evolution law in time and determine the CA eigenstates $\phi_n(t) = e^{-i\omega_n t}/\sqrt{2\pi}$. They form a complete, orthogonal set with harmonic energy spectrum $\hbar\omega_n = n\hbar\omega = n\frac{2\pi\hbar}{T}$ ($n \in \mathbb{Z}$). Hence the CA defines an Hilbert space of basis $|n\rangle$ such that $\langle t|n\rangle = \phi_n(t)$ with induced inner product $\langle n|n'\rangle = \delta_{n,n'}$. Indeed the CA evolution law in time is $|t\rangle \to |t + \delta t + \text{mod } T\rangle$. The CA turns out to be represented by the superposition of these eigenstates $\Phi(t) = \sum_{n\in\mathbb{Z}} a_n\phi_n(t)$. That is, in the related Hilbert space representation, it is described by a point in the Hilbert space $|\Phi\rangle = \sum_n \alpha_n|n\rangle$. Since $i\partial_t\phi_n(t) = \omega_n\phi_n(t)$ the CA ontic time evolution is given, in this Hilbert space formalism, by the Schrödinger equation $i\partial_t|\Phi(t)\rangle = \mathcal{H}|\Phi(t)\rangle$ where the Hermitian operator $\mathcal{H}$ is defined as $\mathcal{H}|n\rangle = \omega_n|n\rangle$. Hence, its evolution is given by the Hilbert operator $\mathcal{U}(dt) = e^{-i\mathcal{H}dt}$. We must bear in mind that the fundamental topology of our CA is the topology of the circle $\mathbb{S}^1$. This determines, by means of the PBCs, the quantum number n.

The formalism and results presented in this paper are a direct generalisation to CA of the Elementary Cycles theory (EC) [3, 4, 5, 6, 7, 8]. EC theory is interpreted in terms of 't Hooft CA in [9] constituting an introduction to this paper. Indeed EC is an effective description of continuous periodic CA in which the CA evolution law is encoded in PBCs of the relativistic coordinates and the CA is thus represented as vibrations associated to this relativistic cyclic dynamics. The powerful formalism of EC theory allows in this way to test 't Hooft proposal in advanced aspects of theoretical physics and condensed matter physics [3, 4]. These results will be here explicitly interpreted in terms of CA theory.

Actually, the dynamics described by this CA model, similarly to the EC model, correspond to those of the time evolution of a Quantum Harmonic Oscillator (QHO) of period T, except for the fact that the quantum number can assume negative values, $n \in \mathbb{Z}$ (positive and negative frequencies). This seems to imply a non positively defined Hamiltonian operator for CA. Nevertheless, as pointed out by 't Hooft, the negative modes can be regarded as describing anti-particles. In confirmation of this we will in fact find that in graphene physics these negative modes correspond to the holes in the Fermi sea, i.e. to the negative modes of the Carbon Nanotubes (CN) energy bands. Indeed, the Elementary Charge Carriers (ECCs) in a CN behaves as "particle on a circle". In particular, a CN with N carbon atoms along the circumference corresponds to a CA of N sites on a circle. They both have the fundamental topology of the circle $\mathbb{S}^1$. We will use the covariant generalisation of this correspondence to derive in a simple way, and in agreement with present literature [10, 11], the essential electronic properties of CNs. CNs can be therefore used to test CA theory and foundational aspects of QM.

Similarly we will show that the CA evolution law, as well as the PBCs in EC theory, yields the most foudantional aspects of SC such as: quantisation of the magnetic flux in unit of $\varphi_0/2 = hc/2e$, the Meissner effect, the Josephson effect, the Little-Park effect, etc. Indeed, the CA evolution law directly implies discretised variations of the Goldstone field which in turn describe the explicit breaking of the ElectroMagnetic (EM) gauge invariance typical of SC [3, 12]. In short, the SC phenomenology will be derived by applying the simple requirement that the wave function of electrons in the superconductor must have closed space-time orbits, in analogy with the closed space-time orbits of electrons in the atomic orbitals (see Bohr atom or Bohr-Sommerfeld quantisation). According to this analogy, the electron in the atomic orbitals can be therefore regarded as being locally in a superconducting regime.

