NEGATIVE POTENTIALS AND COLLAPSING UNIVERSES II

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ABSTRACT. Completing a previous analysis started in [1], we study flat Friedmann–Lemaître–Robertson–Walker (FLRW) models with a perfect fluid matter source and a scalar field nonminimally coupled to matter, self–interacting with a potential that may attain negative values. We prove that the evolution generically forces the Hubble function to diverge to $-\infty$ in a finite time, except in case the potential exhibits a flat plateau at infinity (tending to zero from below); in that case we find conditions which may give rise to ever expanding or recollapsing cosmologies.

1. INTRODUCTION

An accelerating phase of the Universe requires scalar fields with non-negative potentials playing the role of a cosmological term. However, in some cosmological models potentials taking negative values are used, (see [2] for motivations). In these cases several authors conclude that the Universe eventually collapses even if it is flat, [2]-[8]. In a recent paper we rigorously proved that a general class of bounded from above potentials with $\lim_{\phi\to-\infty} V(\phi) = -\infty$, almost always forces the Hubble function H to diverge to $-\infty$ in a finite time, [1]. This means that, up to a non generical choice of initial data, an initially expanding Universe recollapses and develops a singularity in a finite amount of time. The problem of collapsing scalar fields cosmologies from the mathematical point of view has been sporadically treated in the literature, e.g. [9]-[14]. On the other hand, non-negative potentials in FLRW models with mathematically rigorous results, have been studied by several authors (see for example [15, 16, 17, 18]), but to our knowledge there is not a corresponding rigorous treatment of negative potentials, apart from the above mentioned study in [1].

The class considered in [1] contains potentials that fall to $-\infty$ as $\phi \to -\infty$, have a global positive maximum and go to zero from above as $\phi \to +\infty$. The purpose of the present paper is to complete the rigorous treatment of cosmological models with potentials taking negative values. The remaining forms of negative potentials encountered in the literature are listed as follows.

A. Potentials having a negative minimum. Two important examples include the ekpyrotic potentials and those used in models of cyclic Universes; for reviews see Refs. [19], [20].

B. Bounded from below potentials with no minimum. As an example, we mention the potentials

$$V(\phi) = V_0 e^{-\lambda\phi} - C, \quad V_0, C, \lambda > 0,$$

which were considered in the context of supersymmetry theories, see for example Ref. [5].

C. Potentials with $V(\phi)$ decreasing from $+\infty$ to $-\infty$, for example

$$V(\phi) = W_0 - V_0 \sinh(\lambda\phi), \qquad \lambda, V_0 > 0,$$

(see [7] where an exact solution was obtained in the absence of matter).

D. Potentials having a global positive maximum and $\lim_{\phi\to\pm\infty} V(\phi) = -\infty$. Near the maximum, say at $\phi = 0$, they can be represented as

$$V(\phi) = V_0 - \frac{m^2}{2}\phi^2,$$

cf. [2]. An example is the potential

$$V(\phi) = V_0 \left(2 - \cosh\left(\sqrt{2}\phi\right)\right), \quad V_0 > 0,$$

considered in [5]. Potentials of this class appear in cosmological models in $N = 2, 4, 8$ gauged supergravity [21, 22]. For detailed cosmological implications see [3, 5, 6].

From the mathematical point of view, potentials of type A, B and C cannot be studied using the techniques exploited in [1]; on the other hand, for potentials of type D similar arguments as those used in [1] apply almost straightforwardly.

In this paper we study the recollapse problem of scalar-field cosmological models with negative potentials having the general features of the above classes A–D. We consider a nonminimal coupling of the scalar field to matter; the coupling coefficient is assumed to be an arbitrary non-negative bounded function of the scalar field with non-negative limits as $\phi \to \pm\infty$, see (2.6) below. Under quite general assumptions on the potential, initially expanding flat Universes are shown to eventually recollapse, except in case where the potential exhibits a flat plateau at infinity, tending to 0 from below. In this case, generical evolution may also be eternally expanding for some open subsets in the parameter space.

The plan of the paper is as follows. In the next section we write the field equations for flat FLRW models as a constrained four-dimensional dynamical system. We impose a number of assumptions so that our class of potentials includes cases A, B and C. We state our main theorem according to which, the Hubble function, H, almost always diverges to $-\infty$ in a finite time. In section 3 we analyse all possible limit sets of the dynamical system and prove a number of propositions that lead to the proof of the main theorem. Section 4 contains a proposition covering case D, and so we complete our analysis that is discussed in the final section.

