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Calculus of Variations

A variational setting for an indefinite Lagrangian with an affine Noether charge

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Abstract

We introduce a variational setting for the action functional of an autonomous and indefinite Lagrangian on a finite dimensional manifold M. Our basic assumption is the existence of an infinitesimal symmetry whose Noether charge is the sum of a one-form and a function on M. Our setting includes different types of Lorentz–Finsler Lagrangians admitting a timelike Killing vector field.

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1 Introduction

The principle of least or stationary-action in Lagrangian mechanics has been at the heart of the development of the variational calculus. It has given rise to different methods for solving the problem of finding (or at least establishing the existence of) a path of evolution between two points of a dynamical system described by a finite number of variables (see, e.g., [12, 42, 49]). The techniques developed to get solutions have been proved to be useful in the study of general Lagrangian systems with an infinite number of degrees of freedom (see, e.g., [28, 47]). A very classical field of application of these methods is the geodesic problem in Riemannian and Finsler geometry. In this case, completeness of the metric is

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enough to get a solution with fixed end points and topological arguments give multiplicity of geodesics. The landscape is quite different for the analogous problem on a Lorentzian manifold where (geodesic) completeness is not enough to get compactness properties on the space of paths between two points and other geometric conditions as global hyperbolicity have been considered as a replacement [4, 46]. Only recently the features underlying global hyperbolicity, in connection with the geodesic problem and more generally with causality, start to find a field of applications beyond classical Lorentzian geometry (see [11, 27, 35, 41]).

On the other hand, the existence of a symmetry that leaves invariant the action functional of a Lagrangian is a source of information about its stationary points through the Noether's theorem. The impact of this result in variational calculus can be hardly overestimated. A nice application of it to the geodesic problem of a Lorentzian manifold can be found in [31], where a stationary spacetime (M, g_L) (i.e. a spacetime endowed with a timelike Killing vector field) is considered. In this case, the Noether charge associated to the Killing field is used to get a reduction of the Sobolev manifold of paths between two points p and q in M, where the energy functional of the Lorentz metric is defined, to the infinite dimensional submanifold $\mathcal{N}_{p,q}$ of the curves with a.e. constant Noether charge. This reduction resembles the classical Routh reduction for Lagrangian systems (see, e.g. [23, 39]) but it involves merely the paths space and not the phase space. The roots of the idea of this infinite dimensional reduction are in a couple of papers about geodesic connectedness of static and stationary spacetimes admitting a global splitting [9, 30] and, indeed, some local computation in [31] and in the present paper (see Theorem 7.6) are based on those papers.

Our goal is to show that the full variational setting in [31] admits a generalization for an indefinite C^1 Lagrangian L on a smooth finite dimensional manifold M. We assume that L is invariant by a one-parameter group of local diffeomorphisms whose infinitesimal generator is a vector field K and that the associated Noether charge is a C^1 function N on TM, which is affine in each tangent space T_xM:

$$N(x, v) = Q(v) + d(x), \tag{1.1}$$

where Q and d are a one-form and a function on M, respectively. We assume also that d is invariant by the flow of K and $Q(K) < 0$ (see Assumption 2.2). Notice that in the case of a stationary Lorentzian manifold, $d = 0$ and Q coincides with the one-form metrically equivalent to the timelike Killing field K.

In Theorems 5.7 and 6.3 we obtain existence and multiplicity of weak solutions to the Euler-Lagrange equation of the action functional of L connecting two given points on M. The regularity of solutions is analysed in Appendix A. A key assumption in Theorem 5.7 is *c-boundedness* (Definition 5.1) of $\mathcal{N}_{p,q}$. Under conditions contained in Assumptions 2.2–2.9, c-boundedness implies that the reduced action functional $\mathcal{J}$ (differently from the action) is bounded from below (Proposition 5.2) and satisfies the Palais-Smale condition (Theorem 5.6). We show in Sect. 7 that c-boundedness is essentially equivalent to *c-precompactness* of $\mathcal{N}_{p,q}$, a condition introduced in [31] which is a compactness property of the set of paths in a sublevel of the reduced action functional. Actually, on a stationary Lorentzian manifold M, if $\mathcal{N}_{p,q}$ is c-precompact for all $c \in \mathbb{R}$ then M is globally hyperbolic (see [31, Proposition B1] in the case when the timelike Killing vector field is complete and [15, Section 6.4-(a)] for any timelike Killing vector field). On the converse, if M is globally hyperbolic with a complete smooth Cauchy hypersurface then $\mathcal{N}_{p,q}$ is c-precompact for all $c \in \mathbb{R}$ (see [15, Theorem 5.1]). Thus, if c-precompactness is satisfied for all $c \in \mathbb{R}$, the spacetime M cannot be compact. Inspired by Proposition A.3 in [31], we give a condition that implies c-precompactness of $\mathcal{N}_{p,q}$, for all $c \in \mathbb{R}$ and all $p, q \in M$, in our setting, and that cannot be satisfied if M is compact (see Proposition 8.1).

The Lagrangians that we consider (see Sect. 3) include, but are not limited to, C^1 stationary Lorentzian metrics, electromagnetic type Lagrangians on a stationary Lorentzian manifold with a Killing vector field K and K-invariant potentials (see, e.g. [6, 16, 19, 50]) and some stationary Lorentz–Finsler metrics. Loosing speaking, a Lorentz–Finsler metric is an indefinite, positively homogeneous of degree two in the velocities, Lagrangian that generalizes the quadratic form of a Lorentzian metric in the same way as the square of a Finsler metric generalizes the square of the norm of a Riemannian metric. They were studied by K. Beem [7] following some work by H. Busemann. Although considered from time to time in works about anisotropy in special and general relativity (even if they often appear as the square of a more fundamental function, positively homogeneous of degree one in the velocities, see e.g. [13, 34, 45]), there has been a growing interest about them (or their possible generalizations as non-degenerate Lagrangians defined on a cone bundle on M) in the last decade, see for example [1, 10, 17, 29, 32, 33, 36, 38, 40, 44].

Some explicit examples that are covered by our present setting are Beem's Lorentz–Finsler metrics endowed with a timelike Killing vector field K including also their sum with a potential function and a one-form, both invariant by the flow of K (see Example 3.6). In particular, this class includes Lagrangians defined as

$$L = F^2 - \omega^2,$$

introduced in [35], where F and ω are, respectively, a Finsler metric and a one-form on M both invariant by the one-parameter group of local diffeomorphisms generated by K, provided a sign assumption on $F^2(K) - \omega^2(K)$ is satisfied, see Example 3.9. Other examples are given by Lagrangians L that locally, i.e. on a neighborhood of the type $S \times (a, b) \subset M$, can be expressed as

$$L = L_0 + 2(\omega + d/2)\mathrm{d}t - \beta \mathrm{d}t^2, \tag{1.2}$$

where L_0 is a C^1 Tonelli Lagrangian on S, with quadratic growth in the velocities, ω, d and β are respectively a C^1 one-form on S and two C^1 functions on S with $\beta > 0$ (see Example 3.1 and Proposition 7.4). We include the possibility that the Lagrangian L_0 might not be twice differentiable on the zero section of TS, but we require that it is pointwise strongly convex (see Assumption 2.7-(ii)). Notice that the possible lack of twice differentiability of L_0 at the zero section implies that L is not twice differentiable along the line bundle defined by $K = \partial_t$, being t the natural coordinate on the interval (a, b). Lagrangians of the type (1.2) on a global splitting $S \times \mathbb{R}$, with L_0 being the square of a Finsler metric and $d = 0$, were introduced in [36] when $\omega = 0$ (see also [20]) and in [21] for $\omega \neq 0$.

Let us point out a comment about the regularity of the objects we consider in this work. We consider a smooth, finite dimensional manifold M; the Lagrangian L and the vector field K are of class C^1 on TM. Lorentz–Finsler Lagrangians are not twice differentiable at the zero section of TM, hence assuming that L is C^1 is motivated by that wide class of indefinite Lagrangians. We are confident that both the regularity of L and the linearity of the Noether charge can be further relaxed at least for the existence of a global minimizer of the reduced action functional. This is clearly suggested by the fact that L is the sum of a Lagrangian which is strongly convex in the velocities and a C^1 Lagrangian related to the Noether charge (see (2.7)), and that some computations of this work are more related to the sublinearity of the Noether charge than to its expression (1.1).

2 Notations, assumptions and preliminary results

Let M be a smooth, connected, $(m+1)$-dimensional manifold, with $m \geq 1$; let us denote by TM the tangent bundle of M. Throughout the paper, we consider a (auxiliary) complete Riemannian metric g on M and we denote by $\|\cdot\|\colon TM \to \mathbb{R}$ its induced norm, i.e. $\|v\|^2 = g(v,v)$ for all $v \in TM$.

We will often denote an element of TM as a couple (x, v), $x \in M$, $v \in T_xM$ (for example we use such a notation in connection with the variables of an autonomous Lagrangian $L : TM \to \mathbb{R}$, i.e. we will write $L = L(x, v)$). On the other hand, we will avoid specifying the point x where a one-form ω or a vector field K on M is applied, and we will write, for example $\omega(v)$, $v \in TM$ or also $\omega(K)$. Some exceptions are possible for the sake of clarity, and we will write then, e.g., K_x or $\omega_x(v)$, $v \in T_xM$ and also $\omega_x(K)$. We will often explicitly write the variable of a function on M, like in $d(x)$, $C(x)$, $\lambda(x)$, etc. When a vector field K on M is evaluated along a curve $z : [0, 1] \to M$, we will write $K(z)$. In some cases we will look at a one-form Q on M also as a function on TM writing then $Q(x, v)$.

Let $L\colon TM \to \mathbb{R}$ be a Lagrangian on M. For any $(x, v) \in TM$, we denote by $\partial_v L(x, v)[\cdot]$ the vertical derivative of L, i.e. for all $x \in M$ and all $v, w \in T_xM$

$$\partial_v L(x, v)[w] := \frac{\mathrm{d}}{\mathrm{d}s} L(x, v + sw)|_{s=0}.$$

We need also a notion of horizontal derivative of the Lagrangian L (a derivative w.r.t. x). Let $(x^0, \dots, x^m)$ be coordinates on M and let $(x^0, \dots, x^m, v^0, \dots, v^m)$ be the induced ones on TM. Let $(x, v) \in TM$, with coordinates values $(x^0, \dots, x^m, v^0, \dots, v^m)$; we define $\partial_x L(x, v)[\cdot]$ as the v-depending one-form on M locally given by

$$\partial_x L(x, v)[w] := \sum_{i=0}^{m} \frac{\partial L}{\partial x^i}(x, v) w^i.$$

Remark 2.1 Even though, differently from the vertical derivative, this definition is not intrinsic, it fits our purposes (in the following, we will make extensively use of local arguments in computations involving L). In particular, we denote by $\|\partial_x L_c(x, v)\|$ and $\|\partial_v L_c(x, v)\|$ the two scalar fields on TM which are pointwise the norm of the above two linear operators w.r.t. g.

Assumption 2.2 The Lagrangian $L\colon TM \to \mathbb{R}$ satisfies the following conditions:

(i) $L \in C^1(TM)$;

(ii) there exists a C^1 vector field K on M such that L is invariant by the one-parameter group of local C^1 diffeomorphisms generated by K (we call K an *infinitesimal symmetry of* L); moreover the *Noether charge*, i.e. the map $(x, v) \in TM \mapsto \partial_v L(x, v)[K] \in \mathbb{R}$, is a function N on TM which is the sum of a C^1 one-form Q on M and a C^1 function $d\colon M \to \mathbb{R}$, i.e.

$$N(x, v) := \partial_v L(x, v)[K] = Q(v) + d(x); \tag{2.1}$$

(iii) the function d in (2.1) is invariant by the flow of K (in particular the case when d is a constant function is compatible); moreover,

$$Q(K) < 0. \tag{2.2}$$

Remark 2.3 Vector fields K which are infinitesimal symmetries for L can be characterized similarly to Killing vector fields for Finsler metrics (see, e.g., [21]). We denote by K^c the *complete lift* of K to TM, which, using Einstein summation convention, is locally defined as:

$$(K^c)_{(x,v)} = K^h(x)\frac{\partial}{\partial x^h} + \frac{\partial K^h}{\partial x^i}(x)v^i\frac{\partial}{\partial v^h}. \tag{2.3}$$

It follows that, if ψ is a local flow of K, then for any $(x,v)\in TM$ the local flow ψ^c of K^c on TM is given by $\psi^c(t,x,v) = \big(\psi(t,x), \partial_x\psi(t,x)[v]\big)$. Hence,

$$K^c(L)\big(\psi^c(t,x,v)\big) = \frac{\partial\big(L\circ\psi^c\big)}{\partial t}(t,x,v)$$

and, since

$$\frac{\partial\big(L\circ\psi^c\big)}{\partial t}(t,x,v) = 0, \tag{2.4}$$

we get

$$K^c(L)(x,v) = K^h(x)\frac{\partial L}{\partial x^h}(x,v) + \frac{\partial K^h}{\partial x^i}(x)v^i\frac{\partial L}{\partial v^h}(x,v) = 0. \tag{2.5}$$

Remark 2.4 Since K is an infinitesimal symmetry of L, by Noether's theorem, the Noether charge is constant for any weak solution z of the Euler-Lagrange equation

of the Lagrangian L, independently from the boundary conditions. This can be seen by recalling that a weak solution $z = z(s)$ of the Euler-Lagrange equation is a C^1 curve (for fixed end points boundary conditions, see Appendix A) that locally (i.e. in natural local coordinates of TM) satisfies the system of equations

$$\frac{\partial L}{\partial x^i}\big(z(s),\dot z(s)\big) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\right), \quad \forall i = 0,\dots,m, \tag{2.6}$$

hence from (2.5) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)K^i(z(s))\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\right)K^i(z(s)) + \frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\frac{\partial K^i}{\partial x^h}(z(s))\dot z^h(s)\\
&= \frac{\partial L}{\partial x^i}\big(z(s),\dot z(s)\big)K^i(z(s)) + \frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\frac{\partial K^i}{\partial x^h}(z(s))\dot z^h(s) = 0.
\end{aligned}$$

Let us introduce a Lagrangian L_c on TM defined as

$$L_c(x,v) := L(x,v) - \frac{Q^2(v)}{Q(K)}. \tag{2.7}$$

Proposition 2.5 *The following statements hold:*

(i) $L_c \in C^1(TM)$;
and, for all $(x,v)\in TM$:

(ii)

$$Q_x(K) = 2\left(L(x,K) - L(x,0) - d(x)\right); \tag{2.8}$$

(iii)

$$L_c(x,0) = L(x,0),$$
$$L(x,K) + L_c(x,K) = 2\big(L(x,0) + d(x)\big); \tag{2.9}$$

and

$$\partial_v L_c(x,v)[K] = -Q(v) + d(x), \tag{2.10}$$

(iv) the flow of K preserves also L_c, i.e. $K^c(L_c) = 0$.