2. Testing CA theory by means of Carbon Nanotubes: Compton clock vibrational modes and controvariant modulations.

As well-known, the electrons in a graphene monolayer behave as massless charged particles (with pseudo-spin), i.e. as relativistic particles traveling at the Fermi velocity v_F (the analogous of the speed of light for graphene physics). Let us assume that the graphene layer is curled-up in the same direction of the electron motion. The result will be a "particles moving on a circle"

at "light" speed v_F. Since the CN will have a finite number N of carbon atoms along the circumference C_h, such a "circle" will be on a lattice of N sites. In short, the electron moving at velocity v_F along a CN circumference characterised by N carbon atoms can be regarded as a CA with N sites and time period $T_C = C_h/v_F$. Indeed CNs are real physical systems simulating CA. For the sake of simplicity we start by investigating the continuous lattice limit. The effective CN electronic properties will be then obtained by passing to the lattice case. For instance the CA energy spectrum $\hbar\omega_n = n\hbar\omega = \frac{2\pi\hbar n}{T}$ in the lattice case is $\hbar\omega_n = \frac{N\hbar}{2T}\sin(\frac{\pi n}{N})$.

As a consequence of the compactification of one of the two spatial dimensions of the graphene layer, the resulting CN effective dynamics are characterised by a single spatial dimension, i.e. the axial direction $x_\parallel$. Therefore, in such an effective CN one-dimensional space, the electron moving along the circumference is effectively at rest with respect to the spatial dimension. We denote the momentum of the electron along the axial direct with $k_\parallel$ and the corresponding period of the electron moving along the CN circumference, at rest with respect to the axial direction, as $T(k_\parallel = 0) = T_C$. According to the CA model, this effective rest period corresponds to a rest energy $\hbar\omega(k_\parallel = 0)$, such that $T(k_\parallel = 0) = 2\pi/\omega(k_\parallel = 0)$. In relativity it is natural to associate a rest energy to a mass: $\hbar\omega(k_\parallel = 0) = m_* v_F^2$. Indeed the rest period T_C corresponds to the effective Compton time of the electron in the CN, which in turn fixes the effective mass scale of the ECCs in CN according to the Compton relation $m_* = \frac{2\pi\hbar}{T_C v_F^2}$ (that is, the Compton time of a relativistic particle of mass m is $T_C = \frac{2\pi\hbar}{mc^2}$). Thus, we find that the effective mass scale of the electron in the CNs is determined by the CN circumference according to

$$m_* = \frac{2\pi\hbar}{C_h v_F} \tag{1}$$

This effective mass scale is the fundamental rest energy of our CA of rest period $T(k_\parallel = 0) = T_C$. The ECCs at rest can be regarded as rest CA in order to simulate the effective de Broglie internal clock of the electron in the CNs. Notice that the Compton time of the electron $\sim 10^{-21}$s is effectively rescaled to about 10^{-15}s in CNs. Since the latter time scale is accessible to modern timekeepers, CN allows to indirectly test foundational quantum aspects of elementary particles, as pointed out in [4].

The CA is characterised not only by the fundamental energy considered so far, but by a whole spectrum of energy levels. In this rest case the CA rest energy spectrum is $\hbar\omega_n(k_\parallel = 0) = n\hbar\omega(k_\parallel = 0) = nm_*$. Since this is a rest energy spectrum, it is natural to associate it to a mass spectrum $m_n = nm_* = \frac{2\pi n\hbar}{C_h v_F}$, with $n \in \mathbb{Z}$. This yields interesting analogies to the Kaluza-Klein theory as described in [8]. Similarly to the general case, this harmonic spectrum denotes the eigenmodes associated to the PBCs in the proper time τ for the CA describing the ECC in the CN: $\Phi(\tau) = \Phi(\tau + T_C)$, where τ is the proper time of the ECC (i.e. the time coordinate for the electron at rest with respect to the axial direction). In short these PBCs encodes the evolution law of the CA evaluated at rest. In the derivation of the lattice limit we have to consider that, as a consequence of the hexagonal geometry of the graphene layer, two kind of PBCs are possible for the electron moving along the CN circumference. Besides the trivial PBCs described above, it is possible to have configurations symmetric under lattice rotations of $\pm\frac{2}{3}\pi$. In this case the PBCs are thus twisted by a corresponding factor. That is, in general, we can write the PBCs of the ECCs in CNs as