2. FORMULATION OF THE PROBLEM AND MAIN THEOREM

For homogeneous and isotropic flat spacetimes the field equations can be written (see [1]), as an autonomous dynamical system,

$$\dot{\phi} = y, \tag{2.1}$$

$$\dot{y} = -3Hy - V'(\phi) + \alpha\rho, \qquad \alpha = \frac{4-3\gamma}{2}Q(\phi), \tag{2.2}$$

$$\dot{\rho} = -\rho(3\gamma H + \alpha y), \tag{2.3}$$

$$\dot{H} = -\frac{1}{2}\left(y^2 + \gamma\rho\right), \tag{2.4}$$

subject to the constraint,

$$3H^2 = \frac{1}{2}y^2 + V(\phi) + \rho. \tag{2.5}$$

In most quintessence models, the coupling coefficient, Q, is postulated to be a positive constant, see for example [23]; here Q is assumed to be a positive and bounded function of class C^1 such that,

$$Q_{\pm} := \lim_{\phi\to\pm\infty} Q(\phi) > 0. \tag{2.6}$$

For motivation and other couplings see [24, 25]. We recall that the constraint (2.5) is invariant under the flow of (2.1)–(2.4). In the following we will consider solutions to (2.1)–(2.4) which satisfy (2.5) at some initial time, and therefore (2.5) holds throughout the evolution. We will refer to this solution as solutions to (2.1)–(2.5).

In the following we incorporate cases A–C into a large class of potentials $V(\phi) \in \mathcal{C}^2$ satisfying some further assumptions. To begin, let $u(\phi)$ be the function,

$$u(\phi) = \frac{V'(\phi)}{V(\phi)}. \tag{2.7}$$

Assumption 1. *We assume that $V(\phi) \in \mathcal{C}^2$ is such that*

(1) $\lim_{\phi\to-\infty} V(\phi) = +\infty$*,*
(2) There exists a unique $\phi_0 \in \mathbb{R}$: $V(\phi_0) = 0$. Moreover, V is strictly decreasing for all $\phi \leq \phi_0$,
(3) $\lim_{\phi\to\pm\infty} u(\phi) = \lambda_\pm \in \mathbb{R}$*,*
(4) $\lim_{\phi\to+\infty} V(\phi) = V_\infty \leq 0$ *(possibly $V_\infty = -\infty$).*
(5) There exists a $\mathcal{C}^2$–diffeomorphism, $f : (-\infty, \phi_0] \to [0, s_0)$, such that
  (a) The limit $\lim_{\phi\to-\infty} f'(\phi)$*, exists and is equal to zero,*

(b) $\lim_{\phi\to-\infty} f(\phi) = 0$,
(c) $\lim_{\phi\to-\infty} \frac{u'(\phi)}{f'(\phi)} = 0$,
(d) $\lim_{\phi\to-\infty} \frac{f''(\phi)}{f'(\phi)} \in \mathbb{R}$.

(6) If $V_\infty = 0$, then there exists a $\phi_M > 0$, such that V is strictly increasing for $\phi \geq \phi_M$. Moreover, we make a similar hypothesis to (5) *above for $\phi \to +\infty$, assuming the existence of a $\mathcal{C}^2$–diffeomorphism, $g(\phi) : [\phi_M, +\infty) \to (0, s_0]$ such that requests* (5a)–(5d) *hold for $g(\phi)$, as $\phi \to +\infty$.*

Assumptions (5)–(6) are required for situations where the scalar field possibly diverges. In those cases, the above diffeomorphisms are needed to bring a neighbourhood of infinity to a neighbourhood of the origin, [15, 12].

It is easy to verify that cases A, B and C of negative potentials mentioned in the Introduction satisfy Assumption 1. In the next section we will examine the possible $\omega-$limit sets of the system, essentially depending on the asymptotic behavior of the scalar field $\phi(t)$. We will see that, except one case described in Proposition 10, solutions to (2.1)–(2.5) always recollapse to a singularity in a finite amount of time. In particular, the results proved in the next section may be collected in the following main theorem.