Proof Statement (i) comes immediately from (2.7) and Assumption 2.2-(i). Let us prove (ii). Let $x \in M$ be a given point and let $l\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$l(\alpha) = L(x, \alpha K) - L(x,0).$$

Hence, $l(0) = 0$ and, by Assumption 2.2-(ii), we obtain

$$l'(\alpha) = \partial_v L(x, \alpha K)[K] = \alpha Q_x(K) + d(x).$$

As a consequence, the function l is equal to

$$l(\alpha) = \frac{\alpha^2}{2} Q_x(K) + \alpha d(x).$$

Therefore, noticing that $Q_x(K) = 2\big(l(1) - d(x)\big)$, we obtain (2.8). Now (iii) is a simple consequence of (2.7) and (2.8).

Let us prove (iv). From (2.7) it is enough to prove that Q and $Q(K)$ are invariant by the flow of K^c and K, respectively. Let us consider Q as a function on TM, i.e. $Q(x,v) := Q(v)$, thus we have to show that $K^c(Q) = 0$. By (2.2), $K_x \neq 0$ for all $x \in M$, thus for each $\bar{x} \in M$ we can take a neighborhood U of $\bar{x}$ and a coordinate system $(x^0, x^1, \dots, x^m)$ defined in U such that $\frac{\partial}{\partial x^0} = K|_U$. Therefore, in such a coordinate system,

$$Q(x,v) = \frac{\partial L}{\partial v^h}(x,v) K^h - d(x) = \frac{\partial L}{\partial v^0}(x,v) - d(x).$$

Since Q and d are C^1, we know that $\frac{\partial L}{\partial v^0}$ admits continuous partial derivatives w.r.t. the coordinates $(x^0, x^1, \dots, x^m, v^0, v^1, \dots, v^m)$ in TU. Notice also that, from (2.5), $K^c(L) = 0$ is equivalent to $\frac{\partial L}{\partial x^0}(x,v) = 0$. Being then a constant function, $\frac{\partial L}{\partial x^0}$ admits zero partial derivatives w.r.t. the coordinates $(x^0, x^1, \dots, x^m, v^0, v^1, \dots, v^m)$ as well. As d is invariant by the flow of K, we have $\frac{\partial d}{\partial x^0} = 0$ on U. Thus, from (2.3), we then get

$$K^c(Q)(x,v) = \frac{\partial^2 L}{\partial x^0 \partial v^0}(x,v) = \frac{\partial^2 L}{\partial v^0 \partial x^0}(x,v) = 0.$$

Since $Q(x,K) = \frac{\partial L}{\partial v^0}\big(x, (1,0,\dots,0)\big)$, we also have

$$K^c\big(Q(x,K)\big) = K\big(Q(x,K)\big) = \frac{\partial^2 L}{\partial x^0 \partial v^0}\big(x, (1,0,\dots,0)\big) = 0.$$

□

Remark 2.6 From (2.10) and (iv) in Proposition 2.5, we have that, like L, L_c has affine Noether charge as well.

Recalling Remark 2.1, the following assumption ensures some growth conditions on L_c, often used in critical point theory for the action functional of a Lagrangian (see, e.g., [2, 8]), and its pointwise strong convexity.

Assumption 2.7 The Lagrangian $L_c\colon TM \to \mathbb{R}$, defined as in (2.7), satisfies the following assumptions:

(i) there exists a continuous function $C\colon M \to (0, +\infty)$ such that for all $(x, v) \in TM$, the following inequalities hold

$$L_c(x, v) \le C(x)\big(\|v\|^2 + 1\big); \tag{2.11}$$

$$\|\partial_x L_c(x, v)\| \le C(x)\big(\|v\|^2 + 1\big); \tag{2.12}$$

$$\|\partial_v L_c(x, v)\| \le C(x)\big(\|v\| + 1\big); \tag{2.13}$$

(ii) there exists a continuous function $\lambda\colon M \to (0, +\infty)$ such that for each $x \in M$ and for all $v_1, v_2 \in T_x M$, the following inequality holds:

$$\big(\partial_v L_c(x, v_2) - \partial_v L_c(x, v_1)\big)[v_2 - v_1] \ge \lambda(x)\|v_2 - v_1\|^2; \tag{2.14}$$

Remark 2.8 We notice that from (2.14) and (2.10) we obtain

$$\begin{aligned} Q_x(K) &= Q_x(K) - Q_x(0) \\ &= \big(\partial_v L_c(x, 0) - \partial_v L_c(x, K)\big)[K] \le -\lambda(x)\|K\|^2. \end{aligned}$$

Moreover, for all $(x, v) \in TM$ we have

$$\begin{aligned} L_c(x, v) - L_c(x, 0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} L_c(x, sv)\mathrm{d}s = \int_0^1 \partial_v L_c(x, sv)[v]\mathrm{d}s \\ &= \int_0^1 \frac{1}{s}\Big(\partial_v L_c(x, sv)[sv] - \partial_v L_c(x, 0)[sv]\Big)\mathrm{d}s + \partial_v L_c(x, 0)[v] \\ &\ge \frac{1}{2}\lambda(x)\|v\|^2 - \|\partial_v L_c(x, 0)\|\,\|v\|. \end{aligned}$$

Thus, L_c satisfies the growth condition

$$L_c(x, v) \ge L_c(x, 0) - \frac{1}{\lambda(x)}\|\partial_v L_c(x, 0)\|^2 + \frac{\lambda(x)}{4}\|v\|^2.$$

and since $L_c(x, 0) = L(x, 0)$ and $\partial_v L_c(x, 0) = \partial_v L(x, 0)$ we get

$$L_c(x, v) \ge L(x, 0) - \frac{1}{\lambda(x)}\|\partial_v L(x, 0)\|^2 + \frac{\lambda(x)}{4}\|v\|^2. \tag{2.15}$$

The next and final assumption is needed to get a compactness condition on the sublevels of the reduced action functional (see Lemma 5.3) and then in the proof of the Palais-Smale condition for the same functional.

Assumption 2.9 There exist four constants, c_1, c_2, c_3, k_1, k_2 such that, for all $x \in M$, the following inequalities hold:

$$0 < c_1 \le \lambda(x), \tag{2.16}$$

$$L(x, 0) \ge c_2 \quad \text{and} \quad \|\partial_v L(x, 0)\| \le c_3, \tag{2.17}$$

$$0 < k_1 \le -Q_x(K), \tag{2.18}$$

$$|d(x)| \le k_2. \tag{2.19}$$

3 Some classes of examples

In this section we present various type of Lagrangians that satisfy Assumptions 2.2–2.9. We start with a generalization of the Lorentz–Finsler Lagrangians studied in [21].

Example 3.1 Let S be a smooth m–dimensional manifold and $M = S \times \mathbb{R}$. Let g_S be a complete auxiliary Riemannian metric on S, whose associated norm is denoted by $\|\cdot\|_S$, and let g be the product metric $g = g_S \oplus dt^2$. Let $L\colon TM \to \mathbb{R}$ be a Lagrangian on M defined as

$$L\big((x,t),(\nu,\tau)\big) = L_0(x,\nu) + 2\big(\omega(\nu) + d(x)/2\big)\tau - \beta(x)\tau^2, \tag{3.1}$$

where

(i) $L_0\colon TS \to \mathbb{R}$ belongs to $C^1(TS)$ and there exists a continuous positive function $\ell : S \to (0,+\infty)$ such that

$$L_0(x,\nu) \le \ell(x)\big(\|\nu\|_S^2 + 1\big); \tag{3.2}$$

$$\|\partial_x L_0(x,\nu)\|_S \le \ell(x)\big(\|\nu\|_S^2 + 1\big); \tag{3.3}$$

$$\|\partial_\nu L_0(x,\nu)\|_S \le \ell(x)\big(\|\nu\|_S + 1\big); \tag{3.4}$$

(ii) L_0 is pointwise strongly convex, i.e. there exists a continuous function $\lambda_0\colon S \to (0,+\infty)$ such that, for all $x \in S$ and all $\nu_1, \nu_2 \in T_xS$, (2.14) holds with L_c replaced by L_0, λ by λ_0 and $\|\cdot\|$ by $\|\cdot\|_S$;

(iii) ω is a C^1 one-form on S, $d\colon S \to \mathbb{R}$ is a C^1 function and $\beta\colon S \to (0,+\infty)$ is a C^1 positive function.

In this case, the field $K = \partial_t \equiv (0,1)$ is an infinitesimal symmetry of L and d is invariant by the flow of K, because it is a function on S. Notice that if L_0 is the square of a Riemannian norm on S and $d = 0$ then L is the quadratic form associated with the Lorentzian metric of a standard stationary spacetime (see, e.g., [30]). Moreover, if L_0 is the square of the norm of a Riemannian metric plus a one-form ω_0 on S, then they include electromagnetic type Lagrangians on a standard stationary Lorentzian manifold with an exact electromagnetic field on $S \times \mathbb{R}$ having a potential one-form $\omega_0 \oplus d(x)\mathrm{d}t$, see Remark 3.3 below.

In the next result we show that L defined as in (3.1) satisfies Assumptions 2.2–2.7 and we give some further conditions ensuring that it also satisfies Assumption 2.9.

Proposition 3.2 *A Lagrangian L defined as in (3.1), such that (i)–(iii) above hold, satisfies Assumptions 2.2 and 2.7. Moreover, if there exist some constants $b, \ell_1, \ell_2, \ell_3, \ell_4$, such that $\beta(x) \ge b > 0$, $\lambda_0(x) \ge \ell_1 > 0$, $L_0(x,0) \ge \ell_2$, $\|\partial_\nu L_0(x,0)\| \le \ell_3$ and $|d(x)| \le \ell_4$, for every $x \in S$, then L satisfies Assumption 2.9.*

Proof As remarked above, the vector field $\partial_t \equiv (0,1)$ is an infinitesimal symmetry for L; moreover, since by hypotheses L_0, ω and β are of class C^1, $L \in C^1(TM)$ as well. A direct computation shows that

$$\partial_v L\big((x,t),\cdot\big)[(0,1)] = 2\big(\omega_x - \beta(x)\mathrm{d}t\big) + d(x) \tag{3.5}$$

which is an affine function on TM that we denote by N. Let $Q := 2(\omega - \beta\mathrm{d}t)$, hence

$$Q(K) = Q\big((0,1)\big) = -2\beta < 0 \tag{3.6}$$

and thus the conditions in Assumption 2.2 are satisfied. Using (2.7) with (3.6), we see that $L_c\colon TM \to \mathbb{R}$ is given by

$$L_c((x,t),(\nu,\tau)) = L_0(x,\nu) + \left(\frac{1}{\sqrt{\beta(x)}}\omega(\nu) - \sqrt{\beta(x)}\tau\right)^2 + \frac{1}{\beta(x)}\omega^2(\nu) + \frac{d(x)}{2}\tau. \tag{3.7}$$

Let us show that L_c satisfies (2.11), (2.12) and (2.13). By (3.2) we have

$$L_c((x,t),(\nu,\tau)) \leq \ell(x)(\|\nu\|_S^2 + 1) + \frac{2\omega^2(\nu)}{\beta(x)} + 2\beta(x)\tau^2 + \frac{\omega^2(\nu)}{\beta(x)} + \frac{d^2(x)}{2} + \frac{\tau^2}{2},$$

so setting

$$C((x,t)) \equiv C(x) = \max\left\{\ell(x), \frac{3\|\omega_x\|_S^2}{\beta(x)}, 2\beta(x) + \frac{1}{2}, \frac{d^2(x)}{2}\right\}$$

(2.11) holds. Let us compute $\partial_{(x,t)}L_c$:

$$\partial_{(x,t)}L_c\big((x,t),(\nu,\tau)\big)[\xi,\zeta] = \partial_x L_0(x,\nu)[\xi] - 2\partial_x\omega(\xi,\nu)\tau + \mathrm{d}\beta(\xi)\tau^2 + \frac{4}{\beta(x)}\omega(\nu)\partial_x\omega(\xi,\nu) - \frac{2}{\beta^2(x)}\omega^2(\nu)\mathrm{d}\beta(\xi) + \mathrm{d}d(\xi)\tau,$$

Hence,

$$\begin{aligned}\|\partial_{(x,t)}L_c\big((x,t),(\nu,\tau)\big)\| &\leq \|\partial_x L_0(x,\nu)\| + 2\|(\partial_x\omega)_x\|_S\|\nu\|_S|\tau| \\ &+\|(\mathrm{d}\beta)_x\|_S|\tau|^2 + \frac{4}{\beta(x)}\|(\partial_x\omega)_x\|_S\|\omega_x\|_S\|\nu\|_S^2 \\ &+\frac{2}{\beta^2(x)}\|\omega_x\|_S^2\|(\mathrm{d}\beta)_x\|_S\|\nu\|_S^2 + \|(\mathrm{d}d)_x\||\tau|.\end{aligned}$$

By (3.3) and recalling that $\|(\nu,\tau)\|^2 = \|\nu\|_S^2 + |\tau|^2$, we infer the existence of a function $C\colon M \to (0,+\infty)$ such that (2.12) holds. Similarly, using (3.4) we obtain (2.13).