$$\Phi(\tau) = e^{i2\pi\alpha}\Phi(\tau + T_C) \tag{2}$$

where $\alpha = 0, \pm\frac{1}{3}$. In other words the evolution law in the proper time of the CA simulated by CNs is $|\tau_i\rangle \to e^{i2\pi\alpha}|\tau_f + \text{mod } T_C\rangle$. By considering this twist of the PBCs the resulting mass spectrum of the rest CA turns out to have a shift αm_*:

$$m_n = (n+\alpha)\frac{2\pi\hbar}{T_C} = (n+\alpha)m_* = (n+\alpha)\frac{2\pi\hbar}{C_h v_F}\ , \quad (n \in \mathbb{Z})\,. \tag{3}$$

This implies that for $\alpha = 0$ the CN is characterised by a massless mode $m_0 = \bar{m} = 0$, whereas all the other modes are massive: $m_n = nm_*$ with $n \neq 0$. Indeed this case describes metallic CNs. In the case $\alpha = \pm\frac{1}{3}$ all the modes are massive. This in fact corresponds to semiconducting CNs whose characteristic fundamental mode has effective mass

$$\bar{m} = \frac{2\pi\hbar}{3C_h v_F} \,. \tag{4}$$

in agreement with present literature [10]. The lattice limit correctly leads to the effective CN mass spectrum [11]. We find that, for ZigZag (ZZ) and ArmChair (AC) CNs of N carbon atoms in the perimeter, the resulting energy spectra turns out to be, respectively,

$$\text{ZZ)} \quad m_n = m_* \frac{N}{\pi} \sin(\frac{\pi n}{N}) \;; \qquad\qquad \text{AC)} \quad m_n = m_* \frac{N}{3\pi} \left[1 + 2\cos(\frac{\pi n}{N})\right] \,. \tag{5}$$

In order to derive the CN energy bands we must consider the relativistic transformation of our CA of rest period T_C. This can be easily done in analogy with undulatory mechanics. Indeed a periodic continuous CA can be regarded as a "de Broglie periodic phenomenon" and the CA at rest describes the so called "de Broglie internal clock". As originally pointed out by de Broglie, the period and wavelength of a moving particle, and thus its energy and momentum, are determined, through Lorentz transformations, by the rest periodicity, i.e. by the Compton period of the particle. This de Broglie description, at the base of modern undulatory mechanics and QM, directly applies to CNs and CA (or EC).