Theorem 2. *Let $V(\phi)$ satisfy Assumption 1. Then, if either $V_\infty < 0$, or condition,*
(2.8)
$$0 < \gamma < \frac{4}{3}, \quad 0 < Q_+ < \sqrt{6}\frac{2-\gamma}{4-3\gamma}, \quad \lambda_+ < -\frac{4-3\gamma}{2}Q_+ - \frac{3(2-\gamma)\gamma}{(4-3\gamma)Q_+}$$
*does **not** hold, then a solution to* (1)–(2.5)*, up to a zero–measured set of initial data, recollapses to a singularity in a finite amount of time, i.e.,*

$$\exists t_* > 0 : \lim_{t\to t_*^-} H(t) = -\infty. \tag{2.9}$$

Otherwise, if $V_\infty = 0$ and (2.8) *does hold, a solution to* (2.1)–(2.5) *either generically recollapses to a singularity in a finite time or expands forever, with $\phi(t) \to +\infty$ and $y(t), \rho(t)$ and $H(t)$ infinitesimal as $t \to +\infty$.*

3. Qualitative behavior of the solution

Throughout this section we will suppose that $V(\phi)$ satisfies the set of hypotheses collected in Assumption 1. We will call $\phi_\infty \in \mathbb{R} \cup \{\pm\infty\}$ the limit value of $\phi(t)$, if it exists, i.e. $\lim_{t\to\sup\mathbb{I}} \phi(t) = \phi_\infty$, where $\mathbb{I}$ is the maximal interval of definition of a solution to (2.1)–(2.5). Some of the proofs in this section rely on two lemmas proved in [1]; for the convenience of the reader we reproduce them here.

Lemma 3. Let $\gamma(t) = (\phi(t), y(t), \rho(t), H(t))$ be a bounded solution such that $\rho(t_0) > 0$. Then $\gamma(t) \in W^s(q_\pm)$, where $W^s(q)$ is the stable manifold of an equilibrium point q.

It can be shown that the dimension of the stable manifold is always less than the dimension of the phase space, [1]. Therefore, the meaning of the above lemma is that, *future bounded trajectories of the system are not generic*.

Lemma 4. Let $\gamma(t)$ be a solution to the system (2.1)–(2.5). If there exists $t_1 \geq t_0$ and $\bar{V} \in \mathbb{R}$ such that, for all $t \geq t_1$, $V(\phi(t)) \leq \bar{V}$, and either $(\imath)$ $\bar{V} < 0$, or $(\imath\imath)$ $H(t_1) < -\sqrt{\bar{V}/3}$, then $H(t)$ negatively diverges in a finite time, i.e. (2.9) holds.

We are now ready to examine different situations depending on ϕ_∞.

3.1. **Case $\phi_\infty = -\infty$.** We firstly analyse the case C when the scalar field negatively diverges in such a way that $V(\phi(t)) \to +\infty$, see for example [7]. We use expansion normalized variables techniques, first introduced in [26]; see also [15, 12]. Notice that, in the present paper we do not need to assume an a priori estimate on $\lambda_\pm$, unlike for instance in [12, eq.(2.2)].

Proposition 5. If $\phi_\infty = -\infty$ then (2.9) generically holds.

Proof. Since $V(\phi(t)) \to +\infty$, then $H(t)^2 \to +\infty$, and since $H(t)$ is decreasing we conclude that $H\,(t)$ negatively diverges, so we have to prove that this happens in a finite amount of time. Without loss of genericity suppose $H(0) < 0$. We introduce the variables,

$$\phi, \qquad x = \frac{1}{H}, \qquad w = \frac{y}{H}, \qquad z = \frac{\sqrt{\rho}}{H},$$

and a new time coordinate τ, defined by $\mathrm{d}\tau/\mathrm{d}t = -H$, as done in [12]. Using the constraint

$$V(\phi)x^2 + \frac{1}{2}w^2 + z^2 = 3,$$

to eliminate $x(\tau)$, we come to the following system for the triple $(\phi(\tau), w(\tau), z(\tau))$:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -w, \tag{3.1}$$