Let us show that L_c satisfies (2.14). From (3.7) we have

$$\begin{aligned}\partial_{(\nu,\tau)}L_c\big((x,t),(\nu,\tau)\big)[(\nu_1,\tau_1)] &= \partial_v L_0(x,\nu)[\nu_1] \\ &+2\left(\frac{1}{\sqrt{\beta(x)}}\omega(\nu) - \sqrt{\beta(x)}\tau\right)\left(\frac{1}{\sqrt{\beta(x)}}\omega(\nu_1) - \sqrt{\beta(x)}\tau_1\right) \\ &+\frac{2}{\beta(x)}\omega(\nu)\omega(\nu_1) + d(x)\tau_1,\end{aligned}$$

hence using that L_0 is pointwise strongly convex we get

$$\begin{aligned}&\Big(\partial_{(\nu,\tau)}L_c\big((x,t),(\nu_2,\tau_2)\big) - \partial_{(\nu,\tau)}L_c\big((x,t),(\nu_1,\tau_1)\big)\Big)[(\nu_2-\nu_1,\tau_2-\tau_1)] \\ &\geq \lambda_0(x)\|\nu_2-\nu_1\|_S^2 + \frac{4}{\beta(x)}\omega_x^2(\nu_2-\nu_1) \\ &\quad+2\beta(x)(\tau_2-\tau_1)^2 - 4(\tau_2-\tau_1)\omega_x(\nu_2-\nu_1) \\ &\geq \lambda_0(x)\|\nu_2-\nu_1\|_S^2 + \beta(x)(\tau_2-\tau_1)^2,\end{aligned}$$

thus (2.14) holds by taking

$$\lambda(x) := \min\{\lambda_0(x), \beta(x)\}. \tag{3.8}$$

It remains to prove Assumption 2.9. Of course, (2.19) is trivially satisfied and if $\beta \geq b > 0$ then by (3.6) we obtain (2.18). By (3.8), we also have (2.16) with $c_1 = \min\{\ell_1, b\}$. As $L\big((x,t),0\big) = L_0(x,0)$ and $\partial_{(v,\tau)} L\big((x,t),0\big) = \partial_v L_0(x,0)$, (2.17) is satisfied as well with $c_2 = \ell_2$ and $c_3 = \ell_3$. □

Remark 3.3 A special case of a Lagrangian in Example 3.1 that satisfies our assumptions is given by (3.1) with

$$L_0(x,v) = F^2(x,v) + \omega_0(v) + V(x),$$

where $V\colon S \to \mathbb{R}$ is a C^1 function bounded from below, ω_0 is a C^1 one-form on S, such that $\sup_{x\in S} \|(\omega_0)_x\|_S < +\infty$, and $F\colon TS \to [0,+\infty)$ is a C^1 Finsler metric on S, i.e it is a non-negative, C^1 Lagrangian on TS, positively homogeneous of degree 1 w.r.t. v, such that F^2 is pointwise strongly convex, i.e. it satisfies (2.14) on TS. We remark that usually in the definition of a Finsler metric it is assumed that $F^2 \in C^2(TS\setminus 0)$ (where 0 denotes the zero section of TS) and its vertical Hessian, the so-called fundamental tensor g_F,

$$g_F(x,v)[u,w] := \frac{1}{2}\frac{\partial^2 F^2}{\partial s \partial t}(x, v + tu + sw)\Big|_{(s,t)=(0,0)}$$

for all $(x,v) \in TM\setminus 0$ and all $u, w \in T_x M$, is assumed to be positively homogeneous of degree 0 in v and positive definite for all $(x,v) \in TM \setminus 0$ (see, e.g., [5]). Inequality (2.14) for F^2 on TS follows by the mean value theorem applied to the function $v \in TS \mapsto \partial_v F^2(x,v)[v_2 - v_1]$, when v_1, v_2 are not collinear vectors with opposite directions or when one of them is 0; for collinear vectors with opposite directions it follows by a continuity argument. Notice that $\lambda_0(x)$ in (2.14) for F^2 is then equal to

$$\lambda_0(x) = 2 \min_{v\in T_x S\setminus\{0\}} \left(\min_{u \in T_x S\setminus\{0\}} g_F(x,v)\left[\frac{u}{\|u\|_S}, \frac{u}{\|u\|_S}\right]\right),$$

and that (3.2)–(3.4) are ensured by the homogeneity of degree 2 of F^2 w.r.t. v.

We notice that Lagrangians L satisfying Assumptions 2.2–2.7 are generated by Lagrangians L_b satisfying (2.11)–(2.14) and admitting a vector field K as an infinitesimal symmetry with affine Noether charge $N_b = Q_b + d$ such that $Q_b(K) > 0$. Indeed, arguing as in Proposition 2.5-(iv), $L := L_b - \frac{Q_b^2}{Q_b(K)}$ admits K as infinitesimal symmetry and its Noether charge is

$$N = N_b - 2Q_b = -Q_b + d,$$

hence $L_c = L_b$ and then, of course, L_c satisfies Assumption 2.7. This observation gives rise to the following families of examples.

Example 3.4 Let M be a smooth $(m+1)$-dimensional manifold endowed with a complete Riemannian metric g and F, ω_0, V be respectively a Finsler metric, a one-form and function on M, all of C^1 class and invariant by the flow of a nowhere vanishing vector field K on M. Let us assume that the Noether charge associated with F^2 and K is a one-form of class C^1. Let $L : TM \to \mathbb{R}$ be given by

$$L = F^2 + \omega_0 + V - \frac{Q_F^2}{Q_F(K)}, \tag{3.9}$$

and $\lambda(x)$ be the positive continuous function in (2.14) for F^2 on TM.

Proposition 3.5 *Assume that there exist two constants $c_1 > 0$ and $a \geq 0$ such that $\lambda(x) > c_1$, and $\|(\omega_0)_x\| \leq a$, for all $x \in M$, V is bounded from below,* $\inf_{x\in M} \|K_x\| > 0$ *and* $\sup_{x\in M} \|K_x\| < +\infty$. *Then L in (3.9) satisfies Assumptions 2.2–2.9.*

Proof Set $L_b = F^2 + \omega_0 + V$, then L_b admits K as an infinitesimal symmetry with affine Noether charge

$$N_b = N_F + \omega_0(K).$$

Hence, the same holds for L and the one-form appearing in its Noether charge is $Q = -Q_F$. Since ω_0 is invariant by the flow of K, the Lie derivative $\mathcal{L}_K\omega_0$ vanishes. In particular, $0 = \mathcal{L}_K\omega_0(K) = K\big(\omega_0(K)\big)$, i.e. $\omega_0(K)$ is invariant by the flow of K. We also notice that $(Q_F)_x(K) = 2F^2(x, K) > 0$ for all $x \in M$ since, by assumption, $K_x \neq 0$ for all $x \in M$. Thus L satisfies Assumption 2.2. As $L_c = L_b$, it satisfies (2.11)–(2.13) because F^2 is positively homogeneous of degree two; moreover, it satisfies also (2.14) as F^2 is pointwise strongly convex. Since $L(x, 0) = V(x)$, and $\partial_v L(x, 0) = \omega_0$, (2.17) holds; being

$$-Q_x(K) = (Q_F)_x(K) = 2F^2(x, K) \geq c_1\|K\|_x^2 \geq c_1 \inf_{x\in M} \|K\|_x^2 > 0,$$

and $d = \omega_0(K)$, (2.18) and (2.19) hold as well. □

The next class of examples involves Lorentz–Finsler metrics L_F as defined by J. K. Beem in [7] (see also [29, 40, 44]).

Example 3.6 Let M be a smooth manifold of dimension $m + 1$, and g an auxiliary complete Riemannian metric on M. Let $L_F : TM \to \mathbb{R}$ be a Lagrangian which satisfies the following conditions:

(i) $L_F \in C^1(TM) \cap C^2(TM \setminus 0)$, where 0 denotes the zero section of TM;
(ii) $L_F(x, \lambda v) = \lambda^2 L_F(x, v)$ for all $v \in TM$ and all $\lambda > 0$;
(iii) for any $(x, v) \in TM \setminus 0$, the vertical Hessian of L_F, i.e. the symmetric matrix

$$(g_F)_{\alpha\beta}(x, v) := \frac{\partial^2 L_F}{\partial v^\alpha \partial v^\beta}(x, v), \quad \alpha, \beta = 0, 1, \ldots, m,$$

is non-degenerate with index 1.

Let us assume that L_F admits a nowhere vanishing vector field K as an infinitesimal symmetry and that its Noether charge is equal to $N_{L_F} = Q_{L_F}$, where Q_{L_F} is a C^1 one-form such that $Q_{L_F}(K) < 0$. Let $L = L_F + \omega_1 + V$ where ω_1 and V are, respectively, a C^1 one-form on M, such that $\sup_{x\in M} \|(\omega_1)_x\| < +\infty$, and a C^1 function, bounded from below on M. We assume that both ω_1 and V are invariant by the flow of K. Then the Noether charge of L is $N = N_{L_F} + \omega_1(K) = Q_{L_F} + d_{L_F} + \omega_1(K)$. Thus, L satisfies Assumption 2.2. Let $Q := Q_{L_F}$; so L_c is equal to $L_c = L - \frac{Q^2}{Q(K)}$.

Proposition 3.7 *If conditions (i)–(iii) of Example 3.6 hold, then L_c satisfies Assumption 2.7.*

Proof Let us show that L_c admits vertical Hessian at any $(x, v) \in TM\setminus 0$, which is a positive definite bilinear form on T_xM. We observe that for any $(x, v) \in TM \setminus 0$, we have

$$\partial_{vv} L_c(x, v) = \partial_{vv} L(x, v) - \frac{2}{Q(K)} Q \otimes Q$$

$$= \partial_{vv} L_F(x, v) - \frac{2}{Q(K)} Q \otimes Q \tag{3.10}$$

As for each $u \in T_x M$ we have

$$\begin{aligned} \partial_{vv} L(x, v)[K, u] &= \frac{\partial^2 L}{\partial s \partial t}(x, v + tK + su)\Big|_{(s,t)=(0,0)} \\ &= \frac{\partial\big(\partial_v L(x, v + su)[K]\big)}{\partial s}\Big|_{s=0} = \frac{\partial Q(v + su)}{\partial s}\Big|_{s=0} = Q(u), \end{aligned} \tag{3.11}$$

from (2.2) and (3.10), we get $\partial_{vv} L_c(x, v)[K, K] = -Q(K) > 0$. Let $w \in \ker Q$. From (3.11), we have $\partial_{vv} L(x, v)[w, K] = 0$, and since $\partial_{vv} L(x, v)$ has index 1, we also have

$$\partial_{vv} L_c(x, v)[w, w] = \partial_{vv} L(x, v)[w, w] > 0,$$

hence $\partial_{vv} L_c(x, v)[\cdot, \cdot]$ is positive definite. Reasoning as in the last part of Remark 3.3, we deduce that (2.14) holds. From (3.10), since $\partial_{vv} L_F(x, v) - \frac{2}{Q(K)} Q \otimes Q$ is continuous on $TM \setminus 0$ and positively homogeneous of degree 0 in v, we deduce (2.11)

and

$$C(x) = \max\left\{\Lambda(x) + 1, V(x) + \|(\omega_1)_x\|^2\right\},$$

where

$$\Lambda(x) := \max_{\substack{v \in T_x M, \|v\|=1 \\ w \in T_x M}} \left(\frac{1}{2}\partial_{vv} L_F - \frac{Q \otimes Q}{Q(K)}\right)(x, v)\left[\frac{w}{\|w\|}, \frac{w}{\|w\|}\right].$$

Up to redefine $C(x)$, (2.12) and (2.13) can be obtained analogously. □

In particular, Lagrangians in Example 3.6 include the class of C^2 stationary Lorentzian metrics. We also want to consider the C^1 case.

Example 3.8 Let (M, g_L) be a Lorentzian manifold of dimension $m + 1$ with C^1 metric tensor g_L. Let K be a timelike Killing vector field for g_L, i.e. K is a Killing vector field such that $g_L(K, K) < 0$. Then (M, g_L) is called a *stationary Lorentzian manifold*. Let $L(x, v) := g_L(v, v)$; we notice that $L \in C^1(TM)$ and $\partial_v L(x, v)[K] = (g_L)_x(\cdot, K)$, thus $Q(K) = 2g_L(K, K) < 0$. The Lagrangian L_c is equal to

$$L_c(x, v) = g_L(v, v) - \frac{2g_L(K, v)^2}{g_L(K, K)}$$

and then it is equal to the square of the norm of a Riemannian metric g_R (as in [31]). Thus, Assumption 2.7 is satisfied as well (by using the same metric g_R as auxiliary Riemannian metric g), provided that g_R is complete with

$$C(x) = \max\left\{2, (m + 1) \max_{k \in \{0,\dots,m\}} \left(\max_{v \in T_x M \neq 0} \frac{\partial (g_R)_{ij}}{\partial x^k}(x) \frac{v^i}{\|v\|} \frac{v^j}{\|v\|}\right)\right\}.$$

Finally, Assumption 2.9 is satisfied provided that there exists a constant k_1 such that $-g_L(K, K) \geq k_1 > 0$.

The following example of Lagrangians are the Lorentz–Finsler Lagrangians studied in [35] and they can be included in the class of Example 3.6 (see Proposition 3.10).

Example 3.9 Let M be a smooth manifold and

$$L_F = F^2 - \omega^2, \tag{3.12}$$

where F and ω are, respectively, a Finsler metric of class $C^1(TM) \cap C^2(TM \setminus 0)$ and a one-form of class C^1 on M, both invariant by the flow of a nowhere vanishing vector field K and such that the Noether charge N_F associated with F^2 and K is a C^1 one-form, $N_F = Q_F$. Then, L_F admits K as an infinitesimal symmetry and $N_{L_F} = Q_F - 2\omega(K)\omega$. Notice that $(Q_{L_F})_x(K) = 2\big(F^2(x, K) - \omega^2(K)\big)$.