The case of ECC with a non zero velocity along the CN axial direction corresponds to an effective Lorentz boost in the effective CN space-time. That is, the effective period $T(k_\parallel)$ and wave-length $\lambda_\parallel$ of an electron with non-vanishing momentum $\hbar k_\parallel$ in the axial direction of a CN is obtained from the Compton time T_C by means of the Lorentz transformation $T_C = \gamma_\parallel T(k_\parallel) - \beta_\parallel \gamma_\parallel \lambda_\parallel$ in the effective CN space-time, where the Lorentz factor is $\gamma_\parallel = 1/\sqrt{1-\beta_\parallel}$ with $\beta_\parallel = v_\parallel / v_F$ and $v_\parallel$ is the velocity along the axial direction. In short, the motion along the CN is described by boosting the rest CA described above. The effective Compton period $T_C = \frac{C_h}{v_F}$ implies, by means of Lorentz transformations, a time period $T(k_\parallel)$ and a spatial recurrence $\lambda_\parallel$ when when the electron moves along the axial direction — as a consequence, for instance, of an electric potential. These recurrences in time and space determine the effective energy $\hbar\omega(k_\parallel) = \frac{2\pi\hbar}{T(k_\parallel)}$ and the momentum $\hbar k_\parallel = \frac{2\pi\hbar}{\lambda_\parallel}$ of the electron in the CN, according to undulatory mechanics. Notice that the former relation is the relation between time period and fundamental energy for a CA in the rest frame $k_\parallel$, i.e. the CA phase harmony relation in the rest frame of the electron. In analogy to relativistic notations, by introducing the two tangent two-vector $k_\mu = \{\omega/v_F, -k_\parallel\}$ (covariant) and $\lambda^\mu = \{Tv_F, \lambda_\parallel\}$ (controvariant), the CA phase harmony relation can be generalised to the invariant form $m_* v_F^2 T_C/\hbar = k_\mu \lambda^\mu = 2\pi$. Thus, the relativistic constraints for these two vectors are respectively $m_* v_F^2 = \hbar^2 k_\mu k^\mu$ and $\frac{1}{T_C^2} = v_F^2 \frac{1}{\lambda^\mu}\frac{1}{\lambda_\mu}$, denoting the dual relativistic dispersion relations for the fundamental energy and time period, respectively. The covariant CA evolution law for an electron moving in the effective CN space-time $x^\mu = \{t, x_\parallel\}$ is thus given by $|x_i^\mu\rangle \to e^{i2\pi\alpha}|x_f^\mu + \text{mod}\ \lambda^\mu\rangle$ (the CA evolution law in time and space are given by the component $\mu = 0$ and $\mu = 1$, respectively). In analogy to the relativistic Doppler effect, this implies that the time period $T(k_\parallel) = 2\pi/\omega(k_\parallel)$ varies with $k_\parallel$ in such a way that the electron fundamental energy turns out to satisfies the effective relativistic dispersion relation

$$\hbar^2\omega^2(k_\parallel) = \frac{(2\pi\hbar)^2}{T^2(k_\parallel)} = m_*^2 v_F^4 + \hbar^2 k_\parallel^2 v_F^2 \,. \tag{6}$$

In short we have obtained that the motion of the electrons along CNs is the analogous of the relativistic motion, provided that the mass of the electron is replaced by the effective mass m_* and the speed of light by the Fermi velocity v_F.

If we now apply this effective relativistic dispersion relation to every mass eigenmode m_n allowed for the electron effective Compton periodicity T_C we immediately obtain the energy bands structure of CNs in the continuous limit

$$\hbar\omega_n(k_\parallel) = (n+\alpha)\frac{2\pi\hbar}{T(k_\parallel)} = (n+\alpha)\sqrt{m_*^2 v_F^4 + \hbar^2 k_\parallel^2 v_F^2}\ , \quad (n \in \mathbb{Z}) \tag{7}$$

The CNs energy bands follow from this result by performing the lattice limit (and considering that the CN lattice has also a periodicity along the axil direction yielding Brillouin zones). For instance, in agreement we present literature, in the case ZZ CNs we find the energy bands structure — or similarly the density of states [4]:

$$\text{ZZ)} \quad \hbar^2\omega_n(k_\parallel) = m_*^2 v_F^4 \frac{N^2}{3\pi^2}\left(1 + 2\sin\frac{\pi n}{N}\right)^2 - \frac{2}{3}\hbar^2 k_\parallel^2 v_F^2 \cos\frac{\pi n}{N}\ . \tag{8}$$

This description is very interesting to test the validity of CA theory. Indeed one of the main problematics of the CA interpretation of QM is that the negative modes ($n = -1, -2, \ldots$) associated to the CA evolution law implies a non positively defined Hamiltonian operator. 't Hooft as noticed however that these negative modes can be consistently interpreted as antiparticles, i.e. holes in the Dirac sea. This is fully confirmed by CN in which, actually, the negative energy bands describes holes in the Fermi sea.