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = -\left(\frac{1}{2}w^2 - 3\right)(w + u(\phi)) - z^2\left(\frac{\gamma}{2}w + \alpha(\phi) + u(\phi)\right), \tag{3.2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = -\frac{1}{2}z\left[w^2 - \alpha w + \gamma(z^2 - 3)\right], \tag{3.3}$$

where $u(\phi) = V'(\phi)/V\,(\phi)$ was already defined. We recall that we are interested in the dynamics near the critical point "at infinity", $\phi \to -\infty$. Therefore, we introduce the variable $s = f(\phi)$, where f is defined in Assumption 1. In this way we obtain a system in the variables $(w(\tau), z(\tau), s(\tau))$, ruled by equations (3.2), (3.3) and

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = -wf'^{-1}(s)). \tag{3.4}$$

Remembering that, $V(\phi(t))$ is eventually positive, we are interested in solutions to (3.2)–(3.4) such that,

$$\frac{1}{2}w^2 + z^2 < 3. \tag{3.5}$$

We consider critical points of (3.2)–(3.4) such that $s = 0$, which are candidates to be ω–limit points for the solutions we are interested in. We may further restrict ourselves to critical points with $w \geq 0$, since we expect both y and H to be eventually negative and $z \leq 0$. The (w, z)–coordinates of the admissible critical points are then (setting $\lambda = \lambda_-$ and $\alpha = \alpha(f^{-1}(0)) = \frac{4-3\gamma}{2}Q_-$)

$$\mathcal{A} = \left(\sqrt{6}, 0\right), \mathcal{B} = (-\lambda, 0), \mathcal{C} = \left(\frac{2\alpha}{2-\gamma}, \frac{\sqrt{-2\alpha^2 + 3(2-\gamma)^2}}{\gamma - 2}\right),$$
$$\mathcal{D} = \left(-\frac{3\gamma}{\alpha + \lambda}, \frac{\sqrt{3(-3\gamma + \alpha\lambda + \lambda^2)}}{\alpha + \lambda}\right).$$

It is easy to check that all these points, except possibly $\mathcal{B}$, do not coincide with the origin $(0, 0)$. In the particular case when $\lambda = 0$, then $\mathcal{B} = (0, 0)$, but the eigenvalues of the linearised system associated with this critical point are $\{0, 3, \frac{3}{2}\gamma\}$ and so this point is definitely an unstable equilibrium.

The generical situation therefore is that there exists a $\delta > 0 : \frac{1}{2}w^2 + z^2 \geq \delta$ eventually. Then, recalling (2.4),

$$\frac{1}{H(t)} - \frac{1}{H(t_0)} = \int_{t_0}^{t} -\frac{\dot{H}(\sigma)}{H(\sigma)^2}\,\mathrm{d}\sigma = \int_{t_0}^{t} \frac{1}{2}(w^2 + \gamma z^2)\,\mathrm{d}\sigma \geq \delta(t - t_0).$$

Since $H(t_0) < 0$, we conclude that $H(t)$ diverges in a finite amount of time. □

Remark 6. The same dynamics described in the above proposition applies to the more general case $\liminf_{t\to\sup\mathbb{I}} V(\phi(t)) = -\infty$. Indeed, this situation implies again that $H(t) \to -\infty$, and in the above proposition we have proved that the dynamics near the point "at infinity" $\phi \to -\infty$, give necessarily rise to solutions that completely recollapse in a finite time.

3.2. **Case $\phi_\infty \in \mathbb{R}$.** We briefly examine what happens if the scalar field converges to a positive value.

Proposition 7. If $\phi_\infty \in \mathbb{R}$ then (2.9) generically holds.

Proof. If $\phi_\infty \in \mathbb{R}$ then $\phi(t)$ is bounded. Then, if $H(t)$ was bounded too, by (2.5) also $y(t), \rho(t)$ would be bounded so the solution would be bounded which by Lemma 3 is a non generic situation. Then $H(t)$ is unbounded and since it is decreasing by (2.4), it must diverge to $-\infty$. At this point, Lemma 4 applies to give the result. □

3.3. **Case** $\phi_\infty = +\infty$**.** In this situation we must split the argument into two subcases, depending on the value of V_∞. We start by considering shortly the case when this limit value is strictly negative (case C), possibly $-\infty$ (case D).

Proposition 8. If $V_\infty < 0$ and $\phi_\infty = +\infty$ then (2.9) generically holds.