Proposition 3.10 *Assume that $Q_{L_F}(K) < 0$, then $\omega_x(K) \neq 0$ for all $x \in M$ and L_F in (3.12) is a Lagrangian of the type in Example 3.6.*

Proof The non-trivial part of the statement is to prove (iii) in Example 3.6. For all $(x, v) \in TM \setminus 0$ we have:

$$\begin{aligned}
\partial_{vv} L_F(x, v)[K, K] &= \partial_{vv} F^2(x, v)[K, K] - 2\omega^2(K) \\
&= \partial_v\big(\partial_v F^2(x, v)[K]\big)[K] - 2\omega^2(K) = \partial_v\big((Q_F)(v)\big)[K] - 2\omega^2(K) \\
&= Q_F(K) - 2\omega^2(K) = 2(F^2(x, K) - \omega^2(K)) < 0,
\end{aligned}$$

thus in particular we get that $\omega_x(K) \neq 0$, for all $x \in M$. Moreover for all $w \in \ker(\omega_x)$, $w \neq 0$, we have

$$\begin{aligned}
\partial_{vv} L_F(x, v)[w, w] &= \partial_{vv} F^2(x, v)[w, w] - 2\omega^2(w) \\
&= \partial_{vv} F^2(x, v)[w, w] > 0.
\end{aligned}$$

Thus, being K transversal to $\ker(\omega)$, we deduce that $\partial_{vv} L_F(x, v)$ has index 1 for all $(x, v) \in TM \setminus 0$. □

4 The reduced manifold of paths and action

Let $L : TM \to \mathbb{R}$ be a Lagrangian satisfying Assumptions 2.2 and 2.7. Recalling that M is endowed with an auxiliary complete Riemannian metric g, let us consider the set

$$W^{1,2}([0,1], M) := \left\{ z\colon [0,1] \to M : z \text{ is absolutely continuous and } \int_0^1 g(\dot z, \dot z)\mathrm{d}s < +\infty \right\},$$

and, for any two fixed points $p, q \in M$, its subset

$$\Omega^{1,2}_{p,q} := \left\{ z \in W^{1,2}([0,1], M) : z(0) = p,\ z(1) = q \right\}.$$

It is well known that since (M, g) is complete, $W^{1,2}([0,1], M)$ is a smooth, infinite dimensional, complete Riemannian manifold and $\Omega^{1,2}_{p,q}$ is a smooth closed (hence complete) submanifold (see, e.g., [25, Lemma 6.2]). For every $z \in \Omega^{1,2}_{p,q}$, the tangent space $T_z\Omega^{1,2}_{p,q}$ is equal to

$$T_z\Omega^{1,2}_{p,q} = \left\{ \xi \in W^{1,2}_0([0,1], TM) : \xi(s) \in T_{z(s)}M,\ \forall s \in [0,1] \right\}.$$

Weak solutions of (2.6) connecting the points $p, q \in M$ are by definition the critical points of the action functional $\mathcal{A}\colon \Omega^{1,2}_{p,q} \to \mathbb{R}$, defined as

$$\mathcal{A}(z) := \int_0^1 L(z, \dot z)\mathrm{d}s.$$

Remark 4.1 From (2.7), we have that $L = L_c + Q^2/Q(K)$ and hence from Assumption 2.7 we get that $\mathcal{A}$ is C^1 on $\Omega^{1,2}_{p,q}$ (see, e.g., the first part of the proof of Proposition 3.1 in [3]), with differential $\mathrm{d}\mathcal{A}(z)\colon T_z\Omega^{1,2}_{p,q} \to \mathbb{R}$ at a curve $z \in \Omega^{1,2}_{p,q}$ equal to

$$\mathrm{d}\mathcal{A}(z)[\xi] = \int_0^1 \left(\partial_x L(z,\dot z)[\xi] + \partial_v L(z,\dot z)[\dot\xi]\right) \mathrm{d}s. \tag{4.1}$$

Let $\xi \in T_z\Omega_{p,q}$ such that $\xi = X \circ z$, with X a smooth vector field in M, then in natural coordinates $(x^0,\dots,x^m,v^0,\dots,v^m)$, of TM, the integrand function in (4.1) is given by

$$\partial_x L(z,\dot z)[\xi] + \partial_v L(z,\dot z)[\dot\xi] = \frac{\partial L}{\partial x^i}(z,\dot z)X^i(z) + \frac{\partial L}{\partial v^i}(z,\dot z)\frac{\partial X^i}{\partial x^h}(z)\dot z^h.$$

In the following, by an abuse of notation, we also denote by $\dot X$ the derivative of $X(z)$, i.e. $\frac{\partial X^i}{\partial x^h}(z)\dot z^h$.

From (2.5) we then get

$$\partial_x L(z,\dot z)[K] + \partial_v L(z,\dot z)[\dot K] = 0, \tag{4.2}$$

for all $z \in \Omega^{1,2}_{p,q}$.

The main goal of this section is to prove that the critical points of $\mathcal{A}$ lay on the following subset of $\Omega^{1,2}_{p,q}$:

$$\mathcal{N}_{p,q} := \left\{z \in \Omega^{1,2}_{p,q}(M) : N(z,\dot z) \text{ is constant a.e. on } [0,1]\right\}. \tag{4.3}$$

For every $z \in \Omega^{1,2}_{p,q}$, let us define

$$\begin{aligned}\mathcal{W}_z := \Big\{\xi \in T_z\Omega^{1,2}_{p,q} : \exists \mu \in W^{1,2}_0([0,1],\mathbb{R})\\ \text{such that } \xi(s) = \mu(s)K_{z(s)}, \text{ a.e. on } [0,1]\Big\}.\end{aligned}$$

Proposition 4.2

$$\mathcal{N}_{p,q} = \left\{z \in \Omega^{1,2}_{p,q} : d\mathcal{A}(z)[\xi] = 0,\ \forall \xi \in \mathcal{W}_z\right\}.$$

Proof For all $\xi \in \mathcal{W}_z$, from (4.2) we have

$$\begin{aligned}\mathrm{d}\mathcal{A}(z)[\xi] &= \int_0^1 \left(\partial_x L(z,\dot z)[\xi] + \partial_v L(z,\dot z)[\dot\xi]\right) \mathrm{d}s\\ &= \int_0^1 \mu\left(\partial_x L(z,\dot z)[K] + \partial_v L(z,\dot z)[\dot K]\right)\mathrm{d}s + \int_0^1 \mu'\,\partial_v L(z,\dot z)[K]\mathrm{d}s\\ &= \int_0^1 \mu'\,\partial_v L(z,\dot z)[K]\mathrm{d}s.\end{aligned}$$

As a consequence, $\mathrm{d}\mathcal{A}(z)[\xi] = 0$ for all $\xi \in \mathcal{W}_z$ if and only if

$$\int_0^1 \mu' \, \partial_v L(z, \dot{z})[K]\mathrm{d}s = 0, \quad \forall \mu \in W_0^{1,2}([0,1], \mathbb{R}),$$

namely if and only if $\partial_v L(z, \dot{z})[K] = N(z, \dot{z})$ is constant a.e. on $[0, 1]$. □

Proposition 4.3 *The set $\mathcal{N}_{p,q}$ is a C^1 closed submanifold of $\Omega^{1,2}_{p,q}$. Moreover, for every $z \in \mathcal{N}_{p,q}$, the tangent space $T_z\mathcal{N}_{p,q}$ is given by*

$$T_z\mathcal{N}_{p,q} = \left\{ \xi \in T_z\Omega^{1,2}_{p,q} : \partial_x N(z, \dot{z})[\xi] + Q(\dot{\xi}) \text{ is constant a.e. on } [0,1] \right\}. \tag{4.4}$$

Proof Let $F : \Omega^{1,2}_{p,q} \to L^2([0,1], \mathbb{R})$ be defined as

$$F(z) := N(z, \dot{z})$$

and $\mathcal{C} \subset L^2([0,1], \mathbb{R})$ be defined as

$$\mathcal{C} := \left\{ f \in L^2([0,1], \mathbb{R}) : \ f(s) = \text{const. a.e.} \right\}.$$

By the definition of $\mathcal{N}_{p,q}$ given in (4.3), we have

$$\mathcal{N}_{p,q} = F^{-1}(\mathcal{C}).$$

The map F is C^1 and its differential is

$$\mathrm{d}F(z)[\xi] = \partial_x N(z, \dot{z})[\xi] + Q(\dot{\xi}). \tag{4.5}$$

By [37, Proposition 3, p. 28], it is enough to show that for all $z \in \mathcal{N}_{p,q}$ and $h \in L^2([0,1], \mathbb{R})$ there exist $\xi \in T_z\Omega^{1,2}_{p,q}$ and $c \in \mathbb{R}$ such that

$$\mathrm{d}F(z)[\xi] = h + c. \tag{4.6}$$

Therefore, let us fix $z \in \mathcal{N}_{p,q}$ and $h \in L^2([0,1], \mathbb{R})$. Let us consider $\xi \in \mathcal{W}_z \subset T_z\Omega^{1,2}_{p,q}$, so there exists $\mu \in W_0^{1,2}([0,1], \mathbb{R})$ such that $\xi(s) = \mu(s)K(z(s))$. By (4.5), recalling that d is invariant by the flow of K and then $\mathrm{d}d(K) = 0$, we obtain

$$\mathrm{d}F(z)[\xi] = \mu\big(\partial_x Q(\dot{z}, K) + Q(\dot{K})\big) + \mu' Q(K(z)). \tag{4.7}$$

Using (4.7) and recalling that by Assumption 2.2-(iii), $Q_x(K) \neq 0$ for all $x \in M$, (4.6) becomes an ODE in normal form with respect to μ, namely

$$\mu'(s) + a(s)\mu(s) = b_c(s), \tag{4.8}$$

where

$$a(s) = \frac{\partial_x Q(\dot{z}, K) + Q(\dot{K})}{Q(K(z))} \quad \text{and} \quad b_c(s) = \frac{h(s) + c}{Q(K(z))}.$$

Setting $A(s) = \int_0^s a(\tau)\mathrm{d}\tau$, and

$$c = -\left(\int_0^1 \frac{e^{A(s)}}{Q(K(z))} \mathrm{d}s \right)^{-1} \left(\int_0^1 \frac{e^{A(s)} h(s)}{Q(K(z))} \mathrm{d}s \right),$$

a solution of (4.8) which satisfies the boundary conditions $\mu(0) = \mu(1) = 0$ is given by

$$\mu(s) = e^{-A(s)} \int_0^s b_c(s) e^{A(\tau)} \mathrm{d}\tau.$$

Thus, for every $z \in \mathcal{N}_{p,q}$ and $h \in L^2([0,1], \mathbb{R})$, there exist $\xi \in T_z\Omega^{1,2}_{p,q}$ and $c \in \mathbb{R}$ such that (4.6) holds, hence $\mathcal{N}_{p,q}$ is a C^1 submanifold of $\Omega^{1,2}_{p,q}$.

By the previous part of the proof, for all $z \in \mathcal{N}_{p,q}$, $T_z\mathcal{N}_{p,q}$ is identified with the set of all ζ such that $\mathrm{d}F(z)[\zeta] \in T_{F(z)}\mathcal{C}$. Then, (4.4) follows from (4.5) and the fact that $T_{F(z)}\mathcal{C}$ is identified with the set of constant functions on $[0, 1]$.

It remains to show that $\mathcal{N}_{p,q}$ is closed. Let $(z_n)_n \subset \mathcal{N}_{p,q} \subset \Omega^{1,2}_{p,q}$ be a sequence converging to $z \in \Omega^{1,2}_{p,q}$. Up to considering a subsequence, we have that $N(z_n, \dot{z}_n)$ converges pointwise to $N(z, \dot{z})$, so $N(z, \dot{z})$ is constant a.e. on $[0, 1]$, i.e. $z \in \mathcal{N}_{p,q}$. □

Lemma 4.4 *For each $z \in \mathcal{N}_{p,q}$, $T_z\Omega^{1,2}_{p,q} = \mathcal{W}_z \oplus T_z\mathcal{N}_{p,q}$.*

Proof It is enough to show that for each $\zeta \in T_z\Omega^{1,2}_{p,q}$ there exists $\mu \in W^{1,2}_0([0,1], \mathbb{R})$ such that

$$\xi := \zeta - \mu K(z) \in T_z\mathcal{N}_{p,q}.$$

By (4.4), this amounts to prove that there exist $\mu \in W^{1,2}_0([0,1], \mathbb{R})$ and a constant $c \in \mathbb{R}$ such that

$$\partial_x N(z, \dot{z})[\xi] + Q(\dot{\xi}) = c, \quad \text{a.e. on } [0, 1],$$

which is equivalent to

$$\partial_x N(z, \dot{z})[\zeta] + Q(\dot{\zeta}) - \mu\big(\partial_x Q(\dot{z}, K) + Q(\dot{K})\big) - \mu' Q(K(z)) = c, \tag{4.9}$$

a.e. on $[0, 1]$. Arguing as in the proof of Proposition 4.3, we see that (4.9) admits a solution $\mu \in W^{1,2}_0([0,1], \mathbb{R})$ for a certain constant c, and we are done. □

Definition 4.5 The *reduced action functional* $\mathcal{J}$ is the restriction of the functional $\mathcal{A}$ to the manifold $\mathcal{N}_{p,q}$, i.e. $\mathcal{J} \colon \mathcal{N}_{p,q} \to \mathbb{R}$, $\mathcal{J} = \mathcal{A}\big|_{\mathcal{N}_{p,q}}$.

Remark 4.6 Being $\mathcal{A} \in C^1(\Omega^{1,2}_{p,q})$, we get that $\mathcal{J}$ is C^1 on $\mathcal{N}_{p,q}$ as well.

Theorem 4.7 *A curve $z \in \Omega^{1,2}_{p,q}$ is a critical point for $\mathcal{A}$ if and only if $z \in \mathcal{N}_{p,q}$ and z is a critical point for $\mathcal{J}$.*

Proof Let us assume that z is a critical point for $\mathcal{A}$. Then $\mathrm{d}\mathcal{A}(z)[\xi] = 0$ for all $\xi \in \mathcal{W}_z \subset T_z\Omega^{1,2}_{p,q}$ and by Proposition 4.2 we have $z \in \mathcal{N}_{p,q}$. Since $T_z\mathcal{N}_{p,q} \subset T_z\Omega^{1,2}_{p,q}$,

$$\mathrm{d}\mathcal{J}(z)[\xi] = \mathrm{d}\mathcal{A}(z)[\xi] = 0, \quad \forall \xi \in T_z\mathcal{N}_{p,q},$$

so z is a critical point for $\mathcal{J}$.

