Research Article

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Brake orbits for Hamiltonian systems of the classical type via geodesics in singular Finsler metrics

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Abstract: We consider Hamiltonian functions of the classical type, namely, even and convex with respect to the generalized momenta. A brake orbit is a periodic solution of Hamilton's equations such that the generalized momenta are zero on two different points. Under mild assumptions, this paper reduces the multiplicity problem of the brake orbits for a Hamiltonian function of the classical type to the multiplicity problem of orthogonal geodesic chords in a concave Finslerian manifold with boundary. This paper will be used for a generalization of a Seifert's conjecture about the multiplicity of brake orbits to Hamiltonian functions of the classical type.

Keywords: Hamiltonian systems, brake orbits, Finsler metric, variational methods

MSC 2020: 70H12, 70G75, 70H05, 58E10, 53B40

1 Introduction

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be an autonomous Hamiltonian function of class C^2. A curve $(q, p) : [0, T] \to \mathbb{R}^{2n}$ is a solution of Hamilton's equations if

$$\dot{q} = \frac{\partial H}{\partial p}(q, p), \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p). \tag{1.1}$$

Since the Hamiltonian is autonomous, the conservation law of the energy holds. More formally, if $(q, p) : [0, T] \to \mathbb{R}^{2n}$ is a solution of Hamilton's equations, then there exists a real number E, called energy, such that

$$H(q(t), p(t)) = E, \quad \forall t \in [0, T].$$

Let $K : \mathbb{R}^{2n} \to \mathbb{R}$ and $V : \mathbb{R}^n \to \mathbb{R}$ be two functions such that the Hamiltonian can be written as follows:

$$H(q, p) = K(q, p) + V(q),$$

and $K(q, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is even and strictly positive unless $p = 0$. Then, whenever the set $\{q \in \mathbb{R}^n : V(q) < E\}$ is non-empty, it can be thought as a potential well.

This paper concerns the multiplicity of the brake orbits in a bounded potential well. Roughly speaking, a brake orbit is a periodic solution of Hamilton's equations with energy E that oscillates back and forth

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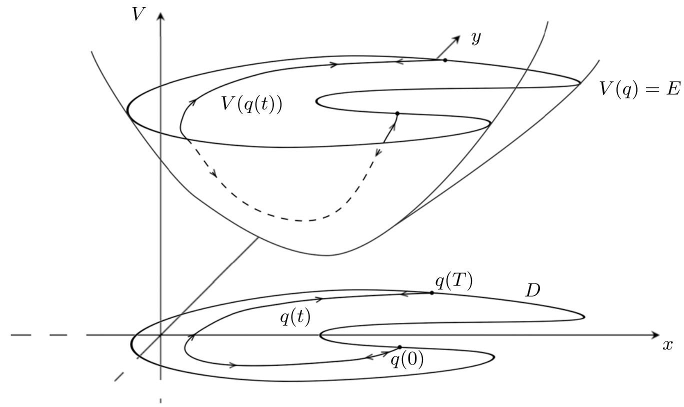

Figure 1: Projection on the configuration space of a break-orbit. The periodic solution $q(t)$ oscillates back and forth between the two points $q(0)$, $q(T)$ that lies on the boundary of the potential well $D = \{q \in \mathbb{R}^2 : V(q) < E\}$.

between two points of the boundary of the potential well (Figure 1). When the Hamiltonian is natural, hence, given by

$$H(q,p) = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(q)p_i p_j + V(q),$$

where $(a^{ij}(q))$ is a positive definite quadratic form on $\mathbb{R}^n$, and the potential well is homeomorphic to the n-dimensional disk in $\mathbb{R}^n$, Seifert conjectured the existence of at least n brake orbits (cf. [1]). This conjecture has motivated an extensive literature on the subject (e.g., [2–10]), and it has been recently proved in [11], exploiting some partial results given by the authors in different previous papers (cf. [12–17]). This work points towards a generalization of Seifert's conjecture, looking for the multiplicity of brake orbits when the Hamiltonian function is of the classical type (see Definition 1.1). Indeed, the present paper includes some results that will be exploited in the future to generalize the Seifert's conjecture for Hamiltonian systems of the classical type. In particular, we show that the brake orbits in a bounded potential well for a Hamiltonian function of the classical type have a one–one correspondence with the orthogonal geodesic chords in a strictly concave Finsler manifold with boundary (see Theorem 1.9). Different generalizations of Seifert's conjecture have been analyzed in the last decades. The papers with the most similar setting to the present one are [18] and [19], where the existence of one brake orbit is proved for Finsler mechanical systems and Hamiltonian systems of the classical type, respectively.

Before formally stating our main result, we need the following definitions (cf. [19]).

Definition 1.1. A Hamiltonian function $H(q,p)$ on $\mathbb{R}^{2n}$ is of the classical type if, for each $q_0 \in \mathbb{R}^n$, the function $p \mapsto H(q_0,p)$ is even and $(\partial^2 H/\partial p^2)(q_0,p)$ is strictly positive definite for all p, namely, there exists a continuous function $\nu : \mathbb{R}^n \to \mathbb{R}$ such that, for all $q \in \mathbb{R}^n$, $\nu(q) > 0$ and

$$\frac{\partial^2 H}{\partial p^2}(q,p)[\xi,\xi] \geq \nu(q)\|\xi\|^2, \quad \forall p,\xi \in \mathbb{R}^n. \tag{1.2}$$

Remark 1.2. If H is a Hamiltonian of the classical type, by (1.2), the inverse of $(\partial H/\partial p)(q,\cdot)$ is well defined for all $q \in \overline{D}$. Hence, with a slight abuse of notation, we will say that a curve $q : [0,T] \to \overline{D}$ is a solution of the Hamilton's equations if $(q,p) : [0,T] \to \mathbb{R}^{2n}$ is a solution of (1.1), where p is implicitly defined by

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t),p(t)), \quad \forall t \in [0,T].$$

Definition 1.3. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian function of the classical type. We define the potential energy function $V : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$V(q) = H(q,0), \quad \forall q \in \mathbb{R}^n,$$

and the kinetic energy function $K : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$K(q,p) = H(q,p) - V(q).$$

By Definitions 1.1 and 1.3, a Hamiltonian of the classical type can be written as follows:

$$H(q,p) = K(q,p) + V(q),$$

where $K(q,p)$ is even with respect to p, strictly positive unless $p = 0$ and, for each $q \in \mathbb{R}^n$,

$$\frac{\partial^2 K}{\partial p^2}(q,p)[\xi,\xi] \geq \nu(q)\|\xi\|^2, \quad \forall p, \xi \in \mathbb{R}^n.$$

Definition 1.4. Let H be a Hamiltonian function of the classical type. A potential well for H is an open set $D \subset \mathbb{R}^n$ with boundary ∂D of class C^2 such that, for some real number E, the followings hold:
- $V(q) < E$ on D;
- $V(q) = E$ on ∂D;
- $\nabla V(q) \neq 0$, for all $q \in \partial D$.

Definition 1.5. Let $D \subset \mathbb{R}^n$ be a potential well for a Hamiltonian H, with $V(q) = E$ on ∂D. A solution $(q(t), p(t))$ of Hamilton's equations for H is called brake orbit if it has energy E, and there exists $T > 0$ such that $q(t) \in D$ for $0 < t < T$, while $q(0), q(T) \in \partial D$.

Following the notation of Remark 1.2, we say that $q : [0,T] \to \overline{D}$ is a brake orbit if it is a solution of (1.1) with energy E, $q(]0,T[) \subset D$ and $q(0), q(T) \in \partial D$.

Remark 1.6. By the conservation law of the energy, if $(q(t), p(t))$ is a brake orbit, then $p(0)$ and $p(T)$ must be zero. Since H is of the classical type (hence, even in p), the solution can be continued so that it will be periodic. In other words, $q(t)$ oscillates back and forth along a curve in D with endpoints in ∂D.

Let us introduce the following notation. Let $(\mathcal{M}, F)$ be a Finsler manifold of class C^3 and let $\Omega \subset \mathcal{M}$ be an open subset with boundary $\partial\Omega \in C^2$ (we refer to [20,21] for a background material about Finsler geometry).

Definition 1.7. A curve $\gamma : [a,b] \to \overline{\Omega}$ is a Finsler geodesic chord if
- It is a geodesic with respect to the Finsler metric F;
- $\gamma(a), \gamma(b) \in \partial\Omega$ and $\gamma(]a,b[) \subset \Omega$.

If $\dot\gamma(a)$ and $\dot\gamma(b)$ are orthogonal, with respect to the Finsler metric F, to $T_{\gamma(a)}\partial\Omega$ and $T_{\gamma(b)}\partial\Omega$, respectively, namely,

$$\frac{\partial F^2}{\partial v}(\gamma(t), \dot\gamma(t))[\xi] := \frac{\mathrm{d}}{\mathrm{d}s}F^2(\gamma(t), \dot\gamma(t) + s\xi)\bigg|_{s=0} = 0, \tag{1.3}$$

for all $\xi \in T_{\gamma(t)}\partial\Omega$, with $t = a, b$, then γ is called the orthogonal Finsler geodesic chord.

This paper reduces the multiplicity problem of the brake orbits in a bounded potential well of a Hamiltonian of the classical type to the related problem of orthogonal geodesic chords in a Finslerian manifold with smooth boundary.

The last ingredient to state our main theorem is the notion of strong concavity of a Finsler manifold with boundary. We say that $\overline{\Omega}$ is strongly concave with respect to the Finsler metric F if every geodesic, which is tangent to $\partial\Omega$ on one point q, lies inside Ω on a neighborhood of q. Thus, differently from the notion of concavity (cf. [22] for the dual notion of convexity), the strong concavity allows the geodesics tangent to the boundary to locally touch the boundary only in one point. We formally define the strong concavity as follows.

Definition 1.8. Let $\overline{\Omega} \subset \mathcal{M}$ be a manifold with smooth (C^2) boundary and let $\psi : \mathcal{M} \to \mathbb{R}$ be a function of class C^2 such that $\psi(\Omega) \subset]0, \infty[$, $\psi(\partial\Omega) = 0$, and $d\psi(q) \neq 0$ for all $q \in \partial\Omega$, where $d\psi$ denotes the differential of ψ. Then, $\overline{\Omega}$ is strongly concave if and only if for all $q \in \partial\Omega$ we have

$$H_\psi(q, v)[v, v] := \frac{d^2}{ds^2}(\psi \circ \gamma)(0) > 0, \quad \forall v \in T_q\partial\Omega, \; v \neq 0, \tag{1.4}$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is the unique geodesic such that $\gamma(0) = q$ and $\dot{\gamma}(0) = v$.

Now we are ready to state our main theorem.

Theorem 1.9. *Let $D \subset \mathbb{R}^n$ be a potential well for a Hamiltonian H of the classical type. If $\overline{D}$ is compact, there exists an open set $\Omega \subset D$, with a Finsler metric F on $\overline{\Omega}$, such that the following statements hold:*
- $\overline{\Omega} \subset D$;
- *$\partial\Omega$ is of class C^2;*
- *$\overline{\Omega}$ is homeomorphic to $\overline{D}$;*
- *$\overline{\Omega}$ is strongly concave with respect to the Finsler metric F;*
- *If $\gamma : [0, 1] \to \overline{\Omega}$ is an orthogonal Finsler geodesic chord, then there exists $[\alpha, \beta] \supset [0, 1]$ and an unique continuous extension $\hat{\gamma} : [\alpha, \beta] \to \overline{D}$ of γ such that*
  - *$\hat{\gamma}$ is a geodesic in $]\alpha, \beta[$;*
  - *Up to a time reparametrization, $\hat{\gamma} : [\alpha, \beta] \to \overline{D}$ is a brake orbit, namely, there exists a diffeomorphism $\sigma : [0, T] \to [\alpha, \beta]$ such that $q = \gamma \circ \sigma : [0, T] \to \overline{D}$ is a brake orbit.*

Theorem 1.9 reduces the study of multiple brake orbits of a Hamiltonian of the classical type to the study of multiple orthogonal geodesic chords in a strongly concave Finsler manifold with boundary. Given a bounded potential well D, we will construct a Finsler manifold $(\overline{\Omega}, F)$, with $\overline{\Omega} \subset D$, such that there exists a bijection between the brake orbits in D and the orthogonal geodesic chords in $(\overline{\Omega}, F)$. This result generalizes the one presented in [17] for natural Hamiltonian functions, where the Finsler metric is actually a Riemannian one. Some results about the multiplicity of orthogonal geodesic chords in the case of convex Finsler manifolds with boundary and some generalizations can be found, for instance, in [23–25].