3. Testing CA theory by means of superconductivity: the role of the temperature

Here we will introduce the role of the temperature in the EC theory, which can be directly generalised to CA. We must bear in mind that in CA, as in undulatory mechanics and EC theory, the time period and the energy are "two faces of the same coin". Therefore interactions, i.e. variations of energy $\hbar\omega = 2\pi\hbar/T$, correspond to local modulations of the period T. Since the temperature implies random collisions among particles (thermal noise), a quantum system at finite temperature is characterised by chaotic (Poissonian) decay of the "complete coherence" of the particles with a characteristic thermal time $\beta = \hbar/k_B\mathcal{T}$, i.e. a dumping $e^{-\hbar\omega/k_B\mathcal{T}} = e^{-\beta/T}$ of the cyclic behaviour of periodicity T of the system, being k_B the Boltzmann constant and $\mathcal{T}$ the temperature. While free CA at zero temperature, as well as the perfect coherence of pure quantum systems, are characterised by perfect periodicity T in the Minkowskian time, thermal quantum systems are characterised by Euclidean time periodicity of period $\beta = \hbar/k_B\mathcal{T}$. These two kinds of temporal periodicities (which can be related by Wick's rotation, as for instance in the Hawking theory of black holes) have fundamental different physical meanings: they are in competition. In the pure quantum systems at zero temperature we have a persistent exact recurrence in time, e.g. described by the phasor $e^{-i\omega_n t}$, whereas in the latter case, corresponding to a non zero temperature, we have a dissipation of the quantum recurrence by Boltzmann factors $e^{-\omega_n/k_B\mathcal{T}} = e^{-n\beta/T}$, i.e. a dumping factor of the Minkowskian periodicity. In other words, while the Minkowskian periodicity T of quantum mechanics tends to form perfect coherent states ("periodic phenomena"), the Euclidean periodicity β describes a dumping associated to the thermal noise which tends to destroy the perfect recurrences of the pure quantum systems. Thus, if the $T \ll \beta$ the system can autocorrelated and give rise to pure quantum phenomena such as superconductivity whereas in the opposite limit the thermal noise breaks the the quantum recurrence before it can give rise to autocorrelation, in this case we have the classical behaviour, e.g. ordinary electric resistance. Another aspect of the temperature which will be used in the following description is the Boltzmann distribution. In a few words, the dumping factor $e^{-\omega_n/k_B\mathcal{T}} = e^{-n\beta/T}$ can be also regarded as the probability to populate the nth vibrational mode of the system. Thus, at very low temperature $T \ll \beta$ only the fundamental mode $n = 1$ will be populated whereas at high temperature many vibrational modes must be considered (e.g.

as in the classical limit in Bohr atom). We will use this aspect to describe the EM symmetry breaking occurring in superconductivity (in analogy with the effective gauge symmetry breaking in extra dimensional extensions of the Standard Model).

These simple arguments, further motivated below, allow us to predict that high temperature SC happens in quantum systems characterised by very small quantum recurrences T, orders of magnitude smaller than in ordinary SC. This is confirmed by CNs in which, as we have seen, the quantum recurrence is explicitly determined by the CN circumference C_h, so that, in agreement with [10, 13], the critical temperature is inversely proportional to the circumference C_h.

Our description of CNs shows that covariant CA provide a model describing the "periodic phenomena" conjectured by de Broglie to describe the quantum behaviour of elementary particles. Indeed, in the continuous limit, a CA (or an EC) of rest period T_C describes the quantum behaviour of a relativistic particle of mass $m = 2\pi\hbar/T_C c^2$, where we have replaced the Fermi velocity with the speed of light c. In general a CA, similarly to the ideal case of a free quantum particle at zero temperature, is characterised by a "complete coherence" in space-time.