Now, let us assume that $z \in \mathcal{N}_{p,q}$ and z is a critical point for $\mathcal{J}$. By Lemma 4.4, for every $\zeta \in T_z\Omega^{1,2}_{p,q}$ there exist $\xi \in T_z\mathcal{N}_{p,q}$ and $\psi \in \mathcal{W}_z$ such that $\zeta = \psi + \xi$. By Proposition 4.2, we have $\mathrm{d}\mathcal{A}(z)[\psi] = 0$, while $\mathrm{d}\mathcal{A}(z)[\xi] = \mathrm{d}\mathcal{J}(z)[\xi] = 0$ because z is a critical point for $\mathcal{J}$. Therefore, $\mathrm{d}\mathcal{A}(z)[\zeta] = \mathrm{d}\mathcal{A}(z)[\psi] + \mathrm{d}\mathcal{A}(z)[\xi] = 0$, namely z is a critical point for $\mathcal{A}$. □

5 Lower boundedness and Palais-Smale condition for the reduced action

Let us give a condition on the manifold $\mathcal{N}_{p,q}$ implying that $\mathcal{J}$ is bounded from below and satisfies the Palais-Smale condition. For every $c \in \mathbb{R}$, we denote by $\mathcal{J}^c$ the sublevel of $\mathcal{J}$, namely

$$\mathcal{J}^c := \left\{z \in \mathcal{N}_{p,q} : \mathcal{J}(z) \leq c\right\}.$$

Definition 5.1 We say that $\mathcal{N}_{p,q}$ is *c-bounded* if $\mathcal{J}^c \neq \emptyset$ and

$$N_c := \sup_{z \in \mathcal{J}^c} |N(z, \dot{z})| < +\infty.$$

Proposition 5.2 *Under Assumptions 2.2–2.9, let $c \in \mathbb{R}$ such that $\mathcal{N}_{p,q}$ is c-bounded. Then, $\mathcal{J}$ is bounded from below.*

Proof By (2.7) and (2.15), we obtain

$$\begin{aligned}\mathcal{J}(z) &= \int_0^1 L(z, \dot{z})\mathrm{d}s \\ &\geq \int_0^1 \left(\frac{\lambda(z)}{4}\|\dot{z}\|^2 + L(z, 0) - \frac{1}{\lambda(z)}\|\partial_v L(x, 0)\|^2\right)\mathrm{d}s + \int_0^1 \frac{Q^2(\dot{z})}{Q(K(z))}\mathrm{d}s \end{aligned} \tag{5.1}$$

Since $\mathcal{N}_{p,q}$ is c-bounded and using (2.19), for every $z \in \mathcal{J}^c$ we have

$$Q^2(\dot{z}) = \left(N(z, \dot{z}) - d(x)\right)^2 \leq 2(N_c^2 + k_2^2) \tag{5.2}$$

thus, using (2.17) and (2.18) we have

$$\mathcal{J}(z) \geq c_2 - \frac{c_3^2}{c_1} - \frac{2(N_c^2 + k_2^2)}{k_1,}$$

and the thesis follows. □

We show now that c-boundedness and Assumptions 2.2–2.9 imply a compactness condition for the sublevels of $\mathcal{J}$.

Lemma 5.3 *Let $c \in \mathbb{R}$ be such that $\mathcal{N}_{p,q}$ is c-bounded. If Assumptions 2.2–2.9 hold, then every sequence $(z_n)_n \subset \mathcal{J}^c$ admits a uniformly convergent subsequence.*

Proof From (5.1) and Assumption 2.9, if $\mathcal{N}_{p,q}$ is c-bounded we have

$$c \geq \mathcal{J}(z_n) \geq \frac{c_1}{4}\int_0^1 \|\dot{z}_n\|^2\mathrm{d}s + c_2 - \frac{c_3^2}{c_1} - \frac{2(N_c^2 + k_2^2)}{k_1},$$

hence the sequence $\|\dot{z}_n\|$ is bounded in $L^2([0, 1])$. Then, denoting by d_g the distance induced by the metric g, by the Cauchy-Schwarz inequality we have

$$d_g(z_n(s_2), z_n(s_1)) \leq \int_{s_1}^{s_2} \|\dot{z}_n\|\mathrm{d}s \leq |s_2 - s_1|^{1/2}\left(\int_0^1 \|\dot{z}_n\|^2\mathrm{d}s\right)^{1/2},$$

for all $0 \leq s_1 \leq s_2 \leq 1$. Thus, (z_n) is uniformly bounded and uniformly equicontinuous and, being (M, g) complete, by the Ascoli-Arzelà theorem there exists a uniformly convergent subsequence. □

Definition 5.4 A sequence $(z_n)_n \subset \mathcal{J}^c$ is said a Palais-Smale sequence for $\mathcal{J}$ if $\mathrm{d}\mathcal{J}(z_n) \to 0$ strongly. We say that $\mathcal{J}$ satisfies the Palais-Smale condition on $\mathcal{J}^c$ if every Palais-Smale sequence $(z_n)_n \subset \mathcal{J}^c$ admits a strongly converging subsequence.

Remark 5.5 We point out that if Assumptions 2.2–2.9 hold and $\mathcal{N}_{p,q}$ is c-bounded, then $\mathcal{J}$ is bounded on any sequence $(z_n) \subset \mathcal{J}^c$ by Proposition 5.2, as it is required in the usual definition of the Palais-Smale condition.

Theorem 5.6 *Under Assumptions 2.2–2.9, assume also that $\mathcal{N}_{p,q}$ is c-bounded. Then $\mathcal{J}$ satisfies the Palais-Smale condition on $\mathcal{J}^c$.*

Proof Let $(z_n)_n \subset \mathcal{J}^c$ be a Palais-Smale sequence for $\mathcal{J}$. By Lemma 5.3, there exists a subsequence, still denoted by $(z_n)_n$, which uniformly converges to a continuous curve $z\colon [0,1] \to M$ such that $z(0) = p$ and $z(1) = q$.

Let us now notice that by Lemma 4.4, and taking into account that the supports of the curves z_n are in a compact subset of M, if $\zeta_n \in T_{z_n}\Omega^{1,2}_{p,q}$ is bounded in H^1 norm then there exist two bounded sequences $\xi_n \in T_{z_n}\mathcal{N}_{p,q}$ and $\mu_n \in H^1_0([0,1],\mathbb{R})$ such that $\zeta_n = \xi_n + \mu_n K_{z_n}$. By Proposition 4.2 and since z_n is a Palais-Smale sequence, we obtain

$$\mathrm{d}\mathcal{A}(z_n)[\zeta_n] = \mathrm{d}\mathcal{A}(z_n)[\xi_n] + \mathrm{d}\mathcal{A}(z_n)[\mu_n K_{z_n}] = \mathrm{d}\mathcal{J}(z_n)[\xi_n] \to 0.$$

We now apply a localization argument as in [2, Appendix A]; thus, we can assume that the Lagrangian L is defined on $[0,1] \times U \times \mathbb{R}^{m+1}$, with U an open neighborhood of 0 in $\mathbb{R}^{m+1}$. Moreover, we can identify $(z_n)_n$ with a sequence in the Sobolev space $H^1([0,1],U)$. By Lemma 5.3, taking into account that the curves z_n have fixed end-points, we get that $(z_n)_n$ is bounded in $H^1([0,1],U)$ and so it admits a subsequence, still denoted by (z_n), which weakly and uniformly converges to a curve $z \in H^1([0,1],\mathbb{R}^{m+1})$ which also satisfies the same fixed end-points boundary conditions. Thus, being $z_n - z$ bounded in H^1, we have $\mathrm{d}\mathcal{A}(z_n)[z_n - z] \to 0$, i.e.

$$\begin{aligned}
&\int_0^1 \partial_x L_c(z_n,\dot z_n)[z_n - z]\mathrm{d}s + \int_0^1 \partial_v L_c(z_n,\dot z_n)[\dot z_n - \dot z]\mathrm{d}s\\
&\quad - \int_0^1 \frac{2Q(\dot z_n)\partial_x Q(\dot z_n, z_n - z)}{\Lambda(z_n)}\mathrm{d}s - \int_0^1 \frac{2Q(\dot z_n)Q(\dot z_n - \dot z)}{\Lambda(z_n)}\mathrm{d}s\\
&\quad + \int_0^1 \frac{Q^2(\dot z_n)\mathrm{d}\Lambda(z_n)[\dot z_n - \dot z]}{\Lambda^2(z_n)}\mathrm{d}s \longrightarrow 0,
\end{aligned}$$

where $\Lambda(x) := -Q_x(K)$. From (2.12),

$$\big|\partial_x L_c(z_n,\dot z_n)[z_n - z]\big| \le C(z_n)\big(\|\dot z_n\|^2 + 1\big)\|z_n - z\|,$$

thus, recalling that C is continuous and $z_n - z$ uniformly converges to 0, the first integral in the above expression converges to 0. Since the sequence $Q(\dot z_n)$ is uniformly bounded on $[0,1]$ (recall (5.2)) and, from (2.18), $0 < 1/\Lambda(z_n) < 1/k_1$, the third term above converges to 0 because $\dot z_n$ is bounded in L^1 and $z_n - z \to 0$ uniformly. Analogously the fourth term goes to 0 since z_n converges uniformly to z and $\dot z_n - \dot z \to 0$ weakly in H^1. For estimating the fifth term, taking into account that $Q^2(\dot z_n)$ is uniformly bounded on $[0,1]$, we observe that $\mathrm{d}\Lambda(z_n) \to \mathrm{d}\Lambda(z)$ in operator norm and then

$$\int_0^1 \frac{\mathrm{d}\Lambda(z_n)[\dot z_n - \dot z]}{\Lambda^2(z_n)}\mathrm{d}s = \int_0^1 \frac{\big(\mathrm{d}\Lambda(z_n) - \mathrm{d}\Lambda(z)\big)[\dot z_n - \dot z]}{\Lambda^2(z_n)}\mathrm{d}s + \int_0^1 \frac{\mathrm{d}\Lambda(z)[\dot z_n - \dot z]}{\Lambda^2(z_n)}\mathrm{d}s,$$

and both the above integrals goes to 0, because $\dot{z}_n - \dot{z}$, in the first one, is bounded in L^1 and, in the second one, weakly converges to 0 in H^1. Thus, we have obtained that

$$\int_0^1 \partial_v L_c(z_n, \dot{z}_n)[\dot{z}_n - \dot{z}]\mathrm{d}s \longrightarrow 0. \tag{5.3}$$

Using that z_n pointwise converges to z and $\dot{z}_n$ is bounded in L^1, from (2.13) and Lebesgue's dominated convergence theorem we get

$$\int_0^1 \partial_v L_c(z_n, \dot{z})[\dot{z}_n - \dot{z}]\mathrm{d}s - \int_0^1 \partial_v L_c(z, \dot{z})[\dot{z}_n - \dot{z}]\mathrm{d}s \longrightarrow 0.$$

As $\dot{z}_n - \dot{z} \to 0$ weakly in H^1, also $\int_0^1 \partial_v L_c(z, \dot{z})[\dot{z}_n - \dot{z}]\mathrm{d}s \to 0$, and then from the above limit

$$\int_0^1 \partial_v L_c(z_n, \dot{z})[\dot{z}_n - \dot{z}]\mathrm{d}s \longrightarrow 0. \tag{5.4}$$

From (2.16), (5.3) and (5.4) we then get

$$\frac{c_1}{4}\int_0^1 |\dot{z}_n - \dot{z}|^2\mathrm{d}s \le \int_0^1 \big(\partial_v L_c(z_n, \dot{z}_n) - \partial_v L_c(z_n, \dot{z})\big)[\dot{z}_n - \dot{z}]\mathrm{d}s \longrightarrow 0,$$

which implies that $z_n \to z$ strongly in H^1. Moreover, there exists a subsequence such that $\dot{z}_n(s) \to \dot{z}(s)$ a.e. on $[0, 1]$ and then

$$N\big(z_n(s), \dot{z}_n(s)\big) \to N\big(z(s), \dot{z}(s)\big), \qquad \text{a.e. on } [0, 1],$$

so that also $N(z, \dot{z})$ is constant a.e. on $[0, 1]$, i.e. $z \in \mathcal{N}_{p,q}$ as required. □

From Propositions 5.2 and Theorem 5.6, $\mathcal{J}$ is bounded from below and satisfies the Palais-Smale condition on $\mathcal{J}^c$. Since $\mathcal{N}_{p,q}$ is only a C^1 submanifold of $\Omega^{1,2}_{p,q}$ (recall Proposition 4.3) then the exponential map of its infinite dimensional Riemannian structure is not well-defined, and we cannot invoke Ekeland's variational principle to conclude that a minimizer of $\mathcal{J}$ exists (see [25, Proposition 5.1]). Anyway, from [48, Theorem 3.1] (which, nevertheless, is based on Ekeland's variational principle) or as a straightforward consequence of the noncritical interval theorem (see [22, Theorem (2.15)]), we actually get the existence of a minimizer of $\mathcal{J}$. Summing up, we have the following result:

Theorem 5.7 *Let $L\colon TM \to \mathbb{R}$ be an indefinite Lagrangian satisfying Assumptions 2.2–2.9. Assume also that $\mathcal{N}_{p,q}$ is c-bounded, for some $c \in \mathbb{R}$. Then there exists a curve $z \in \mathcal{N}_{p,q}$ which minimizes $\mathcal{J}$ and it is then a critical point of $\mathcal{A}$ on $\Omega^{1,2}_{p,q}$.*

Remark 5.8 The critical points of $\mathcal{A}$ on $\Omega^{1,2}_{p,q}$, whose existence is ensured by Theorem 5.7, satisfy the Euler-Lagrange equation (2.6) in weak sense. We will show in Appendix A that they also satisfy it in classical sense.

6 Multiplicity of critical points

In this section we obtain a multiplicity result for critical points of the functional $\mathcal{A}$ by using Ljusternik-Schnirelmann theory, provided that M is a not contractible. Let us recall the definition of Ljusternik-Schnirelmann category. Let A be a non-empty subset of a topological space B; the *Lusternik-Schnirelman category* of a A, denoted by $\mathrm{cat}_B(A)$, is the least integer

n such that A can be covered n closed contractible (in B) subsets of B. If no such a number exists then $\mathrm{cat}_B(A) = +\infty$. If $A = \emptyset$, we set $\mathrm{cat}_B A = 0$. We denote $\mathrm{cat}_B(B)$ with $\mathrm{cat}(B)$.

By [26, Proposition 3.2], we know that if M is a non-contractible manifold then $\mathrm{cat}(\Omega^{1,2}_{p,q}) = +\infty$. This fact can be exploited together with the following proposition, which is a straightforward corollary of [22, Theorem (3.6)] and allows to prove the multiplicity of critical points for a functional of class C^1 defined on a manifold with the same regularity, as it is in our setting.

Theorem 6.1 (Corvellec-Degiovanni-Marzocchi) *Let $\mathcal{M}$ be a (possibly infinite dimensional) C^1 Riemannian manifold and $f : \mathcal{M} \to \mathbb{R}$ be a bounded from below C^1 functional satisfying the Palais-Smale condition.*

Then f has at least $\mathrm{cat}(\mathcal{M})$ critical points. Moreover, if $\mathrm{cat}(\mathcal{M}) = +\infty$ then $\sup f = +\infty$ and there exists a sequence $(c_m)_m$ of critical values such that $c_m \to +\infty$.