This paper is organized as follows. Some standard notations are presented at the end of this introduction. In Section 2, we present and study a Jacobi-Finsler metric F defined on the potential well such that its geodesics are, up to a time reparametrization, the solution of the Hamiltonian system. However, F cannot be defined on the boundary ∂D, since it degenerates to the zero function. Therefore, in Section 3, we analyze the behavior of the solutions of Hamilton equations near the boundary. Following a variational approach, we see the geodesics as critical points of the energy functional $\mathcal{J}$ of the Jacobi-Finsler metric. Through the energy functional, in Section 4, we define the function $\psi : \overline{D} \to R$, which is the infimum of $\mathcal{J}$ among all the geodesics that connect a point to the boundary of the potential well. Hence, $\psi(y) \to 0$ as y approaches the boundary ∂D, and we also prove that, if y is sufficiently near to the boundary, there exists a unique geodesic γ_y connecting y and ∂D such that $\psi(y) = \mathcal{J}(\gamma_y)$. In Section 5, we prove that ψ is of class C^2 near the boundary ∂D and that there exists a $\hat{\delta} > 0$ such that $\overline{\Omega} = \psi^{-1}([\hat{\delta}, \infty[)$ is a strongly concave set with respect to F. In Section 6, we finally give the proof of Theorem 1.9 exploiting all the previous results. In this proof, the main idea is to connect an orthogonal geodesic Finsler chord in $\overline{\Omega}$ with the unique geodesic that realizes $\psi(y) = \mathcal{J}(y)$ to obtain a brake-orbit, up to a time reparametrization (Figure 2).

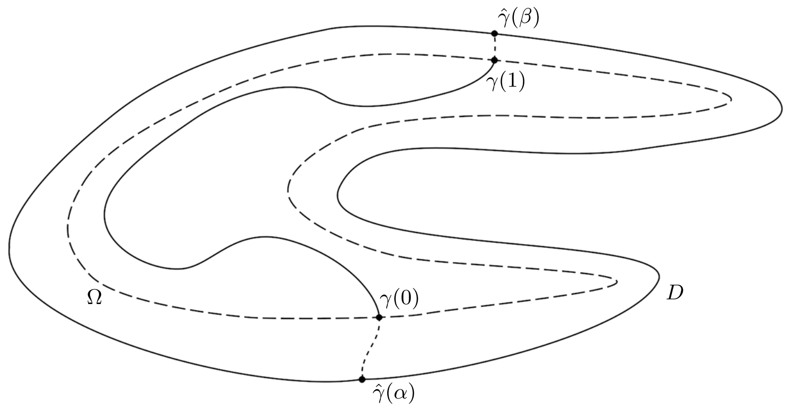

Figure 2: The setting of Theorem 1.9. The orthogonal Finsler geodesic chord γ in $\overline{\Omega}$ can be extended to $\hat{\gamma}$, which is a brake orbit in the potential well D, up to a time reparametrization. The extension is obtained through the unique geodesics that realize $\psi(\gamma(0))$ and $\psi(\gamma(1))$.

1.1 Notation

If f is a real-valued function defined on $\mathbb{R}^{2n}$, then $\partial f/\partial q$ and $\partial f/\partial p$ will denote the differentials of f with respect to q and p, respectively. We denote by f' the differential of f, hence, $f'(q,p) = (\partial f/\partial q, \partial f/\partial p)$. We will denote by v the conjugate variable of p via Legendre transform of a function. Hence, $\partial f/\partial v$ will denote the partial derivative with respect to v. We denote by $\langle \cdot,\cdot \rangle : \mathbb{R}^{2n} \to \mathbb{R}$ the Euclidean scalar product and $\|\cdot\| : \mathbb{R}^{2n} \to \mathbb{R}$ the Euclidean norm. We denote by $J \in M_{2n\times 2n}(\mathbb{R})$ the symplectic matrix

$$J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}.$$

Let $z : [0, T] \to \mathbb{R}^{2n}$ be a curve with $z(t) = (q(t), p(t))$. Using this notation, (1.1) can be written as follows:

$$\dot{z}(t) = JH'(z(t)).$$

For every compact interval $I \subset \mathbb{R}$ and every $A \subset \mathbb{R}^n$, we denote by $W^{1,2}(I, A)$ the Sobolev space:

$$W^{1,2}(I, A) = \{\gamma : I \to A : \gamma \text{ is absolutely continuous and } \dot{\gamma} \in L^2(I, \mathbb{R}^n)\}.$$

2 The Jacobi-Finsler metric

Let H be a Hamiltonian of the classical type and $D \subset \mathbb{R}^n$ be the (open) potential well such that $V(q) \equiv E$ on ∂D and $\overline{D}$ is compact. In this section, following the same construction of [19], we endow the potential well with a Finsler metric whose geodesics are linked to the solution of the Hamiltonian system via time reparametrization. Let us define

$$\Sigma = \{(q,p) \in \mathbb{R}^{2n} : q \in D, H(q,p) = E\}.$$

If $(q_0, p_0) \in \Sigma$, then $p_0 \neq 0$, and this implies that $H'(q_0, p_0)$ is different from zero. As a consequence, Σ is a regular level surface for H.

Lemma 2.1. *There exists a function $U : D \times \mathbb{R}^n \to \mathbb{R}$ such that*
- *U is of class C^1;*
- *U is of class C^2 on $D \times (\mathbb{R}^n \setminus \{0\})$;*
- *$U(q,p)$ is even and homogeneous of degree 2 in p;*
- *$\Sigma = U^{-1}(1)$ and Σ is a regular level surface for U.*

Proof. Since H is convex with respect to p, for every $q_0 \in D$, the set

$$\{p : H(q_0, p) = E\}$$

is a nonempty, convex, compact hypersurface in $\mathbb{R}^n$, symmetric about the origin. As a consequence, there exists a unique function $U : \mathbb{R}^{2n} \to \mathbb{R}$, which is homogeneous of degree 2 in p and which is identically 1 on Σ. Since H is of class C^2, so it is U on $D \times (\mathbb{R}^n \setminus \{0\})$. Moreover, the homogeneity of degree 2 in p implies both the C^1-regularity of U and that $U'(q, p) \neq 0$ for all $(q, p) \in \Sigma$. □

Since Σ is a regular level surface for the Hamiltonian functions H and U, we have the following result.

Lemma 2.2. *A curve $(q, p) : [0 : T] \to D \times \mathbb{R}^n$ is a solution of Hamilton's equations for H if and only if it is a solution of Hamilton's equations for U, up to time reparametrization.*

Proof. See [19, Lemma 2.1]. □

Remark 2.3. Let $x = (q, p) : [0, S] \to \mathbb{R}^{2n}$ be a solution of Hamilton's equations with Hamiltonian U; hence,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = JU'(x(s)).$$

By Lemma 2.2, there exists a function $\lambda : [0, T] \to [0, S]$ such that $z = x \circ \lambda : [0, T] \to \mathbb{R}^{2n}$ is a solution of Hamilton's equations with Hamiltonian H; hence,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = JH'(z(t)).$$

As a consequence, we obtain

$$JH'(x(\lambda(t))) = JH'(z(t)) = \frac{\mathrm{d}z}{\mathrm{d}t}(t) = \frac{\mathrm{d}\lambda}{\mathrm{d}t}(t)\frac{\mathrm{d}x}{\mathrm{d}s}(\lambda(t)) = \frac{\mathrm{d}\lambda}{\mathrm{d}t}(t)JU'(x(\lambda(t))).$$

Imposing that λ is an orientation preserving reparametrization, we obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}(t) = \frac{\langle H'(z(t)), U'(z(t))\rangle}{\|U'(z(t))\|^2} = \frac{\|H'(z(t))\|}{\|U'(z(t))\|}.$$

Hence, we have

$$\frac{\mathrm{d}x}{\mathrm{d}s}(s) = \frac{\|U'(z(t))\|}{\|H'(z(t))\|}\frac{\mathrm{d}z}{\mathrm{d}t}(t), \quad \text{with } \lambda(t) = s. \tag{2.1}$$

The inverse function of λ satisfies

$$\frac{\mathrm{d}(\lambda^{-1})}{\mathrm{d}s}(s) = \frac{\|U'(x(s))\|}{\|H'(x(s))\|}.$$

Hence, we can obtain the time reparametrization solving the following integral:

$$t(s) = \int_0^s \frac{\|U'(x(\sigma))\|}{\|H'(x(\sigma))\|}\mathrm{d}\sigma. \tag{2.2}$$

The following result provides the Finsler metric that we will employ in our study.

Lemma 2.4. *Let $G : D \times \mathbb{R}^n \to \mathbb{R}$ be the Legendre transform of U with respect to p; hence,*

$$G(q, v) = \sup_{p\in\mathbb{R}^n}(\langle v, p\rangle - U(q, p)),$$

and define $\mathcal{L} : D \times \mathbb{R}^n \to D \times \mathbb{R}^n$ *as follows:*

$$\mathcal{L}(q,p) = \left(q, \frac{\partial U}{\partial p}(q,p)\right) = (q,v). \tag{2.3}$$

Then, the function $F : D \times \mathbb{R}^n \to \mathbb{R}$, *defined as* $F = \sqrt{G(q,v)}$, *is a Finsler metric on* D. *Moreover, a curve* $q : [0,S] \to D$ *is a Finsler geodesic parametrized by arc length if and only if* $(q(s), p(s)) = \mathcal{L}^{-1}(q(s), \dot{q}(s))$ *is a solution of Hamilton's equations with Hamiltonian* U *and* $U(q(s), p(s)) \equiv 1$.

Proof. Since U is convex and homogeneous of degree 2 in p, for every $(q,v) \in D \times \mathbb{R}^n$, the function

$$p \mapsto \langle v, p\rangle - U(q,p)$$

has a unique maximum. Therefore, the function G is well defined, convex, and homogeneous of degree 2 in v. Moreover, G is of class C^2 on $D \times \mathbb{R}^n \backslash \{0\}$, while it is of class C^1 on $D \times \mathbb{R}^n$. Thus, the function $F : D \times \mathbb{R}^n \to R$ defined as $F(q,v) = \sqrt{G(q,v)}$ is a Finsler metric on D. Since

$$\frac{\partial^2 U}{\partial p^2}(q,p) > 0, \quad \forall q \in D, \quad \forall p \in \mathbb{R}^n \backslash \{0\},$$

the map $(\partial U/\partial p)(q,\cdot)$ is invertible; thus, $\mathcal{L}$ is a diffeomorphism, and it is homogeneous of degree 1 with respect to p. The equivalence between the Finsler geodesics parametrized by arc length and the solutions of (1.1) with energy E is a direct consequence of the Legendre transform (see, for instance, [20, Chapter I, p. 22]). □

By Lemmas 2.2 and 2.4, if q is a geodesic in D, then $\mathcal{L}^{-1}(q,\dot{q})$ is a solution of Hamilton's equations with the original Hamiltonian H, up to a time reparametrization. As a consequence, finding a Finsler geodesic in D is equivalent to finding a solution of (1.1) in D with energy E. The following result provides the reparametrization that links the geodesics to the solutions of (1.1), combining the time reparametrization (2.2) with the Legendre transform defined in (2.3). For the sake of presentation, we use the following notation:

$$\phi(q,v) := \frac{\|U'(\mathcal{L}^{-1}(q,v))\|}{\|H'(\mathcal{L}^{-1}(q,v))\|}, \quad \forall (q,v) \in \mathcal{L}(\Sigma). \tag{2.4}$$

Remark 2.5. The function ϕ given by (2.4) is well defined. Indeed, since $\overline{\Sigma}$ is compact, $\frac{\partial V}{\partial q}(q) \neq 0$ for all $q \in \partial D$ and H is strictly convex with respect to p, there exist two constants h_1 and h_2 such that

$$0 < h_1 \leq \|H'(q,p)\| \leq h_2, \quad \forall (q,p) \in \overline{\Sigma}. \tag{2.5}$$