Generalising the CN description to the four-dimensional space-time of ordinary relativity, in a given reference frame the free CA as time period $T(\vec{k})$ and wave-length $\vec{\lambda}$ which determines its energy $\hbar\omega(\vec{k}) = 2\pi\hbar/T(\vec{k})$ and momentum $\hbar k_i = 2\pi\hbar/\lambda^i$ $(i = 1, 2, 3)$. In covariant notation we can introduce the four vectors $k_\mu = \{\omega/c, -\vec{k}\}$ and $\lambda^\mu = \{Tc, \vec{\lambda}\}$ related by the phase harmony relation $k_\mu\lambda^\mu = 2\pi\hbar$. This means that the evolution law is $|x_i^\mu\rangle \to |x_f^\mu + \text{mod } \lambda^\mu\rangle$ (notice that in this case a twist fact of $\alpha = 1/2$ reproduces the vacuum energy of the particles $\hbar\omega(\vec{k})/2$). The generic CA solution, in the free case, is $\Phi(x) = \sum_n e^{-ik_{(n)\mu}x^\mu}$ and the PBCs, encoding the evolution law, are $\Phi(x^\mu) = \Phi(x^\mu + \lambda^\mu)$. These imply the (normally ordered) energy-momentum spectrum $\vec{k}_{(n)\mu} = n\vec{k}_\mu$. As can be easily seen from time CA evolution, $\Phi(t) = \sum_n e^{i\omega_n t}$ and $\Phi(t) = \Phi(t+T)$ imply the ordinary (normally ordered) energy spectrum of a relativistic quantum particle $\omega(\vec{k}) = n\hbar\omega(\vec{k})$ (in analogy with a string vibrating with period $T(\vec{k})$). In short, the CA evolution law implies that the physical state Φ describing the wave function of a free relativistic particle at zero temperature must have closed orbits along its space-time evolution. That is, we have a correspondence with the Bohr-Sommerfeld quantisation. Indeed, an interacting CA is characterised by a locally modulated space-time period so that, in the interacting case, the generic solution has the form of a locally modulated wave $\phi(x) = e^{-i\int^x dy^\mu k^\mu(y)}$. The evolution law, i.e. the PBCs for this generic solution, in this case yields the relativistic Bohr-Sommerfeld quantisation condition $\oint_\lambda dy^\mu k_\mu(y) = 2\pi n\hbar$. It is well-known that in the case of a Coulombian potential this yields the Bohr energy levels of the atomic orbitals $E_n = -13.6$ eV$/n^2$.

Let us assume unitary gauge invariance for a CA (representing electrons in equilibrium with the EM field in the superconducting material) $\Phi(\vec{x}, t) = U(\vec{x}, t)\Phi(\vec{x}, t)$, where $U(\vec{x}, t) = e^{-i\frac{e}{\hbar c}\theta(\vec{x},t)}$. The Goldstone $\theta(\vec{x}, t)$ denotes the local invariance of the EM interaction and e is the electric charge. In analogy with the CA edescription, we now impose PBCs $\Phi(\vec{x}, t) = \Phi(\vec{x}, t + T(\vec{k}))$ (for the sake of simplicity we only consider the time component, furthermore we assume here that the topology is the orbifold $\mathbb{S}^1/\mathcal{Z}_2$ associated to the fundamental topology $\mathbb{S}^1$ of the CA, so that the phase invariance is $n\pi$ rather than $2\pi n$). Thus, in this case the PBCs in time of period $T(\vec{k})$ imply the following evolution law for the Goldstone mode

$$\frac{e}{\hbar c}\theta(\vec{x}, t) = \frac{e}{\hbar c}\theta(\vec{x}, t + T(\vec{k})) + \text{mod } \ n\pi. \tag{9}$$

As a consequence of the CA evolution law the Goldstone field can only vary by finite amounts

$$\Delta\theta(\vec{x}, t) = \frac{\varphi_0}{2}\ , \quad \text{with } \ \varphi_0 = \frac{hc}{e}\,. \tag{10}$$