Proof. If $\phi_\infty = +\infty$ then, since $V_\infty < 0$, there exists a $\bar{V} < 0$, such that, $V(\phi(t)) \leq \bar{V} < 0$ eventually and then Lemma 4 applies again to give the result. □

A more subtle case happens when $V_\infty = 0$, as is the case of the ekpyrotic potentials. In this situation the critical point "at infinity" corresponding to $\phi \to +\infty$ must be studied carefully, since it may give rise to ever expanding cosmologies. Before we state the precise theorem, the following preliminary result is needed.

Lemma 9. If $V_\infty = 0$ and $\phi_\infty = +\infty$, then $H_\infty = \lim_{t\to\sup\mathbb{I}} H(t) \leq 0$.

Proof. By contradiction, suppose $H_\infty > 0$. Then $\frac{1}{2}y^2 + \rho \to 3H_\infty^2$ and therefore $\sup\mathbb{I} \in \mathbb{R}$, otherwise it would be

$$H(t) - H(0) = -\frac{1}{2}\int_{t_0}^{t} y(s)^2 + \gamma\rho(s)\,\mathrm{d}s \to -\infty,$$

as $t \to \infty$, a contradiction. By the Cauchy-Schwarz inequality we obtain,

$$\begin{aligned}(\phi(t) - \phi(t_0))^2 &= \left(\int_{t_0}^{t} \dot{\phi}(s)\,\mathrm{d}s\right)^2 \leq (t - t_0)\int_{t_0}^{t} y(s)^2\,\mathrm{d}s \\ &\leq (t - t_0)\int_{t_0}^{t} -2\dot{H}(s)\,\mathrm{d}s = 2(t - t_0)(H(t_0) - H(t)),\end{aligned}$$

that converges to the finite value $2(\sup\mathbb{I} - t_0)(H(t_0) - H_\infty) \in \mathbb{R}$, hence $\phi(t)$ is bounded, which is a contradiction. Therefore, $H_\infty \leq 0$. □

Proposition 10. Suppose that $\phi_\infty = +\infty$ and $V_\infty = 0$. If (2.8) does not hold, then (2.9) generically holds. Otherwise, i.e. if (2.8) holds, either (2.9) generically holds or the solution expands forever, with $\phi(t) \to +\infty$ and $y(t), \rho(t)$ and $H(t)$ infinitesimal as $t \to +\infty$.

Proof. By the preceding Lemma, $H_\infty \leq 0$. If H_∞ is strictly negative then the results follows from Lemma 4. Suppose now it is zero; this means that the solution expands forever and a normalized variables scheme can be used to study the critical point "at infinity". We use variables (ϕ, x, w, z) as in Proposition 5, which are functions of a new time τ coordinate defined by $\mathrm{d}\tau/\mathrm{d}t = H$, (note that unlike the case treated in Proposition 5, now $H > 0$). Using the function g defined in Assumption 1 and arguing as in the proof of Proposition 5, we arrive at the following system,

(3.6)
$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = \left(\frac{1}{2}w^2 - 3\right)(w + u(g^{-1}(s))) + z^2\left(\frac{\gamma}{2}w + \alpha(g^{-1}(s)) + u(g^{-1}(s))\right),$$

(3.7)
$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{2}z\left[w^2 - \alpha(g^{-1}(s))w + \gamma(z^2 - 3)\right],$$

(3.8)
$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = wg'(g^{-1}(s)).$$

We are interested in solutions such that $s \to 0$ and $\frac{1}{2}w^2 + z^2 > 3$, with $w, z \geq 0$ eventually. Therefore, the (w, z)–coordinates of the critical points that are admissible candidates to be ω–limit points are, (setting $\lambda = \lambda_+$ and $\alpha = \alpha(g^{-1}(0)) = \frac{4-3\gamma}{2}Q_+$)

$$\mathcal{A} = \left(\sqrt{6}, 0\right), \mathcal{B} = (-\lambda, 0), \mathcal{C} = \left(\frac{2\alpha}{2-\gamma}, \frac{\sqrt{-2\alpha^2 + 3(2-\gamma)^2}}{2-\gamma}\right),$$
$$\mathcal{D} = \left(-\frac{3\gamma}{\alpha+\lambda}, -\frac{\sqrt{3(-3\gamma + \alpha\lambda + \lambda^2)}}{\alpha+\lambda}\right).$$

The analysis of these critical points reveals that the only sink can be $\mathcal{C}$, and this happens precisely when (2.8) holds. In this case, we obtain ever expanding solutions such that $H_\infty = 0$, and consequently, both y and ρ tend to zero; since the solution is defined for $\tau \to +\infty$, and recalling that t is an increasing function of τ, we get $\sup \mathbb{I} = +\infty$, i.e. also the corresponding solution to (2.1)–(2.5) is defined for $t \to +\infty$.