Lemma 2.6. *Let* $\gamma : [0,1] \to D$ *be a Finsler geodesic such that*

$$G(\gamma(s), \dot{\gamma}(s)) = c_\gamma, \quad \forall s \in [0,1],$$

and let $\lambda : [0,T] \to [0,1]$ *be the reparametrization such that* $\gamma \circ \lambda$ *is a solution of* (1.1) *with energy* E. *Then, the inverse of* λ *is given by*

$$t(s) = \sqrt{c_\gamma} \int_0^s \phi\left(\gamma(\sigma), \frac{\dot{\gamma}(\sigma)}{\sqrt{c_\gamma}}\right) d\sigma.$$

Proof. If $G(\gamma(s), \dot{\gamma}(s)) = c_\gamma$ for all s, then the reparametrization

$$\lambda_1 : [0, \sqrt{c_\gamma}] \to [0,1], \quad \lambda_1(\tau) = \frac{\tau}{c_\gamma}$$

is such that the curve $\hat{\gamma} = \gamma \circ \lambda_1 : [0, \sqrt{c_\gamma}] \to D$ is a geodesic parametrized by arc length. By Lemma 2.4, the curve $x : [0, \sqrt{c_\gamma}] \to D$, which is defined as follows:

$$x(\tau) = (q(\tau), p(\tau)) = \mathcal{L}^{-1}(\hat{\gamma}(\tau), \dot{\hat{\gamma}}(\tau)),$$

is a solution of Hamilton's equations with respect to U. Let $\lambda_2 : [0, T] \to [0, \sqrt{c_\gamma}]$ be the inverse of

$$\tau \mapsto \int_0^\tau \frac{\|U'(x(u))\|}{\|H'(x(u))\|} \mathrm{d}u = \int_0^\tau \phi(\hat{\gamma}(u), \dot{\hat{\gamma}}(u)) \mathrm{d}u.$$

With the change of variable $\sigma = u/\sqrt{c_\gamma}$, $\sigma \in [0, 1]$, we have

$$\lambda_2^{-1}(\tau) = \sqrt{c_\gamma} \int_0^{\tau/\sqrt{c_\gamma}} \phi(\hat{\gamma}(\sqrt{c_\gamma}\sigma), \dot{\hat{\gamma}}(\sqrt{c_\gamma}\sigma)) \mathrm{d}\sigma = \sqrt{c_\gamma} \int_0^{\tau/\sqrt{c_\gamma}} \phi\left(\gamma(\sigma), \frac{\dot{\gamma}(\sigma)}{\sqrt{c_\gamma}}\right) \mathrm{d}\sigma.$$

By Remark 2.3, in particular by (2.2), $x \circ \lambda_2$ is a solution of (1.1). As a consequence, since $\mathcal{L}^{-1}$ is the identity map with respect to the first variable, the curve $\gamma \circ \lambda_1 \circ \lambda_2 : [0, T] \to D$ is the reparametrization of γ such that it is a solution of (1.1) with energy E. Hence, the desired reparametrization $\lambda : [0, T] \to [0, 1]$ is given by $\lambda = \lambda_1 \circ \lambda_2$ and its inverse $t : [0, 1] \to [0, T]$ is given by

$$t(s) = \lambda_2^{-1}(\lambda_1^{-1}(s)) = \lambda_2^{-1}(\sqrt{c_\gamma} s) = \sqrt{c_\gamma} \int_0^s \phi\left(\gamma(\sigma), \frac{\dot{\gamma}(\sigma)}{\sqrt{c_\gamma}}\right) \mathrm{d}\sigma,$$

and we are done. □

Remark 2.7. When H is a Hamiltonian of natural type, the previous construction leads to the well-known Maupertuis principle (cf. [17]). Indeed, set

$$H(q, p) = \frac{1}{2} \sum_{i,j=1}^n a^{ij}(q) p_i p_j + V(q),$$

where $(a^{ij}(q))$ is a positive definite quadratic form on $\mathbb{R}^n$. Then, using the aforementioned construction,

$$U(q, p) = \frac{1}{2(E - V(q))} \sum_{i,j=1}^n a^{ij}(q) p_i p_j,$$

and its Legendre transform is expressed as follows:

$$G(q, v) = \frac{1}{2}(E - V(q)) \sum_{i,j=1}^n a_{ij}(q) v^i v^j, \tag{2.6}$$

where $(a_{ij}(q))$ is the inverse of $(a^{ij}(q))$. We observe that $G(q, v)$ degenerates on the boundary ∂D, where, by continuity, it can be extended to 0. Since

$$U'(q, p) = \frac{1}{E - V(q)} H'(q, p), \quad \forall (q, p) \in \Sigma,$$

then

$$\frac{\|U'(q, p)\|}{\|H'(q, p)\|} = \frac{1}{E - V(q)}, \quad \forall (q, p) \in \Sigma.$$

Using Lemma 2.6, if $\gamma : [0, 1] \to D$ is a geodesic of constant speed with respect to the Riemannian metric $\sqrt{G}$, then we can obtain the reparametrization $\lambda : [0, T] \to [0, 1]$ such that $q = \gamma \circ \lambda : [0, T] \to D$ is a solution of (1.1) for H. Using (2.2), the inverse of λ is given by

$$t(s) = \sqrt{c_\gamma} \int_0^s \frac{1}{E - V(\gamma(\sigma))} \mathrm{d}\sigma,$$

where $G(\gamma_\gamma(s), \dot{\gamma}(s)) \equiv c_\gamma$.

3 Jacobi-Finsler metric near the boundary

Since U and G are not defined on ∂D, the aforementioned construction does not allow to see the brake orbits in $\overline{D}$ as Finsler geodesics. In this section, we estimate the behavior of U near the boundary ∂D, and we will show that G degenerates on ∂D to the zero function, as it can be seen in (2.6) for the case of natural Hamiltonian systems. Differently from [19], we are interested in the multiplicity of the brake orbits, not only in their existence. Hence, in addition to the construction given in [19], we give an upper and a lower bound for the Finsler metric that depend only on H and the potential well D, and these bounds will be exploited to obtain the one-one correspondence between the brake orbits and the orthogonal geodesic chords.

As a preliminary step, we give the following result, which is available up to a modification of $H(q,p) = K(q,p) + V(q)$ far away from Σ.

Lemma 3.1. *There exist two constants $\nu_1, \nu_2 > 0$ such that the followings hold for every $q \in \overline{D}$ and $p \in \mathbb{R}^n$:*

$$\nu_1\|\xi\|^2 \leq \frac{\partial^2 K}{\partial p^2}(q,p)[\xi,\xi] \leq \nu_2\|\xi\|^2, \quad \forall \xi \in \mathbb{R}^n; \tag{3.1}$$

$$\nu_1\|p\| \leq \left\|\frac{\partial K}{\partial p}(q,p)\right\| \leq \nu_2\|p\|; \tag{3.2}$$

$$\frac{1}{2}\nu_1\|p\|^2 \leq K(q,p) \leq \frac{1}{2}\nu_2\|p\|^2. \tag{3.3}$$

Proof. Since we are interested in the solutions of Hamilton's equations for H in Σ, which is a bounded set, we can modify H far away from Σ. Hence, we may assume that H is fiber-wise quadratic for $\|p\|$ sufficiently large. By (1.2) and the compactness of $\overline{D}$, there exist $\nu_1, \nu_2 > 0$ such that (3.1) holds. Since $(\partial K/\partial p)(q,0) = 0$ and $K(q,0) = 0$ for all q, from (3.1), we infer (3.2) and (3.3) by integration. □

Lemma 3.2. *Let ν_1 and ν_2 be the constants defined by* Lemma 3.1. *Then, the followings hold*:

$$\frac{\nu_1}{2(E - V(q))}\|p\|^2 \leq U(q,p) \leq \frac{\nu_2}{2(E - V(q))}\|p\|^2, \quad \forall (q,p) \in D \times \mathbb{R}^n, \tag{3.4}$$

$$\frac{(E - V(q))}{2\nu_2}\|v\|^2 \leq G(q,v) \leq \frac{(E - V(q))}{2\nu_1}\|v\|^2, \quad \forall (q,v) \in D \times \mathbb{R}^n. \tag{3.5}$$

Moreover, there exists a constant $\nu_3 > 0$ such that

$$\|U'(q,p)\| \geq \frac{\nu_3}{E - V(q)}, \quad \forall (q,p) \in \Sigma. \tag{3.6}$$

Proof. Set $\mathbb{S}^{n-1} = \{\theta \in \mathbb{R}^n : \|\theta\| = 1\}$. We define $\widetilde{H} : D \times \mathbb{S}^{n-1} \times \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$\widetilde{H}(q,\theta,\omega) = H(q,\omega\theta) - E.$$

Since $\widetilde{H}(q,\theta,0) < 0$ for all $(q,\theta) \in D \times \mathbb{S}^{n-1}$, exploiting also the convexity of H, we obtain that for all $(q,\theta) \in D \times \mathbb{S}^{n-1}$, there exists an unique $\omega > 0$ such that $\widetilde{H}(q,\theta,\omega) = 0$. As a consequence, the function $\omega : D \times \mathbb{S}^{n-1} \to \mathbb{R}^+$ such that

$$\widetilde{H}(q,\theta,\omega(q,\theta)) = 0, \quad \forall (q,\theta) \in D \times \mathbb{S}^{n-1}$$

is well defined. Moreover, by (3.2), we have

$$\frac{\partial \widetilde{H}}{\partial \omega}(q,\theta,\omega) = \left\langle \frac{\partial H}{\partial p}(q,\omega\theta), \theta \right\rangle = \frac{1}{\omega}\left\langle \frac{\partial K}{\partial p}(q,\omega\theta), \omega\theta \right\rangle \geq \omega\nu_1 > 0.$$

So we can apply the implicit function theorem to obtain that the function ω is of class C^2 and it satisfies

$$\frac{\partial\omega}{\partial q}(q,\theta) = -\left(\left\langle \frac{\partial H}{\partial p}(q,\omega(q,\theta)\theta),\theta\right\rangle\right)^{-1}\frac{\partial H}{\partial q}(q,\omega(q,\theta)) = -\left(\left\langle \frac{\partial K}{\partial p}(q,\omega(q,\theta)\theta),\theta\right\rangle\right)^{-1}\frac{\partial H}{\partial q}(q,\omega(q,\theta)). \tag{3.7}$$

By definition of $\omega(q,\theta)$, $K(q,\omega(q,\theta)\theta) = E - V(q)$ for all $(q,\theta) \in D\times\mathbb{S}^{n-1}$. By (3.3), we have

$$\frac{1}{2}\nu_1\omega^2(q,\theta) \le K(q,\omega(q,\theta)\theta) = E - V(q) \le \frac{1}{2}\nu_2\omega^2(q,\theta),$$

and hence,

$$0 < \frac{2(E-V(q))}{\nu_2} \le \omega^2(q,\theta) \le \frac{2(E-V(q))}{\nu_1}, \quad \forall q\in D, \quad \forall\theta\in\mathbb{S}^{n-1}. \tag{3.8}$$

By definition of $\omega(q,\theta)$, we have also

$$U(q,\omega(q,\theta)\theta) = 1, \quad \forall q\in D, \quad \forall\theta\in\mathbb{S}^{n-1}.$$

Since U is homogeneous of degree 2 in p, for all $q\in D$ and $p\neq 0$, we obtain

$$U(q,p) = \frac{\|p\|^2}{\omega^2(q,p/\|p\|)}U\left(q,\omega\left(q,\frac{p}{\|p\|}\right)\frac{p}{\|p\|}\right) = \frac{\|p\|^2}{\omega^2(q,p/\|p\|)}. \tag{3.9}$$

Using (3.8) and (3.9), we obtain (3.4). Since the Legendre transform inverts the order relation, (3.4) implies (3.5).