Remark 6.2 Actually [22, Theorem (3.6)] is stated for a continuous functional f on a complete metric space X with a critical point defined by using the notion of *weak slope* introduced in [24]. Points with vanishing weak slope are standard critical points if f is a C^1 functional on a Riemannian manifold. The metric space must also be *weakly locally contractible*, meaning that each $x \in X$ admits a neighborhood contractible in X. Notice that if X is weakly locally contractible then, for each $x \in X$, $\mathrm{cat}_X(\{x\}) = 1$. A C^1 Riemannian manifold is clearly weakly locally contractible (it is enough to take a small neighborhood of x diffeomorphic to a ball in the model Hilbert space). Thus, for example, both $\Omega^{1,2}_{p,q}$ and $\mathcal{N}_{p,q}$ are weakly locally contractible, the latter a fortiori being also a strong deformation retract of $\Omega^{1,2}_{p,q}$ if K is complete (see Proposition 6.4). Finally, we notice that in [22] the definition of Ljusternik-Schnirelman category is given with open coverings instead of closed one. This is equivalent to the definition with closed coverings in every ANR space; since metrizable manifolds are ANR (see [43, Theorem 5]), the two definitions are then equivalent for $\mathcal{N}_{p,q}$.

Let us now state the main result of this section.

Theorem 6.3 *Let M be a non-contractible manifold and $L\colon TM \to \mathbb{R}$ a Lagrangian that satisfies Assumptions 2.2–2.9. If K is a complete vector field and $\mathcal{N}_{p,q}$ is c-bounded for all $c \in \mathbb{R}$, then there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset \Omega^{1,2}_{p,q}$ of critical points of $\mathcal{A}$ such that*

$$\lim_{n\to\infty} \mathcal{A}(z_n) = +\infty.$$

Like in the existence result given in Theorem 5.7, we cannot work directly on $\Omega^{1,2}_{p,q}$ to prove Theorem 6.3, where $\mathcal{A}$ is not bounded from below and does not satisfy the Palais-Smale condition, but we have to restrict our analysis on $\mathcal{N}_{p,q}$.

Let us first show that when K is complete then $\mathcal{N}_{p,q}$ is a strong deformation retract of $\Omega^{1,2}_{p,q}$ (so that the Ljusternik-Schnirelmann category is preserved), namely there exists a homotopy $H\colon \Omega^{1,2}_{p,q} \times [0, 1] \to \Omega^{1,2}_{p,q}$ such that, for all $z \in \Omega^{1,2}_{p,q}$, $w \in \mathcal{N}_{p,q}$ and $t \in [0, 1]$, we have $H(z, 0) = z$, $H(z, 1) \in \mathcal{N}_{p,q}$ and $H(w, t) = w$. Next proposition extends [31, Proposition 5.9] from stationary Lorentzian manifold to our setting.

Proposition 6.4 *Assume that K is a complete vector field, then $\mathcal{N}_{p,q}$ is a strong deformation retract of $\Omega^{1,2}_{p,q}$.*

In the proof of Proposition 6.4, it will be useful the following preliminary result.

Lemma 6.5 *Let the vector field K be complete and let $\psi\colon \mathbb{R}\times M \to M$ be its flow. Then, for every $z \in \Omega^{1,2}_{p,q}$ there exists a uniquely defined function $\phi \in H^1_0([0,1],\mathbb{R})$ such that*

$$\psi\big(\phi(\cdot), z(\cdot)\big) \in \mathcal{N}_{p,q}. \tag{6.1}$$

Moreover, defining $\Psi\colon \Omega^{1,2}_{p,q} \to \mathcal{N}_{p,q}$ as

$$\big(\Psi(z)\big)(s) := \psi\big(\phi(s), z(s)\big),$$

the function Ψ is C^1.

Proof Let $z \in \Omega^{1,2}_{p,q}$ and, for each $\phi \in H^1_0([0,1],\mathbb{R})$, let us denote by $w\colon [0,1] \to M$ the curve

$$w(s) = \psi(\phi(s), z(s)) \tag{6.2}$$

We want to find $\phi \in H^1_0([0,1],\mathbb{R})$ such that $w \in \mathcal{N}_{p,q}$, hence $w(0) = p$, $w(1) = q$ and

$$N(w, \dot w) = C, \quad \text{a.e. on } [0,1], \tag{6.3}$$

for some constant $C \in \mathbb{R}$. By differentiating (6.2), we get

$$\dot w(s) = \partial_t\psi(\phi(s), z(s))\phi'(s) + \partial_x\psi(\phi(s), z(s))[\dot z(s)].$$

Substituting this expression in (6.3) and recalling that $N = Q + d$, we get

$$\begin{aligned}&\phi'(s)Q_{w(s)}\big(\partial_t\psi\big(\phi(s), z(s)\big)\big) + Q_{w(s)}\big(\partial_x\psi(\phi(s), z(s))[\dot z(s)]\big) + d\big(w(s)\big)\\ &\quad= \phi'(s)Q_{w(s)}\big(\partial_t\psi\big(\phi(s), z(s)\big)\big) + N\big(w(s), \partial_x\psi(\phi(s), z(s))[\dot z]\big) = C\end{aligned} \tag{6.4}$$

which, for each $z \in \Omega^{1,2}_{p,q}$, can be seen as a differential equation for ϕ.

Let us rewrite (6.4) in order to get a simpler equation. Since $\psi = \psi(t,x)$ is the flow generated by K, we have

$$\partial_t\psi(\phi, z) = K(\psi(\phi, z)) = K(w). \tag{6.5}$$

Using the group property $\psi(t_1, \psi(t_2, x)) = \psi(t_1 + t_2, x)$, and (6.5) we also obtain

$$\partial_x\psi(\phi, z)[K(z)] = K(w). \tag{6.6}$$

Moreover, recalling (2.4), for every $v \in T_{z(s)}M$ we have

$$L(z(s), v) = L(w(s), \partial_x\psi(\phi(s), z(s))[v]),$$

thus

$$\partial_v L(z(s), v)[K] = \partial_v L(w(s), \partial_x\psi\big(\phi(s), z(s))[v]\big)\big[\partial_x\psi(\phi(s), z(s))[K]\big].$$

By (2.1) and (6.6), the last equality becomes

$$N(z(s), v) = N\big(w(s), \partial_x\psi(\phi(s), z(s))[v]\big), \quad \forall v \in T_{z(s)}M. \tag{6.7}$$

Substituting v with $\dot z$ in (6.7), we get

$$N(z(s), \dot z(s)) = N\big(w(s), \partial_x\psi(\phi(s), z(s))[\dot z]\big). \tag{6.8}$$

Recalling that $Q(K)$ is invariant by the flow (see the proof of Proposition 2.5-(iv)), by (6.5) we have

$$Q_{w(s)}\big(\partial_t\psi\big(\phi(s), z(s)\big)\big) = Q_{w(s)}(K) = Q_{z(s)}(K). \tag{6.9}$$

Thus, from (6.8) and (6.9), (6.4) becomes

$$\phi' Q(K(z)) + N(z,\dot z) = C$$

and since by Assumption 2.2, $Q(K(z))$ is different from 0, we get

$$\phi' = \frac{C - N(z,\dot z)}{Q(K(z))}. \tag{6.10}$$

Hence, ϕ can be obtained as the solution of (6.11) with initial condition $\phi(0) = 0$ and, by setting $\int_0^1 \phi'(s)\mathrm{d}s = 0$, we can ensure that $\phi(1) = 0$ by taking

$$C = \left(\int_0^1 \frac{N(z,\dot z)}{Q(K(z))}\mathrm{d}s\right)\left(\int_0^1 \frac{\mathrm{d}s}{Q(K(z))}\right)^{-1}. \tag{6.11}$$

The fact that Ψ is C^1 is a simple consequence of the C^1-regularity of N and (6.10). □

Proof of Proposition 6.4 By Lemma 6.5, for every $z \in \Omega^{1,2}_{p,q}$, we consider $\phi \in H^1_0([0,1],\mathbb{R})$ (depending on z and univocally defined as shown in Lemma 6.5), such that (6.1) holds. Then, let us define $H\colon \Omega^{1,2}_{p,q} \times [0,1] \to \Omega^{1,2}_{p,q}$ as

$$H(z,t) := \psi(t\phi, z).$$

Notice that $H(\cdot,0)$ is the identity map on $\Omega^{1,2}_{p,q}$, and $H(\Omega^{1,2}_{p,q},1) \subset \mathcal{N}_{p,q}$. If $w \in \mathcal{N}_{p,q}$, then $N(w,\dot w)$ is constant and recalling that ϕ satisfies (6.10) with C given by (6.11), we get that the corresponding ϕ is the zero function, hence $H(w,t) = w$ for all $t \in [0,1]$. □

We can now prove Theorem 6.3.

Proof of Theorem 6.3 Since the Ljusternik-Schnirelmann category is a homotopy invariant, by Proposition 6.4 and [26, Proposition 3.2], we have $\mathrm{cat}(\mathcal{N}_{p,q}) = \mathrm{cat}(\Omega^{1,2}_{p,q}) = +\infty$. From Theorem 5.6, $\mathcal{J}$ satisfies the Palais-Smale condition on $\mathcal{J}^c$ for every $c \in \mathbb{R}$ and, then, on $\mathcal{N}_{p,q}$ (recall Remark 5.5). Hence, by Theorem 6.1, there exists a sequence $(z_n)_n \subset \mathcal{N}_{p,q}$ of critical points of $\mathcal{J}$ such that $\mathcal{J}(z_n) \to +\infty$. By Theorem 4.7, every critical point of $\mathcal{J}$ is a critical point of $\mathcal{A}$, and $\mathcal{A}(z_n) = \mathcal{J}(z_n)$. □

7 *c*-precompactness and *c*-boundedness

In light of Theorems 5.7 and 6.3, it becomes important to give conditions ensuring the c-boundedness of $\mathcal{N}_{p,q}$. We firstly need the following definition, introduced in [31].

Definition 7.1 Let c be a real number. The set $\mathcal{N}_{p,q}$ is said to be *c-precompact* if every sequence $(z_n)_n \subset \mathcal{J}^c$ has a uniformly convergent subsequence. We say that $\mathcal{J}$ is *pseudocoercive* if $\mathcal{N}_{p,q}$ is c-precompact for all $c \in \mathbb{R}$.

We are going to show that c-boundedness and c-precompactness are essentially equivalent properties for Lagrangians admitting a local expression of "product" type (see (7.1) below). As a first step, we notice that Lemma 5.3 immediately gives one of the implications in the equivalence.

Proposition 7.2 *Let Assumptions 2.2–2.9 hold. If $\mathcal{N}_{p,q}$ is c-bounded, then it is c-precompact.*

The converse implication holds if L admits a local structure of the type in (3.1), so we give the following definition.

Definition 7.3 We say that L admits a *stationary product type local structure* if for every point $p \in M$ there exist an open precompact neighborhood $U_p \subset M$ of p, a manifold with boundary S_p, an open interval $I_p = (-\epsilon_p, \epsilon_p) \subset \mathbb{R}$, and a diffeomorphism $\phi\colon S_p \times I_p \to U_p$ such that, named t the natural coordinate of I_p,

$$\phi_*(\partial_t) = K\big|_{U_p},$$

and for all $\big((x,t),(\nu,\tau)\big) \in T(S_p \times I_p)$ we have

$$L \circ \phi_*\big((x,t),(\nu,\tau)\big) = L_0(x,\nu) + 2\big(\omega(\nu) + d(x)/2\big)\tau - \beta(x)\tau^2, \tag{7.1}$$

where

- $L_0 \in C^1(TS_p)$ is a Lagrangian on S_p which satisfies (3.2)–(3.4) with respect to the norm $\|\cdot\|_{S_p}$ of the metric induced on S_p by the auxiliary Riemannian metric on M, and it is pointwise strongly convex, i.e. it satisfies (2.14) on TS_p (with L_c replaced by L_0 and $\|\cdot\|$ by $\|\cdot\|_{S_p}$), for a continuous function $\lambda\colon S_p \to (0,+\infty)$;
- ω is a C^1 one-form on S_p;
- $d\colon S_p \to \mathbb{R}$ is a C^1 function;
- $\beta\colon S_p \to (0,+\infty)$ is a positive C^1 function.

Notice that Definition 7.3 is satisfied for $L(v) = g_L(v,v)$, where g_L is a C^1 Lorentzian metric on M having a timelike Killing vector field K; in such a case S_p is a spacelike hypersurface in M, L_0 is the Riemannian metric induced on it by g_L, $\beta(x) = -g_L(K_x, K_x)$ and ω is the one-form metrically equivalent to the orthogonal projection of K on TS_p and $d \equiv 0$ (see, e.g., [31, Appendix C]). The next result shows that it is satisfied as well by a Lagrangian fulfilling Assumptions 2.2–2.7.