If the solution does not start into the basin of attraction of $\mathcal{C}$, it is unbounded, thus, $\frac{1}{2}w^2 + z^2 \to +\infty$. Suppose by contradiction that they also correspond to ever expanding cosmologies with $H_\infty = 0$. Then $\sup \mathbb{I} = +\infty$. Set

$$\widetilde{x} = \frac{\sqrt{|V|}}{H},$$

and observe that

$$\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau} = \frac{1}{2}\widetilde{x}(u(g^{-1}(s))w + w^2 + \gamma z^2) \approx \frac{1}{2}\widetilde{x}(w^2 + \gamma z^2),$$

where the symbol $\approx$ is used to denote the dominant terms. Now,

$$\widetilde{x}^2 = \frac{1}{2}w^2 + z^2 - 3 \approx \frac{1}{2}w^2 + z^2 \approx K(w^2 + \gamma z^2),$$

for some constant $K > 0$, so $\mathrm{d}\widetilde{x}/\mathrm{d}\tau \approx A\widetilde{x}^3$ for some $A > 0$, which implies $\widetilde{x}(\tau) \approx (a - b\tau)^{-1/2}$ for suitable $a, b > 0$.

Then,

$$\frac{1}{2}(w^2 + \gamma z^2) \approx \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau}\frac{1}{\widetilde{x}} \approx \frac{b}{2}(a - b\tau)^{-1},$$

hence,

$$\frac{\mathrm{d}H}{\mathrm{d}\tau} = -H\frac{1}{2}(w^2 + \gamma z^2) \approx -H(\tau)\frac{b}{2}(a - b\tau)^{-1},$$

from which $H(\tau) \approx H_0\sqrt{a - b\tau}$. This implies that

$$t = \int_{\tau_0}^{\tau} \frac{1}{H(\sigma)}\mathrm{d}\sigma \approx \int_{\tau_0}^{\tau} \frac{1}{H_0\sqrt{a - b\sigma}}\,\mathrm{d}\sigma,$$

which converges as $\tau \to a/b$. This means that $\sup \mathbb{I} \in \mathbb{R}$, that is a contradiction. Therefore, $H(t) < 0$ eventually. Since $V(\phi(t)) < 0$ eventually, the conclusion follows from Lemma 4. □

Example 11. To illustrate the situation depicted in Proposition 10, let us consider the double exponential potential,

$$V(\phi) = e^{\lambda_- \phi} - e^{\lambda_+ \phi},$$

that falls into case A, provided that $\lambda_- < \lambda_+ < 0$.

Choose parameters, for example, $\lambda_+ = -4$, $\lambda_- = -5$, and $Q(\phi) = 1$, a constant. Then condition (2.8) is satisfied for a non empty set of the admissible values of γ, for instance $\gamma = 1$, (dust). With this set of parameters, both expansion and recollapse may take place, depending on the initial conditions.

With initial conditions for instance, $H(0) = 1$, $\phi(0) = 2$ and $y(0) = -1$ (the initial value $\rho(0)$ is not arbitrary, but is determined by (2.5)), the scalar field positively diverges in an infinite time and the Hubble function remains always positive, tending asymptotically to zero; therefore the Universe expands forever, Figures 1(a)–1(b).

Simply changing the initial conditions, for instance $y(0) = -2$, then the scalar positively diverges again, but now in a finite amount of time. Indeed, $H(t)$ changes sign and once it becomes negative, the solution is forced to recollapse and develop a singularity, Figures 1(c)–1(d).