It remains to prove (3.6). Let us fix $\delta > 0$ and set

$$D_\delta = \{q\in D : V(q) \le E-\delta\},$$

so every point in D_δ is far away from the boundary ∂D. By the bounds on the function U given by (3.4) and recalling that U is homogeneous of degree 2 in p, there exists a constant c_δ such that

$$\|U'(q,p)\| \ge \left\|\frac{\partial U}{\partial p}(q,p)\right\| \ge c_\delta > 0, \quad \forall (q,p)\in\Sigma, \quad q\in D_\delta. \tag{3.10}$$

By the arbitrariness of δ, we can obtain (3.6) by proving it for all $(q,p)\in\Sigma$ with q sufficiently near the boundary. More precisely, we prove the existence of a constant c_1 such that

$$\|U'(q,p)\| \ge \left\|\frac{\partial U}{\partial q}(q,p)\right\| \ge \frac{c_1}{E-V(q)}, \tag{3.11}$$

for all $(q,p)\in\Sigma$ with q sufficiently near the boundary. For every $(q,p)\in\Sigma$, $U(q,p)=1$, so by (3.9), we obtain

$$\frac{\partial U}{\partial q}(q,p) = -\frac{2\|p\|^2}{\omega^3(q,p/\|p\|)}\frac{\partial\omega}{\partial q}\left(q,\frac{p}{\|p\|}\right) = -\frac{2}{\omega(q,p/\|p\|)}\frac{\partial\omega}{\partial q}\left(q,\frac{p}{\|p\|}\right), \quad \forall (q,p)\in\Sigma.$$

As a consequence, using also (3.7) and denoting $p/\|p\|$ by θ, we have

$$\frac{\partial U}{\partial q}(q,p) = \frac{2}{\omega(q,\theta)}\left(\left\langle\frac{\partial K}{\partial p}(q,\omega(q,\theta)\theta),\theta\right\rangle\right)^{-1}\frac{\partial H}{\partial q}(q,\omega(q,\theta)), \tag{3.12}$$

for all $(q,p)\in\Sigma$. By (3.2), we have

$$\left\|\left\langle\frac{\partial K}{\partial p}(q,\omega(q,\theta)\theta),\theta\right\rangle\right\|^{-1} \ge \frac{1}{\nu_2\omega(q,\theta)}.$$

Hence, by (3.12) and using again (3.8), we obtain

$$\left\|\frac{\partial U}{\partial q}(q,p)\right\| \ge \frac{2}{\nu_2\omega^2(q,\theta)}\left\|\frac{\partial H}{\partial q}(q,\omega(q,\theta)\theta)\right\| \ge \frac{\nu_1}{\nu_2(E-V(q))}\left\|\frac{\partial H}{\partial q}(q,\omega(q,\theta)\theta)\right\|, \quad \forall (q,p)\in\Sigma. \tag{3.13}$$

The existence of a strictly positive constant c_1 such that (3.11) holds for all $(q, p) \in \Sigma$ with q sufficiently near the boundary ∂D can be obtained by (3.13), recalling that $\overline{D}$ is compact and $(\partial V/\partial q)(q) \neq 0$ in ∂D. Finally, we obtain (3.6) by (3.10) and (3.11), recalling the arbitrariness of δ. □

Remark 3.3. By (3.5), we can extend G on the boundary ∂D by continuity. Denoting this extension again with G, we have

$$G(q, v) = 0, \quad \forall q \in \partial D, \ \forall v \in \mathbb{R}^n.$$

3.1 Behaviour of the solutions near the boundary

In this section, we present some preliminary results about the behavior of the solutions of Hamilton's equations near the boundary of the potential well D. These results are required to analyze the time reparametrization of the Finsler geodesics that correspond to the brake orbits and to study the strong concavity of the set $\overline{\Omega}$ described in Theorem 1.9.

Lemma 3.4. *There exists $\bar{\varepsilon} > 0$ such that, if $(q(t), p(t))$ is a solution of* (1.1) *with Hamiltonian H and energy E such that $V(q(t)) \geq E - \bar{\varepsilon}$ for $t \in [a, b]$, then*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} V(q(t)) \leq -\bar{\varepsilon}, \quad \forall t \in [a, b].$$

Proof. See [19, Lemma 5.2]. □

The following result provides an upper bound for the length of a time interval in which a solution of Hamilton's equations with energy E can be uniformly near the boundary.

Lemma 3.5. *Let $\bar{\varepsilon}$ given by* Lemma 3.4. *If $(q(t), p(t))$ is a solution of Hamilton's equations with total energy E and $V(q(t)) \geq E - \bar{\varepsilon}/2$ for $t \in [a, b]$, then $b - a \leq 2$.*

Proof. See [19, Corollary 5.3]. □

For every $Q \in \partial D$, we denote by $z(t, Q) = (q(t, Q), p(t, Q))$ the solution of Hamilton's equations for H with total energy E and such that $q(0) = Q$. Since $z(t, Q)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{z}(t, Q) = JH'(z(t, Q)), \\ z(0, Q) = (Q, 0), \end{cases}$$

it is well defined and of class C^1.

Remark 3.6. Since JH' is a function of class C^1, also $\dot{z}(t, Q)$ is of class C^1 with respect to the variables t and Q.

Lemma 3.7. *For every $Q_0 \in \partial D$, there exists a function $\rho : [0, +\infty[\times \partial D \to \mathbb{R}^n$ of class C^1 such that* $\mathrm{d}\rho(0, Q_0) = 0$ *and*

$$\dot{q}(t, Q) = -t \frac{\partial^2 H}{\partial p^2}(Q_0, 0) \nabla V(Q_0) + \rho(t, Q), \quad \forall t \in [0, +\infty[, \quad \forall Q \in \partial D. \tag{3.14}$$

Proof. We define $\rho_0 : [0, +\infty[\times \partial D \to \mathbb{R}^n$ as follows:

$$\rho_0(t, Q) = \ddot{q}(t, Q) - \ddot{q}(0, Q_0).$$

Recalling that $z(t, Q)$ is of class C^1 both respect to t and Q and taking the derivative with respect to t of $\dot{z}(t, Q) = JH'(z(t, Q))$, we obtain that $\ddot{z}(t, Q)$ is a continuous function, so $\rho_0(t, Q)$ is a continuous function. Since $p(0, Q) = 0$ for all $Q \in \partial D$, then $\dot{q}(0, Q) = 0$ and

$$\begin{aligned}\ddot{q}(0, Q) &= \frac{\mathrm{d}}{\mathrm{d}t}\dot{q}(t, Q)\Big|_{t=0}\\ &= \left(\frac{\partial^2 H}{\partial q \partial p}(q(t, Q), p(t, Q))\dot{q}(t, Q) + \frac{\partial^2 H}{\partial p^2}(q(t, Q), p(t, Q))\dot{p}(t, Q)\right)\Big|_{t=0}\\ &= -\frac{\partial^2 H}{\partial p^2}(Q, 0)\nabla V(Q), \quad \forall Q \in \partial D.\end{aligned}$$

By definition of ρ_0, we have

$$\ddot{q}(t, Q) = \ddot{q}(0, Q_0) + \rho_0(t, Q) = -\frac{\partial^2 H}{\partial p^2}(Q_0, 0)\nabla V(Q_0) + \rho_0(t, Q).$$

Integrating the previous equation, recalling that $\dot{q}(0, Q) = 0$ and setting $\rho(t, Q) = \int_0^t \rho_0(\tau, Q)\mathrm{d}\tau$, we obtain (3.14). Since $\rho_0(0, Q_0) = 0$, we have $\mathrm{d}\rho(0, Q_0) = 0$. □

4 The Jacobi-Finsler energy function

In this section, we introduce the function $\psi : \overline{D} \to \mathbb{R}$, which will be exploited to define the strongly concave set $\overline{\Omega}$ with the properties required by Theorem 1.9. Given a point $y \in \overline{D}$, the function $\psi(y)$ is the infimum of the energy of the curves connecting y with ∂D. We prove that, if y is sufficiently near the boundary, $\psi(y)$ is attained on exactly one curve that is a solution of (1.1) with energy E, up to time reparametrization.

Let us define the functional $\mathcal{J} : W^{1,2}([0, 1], \overline{D}) \to \mathbb{R}$ as follows:

$$\mathcal{J}(\gamma) = \int_0^1 G(\gamma(s), \dot{\gamma}(s))\mathrm{d}s.$$

If $\gamma([0, 1]) \subset D$, then $\mathcal{J}$ is differentiable at γ and its differential

$$\mathrm{d}\mathcal{J}(\gamma) : W^{1,2}([0, 1], \mathbb{R}^n) \to \mathbb{R}$$

is given by

$$\mathrm{d}\mathcal{J}(\gamma)[\xi] = \int_0^1 \left(\frac{\partial G}{\partial q}(\gamma(s), \dot{\gamma}(s))[\xi(s)] + \frac{\partial G}{\partial v}(\gamma(s), \dot{\gamma}(s))[\dot{\xi}(s)]\right)\mathrm{d}s.$$

For every $y \in D$, we define X_y as follows:

$$X_y := \{\gamma \in W^{1,2}([0, 1], \overline{D}) : \gamma(0) = y, \quad \gamma([0, 1[) \subset D \quad \text{and} \quad \gamma(1) \in \partial D\}.$$

Definition 4.1. We define the function $\psi : \overline{D} \to \mathbb{R}$ as follows:

$$\psi(y) := \inf_{\gamma \in X_y} \mathcal{J}(\gamma). \tag{4.1}$$

The function ψ will be the main focus of our analysis. Indeed, from now on, we will state and prove some results that will lead to define the set Ω described in Theorem 1.9 as $\psi^{-1}(]\delta, \infty[)$, for some δ sufficiently small.

Lemma 4.2. *There exists a constant $\bar{d} > 0$ such that*

$$\psi(y) \le \bar{d}, \quad \forall y \in \overline{D}.$$

Proof. The thesis directly follows from the upper bound given in (3.5) and the compactness of $\overline{D}$. □

Proposition 4.3. *For every $y \in D$, $\psi(y)$ is attained on at least one curve $\gamma_y \in X_y$. Moreover, γ_y satisfies*

$$\int_0^1 \left(\frac{\partial G}{\partial q}(\gamma_y(s), \dot{\gamma}_y(s))[\xi(s)] + \frac{\partial G}{\partial v}(\gamma_y(s), \dot{\gamma}_y(s))[\dot{\xi}(s)] \right) ds = 0, \quad \forall \xi \in W_0^{1,2}([0,1], \mathbb{R}^n), \tag{4.2}$$

and there exist a $T > 0$ and a diffeomorphism $\sigma : [0, T] \to [0, 1]$ such that, setting $\hat{\gamma}_y = \gamma_y \circ \sigma : [0, T] \to \overline{D}$, the pair $(q, p) : [0, T] \to \overline{D} \times \mathbb{R}^n$ given by

$$(q(t), p(t)) = \mathcal{L}^{-1}(\hat{\gamma}_y(t), \dot{\hat{\gamma}}_y(t))$$

is a solution of (1.1) *with energy E, $q(0) = y$ and $q(T) \in \partial D$.*

To prove Proposition 4.3, we obtain γ_y as the weak limit of a sequence of Finsler geodesics $(\gamma_k) \subset W^{1,2}([0, 1], D)$. We exploit the fact that (T_k) is uniformly bounded, where T_k are given by

$$T_k = t_k(1) = \sqrt{\mathcal{J}(\gamma_k)} \int_0^1 \phi\left(\gamma(\sigma), \frac{\dot{\gamma}(\sigma)}{\sqrt{\mathcal{J}(\gamma_k)}}\right) d\sigma.$$

Recalling the reparametrization given by Lemma 2.6, T_k is the final time of the reparametrization of γ_k, which is a solution of (1.1) with energy E. More formally, we require the following lemma.