This simple condition yields the characteristic behaviours of a superconducting regime, as described in [12]. In short, according to our ansatz, as a consequence of the closed orbits of

the wave-function imposed by the CA evolution law, $\theta(\mathbf{x},t)$ can only vary by discrete amounts running along the contour Σ in which the EM field is a pure gauge $A_\mu(\vec{x},t) = \partial_\mu\theta(\vec{x},t)$. The Stokes theorem gives a quantization of the magnetic flux through the area $\mathcal{S}_\Sigma$ limited by Σ,

$$\int_{\mathcal{S}_\Sigma} \mathbf{B}(\mathbf{x},t)\cdot d\mathbf{S} = \oint_\Sigma \mathbf{A}(\mathbf{x},t)\cdot d\mathbf{x} = \oint_\Sigma \nabla\theta(\mathbf{x},t)\cdot d\mathbf{x} = n\frac{\varphi_0}{2}\,. \tag{11}$$

Since the magnetic flux is quantised, the electric current cannot smoothly decay while flowing around the torus, and there is not electric resistance, i.e. we have SC, [12] and [5, 6, 7]. The quantisation (10) means that the Goldstone field $\theta(\mathbf{x},t)$ transforms as the phase of a condensate of a fermionic pair operator $\left\langle \epsilon_{\alpha\beta}\Phi^\alpha\Phi^\beta \right\rangle$ of charge $-2e$. That is, it plays the role of the usual Cooper pair of the BCS microscopic theory of SC. From (10) it is also manifest that the quantisation of the Goldstone field implies a breaking of the EM gauge $U_{EM}(1)$ to $\mathcal{Z}_2$.

Now we consider a EM gauge connection to the CA physical state $\Phi(x') = e^{i\frac{e}{\hbar c}\int_x^{x'} A_\mu dx^\mu}\Phi(x')$. Indeed, gauge interaction has a particular geometrodynamical meaning in CA, as revealed by using the close analogy with the EC theory [7]. This can be alternatively obtained by using the minimal substitution $\hbar k_\mu(x) \to \hbar k_\mu(x) - eA_\mu(x)$ in the solution of the interacting CA. By imposing the CA evolution law in space-time with orbifold topology $\mathbb{S}^1/\mathcal{Z}_2$, i.e. the PBCs in space-time, this yields the Dirac quantisation condition for magnetic monopoles (in analogy with the Bohr-Sommerfeld quantisation)

$$\oint A_\mu dx^\mu = n\frac{\varphi_0}{2}\,. \tag{12}$$

The spatial component yields the quantisation of the magnetic flux described above whereas the time component yields the Josephson effect. Let us consider a junction between superconductors with a voltage difference ΔV (the BCs in this case is determined by the isolating barrier). Since the Maxwell equation in this case implies that $\Delta V = -\partial_t\theta$, the Stokes' theorem yields $T_{junct}\Delta V/c = \varphi_0/2$ corresponding to the frequency $f_{junct} = 1/T_{junct}$ of the Josephson effect. Alternatively this can be derived by considering that Lagrangian at the junction can only depend on the phase difference of the Goldstone fields $\mathcal{L}_{junct} = \mathcal{F}(\Delta\theta)$. The Josephson effect follows from the fact that the evolution law implies that the Goldstone can only vary by finite steps, see eq.(10), so that $\mathcal{F}(\Delta\theta) = \mathcal{F}(\Delta\theta + n\varphi_0/2)$ [12].

Due to the quantum recurrence, which in our description originates from the CA evolution law or by the PBCs, the EM gauge field A_μ, similarly to the Goldstone field, is constrained to have a periodic behaviour. This means that it can be expanded in vibrational eigenmode which, for the sake of simplicity, we assume here to be harmonic $A_\mu(\vec{x},t) = \sum_{n\in\mathbb{Z}} e^{-in\omega^\gamma t}A_{(n)\mu}(\vec{x})$. This implies that the EM Lagrangian can be expanded in vibrational eigenmodes as well. In the temporal (unitary) gauge

$$\mathcal{L}_{YM} = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \sum_{n\in\mathbb{Z}}\int d^3x\left[-\frac{1}{4}F_{(n)\mu\nu}F^{\mu\nu}_{(n)} - \frac{1}{2}\frac{(\omega_n^\gamma)^2}{c^2}A_{(n)\mu}A^\mu_{(n)}\right] \tag{13}$$