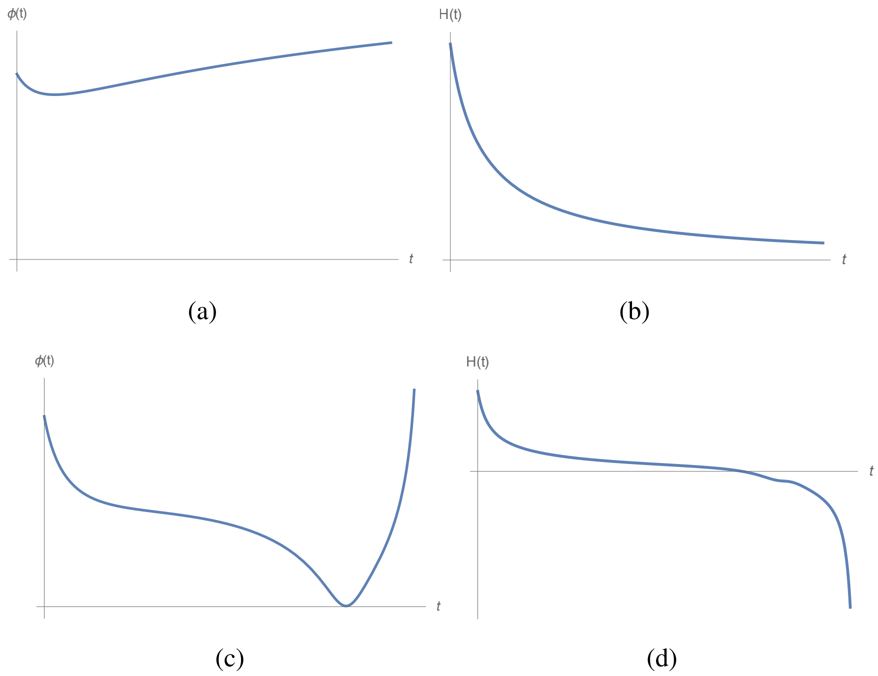

FIGURE 1. The cases studied in Proposition 10. In the top figures 1(a)–1(b), the scalar field positively diverges and the Universe expands forever. I the bottom figures 1(c)–1(d), the scalar field positively diverges again, but in a finite time, resulting in recollapse and development of a singularity.

3.4. **Case ϕ_∞ does not exist.** In this subsection we study the case when $\phi(t)$ neither converges nor diverges.

Proposition 12. If ϕ_∞ does not exist, then (2.9) generically holds.

Proof. First we claim that

$$H_\infty = \lim_{t\to\sup\mathbb{I}} H(t) = -\infty, \tag{3.9}$$

generically holds, by considering the following subcases.

(1) Suppose $\liminf_{t\to\sup\mathbb{I}} V(\phi(t)) \geq 0$. If by contradiction, $H(t)$ was bounded, then from (2.5) we could conclude that $y(t), \rho(t)$ were bounded too. Then, for the solution to be generic (recall again Lemma 3), $\phi(t)$ should be unbounded. But since $V(\phi(t))$ must be eventually non negative, this would imply that $\limsup_{t\to\sup\mathbb{I}} V(\phi(t)) = +\infty$. Then a sequence $t_n \to \sup\mathbb{I}$ exists, such that $H(t_n)^2 \to +\infty$, which means that $H(t_n) \to -\infty$, which is a contradiction. Thus, $H(t)$ cannot be bounded and therefore (3.9) must hold.

(2) Suppose $\liminf_{t\to\sup\mathbb{I}} V(\phi(t)) < 0$. In this case there exist sequences $\{t_n\}, \{s_n\}$, such that

$$t_n, s_n \to \sup\mathbb{I}, \quad t_n < s_n < t_{n+1},$$

with $V(\phi(t_n)), V(\phi(s_n)) < 0$ and $\phi(t)$ lies between $\phi(t_n)$ and $\phi(s_n)$, $\forall t \in [t_n, s_n]$. Using Cauchy-Schwarz inequality as in the proof of Lemma 9 we get

$$(\phi(t_n) - \phi(s_n))^2 \leq (s_n - t_n) \int_{t_n}^{s_n} -2\dot{H}(s)\,\mathrm{d}s = 2(s_n - t_n)(H(t_n) - H(s_n)),$$

and therefore

$$s_n - t_n \geq \frac{(\phi(t_n) - \phi(s_n))^2}{2(H(t_n) - H(s_n))}. \tag{3.10}$$

Now, if by contradiction $H_\infty \in \mathbb{R}$ then (3.10) would imply that $s_n - t_n \to +\infty$ and as a consequence $\sup\mathbb{I} = +\infty$. Moreover comparison theorems in ODE would say that $H(t) \leq z(t)$ in $[t_n, s_n]$, where $z(t)$ solves the Cauchy problem

$$\dot{z}(t) = \frac{\gamma}{2}(-3z(t)^2 + \bar{V}), \qquad z(t_n) = H(t_n),$$

and $\bar{V}$ is a (negative) constant such that $V(\phi) < \bar{V}$, for every ϕ between ϕ_t and ϕ_s. Now, observe that the solution $z(t)$ to the Cauchy problem above negatively diverges for some $t_n + \delta_n$, where δ_n is uniformly bounded with respect to n, whereas $s_n - t_n \to +\infty$, and this is a contradiction. Hence $H_\infty = -\infty$, i.e. (3.9) holds.