Lemma 4.4. *Let $(\gamma_k) \subset W^{1,2}([0, 1], D)$ be a sequence of Finsler geodesics. If there exist two constants c_1, c_2 such that*

$$0 < c_1 \le \mathcal{J}(\gamma_k) \le c_2, \quad \forall k \in \mathbb{N}, \tag{4.3}$$

then there exist two constants c_3, c_4 such that

$$0 < c_3 \le T_k \le c_4, \quad \forall k \in \mathbb{N}. \tag{4.4}$$

Proof. Since $(E - V(q)) \to 0$ when $q \to \partial D$ and $\overline{D}$ is compact, there exists a strictly positive constant c_5 such that

$$\frac{1}{E - V(q)} \ge c_5, \quad \forall q \in D.$$

By the definition of ϕ given by (2.4), using (2.5) and (3.6), we have

$$\int_0^1 \phi\left(\gamma_k(s), \frac{\dot{\gamma}_k(s)}{\sqrt{\mathcal{J}(\gamma_k)}}\right) ds \ge \int_0^1 \frac{\nu_3}{h_2(E - V(\gamma_k(s)))} ds \ge \frac{c_5 \nu_3}{h_2} > 0.$$

By using also (4.3), we obtain

$$T_k = \sqrt{\mathcal{J}(\gamma_k)} \int_0^1 \phi\left(\gamma(\sigma), \frac{\dot{\gamma}(\sigma)}{\sqrt{\mathcal{J}(\gamma_k)}}\right) d\sigma \ge \sqrt{c_1} \frac{c_5 \nu_3}{h_2} =: c_3 > 0.$$

To prove the existence of a constant c_4 such that (4.4) holds, we work directly on the reparametrizations of γ_k. Following the construction given in [19, Lemma 5.1], let $\bar{\varepsilon} > 0$ be given by Lemma 3.4 and let us divide $\overline{D}$ into (Figure 3):

- The rim $\{q \in \overline{D} : V(q) \geq E - \bar{\varepsilon}/2\}$;
- The band $\{q \in \overline{D} : E - \bar{\varepsilon} \leq V(q) \leq E - \bar{\varepsilon}/2\}$;
- The core $\{q \in \overline{D} : V(q) \leq E - \bar{\varepsilon}\}$.

Let us set

$$\overline{\Sigma}_{\bar{\varepsilon}/2} = \{(q, p) \in \Sigma : V(q) \leq E - \bar{\varepsilon}/2\}.$$

Since $\overline{\Sigma}_{\bar{\varepsilon}/2}$ is compact, there exists a constant $\bar{\phi} > 0$ such that $\phi(q, p) \leq \bar{\phi}$ for all $(q, p) \in \overline{\Sigma}_{\bar{\varepsilon}/2}$. For every k, we set $I_k = \{s \in [0, 1] : V(\gamma_k(s)) < E - \bar{\varepsilon}/2\}$ and $C_k = [0, 1] \setminus I_k$. Hence,

$$\begin{aligned}\int_0^1 \phi\left(\gamma_k(s), \frac{\dot{\gamma}_k(s)}{\sqrt{\mathcal{J}(\gamma_k)}}\right)\mathrm{d}s &= \int_{I_k} \phi\left(\gamma_k(s), \frac{\dot{\gamma}_k(s)}{\sqrt{\mathcal{J}(\gamma_k)}}\right)\mathrm{d}s + \int_{C_k} \phi\left(\gamma_k(s), \frac{\dot{\gamma}_k(s)}{\sqrt{\mathcal{J}(\gamma_k)}}\right)\mathrm{d}s \\ &\leq \bar{\phi} + \int_{C_k} \phi\left(\gamma_k(s), \frac{\dot{\gamma}_k(s)}{\sqrt{\mathcal{J}(\gamma_k)}}\right)\mathrm{d}s.\end{aligned} \tag{4.5}$$

For every k, the set C_k is the union of closed and disjoint intervals in which the orbit γ_k is in the rim. The orbit can enter the rim many times, but, as a consequence of Lemma 2.6, each pair of passages into the rim must be separated by a dip into the core, and this requires the solution to cross the band twice. This bounds the number of closed disjoint intervals that constitute C_k, independently of k. Indeed, let us set

$$\bar{d} = \min\{\mathcal{J}(\gamma) : \gamma \in W^{1,2}([0, 1], D),\ V(\gamma(0)) = E - \bar{\varepsilon},\ V(\gamma(1)) = E - \bar{\varepsilon}/2\}.$$

Since (4.3) holds, we have that γ_k can cross the band at most N times, where N is a positive integer strictly greater than $c_2/(2\bar{d})$, independent of k. As a consequence, by Lemma 3.5, we have

$$\sqrt{\mathcal{J}(\gamma_k)} \int_{C_k} \phi\left(\gamma_k(s), \frac{\dot{\gamma}_k(s)}{\sqrt{\mathcal{J}(\gamma_k)}}\right)\mathrm{d}s \leq 2N,$$

and, by (4.3) and (4.5), we have

$$T_k \leq \sqrt{c_2} \quad \bar{\phi} + 2N =: c_4,$$

so (4.4) holds. □

The next lemma provides the sequence of geodesics that will be exploited in the proof of Proposition 4.3. For every $k \in \mathbb{N}$, let $D_k = V(]-\infty, E - 1/k[) \cap D$ and

$$X_y^k = \{\gamma \in W^{1,2}([0, 1], \overline{D}_k) : \gamma(0) = y, \gamma(1) \in \partial D_k\}.$$

Lemma 4.5. *For k sufficiently large, the functional $\mathcal{J}$ restricted to X_y^k has a minimum γ_k that is a Finsler geodesic and such that $\gamma_k([0, 1[) \subset D_k$.*

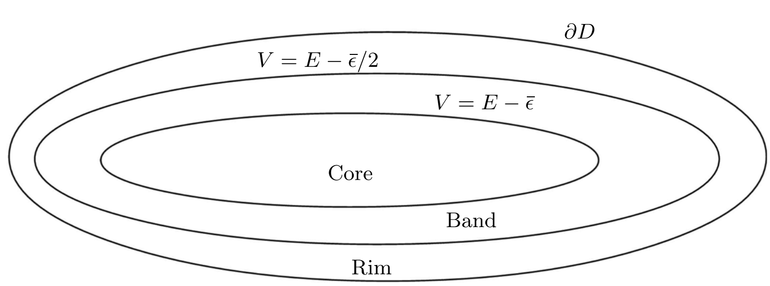

Figure 3: The division of the potential well D into the rim, the band and the core.

Proof. By the continuity of the function V, for k sufficiently large, we have $y \in D_k$, so $X_y^k \neq \emptyset$, and D_k is homeomorphic to D; hence, $\overline{D}_k$ is a closed and bounded set; thus it is compact. In the following, let us denote by $\mathcal{J}$ the restriction of it to X_y^k. Since $\mathcal{J}$ is bounded from below and $\overline{D}_k$ is compact, a minimizing sequence $(\gamma_m)_m \subset X_y^k$ is equibounded with respect to norm of $W^{1,2}([0,1], \overline{D}_k)$. By the Ascola-Arzelà theorem, unless to consider a subsequence, $(\gamma_m)_m$ converges uniformly to a curve γ_k and, since $\overline{D}_k$ is closed, $\gamma_k \in X_y^k$. Moreover, $(\gamma_m)_m$ converges weakly to γ_k in $W^{1,2}([0,1], \overline{D}_k)$. By Lemma 2.4, the function $G(q,v)$ is strongly convex in $\overline{D}_k$ with respect to the variable v. As a consequence, $\mathcal{J}$ is lower weakly semi-continuous; hence,

$$\mathcal{J}(\gamma_k) \leq \liminf_{m\to\infty} \mathcal{J}(\gamma_m) = \inf_{\gamma \in X_y^k} \mathcal{J}(\gamma),$$

so γ_k is a minimum for $\mathcal{J}$ in X_y^k. Let us prove that $\gamma_k([0,1[) \in D_k$. Assume, by contradiction, that there exists $\bar{s} \in \,]0,1[$ such that $\gamma_k(\bar{s}) \in \partial D_k$. Then, defining the curve $\tilde{\gamma}_k \in X_y^k$ as $\tilde{\gamma}_k(s) = \gamma_k(s/\bar{s})$, we obtain

$$\mathcal{J}(\gamma_k) = \int_0^1 G(\gamma_k, \dot{\gamma}_k)\mathrm{d}s \geq \int_0^{\bar{s}} G(\gamma_k, \dot{\gamma}_k)\mathrm{d}s = \frac{1}{\bar{s}}\int_0^1 G(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\mathrm{d}s > \int_0^1 G(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\mathrm{d}s = \mathcal{J}(\tilde{\gamma}_k),$$

which contradicts the minimality of γ_k. Since γ_k is a curve, which minimizes the energy functional $\mathcal{J}$ and $\gamma_k([0,1[) \subset D_k$, we obtain that γ_k is a geodesic. □

Proof of Proposition 4.3. For k sufficiently large, let $(\gamma_k)_k$ be a sequence of geodesics obtained with Lemma 4.5. Setting $\ell_k = \mathcal{J}(\gamma_k)$, by definition of ψ in (4.1), we have

$$\liminf_{k\to\infty} \ell_k \geq \psi(y).$$

We claim that

$$\liminf_{k\to\infty} \ell_k = \psi(y). \tag{4.6}$$

By absurd, if it was $\liminf_{k\to\infty}\ell_k > \psi(y)$, then we could find a curve $x \in X_y$ such that $\mathcal{J}(x) < \liminf_{k\to\infty}\ell_k$ and a suitable reparametrization of x would yield a curve $x_k \in X_y^k$ such that $\mathcal{J}(x_k) < \ell_k$, which contradicts the minimality of ℓ_k. Hence, (4.6) holds. Since γ_k minimizes $\mathcal{J}$ on X_y^k, it is a geodesic with constant speed; hence,

$$G(\gamma_k(s), \dot{\gamma}_k(s)) = \ell_k, \quad \forall s \in [0,1].$$

Using (3.5) and Lemma 4.2, there are two constants c_1, c_2 such that

$$0 < c_1 \leq \ell_k \leq c_2, \tag{4.7}$$

for all k sufficiently large. As a consequence, we can apply Lemma 4.4, so there exist two constants c_3, c_4 such that (4.4) holds for every k sufficiently large. Using also (2.5), (3.6), and (4.7), we have

$$c_4 \geq T_k \geq \sqrt{c_3}\int_0^1 \frac{\nu_3}{h_2(E - V(\gamma_k(s)))}\mathrm{d}s.$$

Then, the sequence

$$\int_0^1 \frac{1}{E - V(\gamma_k(s))}\mathrm{d}s$$

is bounded. By (3.5) and (4.7), we have

$$\int_0^1 \|\dot{\gamma}_k(s)\|^2 \mathrm{d}s \leq \int_0^1 \frac{2\nu_2}{E - V(\gamma_k(s))} G(\gamma_k(s), \dot{\gamma}_k(s))\mathrm{d}s = \ell_k \int_0^1 \frac{2\nu_2}{E - V(\gamma_k(s))}\mathrm{d}s \leq 2\nu_2 c_2 \int_0^1 \frac{1}{E - V(\gamma_k(s))}\mathrm{d}s,$$

so (γ_k) is bounded in $W^{1,2}([0,1],\overline{D})$. By the Ascoli Arzelà theorem, γ_k uniformly converges to a curve γ_y, up to a subsequence. We claim that γ_y is a minimizer for $\mathcal{J}$ in X_y. Let us show that $\gamma_y \in X_y$. Since γ_k converges uniformly to γ_y, $\gamma_y(0) = y$ and $\gamma(1) \in \partial D$. We show that $\gamma([0,1[) \subset D$ arguing by contradiction. Let $\bar{s} \in (0,1)$ be the first instant, where $\gamma_y(\bar{s}) \in \partial D$. By the minimality of γ_k, we have that $\gamma_y([\bar{s},1]) \subset \partial D$. Thus, we obtain

$$\lim_{k\to\infty}(1-\bar{s})\ell_k = \lim_{k\to\infty}\int_{\bar{s}}^{1} G(\gamma_k(s),\dot{\gamma}_k(s))\mathrm{d}s = \int_{\bar{s}}^{1} G(\gamma_y(s),\dot{\gamma}_y(s))\mathrm{d}s = 0,$$

in contradiction with $\ell_k \geq c_1 > 0$, given by (4.7). Hence, γ_y belongs to X_y and, since $\mathcal{J}(\gamma_y) \leq \liminf_{k\to\infty}\ell_k$, by (4.6), we obtain $\mathcal{J}(\gamma_y) = \psi(y)$. Being a minimizer, γ_y satisfies (4.2). By Lemma 2.6, the diffeomorphism $\sigma : [0,T] \to [0,1]$ such that $\gamma_y \circ \sigma$ is a solution of (1.1) with energy E has inverse

$$t(s) = \sqrt{\mathcal{J}(\gamma_y)}\int_0^s \phi\left(\gamma_y(\sigma), \frac{\dot{\gamma}_y(\sigma)}{\sqrt{\mathcal{J}(\gamma_y)}}\right)\mathrm{d}\sigma.$$

By (4.4),

$$T = t(1) = \sqrt{\mathcal{J}(\gamma_y)}\int_0^1 \phi\left(\gamma_y(\sigma), \frac{\dot{\gamma}_y(\sigma)}{\sqrt{\mathcal{J}(\gamma_y)}}\right)\mathrm{d}\sigma$$

is bounded and strictly greater than 0. □

The next main step is proving that if y is sufficiently near ∂D, then the minimizer of $\mathcal{J}$ in X_y is unique. To this aim, we require the following lemma, which provides a coordinate system of a neighborhood of ∂D through the solutions of (1.1) that start from the boundary.