However we must consider that at very low temperature $T \ll \beta$ only the lower mode of the system, that here we will denote by the bar sign, is populated. This means that the summation of the above Lagrangian must be truncated to the fundamental mode. As well-known for gauge invariance in extra dimensional theories (see Higgsless, Composite Higgs or unified gauge-Higgs models), this implies that the low temperature is no more gauge invariance. So we find again an effective EM gauge breaking. Roughly speaking, at very low temperature we can imagine that, similarly to a Bose-Einstein condensation, all the vibrational modes of energy $\omega_n^\gamma = n\omega^\gamma$ condensate on the fundamental mode.

This symmetry breaking can also be seen from the effective matter Lagrangian $\mathcal{L}_S$ obtained by integrating out from the equations of motion the contributions of the higher modes as a function of the fundamental one. By expanding $\mathcal{L}_S$ for small perturbations around the fundamental mode, which can be effectively written as $\hat{\Phi} = \rho e^{i2e\theta}$, we get

$$\begin{aligned} \mathcal{L}_S &\simeq \int d^3x \left[-\frac{1}{2}\hat{\Phi}^*|(\vec{\nabla} - i2e\vec{A})|^2\hat{\Phi} - \frac{1}{2}\alpha|\hat{\Phi}|^2 - \frac{1}{4}\beta|\hat{\Phi}|^4 \right] \\ &= \int d^3x \left[-2e^2\rho^2(\vec{\nabla} - i2e\vec{A})^2 - \frac{1}{2}(\vec{\nabla}\rho)^2 - \frac{1}{2}\alpha\rho^2 - \frac{1}{4}\beta\rho^4 \right] . \end{aligned} \tag{14}$$

This describes (with appropriate normalisation) the BCS theory of ordinary SC. The first term describe the fundamental vibrational mode without the perturbation of the higher modes. Due to gauge transformations it can be written as $\bar{\mathcal{L}}_S[A_\mu - \partial_\mu\phi] \sim \frac{V}{\Lambda^2}(A_\mu - \partial_\mu\theta)$, V is the volume and $\Lambda = \frac{1}{\sqrt{4e^2\langle\rho^2\rangle}}$ is the penetration length of the Meissner effect. Indeed, deep inside the superconductor where the EM field is pure gauge, we have $F_{\mu\nu}(\partial_\chi\theta) = 0$ and the magnetic field is vanishing $\vec{B} = 0$. $\bar{\mathcal{L}}_S$ describes the fact that the energy cost to expel the magnetic field is small with respect to the that of the classical configuration. By considering that the minimum of ρ is $\langle\rho^2\rangle = -\frac{\alpha}{\beta}$ we get $\Lambda = \frac{1}{2e}\sqrt{-\frac{\alpha}{\beta}}$, in agreement with ordinary SC. Finally, by expanding ρ around its minimum $\rho = \langle\rho\rangle + \rho'$ we get $\vec{\nabla}^2\rho' = -2\alpha\rho'$. This allows us to introduce the coherence length $\xi = \frac{1}{-2\alpha}$. This describes the stability of the vortices and of the flux-quantised magnetic field. Indeed the energy density of the superconducting state is lower than the energy density of the normal state. We have $\frac{\alpha^2}{4\beta} = \frac{1}{32e^2\lambda^2\xi^2}$. Thus, in agreement with the ordinary description, the case $\xi > \Lambda$ and $\xi < \Lambda$ describe Type I SC and Type II SC, respectively.

4. Conclusions

In this paper we have shown that the interest of the 't Hooft CA theory, or similarly of the EC theory, is not only limited to our knowledge of foundational aspects of QM: it also represents a powerful tool to derive and describe in a very simple way nontrivial quantum phenomena such as the behaviour of the ECCs in CNs and the essential phenomenology of SC. Indeed we have derived the CN electronic properties, and essential quantum effects associated to SC, by simply considering the relativistic modulations of the vibrational modes, and the role the temperature in the cyclic dynamics, characterising the evolution law of CA.

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