Proposition 7.4 *Let $L : TM \to \mathbb{R}$ satisfy Assumptions 2.2 and 2.7. Then it admits a stationary product type local structure.*

Proof Let us denote by $\mathcal{D}$ the distribution in TM generated by the kernel of Q, i.e. for all $z \in M$, $\mathcal{D}_z = \ker Q_z$. Notice that by (2.2), $\mathcal{D}$ has constant rank equal to m (recall that $\dim(M) = m+1$). Let $\bar z \in M$ and $S_{\bar z}$ be a smooth hypersurface (with boundary) in M such that $\bar z \in S_{\bar z}$ and $T_{\bar z}S_{\bar z} = \mathcal{D}_{\bar z}$. We endow $S_{\bar z}$ with the Riemannian metric induced by the auxiliary Riemannian metric g on M and let us denote its norm with $\|\cdot\|_{S_{\bar z}}$. From (2.2), up to shrink $S_{\bar z}$, we can assume that for all $x \in S_{\bar z}$, K_x is transversal to $S_{\bar z}$, i.e. $T_xM = T_xS_{\bar z} \oplus [K_x]$. Using (2.7), we get

$$\partial_v L(x,\nu) = \partial_v L_c(x,\nu) + \frac{2}{Q(K)} Q(\nu) Q_x,$$

for all $(x,\nu) \in TS_{\bar z}$. In particular, $\partial_v L(\bar z,\nu) = \partial_v L_c(\bar z,\nu)$ for all $\nu \in T_{\bar z}S_{\bar z}$. Considering a smaller hypersurface $S_{\bar z}$ such that

$$\lambda_0 := \min_{x \in S_{\bar z}} \left(\lambda(x) + \frac{2}{Q(K)} \max_{\|\nu\|_{S_{\bar z}}=1} Q_x^2(\nu)\right) > 0, \tag{7.2}$$

for all $(x,\nu_1), (x,\nu_2) \in TS_{\bar z}$ we have

$$\begin{aligned}
&\big(\partial_v L(x,\nu_2) - \partial_v L(x,\nu_1)\big)[\nu_2 - \nu_1] \\
&\quad = \big(\partial_v L_c(x,\nu_1) - \partial_v L_c(x,\nu_2)\big)[\nu_2 - \nu_1] \\
&\qquad + \frac{2}{Q(K)} Q^2(\nu_2 - \nu_1) \geq \lambda_0 \|\nu_2 - \nu_1\|^2_{S_{\bar z}}.
\end{aligned}$$

Let $L_0 = L|_{TS_{\bar{z}}}$; the above inequality gives then (2.14) for L_0 on $TS_{\bar{z}}$. Since $L_0(x, v) = L_c(x, v) + \frac{Q_x^2(v)}{Q_x(K_x)}$ and L_c satisfies (2.11)–(2.13), we also have that L_0 satisfies (3.2)–(3.4). Let us now evaluate $\frac{\mathrm{d}}{\mathrm{d}s}L(x, y + s\tau K)$, for any $y \in T_xM$, $x \in M$, and $\tau \in \mathbb{R}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}L(x, y + s\tau K) &= \partial_v L(x, y + s\tau K)[\tau K] = \tau\partial_v L(x, y + s\tau K)[K]\\ &= \tau\big(Q(y + s\tau K) + d(x)\big) = \tau\big(Q(y) + s\tau Q(K) + d(x)\big).\end{aligned}$$

Hence, integrating w.r.t. s between 0 and 1 we get

$$L(x, y + \tau K) - L(x, y) = \tau\big(Q(y) + d(x)\big) + \frac{1}{2}\tau^2 Q(K). \tag{7.3}$$

Let now $w \in T_xM$, $x \in S_{\bar{z}}$, and $w_S \in T_xS_{\bar{z}}$, $\tau_w \in \mathbb{R}$ such that $w = w_S + \tau_w K_x$. From (7.3) we get

$$\begin{aligned}L(x, w) &= L(x, w_S + \tau_w K)\\ &= L(x, w_S) + \tau_w\big(Q(w_S) + d(x)\big) + \frac{1}{2}\tau_w^2 Q(K)\\ &= L_0(x, w_S) + \tau_w\big(Q(w_S) + d(x)\big) + \frac{1}{2}\tau_w^2 Q(K).\end{aligned} \tag{7.4}$$

Thus, we get an expression of the type at the right-hand side of (7.1) on $S_{\bar{z}}$ by defining ω as the one-form induced by $Q/2$ on $S_{\bar{z}}$ and $\beta(x) := -Q(K)/2$. Since L is invariant by the flow of K^c we then obtain (7.1) on $S_{\bar{z}} \times I_{\bar{z}}$, for some open interval $I_{\bar{z}}$ containing 0, by taking ϕ as the restriction to $S_{\bar{z}} \times I_{\bar{z}}$ of the flow ψ of K adapted to $S_{\bar{z}}$, i.e. such that $S_{\bar{z}} = \psi(S_{\bar{z}} \times \{0\})$. □

Remark 7.5 Notice that if the distribution $\mathcal{D}$ generated by the kernel of Q is integrable then we can take in the above proof $S_{\bar{z}}$ equal to an integral manifold of $\mathcal{D}$. In this case the local expression of L simplifies to

$$L \circ \phi_*\big((x, t), (v, \tau)\big) = L_0(x, v) + d(x)\tau - \beta(x)\tau^2.$$

This can be considered as a generalization of the notion of a static Lorentzian metric to an indefinite Lagrangian admitting an infinitesimal symmetry satisfying Assumptions 2.2–2.7 (compare also with [20, 21]).

By Proposition 7.4 we obtain the following generalization of [31, Lemma 4.1].

Theorem 7.6 *Let Assumptions 2.2 and 2.7 hold. If $\mathcal{N}_{p,q}$ is c-precompact then it is c-bounded.*

Proof Let $(z_n)_n \subset \mathcal{J}^c$ be a sequence such that

$$\lim_{n\to\infty} |N(z_n, \dot{z}_n)| = \sup_{z\in\mathcal{J}^c} |N(z, \dot{z})|.$$

Moreover, let $(C_{z_n})_n \subset \mathbb{R}$ be the sequence of real numbers such that for all n

$$C_{z_n} = \frac{1}{2}N\big(z_n(s), \dot{z}_n(s)\big), \quad \text{a.e. in } [0, 1].$$

To obtain the thesis, it suffices to prove that C_{z_n} is bounded. Since $\mathcal{N}_{p,q}$ is c-precompact we can assume, up to pass to a subsequence, that z_n converges uniformly to a curve $z \in \mathcal{J}^c$. We can then assume that there exists a finite number of neighborhoods U_k, with $k = 1, \dots, N$, that cover $z([0, 1])$ such that, for some finite sequence $0 = a_0 < a_1 < \cdots < a_N = 1$, $z_n([a_{k-1}, a_k]) \subset U_k$, for all n sufficiently large and for all $k = 1, \dots, N$. Moreover, by

Proposition 7.4, in each domain U_k we can identify L with $L \circ (\phi_k)_*$ so that L, evaluated along a curve $z(s) = \big(x(s), t(s)\big)$ contained in U_k, is given by

$$L(z, \dot z) = L_{0,k}(x, \dot x) + 2\big(\omega_k(\dot x) + d_k(x)/2\big)\dot t - \beta_k(x)\dot t^2,$$

(here we are not writing the point where the one-forms ω_k are applied). Up to replace each U_k by a precompact open subset, we can assume that

$$\max_k(\|\omega_k\|) = \max_k \Big(\sup_{\substack{\|y\|=1 \\ y \in TU_k}} |\omega_k(y)|\Big) = D_0 < +\infty, \tag{7.5}$$

and

$$\max_k \big(\sup_{x \in U_k} |d_k(x)|\big) = D_1 < +\infty, \tag{7.6}$$

reminding that $\|y\| = \sqrt{g(y, y)}$, where g is the auxiliary Riemannian metric. Analogously, we have

$$\Delta = \max_k \Big(\sup_{m_1, m_2 \in U_k} |t^k(m_1) - t^k(m_2)|\Big) < +\infty$$

and there also exist two constants ν, μ such that

$$0 < \nu \le \beta_k \le \mu, \quad \text{for all } k \in \{1, \dots, N\}.$$

In the following, we write $L_{0,k}$, ω_k, d_k and β_k without the index k. In this local charts, let $z_n(s) = \big(x_n(s), t_n(s)\big)$. As for (3.1) and (3.5), we have $N(z_n, \dot z_n) = 2\omega(\dot x_n) - 2\beta \dot t_n + d(x_n)$. Hence,

$$\dot t_n = \frac{\omega(\dot x_n) + d(x_n)/2 - C_{z_n}}{\beta(x_n)}. \tag{7.7}$$

Defining $T_n^k = t_n(a_k) - t_n(a_{k-1})$, we have

$$T_n^k = \int_{a_{k-1}}^{a_k} \dot t_n \mathrm{d}s = \int_{a_{k-1}}^{a_k} \frac{\omega(\dot x_n) + d(x_n)/2 - C_{z_n}}{\beta(x_n)} \mathrm{d}s. \tag{7.8}$$

Therefore, the quantity

$$b_n^k := \int_{a_{k-1}}^{a_k} \frac{\mathrm{d}s}{\beta(x_n(s))}.$$

is well-defined and finite. Moreover,

$$\frac{a_k - a_{k-1}}{\mu} \le b_n^k \le \frac{a_k - a_{k-1}}{\nu}. \tag{7.9}$$

From (7.8) we obtain

$$C_{z_n} = \frac{1}{b_n^k} \left(\int_{a_{k-1}}^{a_k} \frac{\omega(\dot x_n) + d(x_n)/2}{\beta(x_n)} \mathrm{d}s - T_n^k \right). \tag{7.10}$$

By (7.5), we have $|w(\dot x_n)| \le D_0 \|\dot x_n\|$. As a consequence, using also (7.6), $|T_n^k| \le \Delta$ and (7.9), from (7.10) we have

$$|C_{z_n}| < \frac{(D_0 + D_1)\mu}{\nu(a_{k-1} - a_k)} \int_{a_{k-1}}^{a_k} \|\dot x_n\| \mathrm{d}s + \frac{\mu \Delta}{a_{k-1} - a_k}. \tag{7.11}$$

By (7.11), to prove that C_{z_n} is bounded, and thus to prove the theorem, it suffices to show that

$$\sup_n \int_0^1 \|\dot{x}_n\| \mathrm{d}s < +\infty. \tag{7.12}$$

To this end, recall that by (7.7) we have

$$\begin{aligned} L\big((x_n, t_n), (\dot{x}_n, \dot{t}_n)\big) &= L_0(x_n, \dot{x}_n) + 2\big(\omega(\dot{x}_n) + d(x_n)/2\big)\dot{t}_n - \beta(x_n)\dot{t}_n^2 \\ &= L_0(x_n, \dot{x}_n) + \frac{\big(\omega(\dot{x}_n) + d(x_n)/2\big)^2 - C_{z_n}^2}{\beta(x_n)}, \end{aligned}$$

therefore, using also (7.10) we obtain

$$\begin{aligned} &\int_{a_{k-1}}^{a_k} L\big((x_n, t_n), (\dot{x}_n, \dot{t}_n)\big)\mathrm{d}s = \int_{a_{k-1}}^{a_k} L_0(x_n, \dot{x}_n)\mathrm{d}s \\ &+ \int_{a_{k-1}}^{a_k} \frac{\big(\omega(\dot{x}_n) + d(x_n)/2\big)^2}{\beta(x_n)}\mathrm{d}s - \frac{1}{b_n^k}\left(\int_{a_{k-1}}^{a_k} \frac{\omega(\dot{x}_n) + d(x_n)/2}{\beta(x_n)}\mathrm{d}s\right)^2 \\ &+ \frac{2T_n^k}{b_n^k}\int_{a_{k-1}}^{a_k} \frac{\omega(\dot{x}_n) + d(x_n)/2}{\beta(x_n)}\mathrm{d}s - \frac{(T_n^k)^2}{b_n^k}. \end{aligned} \tag{7.13}$$

By the Schwartz inequality in L^2, we obtain

$$\begin{aligned} &\left(\int_{a_{k-1}}^{a_k} \frac{\omega(\dot{x}_n) + d(x_n)/2}{\beta(x_n)}\mathrm{d}s\right)^2 \\ &\quad \leq \left(\int_{a_{k-1}}^{a_k} \frac{\mathrm{d}s}{\beta(x_n)}\right) \int_{a_{k-1}}^{a_k} \frac{\big(\omega(\dot{x}_n) + d(x_n)/2\big)^2}{\beta(x_n)}\mathrm{d}s \\ &\quad = b_n^k \int_{a_{k-1}}^{a_k} \frac{\big(\omega(\dot{x}_n) + d(x_n)/2\big)^2}{\beta(x_n)}\mathrm{d}s. \end{aligned}$$

Hence, from (7.13) we obtain

$$\begin{aligned} &\int_{a_{k-1}}^{a_k} L\big((x_n, t_n), (\dot{x}_n, \dot{t}_n)\big)\mathrm{d}s \geq \int_{a_{k-1}}^{a_k} L_0(x_n, \dot{x}_n)\mathrm{d}s \\ &\quad + \frac{2T_n^k}{b_n^k}\int_{a_{k-1}}^{a_k} \frac{\omega(\dot{x}_n) + d(x_n)/2}{\beta(x_n)}\mathrm{d}s - \frac{(T_n^k)^2}{b_n^k}. \end{aligned} \tag{7.14}$$

Since L_0 is the Lagrangian in a stationary product type local structure, as for (2.15), we deduce that there exist two positive constants $\ell_1, \ell_2 \in \mathbb{R}$ such that, for all the domains U_k of the charts, we have

$$L_0(x_n, \dot{x}_n) \geq \ell_1 \|\dot{x}_n\|^2 - \ell_2.$$

Since d_k, T_n^k and $1/b_n^k$ are bounded for each k, we obtain the existence of two positive constants E_1, E_2 (depending on $\nu, \mu, \Delta, D_0, D_1, \ell_2$) such that

$$\begin{aligned} c \geq \mathcal{J}(z_n) &= \int_0^1 L(x_n, \dot{z}_n)\mathrm{d}s = \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} L\big((x_n, t_n), (\dot{x}_n, \dot{t}_n)\big)\mathrm{d}s \\ &\geq \ell_1 \int_0^1 \|\dot{x}_n\|^2 \mathrm{d}s - E_1 \int_0^1 \|\dot{x}_n\| \mathrm{d}s - E_2. \end{aligned}$$

As a consequence, (7.12) holds and by (7.10) we conclude that $\mathcal{N}_{p,q}$ is c-precompact. □

Remark 7.7 If $\mathcal{N}_{p,q}$ is c-precompact, then there exists a compact subset of M that contains the images of all curves in $\mathcal{J}^c$. Therefore, Assumption 2.9 holds on such a compact set.

From Theorem 7.6, Remark 7.7, Theorems 5.7, 6.3 we deduce the following corollary.

Corollary 7.8 *Let $L\colon TM \to \mathbb{R}$ satisfy Assumptions 2.2 and 2.7. If $\mathcal{N}_{p,q}$ is c-precompact for some $c \in \mathbb{R}$ such that $\mathcal{J}^c \neq \emptyset$, then $\mathcal{J}^c$ is bounded from below and it admits a minimizer which is critical point of $\mathcal{A}$.*

Moreover, if $\mathcal{J}$ is pseudocoercive, K is complete, and M is a non-contractible manifold, then $\mathcal{N}_{p,q} \neq \emptyset$ and there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset \Omega^{1,2}_{p,q}$ of critical points of $\mathcal{A}$ such that $\lim_{n\to\infty} \mathcal{A}(z_n) = +\infty$.

Recalling Example 3.8, by Corollary 7.8 we then obtain the following extension of [31, Theorems 1.2 and 1.3] to C^1 stationary Lorentzian manifolds.