Lemma 4.6. *There exists a constant $\bar{\delta} > 0$ such that the following property holds*:

$$\begin{gathered}\forall y \in D \text{ with } \psi(y) \leq \bar{\delta} \text{ there exists a unique solution } (q_y, p_y) \text{ of (1.1)} \\ \text{with energy } E \text{ and a unique } t_y > 0 \text{ such that } q_y(0) \in \partial D,\ q_y(t_y) = y \text{ and} \\ \psi(q_y(t)) \leq \bar{\delta}, \text{ for all } t \in [0, t_y].\end{gathered} \tag{4.8}$$

A representation of Property (4.8) is given in Figure 4.

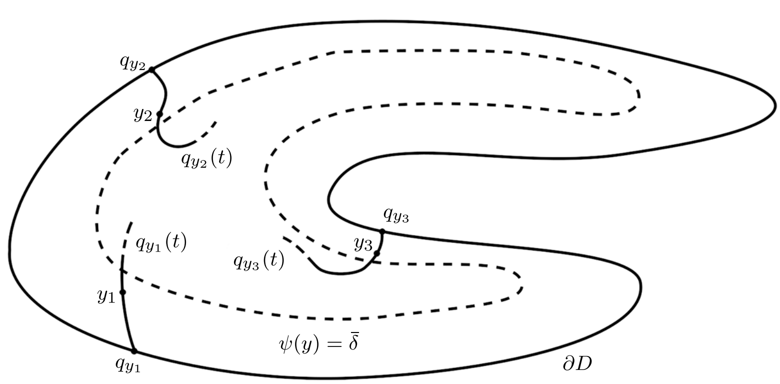

Figure 4: A representation of Property (4.8). Each point y_i, $i = 1, 2, 3$, is such that $\psi(y_i) \leq \bar{\delta}$. Hence, there exists a unique $q_{y_i} \in \partial D$ and a unique t_{y_i} such that $q(0) = q_{y_i}$, $q(t_{y_i}) = y_i$ and $\psi(q(t)) \leq \bar{\delta}$ for every $t \in [0, t_{y_i}]$, where q is the unique solution of Hamilton's equations with energy E starting from q_{y_i}.

Proof. By Lemma 3.7, for every $Q_0 \in \partial D$, there exists a function $\rho : \partial D \times \mathbb{R} \to \mathbb{R}^n$ such that

$$q(t, Q) = Q - \frac{t^2}{2}\frac{\partial^2 H}{\partial p^2}(Q_0, 0)\frac{\partial V}{\partial q}(Q_0) + \int_0^t \rho(\tau, Q)\mathrm{d}\tau, \quad \forall Q \in \partial D,$$

where the vector $\frac{\partial^2 H}{\partial p^2}(Q_0, 0)\frac{\partial V}{\partial q}(Q_0)$ is not tangent to ∂D for every $Q_0 \in \partial D$. Indeed, $\frac{\partial V}{\partial q}(Q_0)$ is orthogonal to ∂D by definition of D and by (3.1), we have

$$\left\langle \frac{\partial^2 H}{\partial p^2}(Q_0, 0)\frac{\partial V}{\partial q}(Q_0), \frac{\partial V}{\partial q}(Q_0) \right\rangle \geq \nu_1 \left\| \frac{\partial V}{\partial q}(Q_0) \right\|^2 > 0, \quad \forall Q_0 \in \partial D.$$

As a consequence, if $\{y_1, \dots, y_{n-1}\}$ is a coordinate system of ∂D in a neighborhood of Q_0, then $\{y_1, \dots, y_{n-1}, t\}$ is a local coordinate system on the manifold with boundary ∂D and $(t, Q) \mapsto q(t, Q)$ defines a local chart. By the compactness of ∂D, we can construct a neighborhood $N \subset \overline{D}$ of ∂D as union of a finite number of such local charts. By the upper bound on G given by (3.5), $\psi(y) \to 0$ as $y \to \partial D$, and there exists a $\bar{\delta} > 0$ such that $\psi^{-1}([0, \bar{\delta}]) \subset N$. As a consequence, if $y \in \overline{D}$ satisfies $\psi(y) \leq \bar{\delta}$, then $y \in N$, and it is uniquely represented by a coordinate of the constructed local chart, so there exists a unique solution (q_y, p_y) of (1.1) with energy E and a unique $t_y > 0$ such that $q_y(0) \in \partial D$, $q_y(t_y) = y$ and $q_y(t) \in N$ for each $t \in [0, t_y]$. It remains to prove that $\psi(q_y(t)) \leq \bar{\delta}$ for every $t \in [0, t_y]$. By Proposition 4.3, there exists γ_y such that $\mathcal{J}(\gamma_y) = \psi(y)$ and a reparametrization of it is a solution of (1.1) with energy E and with an endpoint in ∂D. We recall that, since H is even with respect to p, a backward parametrization of a solution is still a solution. Thus, there exists a solution $q_{\gamma_y} : [0, T] \to \overline{D}$ of (1.1) and energy E such that $q_{\gamma_y}(0) \in \partial D$, $q_{\gamma_y}(T) = y$, and for each $t \in [0, T]$, there exists an $s \in [0, 1]$ such that $q_{\gamma_y}(t) = \gamma_y(s)$. Using the reparametrizations of γ_y, it can be proved that for each $s \in [0, 1]$, we have $\psi(\gamma_y(s)) \leq \bar{\delta}$, so $\psi(q_{\gamma_y}(t)) \leq \bar{\delta}$ for every $t \in [0, T]$. By definition of $\bar{\delta}$, this implies that $q_{\gamma_y}(t) \in N$ for every $t \in [0, T]$. By the uniqueness of q_y, we obtain that $T = t_y$ and $q_{\gamma_y}(t) = q_y(t)$ for every $t \in [0, t_y]$, so we are done. ☐

Notation: If $y \in D$ is such that $\psi(y) \leq \bar{\delta}$, we denote by (t_y, Q_y) the unique element in $\mathbb{R}^+ \times \partial D$ such that $q(t_y, Q_y) = y$ and $\psi(q(t, Q_y)) \leq \bar{\delta}$, for all $t \in [0, t_y]$.

Remark 4.7. Both t_y and Q_y are functions of class C^1 with respect to y, since they are implicitly defined by the coordinate system given by the proof of Lemma 4.6.

Proposition 4.8. *For every $y \in D$ such that $\psi(y) \leq \bar{\delta}$, the minimizer of $\mathcal{J}$ in the space X_y is unique.*

Proof. By contradiction argument, let us assume the existence of two different curves, $\gamma_1, \gamma_2 \in X_y$, such that $\psi(y) = \mathcal{J}(\gamma_1) = \mathcal{J}(\gamma_2)$. Since γ_1 and γ_2 are two minimizers, by Proposition 4.3, they are reparametrizations of two solutions of (1.1) with energy E and final points on ∂D. Moreover, $\psi(\gamma_1(s)) \leq \bar{\delta}$ and $\psi(\gamma_2(s)) \leq \bar{\delta}$ for every $s \in [0, 1]$. Hence, if $\gamma_1(1) \neq \gamma_2(1)$, then setting $Q_1 = \gamma_1(1)$ and $Q_2 = \gamma_2(1)$, we have

$$y = q(t_1, Q_1) \quad \text{and} \quad y = q(t_2, Q_2),$$

for some $t_1, t_2 > 0$,

$$\psi(q(t, Q_1)) \leq \bar{\delta} \quad \forall t \in [0, t_1] \quad \text{and} \quad \psi(q(t, Q_2)) \leq \bar{\delta} \quad \forall t \in [0, t_2].$$

As a consequence, y is given by two different coordinates of the local chart constructed in Lemma 4.6, which is a contradiction. If $\gamma_1(1) = \gamma_2(1)$, by the uniqueness of the solution $q(t, Q_y)$ of (1.1) with energy E, we infer that $\gamma_1(s) = \gamma_2(s)$ for all $s \in [0, 1]$, which is a contradiction. ☐

Remark 4.9. By Propositions 4.3 and 4.8, for every $y \in D$ such that $\psi(y) \leq \bar{\delta}$, the minimizer γ_y and the curve $q(t, Q_y)$ are linked by a reparametrizations, which invert the orientation.

5 Differentiability and concavity

In this section, we prove that ψ is of class C^2 near the boundary and that, if $\hat{\delta}$ is sufficiently small, the set $\psi^{-1}([\hat{\delta}, +\infty[)$ is strongly concave with respect to the Finsler metric F.

Let $\bar{\delta}$ satisfies property (4.8), and set

$$D_{\bar{\delta}} = \{y \in D : \psi(y) \le \bar{\delta}\}.$$

Proposition 5.1. *For every $y \in D_{\bar{\delta}}$, ψ is differentiable at y and*

$$d\psi(y)[\xi] = -\frac{\partial G}{\partial v}(y, \dot{\gamma}_y(0))[\xi], \quad \forall \xi \in \mathbb{R}^n. \tag{5.1}$$

Proof. Let us fix $\xi \in \mathbb{R}^n$, $\|\xi\| \le 1$ and set

$$v_\xi(s) = \max\{0, 1 - 2s\}\xi,$$

and hence, $v_\xi(s) = 0$ for all $s \in [1/2, 1]$. Let us define $\tilde{\mathcal{J}} : W^{1,2}([0,1], D) \to \mathbb{R}$ as follows:

$$\tilde{\mathcal{J}}(\gamma) = \int_0^{\frac{1}{2}} G(\gamma(s), \dot{\gamma}(s))ds.$$

Since the curve $\gamma_y|_{[0,1/2]}$ is uniformly far from ∂D, so are the curves $(\gamma_y + \varepsilon v_\xi)|_{[0,1/2]}$ for ε sufficiently small. Moreover, by Proposition 4.3, there exists a constant $c_y > 0$ such that $G(\gamma_y(s), \dot{\gamma}_y(s)) = c_y$, for all $s \in [0, 1[$. As a consequence, for ε sufficiently small, we can assume that

$$\dot{\gamma}_y(s) + \sigma\varepsilon\dot{v}_\xi(s) \neq 0, \quad \forall s \in \left[0, \frac{1}{2}\right], \quad \forall \sigma \in [0,1], \tag{5.2}$$

and thus, we shall work in a region where G is of class C^2. By definition of ψ, we have

$$\psi(y + \varepsilon\xi) \le \mathcal{J}(\gamma_y + \varepsilon v_\xi).$$

Since $\psi(y) = \mathcal{J}(\gamma_y)$, we obtain

$$\psi(y + \varepsilon\xi) - \psi(y) \le \mathcal{J}(\gamma_y + \varepsilon v_\xi) - \mathcal{J}(\gamma_y) = \tilde{\mathcal{J}}(\gamma_y + \varepsilon v_\xi) - \tilde{\mathcal{J}}(\gamma_y),$$

and hence,

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}(\psi(y + \varepsilon\xi) - \psi(y)) \le \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon}(\tilde{\mathcal{J}}(\gamma_y + \varepsilon v_\xi) - \tilde{\mathcal{J}}(\gamma_y)).$$