In both cases (2a) and (2b) we have shown that (3.9) holds. Let us prove that this happens in a finite amount of time. If $\liminf_{t\to\sup\mathbb{I}} \phi(t) \in \mathbb{R}$ then there exists a $\bar{V} \in \mathbb{R}$, such that $V(\phi(t)) \leq \bar{V}$ eventually, and the result follows from Lemma 4.

If $\liminf_{t\to\sup\mathbb{I}} \phi(t) = -\infty$, i.e., $\limsup_{t\to\sup\mathbb{I}} V(\phi(t)) = +\infty$, then we can consider the same system in normalized variables used in case (1c) before, see Remark 6 after Proposition 5. □

4. Potentials of class D

The treatment of potentials of class D is similar to the methods used in [1, Theorem 1]. In fact, we have the following result.

Proposition 13. Let $V(\phi) \in \mathcal{C}^2$ such that

(1) $\lim_{\phi\to\pm\infty} V(\phi) = -\infty$,
(2) V has a unique nondegenerate critical point (that has to be, in view of (1), the global maximum),

(3) There exist $\lambda > 0$ and $M > 0$ such that, $|V'(\phi)| \leq -\lambda V(\phi)$, for all $\phi : |\phi| > M$.

Then a solution to (2.1)–(2.5) generically recollapses in a finite time, i.e. (2.9) holds.

The argument follows the same line of the proof used for left unbounded potentials treated in [1]. Indeed, in that paper a class of potentials was considered such that hypotheses (1) and (3) hold only for $\phi \to -\infty$, whereas $V(\phi) \to 0^+$ as $\phi \to +\infty$. A critical value $\dot{\phi}_{crit}$ is shown to exist such that, if $y(t_0) < \dot{\phi}_{crit}$, the scalar field is eventually forced to take values to the left of the global maximum. The proof of Theorem 1 in [1], which is the analogue of the above Proposition 13, relies precisely on these hypotheses, in such a way that one can be sure that the scalar field does not positively diverge to the critical point at infinity, a case examined in a previous paper [25]. In the present case, potentials of class D do not have that complication, diverging to $-\infty$ on both directions, and therefore, regardless of the behavior of the scalar field, and recalling Lemma 3, it can be proved with exactly the same argument that, the solution recollapses almost always and the Hubble function negatively diverges in a finite time.

5. Discussion

With Proposition 13 we complete the analysis, carried on in Ref. [1], of the class of potentials falling to minus infinity as $\phi \to -\infty$, having a global positive maximum and going to zero from above as $\phi \to +\infty$. In that study, assuming that the growth of $|V(\phi)|$ to infinity is at most exponential, the corresponding initially expanding Universes, eventually collapse in a finite time, up to a set of initial data of measure zero. In the present paper we extend the analysis to situations where the negative branch of the potential function may possibly not diverge, whereas the positive branch diverges to $+\infty$. We have investigated the qualitative behavior of the Hubble function, examining all possible cases for the asymptotic behavior of the scalar field. We have found that the recollapse and the formation of a future singularity always take place in a *generic* way, i.e. stable with respect to perturbations of the initial data of the system. Moreover, recollapse is the only generical situation allowed, except in case the potential goes to zero from below as $\phi \to +\infty$ and (2.8) holds; in this case there also exists generical choices of initial data that do not lead to recollapse, producing an ever–expanding cosmology where the scalar field positively diverges. Our conclusions are valid for scalar fields coupled to matter, as well as for uncoupled models studied so far in the literature.

Cosmology with negative potentials is the basis of the cyclic Universes in the context of the ekpyrotic scenario. Our results may be helpful in building solid models of cyclic cosmologies and therefore avoid the fragility of this scenario with respect to the unknown physics at the singularity.

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