Then, using the dominated convergence theorem, an integration by parts and recalling that γ_y satisfies (4.2), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\tilde{\mathcal{J}}(\gamma_y + \varepsilon v_\xi) - \tilde{\mathcal{J}}(\gamma_y)) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\frac{1}{2}}(G(\gamma_y + \varepsilon v_\xi, \dot{\gamma}_y + \varepsilon \dot{v}_\xi) - G(\gamma_y, \dot{\gamma}_y))ds \\
&= \int_0^{\frac{1}{2}}\left(\frac{\partial G}{\partial q}(\gamma_y, \dot{\gamma}_y)[v_\xi] + \frac{\partial G}{\partial v}(\gamma_y, \dot{\gamma}_y)[\dot{v}_\xi]\right)ds \\
&= \left[\frac{\partial G}{\partial v}(\gamma_y, \dot{\gamma}_y)[v_\xi]\right]_0^{1/2} \\
&= -\frac{\partial G}{\partial v}(y, \dot{\gamma}_y(0))[\xi],
\end{aligned}$$

and hence,

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}(\psi(y+\varepsilon\xi)-\psi(y)) \le -\frac{\partial G}{\partial v}(y,\dot\gamma_y(0))[\xi].$$

It remains to prove that

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}(\psi(y+\varepsilon\xi)-\psi(y)) \ge -\frac{\partial G}{\partial v}(y,\dot\gamma_y(0))[\xi]. \tag{5.3}$$

Since $\psi(y+\varepsilon\xi)=\mathcal{J}(\gamma_{y+\varepsilon\xi})$ and $\psi(y)\le \mathcal{J}(\gamma_{y+\varepsilon\xi}-\varepsilon v_\xi)$, we have

$$\psi(y+\varepsilon\xi)-\psi(y)\ge \mathcal{J}(\gamma_{y+\varepsilon\xi})-\mathcal{J}(\gamma_{y+\varepsilon\xi}-\varepsilon v_\xi)=\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi})-\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi}-\varepsilon v_\xi). \tag{5.4}$$

By (5.2), $\tilde{\mathcal{J}}$ is of class C^2 in a neighborhood of γ_y. Hence, there exists some $\sigma_\varepsilon\in\,]0,1[$ such that

$$\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi})-\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi}-\varepsilon v_\xi)=\varepsilon \mathrm{d}\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi})[v_\xi]-\frac{\varepsilon^2}{2}\mathrm{d}^2\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi}-\sigma_\varepsilon\varepsilon v_\xi)[v_\xi,v_\xi]. \tag{5.5}$$

Now, we are going to prove that

$$\lim_{\varepsilon\to 0}\varepsilon \mathrm{d}^2\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi}-\sigma_\varepsilon\varepsilon v_\xi)[v_\xi,v_\xi]=0. \tag{5.6}$$

Since γ_y is uniformly far from ∂D on the interval $[0,1/2]$, the same holds for $\gamma_{y+\varepsilon\xi}$ whenever ε is sufficiently small. As a consequence, there exists a constant $c_1>0$ such that

$$\frac{1}{E-V(\gamma_{y+\varepsilon\xi}(s))}\le c_1,\quad \forall s\in\left[0,\frac{1}{2}\right].$$

Since $\gamma_{y+\varepsilon\xi}$ is a minimal geodesic, we also have

$$G(\gamma_{y+\varepsilon\xi}(s),\dot\gamma_{y+\varepsilon\xi}(s))=\psi(y+\varepsilon\xi),\quad \forall s\in\left[0,\frac{1}{2}\right].$$

Moreover, using (3.5), there exists a constant $c_2>0$ such that

$$\int_0^{\frac{1}{2}}\|\dot\gamma_{y+\varepsilon\xi}(s)\|^2\mathrm{d}s\le \int_0^{\frac{1}{2}}\frac{2\nu_2}{E-V(\gamma_{y+\varepsilon\xi}(s))}G(\gamma_{y+\varepsilon\xi}(s),\dot\gamma_{y+\varepsilon\xi}(s))\mathrm{d}s\le c_1\nu_2\psi(y+\varepsilon\xi)\le c_2. \tag{5.7}$$

Hence, $\gamma_{y+\varepsilon\xi}$ is uniformly bounded in $W^{1,2}\left(\left[0,\frac{1}{2}\right],D\right)$. Since $v_\xi(s)=0$ on $\left[0,\frac{1}{2}\right]$, we have that $\mathrm{d}^2\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi}-\sigma_\varepsilon\varepsilon v_\xi)[v_\xi,v_\xi]$ is uniformly bounded with respect to ε sufficiently small; hence, (5.6) holds. By (5.5) and (5.6), we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi})-\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi}-\varepsilon v_\xi))=\lim_{\varepsilon\to 0}\mathrm{d}\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi})[v_\xi]. \tag{5.8}$$

Since $\gamma_{y+\varepsilon\xi}$ satisfies (4.2), integration by parts leads to

$$\mathrm{d}\tilde{\mathcal{J}}(\gamma_{y+\varepsilon\xi})[v_\xi]=-\frac{\partial G}{\partial v}(y+\varepsilon\xi,\dot\gamma_{y+\varepsilon\xi}(0))[\xi]. \tag{5.9}$$

To obtain (5.3) and conclude the proof, it suffices to show that

$$\lim_{\varepsilon\to 0}\dot\gamma_{y+\varepsilon\xi}(0)=\dot\gamma_y(0). \tag{5.10}$$

To this aim, we exploit the uniqueness of γ_y ensured by Proposition 4.8. Arguing by contradiction, let (ε_n) be a sequence such that $\varepsilon_n\to 0$ and

$$\lim_{n\to\infty}\dot\gamma_{y+\varepsilon_n\xi}(0)\ \ne\ \dot\gamma_y(0).$$

By (5.7), $\gamma_{y+\varepsilon\xi}$ are uniformly bounded in $W^{1,2}([0,1],\overline{D})$; hence, there exists $v \in \mathbb{R}^n$ such that $\lim_{n\to\infty}\dot{\gamma}_{y+\varepsilon_n\xi}(0) = v \neq \dot{\gamma}_y(0)$. Since $(\gamma_{y+\varepsilon_n\xi})$ is a sequence of geodesics, it converges with respect to the C^1 norm to a minimum. Since the minimum is unique by Proposition 4.8, then $\gamma_{y+\varepsilon_n\xi} \to \gamma_y$ in C^1, so $\dot{\gamma}_{y+\varepsilon_n\xi}(0) \to \dot{\gamma}_y(0)$, which is a contradiction.

Therefore, (5.10) holds, and using also (5.9), we have

$$\lim_{\varepsilon\to 0} \mathrm{d}\mathcal{J}(\gamma_{y+\varepsilon\xi})[v_\xi] = -\frac{\partial G}{\partial v}(y, \dot{\gamma}_y(0))[\xi]. \tag{5.11}$$

Finally, combining (5.4), (5.8), and (5.11), we obtain (5.3), and we are done. □

Lemma 5.2 will play a central role because it links the initial velocity of the curve γ_y with $\dot{q}(t_y, Q_y)$ through a function of class C^1.

Lemma 5.2. *There exists a function $\varphi : D_{\bar\delta} \times \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that*

$$\sqrt{\psi(y)}\,\varphi(y, \dot{q}(t_y, Q_y))\dot{q}(t_y, Q_y) = -\dot{\gamma}_y(0). \tag{5.12}$$

Proof. Let $\zeta_y : [0, \sqrt{\psi(y)}\,] \to \overline{D}$ the backward arc-length reparametrization of γ_y, namely,

$$\zeta_y(s) = \gamma_y\left(1 - \frac{s}{\sqrt{\psi(y)}}\right).$$

As a consequence,

$$\dot{\zeta}_y(\sqrt{\psi(y)}\,) = -\frac{1}{\sqrt{\psi(y)}}\dot{\gamma}_y(0). \tag{5.13}$$

By Lemma 2.4, the curve $x : [0, \sqrt{\psi(y)}\,] \to \mathbb{R}^{2n}$ given by

$$x(s) = \mathcal{L}^{-1}(\zeta_y(s), \dot{\zeta}_y(s)),$$

is a solution of Hamilton's equations with respect to U and $U(x(s)) \equiv 1$. Since $y \in D_{\bar\delta}$, By Lemma 4.6 and Remark 4.9, $x(s)$ is actually a reparametrization of $z(t, Q_y) = (q(t, Q_y), p(t, Q_y))$, with $x(\sqrt{\psi(y)}\,) = z(t_y, Q_y)$. Hence, using (2.1) and recalling that $\mathcal{L}$ is the identity map with respect to the first variable, we have

$$\dot{\zeta}_y(\sqrt{\psi(y)}\,) = \frac{\|U'(z(t_y, Q_y))\|}{\|H'(z(t_y, Q_y))\|}\dot{q}(t_y, Q_y) = \frac{\|U'(y, p(t_y, Q_y))\|}{\|H'(y, p(t_y, Q_y))\|}\dot{q}(t_y, Q_y). \tag{5.14}$$

Since the map

$$p(t_y, Q_y) \mapsto \dot{q}(t_y, Q_y) = \frac{\partial H}{\partial p}(y, p(t_y, Q_y))$$

is invertible, there exists $\varphi : D_{\bar\delta} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\varphi(y, \dot{q}(t_y, Q_y)) = \frac{\|U'(y, p(t_y, Q_y))\|}{\|H'(y, p(t_y, Q_y))\|}. \tag{5.15}$$

Combining (5.13)–(5.15), we obtain (5.12). Recalling that both t_y and Q_y are of class C^1 by Remark 4.7, and that $\dot{q}(t_y, Q_y) \neq 0$ for every $y \in D_{\bar\delta}$, the function φ is of class C^1 as a composition of the derivatives of H and U, □

Lemma 5.3. *The function ψ is of class C^2 in $D_{\bar\delta}$.*

Proof. By (5.12), we deduce that $\dot{\gamma}_y(0)$ is continuous as a function of $y \in D_{\bar{\delta}}$. Hence, by (5.1), ψ is of class C^1 in $D_{\bar{\delta}}$. Using again (5.12) and the C^1-regularity of φ and $\dot{q}(t_y, Q_y)$, we deduce that $\dot{\gamma}_y(0)$ is of class C^1. By using (5.1), we obtain the thesis. □

Recalling the notion of strong concavity given in Definition 1.8 and the definition of $H_\psi(y, v)[v, v]$ in (1.4), the next proposition shows that the set $\psi([\hat{\delta}, \infty[)$ is strongly concave, provided $\hat{\delta} > 0$ sufficiently small.

Proposition 5.4. *There exists $\hat{\delta} \in]0, \bar{\delta}]$ such that for every $y \in D$ with $0 < \psi(y) \le \hat{\delta}$, we have*

$$H_\psi(y, \xi)[\xi, \xi] > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} : \mathrm{d}\psi(y)[\xi] = 0.$$

Proof. For every $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\mathrm{d}\psi(y)[\xi] = 0$, we denote by η the unique Finsler geodesic such that $\eta(0) = y$ and $\dot{\eta}(0) = \xi$. We have to prove that, for y sufficiently near the boundary ∂D,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi \circ \eta)(0) > 0.$$

Let ζ be the reparametrization of η, which is a solution of (1.1) with energy E. By Remark 2.3, there exists a function λ of class C^2 such that $\zeta(t) = \eta(\lambda(t))$, $\lambda(0) = 0$, and $\dot{\lambda}(t) > 0$. Hence,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi \circ \zeta)(0) = \dot{\lambda}(0)\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi \circ \eta)(0) + \ddot{\lambda}(0)\mathrm{d}\psi(y)[\xi] = \dot{\lambda}(0)\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi \circ \eta)(0).$$

As a consequence, it suffices to prove that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi \circ \zeta)(0) > 0.$$

By using (5.1) and (5.12), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\psi(\zeta(s)) &= \mathrm{d}\psi(\zeta(s))[\dot{\zeta}(s)] \\
&= -\frac{\partial G}{\partial v}(\zeta(s), \dot{\gamma}_{\zeta(s)}(0))[\dot{\zeta}(s)] \\
&= \sqrt{\psi(\zeta(s))}\,\varphi(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))\frac{\partial G}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)].
\end{aligned}$$

Let us set $w = \dot{\zeta}(0)$. Since $\dot{q}(t_y, Q_y)$ is parallel to $\dot{\gamma}_y(0)$, we have

$$\frac{\partial G}{\partial v}(y, \dot{q}(t_y, Q_y))[w] = 0,$$

and thus, we obtain

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\psi \circ \zeta)(0) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\sqrt{\psi(\zeta(s))}\,\varphi(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))\right)\bigg|_{s=0} \frac{\partial G}{\partial v}(y, \dot{q}(t_y, Q_y))[w] \\
&\quad + \sqrt{\psi(y)}\,\varphi(y, \dot{q}(t_y, Q_y))\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial G}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0} \\
&= \sqrt{\psi(y)}\,\varphi(y, \dot{q}(t_y, Q_y))\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial G}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0}.
\end{aligned}$$

Since $\psi(y)$ and $\varphi(y, \dot{q}(t_y, Q_y))$ are two strictly positive functions with respect to $y \in D$, it remains to prove that, for y sufficiently near the boundary ∂D, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial G}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0} > 0. \tag{5.16}$$

Let us define the function $\Gamma : D \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\Gamma(q, v) = \frac{G(q, v)}{E - V(q)}.$$

Since $\mathrm{d}\psi(y)[w] = 0$, we have

$$\frac{\partial \Gamma}{\partial v}(y, \dot{q}(t_y, Q_y))[w] = \frac{1}{E - V(y)} \frac{\partial G}{\partial v}(y, \dot{q}(t_y, Q_y))[w] = 0.$$

As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial G}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0} = (E - V(y))\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \Gamma}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0}.$$

Hence, to obtain (5.16), it suffices to prove that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \Gamma}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0} > 0, \tag{5.17}$$

for y sufficiently near the boundary ∂D. Setting

$$I_1(y) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \Gamma}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))\right)\bigg|_{s=0}[w],$$
$$I_2(y) = \frac{\partial \Gamma}{\partial v}(y, \dot{q}(t_y, Q_y))[\ddot{\zeta}(0)],$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \Gamma}{\partial v}(\zeta(s), \dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))[\dot{\zeta}(s)]\right)\bigg|_{s=0} = I_1(y) + I_2(y).$$

Let us study $I_1(y)$ and $I_2(y)$ separately. We have

$$I_1(y) = \frac{\partial^2 \Gamma}{\partial q \partial v}(y, \dot{q}(t_y, Q_y))[w, w] + \frac{\partial^2 \Gamma}{\partial v^2}(y, \dot{q}(t_y, Q_y))\left[\frac{\mathrm{d}}{\mathrm{d}s}(\dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))\bigg|_{s=0}, w\right].$$

Since $q(t, Q_y)$ is a solution of (1.1) with energy E, by (3.2), we have

$$\nu_1\|p(t_y, Q_y)\| \le \left\|\frac{\partial K}{\partial p}(y, p(t_y, Q_y))\right\| = \|\dot{q}(t_y, Q_y)\| \le \nu_2\|p(t_y, Q_y)\|,$$

and by (3.3), we obtain that there exist two constants $c_1, c_2 > 0$ such that

$$c_1(E - V(y)) \le \|\dot{q}(t_y, Q_y)\|^2 \le c_2(E - V(y)). \tag{5.18}$$

Similarly, since $w = \dot{\zeta}(0)$, we have

$$c_1(E - V(y)) \le \|w\|^2 \le c_2(E - V(y)). \tag{5.19}$$

Since Γ is homogeneous of degree two with respect to v and recalling the bounds for G given by (3.5), there exists a constant c_3 such that

$$\left\|\frac{\partial^2 \Gamma}{\partial q \partial v}(q, v)[\omega, \omega]\right\| \le c_3\|v\|\|\omega\|^2, \quad \forall (q, v) \in D \times \mathbb{R}^n.$$

As a consequence, using (5.18) and (5.19), we have

$$\left\|\frac{\partial^2 \Gamma}{\partial q \partial v}(y, \dot{q}(t_y, Q_y))[w, w]\right\| \le c_3(E - V(y))^{\frac{3}{2}}. \tag{5.20}$$

Let us set

$$v_y := \frac{\partial^2 H}{\partial p^2}(y, 0)\frac{\partial V}{\partial q}(y).$$

We remark that, since $(\partial V/\partial q)(q) \neq 0$ for every $q \in \partial D$, and by the strictly convexity of H given by (3.1), $\|v_y\|$ is uniformly greater than zero if y is sufficiently near the boundary. Since $\lim_{s\to 0} Q_{\zeta(s)} = Q_y$, using (3.14), we have

$$\dot{q}(t, Q_{\zeta(s)}) = -tv_y + \rho(t, Q_{\zeta(s)}), \quad \forall t \in [0, +\infty[,$$

with $\mathrm{d}\rho(0, Q_y) = 0$. By Remark 4.7, t_y and Q_y are functions of class C^1 with respect to y. Moreover, by Remark 3.6, $\dot{q}(t, Q)$ is of class C^1. Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))|_{s=0} = -\mathrm{d}t_y[w]v_y + \frac{\partial\rho}{\partial t}(t_y, Q_y)\mathrm{d}t_y[w] + \frac{\partial\rho}{\partial Q}(t_y, Q_y)\frac{\partial Q}{\partial y}[w]. \tag{5.21}$$

Since $q(t_y, Q_y) = y$, we get

$$\mathrm{d}t_y[v]\dot{q}(t_y, Q_y) + \frac{\partial q}{\partial Q}(t_y, Q_y)\frac{\partial Q_y}{\partial y}[v] = v, \quad \forall v \in \mathbb{R}^n. \tag{5.22}$$

We recall that (t, Q) is a coordinate system in a neighborhood of ∂D, where $y = q(t_y, Q_y)$. Hence, if y tends to ∂D, then $t_y \to 0$ and $(\partial q/\partial Q)(t_y, Q_y)$ goes to the identity map. Similarly, when $y \to \partial D$, $(\partial Q_y/\partial y)[v]$ tends to v uniformly as $\|v\| \leq 1$. Then, by (5.22), $\mathrm{d}t_y[v]\dot{q}(t_y, Q_y) \to 0$ uniformly in v, as $y \to \partial D$. Therefore, since by (5.18) and (5.19) w and $\dot{q}(t_y, Q_y)$, we have

$$0 < \sqrt{\frac{c_1}{c_2}} \leq \frac{\|\dot{q}(t_y, Q_y)\|}{\|w\|}\sqrt{\frac{c_2}{c_1}},$$

and we obtain

$$\lim_{y\to\partial D} \mathrm{d}t_y[w] = 0. \tag{5.23}$$

Since $\mathrm{d}\rho(0, Q_y) = 0$, by (5.21) and (5.23), we infer

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))\Big|_{s=0} = o(1), \tag{5.24}$$

as $y \to \partial D$. Since Γ is homogeneous of degree 2 with respect to v and using (3.5), there exist a constant $c_3 > 0$ such that

$$\left\|\frac{\partial^2\Gamma}{\partial v^2}(q, \xi)[v_1, v_2]\right\| \leq c_4\|v_1\|\|v_2\|, \quad \forall q \in D, \quad \forall \xi \in \mathbb{R}^n\backslash\{0\}, \quad \forall v_1, v_2 \in \mathbb{R}^n.$$

As a consequence, by (5.19) and (5.24), we obtain

$$\lim_{y\to\partial D} \frac{\partial^2\Gamma}{\partial v^2}(y, \dot{q}(t_y, Q_y))\left[\frac{\mathrm{d}}{\mathrm{d}s}(\dot{q}(t_{\zeta(s)}, Q_{\zeta(s)}))|_{s=0}, w\right] = o(\sqrt{E - V(y)}). \tag{5.25}$$

By (5.20) and (5.25), we obtain

$$I_1(y) = o(\sqrt{E - V(y)}), \tag{5.26}$$

as $y \to \partial D$.

Let us analyze $I_2(y)$. Since (ζ, p) is a solution of Hamilton's equations, with p implicitly defined by

$$\dot{\zeta}(s) = \frac{\partial H}{\partial p}(\zeta(s), p(s)),$$

we have

$$\ddot{\zeta}(0) = \frac{\partial^2 H}{\partial q \partial p}(y, p(0))w - \frac{\partial^2 H}{\partial p^2}(y, p(0))\left(\frac{\partial K}{\partial q}(y, p(0)) + \frac{\partial V}{\partial q}(y)\right).$$

If $y \to \partial D$, then $\|p(0)\| \to 0$. Therefore,

$$\frac{\partial^2 H}{\partial q \partial p}(y, p(0))w \to 0, \quad \frac{\partial K}{\partial q}(y, p(0)) \to 0,$$

and we obtain $\lim_{y\to\partial D}\ddot{\zeta}(0) = -v_y$. Hence, using also (3.5) and (5.21), we obtain

$$\lim_{y\to\partial D} \frac{\partial \Gamma}{\partial v}\left(y, \frac{\dot{q}(t_y, Q_y)}{\|\dot{q}(t_y, Q_y)\|}\right)[\ddot{\zeta}(0)] = \frac{\partial \Gamma}{\partial v}(y, -v_y)[-v_y] = \Gamma(y, v_y) \geq \frac{1}{2v_2}\|v_y\|^2. \tag{5.27}$$

As a consequence, by (5.18) and (5.27), we obtain

$$\lim_{y\to\partial D} \frac{I_2(y)}{\sqrt{E - V(y)}} > 0. \tag{5.28}$$

Finally, by (5.26) and (5.28), we obtain (5.17), and we are done. □

6 Proof of the main theorem

Finally, we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. Let $\hat{\delta}$ be as in Proposition 5.4 and set

$$\Omega = \psi^{-1}(]\hat{\delta}, +\infty[).$$

By continuity, Ω is an open subset of D and $\partial\Omega = \psi^{-1}(\hat{\delta})$. By Lemma 5.3, ψ is of class C^2 in $D_{\bar{\delta}}$, and since $d\psi \neq 0$ on $\partial\Omega$, we have that $\partial\Omega$ is of class C^2. Since $\bar{\delta}$ satisfies property (4.8) and $\hat{\delta} \leq \bar{\delta}$, $\bar{\Omega}$ is homeomorphic to $\bar{D}$. Since $\partial\Omega$ is a level hyper-surface of ψ, for every $y \in \partial\Omega$, $v \in T_y\partial\Omega$ if and only if $d\psi(y)[v] = 0$. Recalling Definition 1.8, Proposition 5.4 implies that $\bar{\Omega}$ is strongly concave with respect to the Finsler metric F.

Let $\gamma : [0, 1] \to \bar{\Omega}$ be an orthogonal Finsler geodesic chord. We will prove the desired properties of the extension $\hat{\gamma} : [\alpha, \beta] \to \bar{D}$ only in the interval $[1, \beta]$. The case $[\alpha, 0]$ is analog. Set $y = \gamma(1)$. Since γ is an orthogonal Finsler geodesic chord, it satisfies (1.3); hence,

$$\frac{\partial G}{\partial v}(y, \dot{\gamma}(1))[v] = 0, \quad \forall v \in T_{\gamma(1)}\partial\Omega.$$

The minimizer curve γ_y satisfies

$$\frac{\partial G}{\partial v}(y, \dot{\gamma}_y(0))[v] = 0, \quad \forall v \in T_{\gamma(1)}\partial\Omega,$$

and thus, $\dot{\gamma}(1)$ and $\dot{\gamma}_y(0)$ are parallel. As a consequence, the curve $\bar{\gamma} : [0, 2] \to \bar{D}$ defined as

$$\bar{\gamma}(s) = \begin{cases} \gamma(s), & \text{if } s \in [0, 1], \\ \gamma_y(s - 1), & \text{if } s \in]1, 2], \end{cases}$$

is of class C^1 and it is a geodesic with respect to F, up to a suitable time reparametrization. With the analog extension in $[\alpha, 0]$, we obtain a geodesic $\hat{\gamma} : [\alpha, \beta] \to \bar{D}$ such that $\hat{\gamma}(\alpha), \hat{\gamma}(\beta) \in \partial D$ and $\hat{\gamma}(]\alpha, \beta[) \subset D$. By Lemmas 2.2 and 2.4, we have that

$$(q(t), p(t)) = \mathcal{L}^{-1}(\hat{\gamma}(t), \dot{\hat{\gamma}}(t)) \quad \forall t \in]\alpha, \beta[$$

is a solution of (1.1) with energy E for H, up to time reparametrization. By using also Lemma 4.4 to ensure that the time reparametrization is finite, we obtain the existence of a diffeomorphism $\sigma : [0, T] \to [\alpha, \beta]$, with $\sigma(0) = \alpha$ and $\sigma(T) = \beta$, such that

$$(q, p) \circ \sigma : [0, T] \to \bar{\bar{\Sigma}}$$

is a brake orbit. □

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