Corollary 7.9 *Let (M, g) be a Lorentzian manifold such that g is a C^1 metric endowed with a timelike Killing vector field K. If $\mathcal{N}_{p,q}$ is c-precompact for some $c \in \mathbb{R}$ such that $\mathcal{J}^c \neq \emptyset$, then there exists a geodesic connecting p to q. Moreover, if $\mathcal{J}$ is pseudocoercive, K is complete, and M is a non-contractible manifold, then $\mathcal{N}_{p,q} \neq \emptyset$ and there exists a sequence of geodesics $(z_n)_{n\in\mathbb{N}} \subset \Omega^{1,2}_{p,q}$ with unbounded energy.*

Remark 7.10 Apart from completeness of K, whenever $d = 0$, a condition ensuring that $\mathcal{N}_{p,q}$ is non-empty for all p and q in M is that the distribution $\mathcal{D}$ defined by the kernel of Q is not integrable through any point in M. Indeed by Chow-Rashevskii Theorem, there exists then a horizontal C^1 curve γ connecting p to q. Hence, such curve belongs to $\mathcal{N}_{p,q}$ with constant $Q(\dot\gamma) = 0$. We recall that, in the case when L is the quadratic form associated with a stationary Lorentzian metric g_L with Killing vector field K, the non-integrability of $\mathcal{D}$ through any point is equivalent to the fact that K is not static in any region of M. Geodesic connectedness of a smooth static Lorentzian manifold was studied in [18]; we point out that, thanks to Theorem 6.1, the results in [18] can be extended to a C^1 static Lorentzian metric.

8 Dynamic conditions for pseudocoercivity

Inspired by Appendix A in [31], we give some conditions that ensure that $\mathcal{J}$ is pseudocoercive.

Let us assume that there exists a C^1 function $\varphi\colon M \to \mathbb{R}$ which satisfies the monotonicity condition $\mathrm{d}\varphi(K) > 0$.

If K is complete, this implies that M is foliated by level sets of the function φ, and it splits as $\Sigma \times \mathbb{R}$, where Σ is one of this level set. Notice that [31, Assumption (4.11)] implies the completeness of the timelike Killing vector field there, so the setting leading to [31, Proposition A.3] is actually analogous to ours (compare also with [15, Theorem 2.3]). Some differences, on the other hand, are that the splitting $\Sigma \times \mathbb{R}$ is only C^1 and there is no simple link between convexity properties of the induced Lagrangian L_0 and the level set Σ (see Remark 8.3).

Since Σ is transversal to K, using Assumption 2.2 and arguing as in the proof of Proposition 7.4, we get that L is given by (3.1) in $\Sigma \times \mathbb{R}$ for a C^1 Lagrangian $L_0\colon T\Sigma \to \mathbb{R}$. Let us denote by g_Σ the C^1 Riemannian metric on Σ induced by g. We assume that the one-form ω induced by Q on Σ has sublinear growth w.r.t. the distance d_Σ induced by g_Σ, i.e. there exist $\alpha \in [0, 1)$ and two non-negative constants k_0 and k_1 such that

$$\|\omega\|_\Sigma \leq k_0 + k_1\big(d_\Sigma(x, x_0)\big)^\alpha, \tag{8.1}$$

for some $x_0 \in \Sigma$ and all $x \in \Sigma$. We recall that β in an expression like (3.1) for L is equal to $-Q(K)/2$ (see (7.4)).

Proposition 8.1 *Let L satisfy Assumption 2.2 with d in (2.1) bounded and K complete. Let $\varphi\colon M \to \mathbb{R}$ be a C^1 function such that $\mathrm{d}\varphi(K) > 0$. Let Σ a level set of φ and L_0 be the Lagrangian induced by L on Σ. Assume that*

- *L_0 satisfies (i) and (ii) in Example 3.1 (namely it satisfies the growth conditions and the pointwise convexity) and there exist three constants ℓ_1, ℓ_2, ℓ_3 such that $\lambda_0(x) \geq \ell_1 > 0$, $L_0(x, 0) \geq \ell_2$ and $\|\partial_v L_0(x, 0)\|_\Sigma \leq \ell_3$;*
- *ω satisfies (8.1);*
- *there exist two constant b_1 and b_2 such that $0 < b_1 \leq \beta(x) \leq b_2$, for all $x \in \Sigma$.*

Then $\mathcal{J}$ is pseudocoercive.

Proof Recalling that, by Definition 7.1, $\mathcal{J}$ is pseudocoercive if $\mathcal{N}_{p,q}$ is c-precompact for all $c \in \mathbb{R}$, the thesis follows from Proposition 7.2 by showing that that $\mathcal{N}_{p,q}$ is c-bounded for all $c \in \mathbb{R}$.

Let us set $\Delta := t(q) - t(p)$ and let $z_n = z_n(s) = \big(x_n(s), t_n(s)\big) \in \mathcal{J}^c$ be a sequence such that $|N(z_n, \dot{z}_n)| \to \sup_{z \in \mathcal{J}^c} |N(z, \dot{z})|$. As for (7.14), we then get

$$\begin{aligned} c &\geq \int_0^1 L\big((x_n, t_n), (\dot{x}_n, \dot{t}_n)\big)\mathrm{d}s \\ &\geq \int_0^1 L_0(x_n, \dot{x}_n)\mathrm{d}s + \frac{2\Delta}{b_n}\int_0^1 \frac{\omega(\dot{x}_n) + d(x_n)/2}{\beta(x_n)}\mathrm{d}s - \frac{\Delta^2}{b_n}, \end{aligned} \tag{8.2}$$

where $b_n = \int_0^1 \frac{1}{\beta(x_n(s))}\mathrm{d}s$. Then taking into account that d is bounded, $0 < b_1 \leq \beta(x) \leq b_2$, $\lambda_0(x) \geq \ell_1 > 0$, $L_0(x, 0) \geq \ell_2$ and $\|\partial_v L_0(x, 0)\|_\Sigma \leq \ell_3$, for all $x \in \Sigma$, using (2.15) for L_0 and (8.1), we obtain from (8.2) that $\int_0^1 \|\dot{x}_n\|^2_\Sigma \mathrm{d}s$ is bounded. Analogously to (7.7) we have then

$$N(z_n, \dot{z}_n) = 2C_{z_n} = \frac{2}{b_n}\left(\int_0^1 \frac{\omega(\dot{x}_n) + d(x_n)/2}{\beta}\mathrm{d}s - \Delta\right)$$

and hence $N(z_n, \dot{z}_n)$ is bounded as well. □

Remark 8.2 The proof of Proposition 8.1 also shows that the manifold $\mathcal{N}_{p,q}$ associated to the Lagrangian in Example 3.1 is c-bounded for all $c \in \mathbb{R}$ provided that L_0 satisfies (i) and (ii) in Example 3.1, d is bounded, ω has sublinear growth on S (hence (8.1) holds) and there exist some constants $b_1, b_2, \ell_1, \ell_2, \ell_3$ such that $0 < b_1 \leq \beta(x) \leq b_2$, $\lambda_0(x) \geq \ell_1 > 0$, $L_0(x, 0) \geq \ell_2$ and $\|\partial_v L_0(x, 0)\|_\Sigma \leq \ell_3$, for all $x \in S$.

Remark 8.3 The strong convexity condition for L_0 holds if L_0, satisfying (2.14) on $T\Sigma$, satisfies also (7.2) on Σ. This condition can be considered as a replacement of being Σ a spacelike and complete hypersurface when L is the quadratic form of a stationary Lorentzian manifold (in our setting the Riemannian metric on Σ, induced by the auxiliary one g, is complete because g is complete by assumptions). Indeed, in such a case, it is enough to assume that $\nabla\varphi$ is timelike (i.e. φ is a C^1 time function) to get that a level set Σ of φ is spacelike. The existence of such a φ is guaranteed if there exists a spacelike hypersurface that intersects once every flow line of the complete timelike Killing vector field K (see [31, Appendix A]).

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Appendix A: Regularity of the critical points

In this section we show that a critical point of the action functional $\mathcal{A}$ on $\Omega_{p,q}$ is actually a curve of class C^1. This is a quite standard result in relation with Assumptions 2.2 and 2.7, but we give the details for the reader convenience.

Proposition A.1 *Let $L\colon TM \to \mathbb{R}$ be a Lagrangian satisfying Assumptions 2.2 and 2.7 and let z be a critical point of the action functional $\mathcal{A}\colon \Omega^{1,2}_{p,q} \to \mathbb{R}$, $\mathcal{A}(z) = \int_0^1 L(z,\dot z)ds$. Then, both z and $\partial_v L(z,\dot z)$ are of class C^1, the Euler-Lagrange equation (2.6) holds in classical sense, namely for all $i \in \{0,\dots,m\}$,*

$$\frac{\partial L}{\partial z^i}\big(z(s),\dot z(s)\big) = \frac{d}{ds}\left(\frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\right), \quad \text{for all } s\in[0,1], \tag{A.1}$$

and z satisfies the conservation law

$$\partial_v L(z,\dot z)[\dot z] - L(z,\dot z) = E, \tag{A.2}$$

for some constant $E \in \mathbb{R}$.

Proof As regularity of a critical curve is a local result, by Proposition 7.4 we can assume, without loosing generality, that L is a Lagrangian on $U\times I$, where U is a precompact open neighborhood of $\mathbb{R}^m$ and $I\subset\mathbb{R}$ an open interval, defined as

$$L\big((x,t),(\nu,\tau)\big) = L_0(x,\nu) + 2\big(\omega(\nu) + d(x_n)/2\big)\tau - \beta(x)\tau^2, \tag{A.3}$$

for all $\big((x,t),(\nu,\tau)\big) \in (U\times I)\times(\mathbb{R}^m\times\mathbb{R})$. Arguing as in the proof of Proposition 7.4, for any point $\bar z \in M$, we can take U as a hypersurface in M passing through $\bar z$ such that ω vanishes at $\bar z$. Let $z : [0,1] \to U\times I$, $z(s) = (x(s), t(s))$ be a critical point for $\mathcal{A}$ then for all $(\xi,\eta) \in H^1_0([0,1],\mathbb{R}^m)\times H^1_0([0,1],\mathbb{R})$ we have

$$\begin{aligned} 0 = \mathrm{d}\mathcal{A}(z)[(\xi,\eta)] &= \int_0^1 \big(\partial_x L_0(x,\dot x)[\xi] + \partial_v L_0(x,\dot x)[\dot\xi]\big)\,\mathrm{d}s \\ &\quad +2\int_0^1 \left(\partial_x\omega(\xi,\dot x)\dot t + \omega(\dot\xi)\dot t + \omega(\dot x)\dot\eta + \frac{1}{2}\mathrm{d}d(\xi)\dot t + \frac{d(x)}{2}\dot\eta\right)\mathrm{d}s \\ &\quad -\int_0^1 \big(\mathrm{d}\beta(\xi)\dot t^2 + 2\beta(x)\dot t\dot\eta\big)\,\mathrm{d}s. \end{aligned} \tag{A.4}$$

Since $z = (x,t)$ is a critical point of $\mathcal{A}$, there exists a constant $C_z \in \mathbb{R}$ such that

$$N(z,\dot z) = 2\omega(\dot x) - 2\beta(x)\dot t + d(x) = 2C_z,$$

hence we have

$$\dot t = \frac{\omega(\dot x) + d(x)/2 - C_z}{\beta(x)}. \tag{A.5}$$

Moreover, from (A.4), for all $\xi \in C_0^\infty([0,1],\mathbb{R}^m)$ we have

$$\mathrm{d}A(z)[\xi,0] = \int_0^1 \big(\partial_x L_0(x,\dot x)[\xi] + \partial_v L_0(x,\dot x)[\dot\xi]\big)\,\mathrm{d}s$$
$$+2\int_0^1 \left(\partial_x\omega(\xi,\dot x) + \omega(\dot\xi) + \frac{1}{2}\mathrm{d}d(\xi)\right)\dot t\,\mathrm{d}s - \int_0^1 \mathrm{d}\beta(\xi)\dot t^2\mathrm{d}s = 0,$$

hence,

$$\int_0^1 \big(\partial_v L_0(x,\dot x)[\dot\xi] + 2\omega(\dot\xi)\dot t\big)\,\mathrm{d}s = -\int_0^1 \partial_x L_0(x,\dot x)(\xi)\mathrm{d}s$$
$$-\int_0^1 \big(2\partial_x\omega(\xi,\dot x)\dot t + \mathrm{d}d(\xi)\dot t - \mathrm{d}\beta(\xi)\dot t^2\big)\mathrm{d}s.$$

Then, there exists an L^1 map $h\colon [0,1]\to(\mathbb{R}^m)^*$ such that

$$\int_0^1 \big(\partial_v L_0(x,\dot x)[\dot\xi] + 2\omega(\dot\xi)\dot t\big)\,\mathrm{d}s = \int_0^1 h(s)[\xi]\mathrm{d}s.$$

Denoting by H a primitive of $-h$, we obtain the existence of a constant $A\in(\mathbb{R}^m)^*$ such that

$$\partial_v L_0(x,\dot x) + 2\dot t\omega_x = A + H \qquad \text{a.e. on } [0,1]. \tag{A.6}$$

Using (A.5) we obtain

$$\partial_v L_0(x,\dot x) + 2\frac{\omega(\dot x)}{\beta(x)}\omega_x = A + H + \frac{2C_z - d(x)}{\beta(x)}\omega_x, \qquad \text{a.e. on } [0,1],$$

where the right-hand side is an absolute continuous function.

Let us consider the continuous maps $\mathcal{L}\colon U\times\mathbb{R}^m \to (\mathbb{R}^m)^*$ and $\mathcal{P}\colon U\times\mathbb{R}^m \to U\times(\mathbb{R}^m)*$ defined respectively as

$$\mathcal{L}(x,v) := \partial_v L_0(x,v) + 2\frac{\omega(v)}{\beta(x)}\omega_x$$

and

$$\mathcal{P}(x,v) := (x,\mathcal{L}(x,v)).$$

As in the proof of Proposition 7.4, recalling how U has been chosen, and up to take a smaller U, we can state that there exists $C>0$ such that for any $x\in U$ and for all $v_1, v_2\in\mathbb{R}^m$

$$\big(\mathcal{L}(x,v_2) - \mathcal{L}(x,v_1)\big)[v_2 - v_1] \geq C\|v_2 - v_1\|^2. \tag{A.7}$$

Notice that (A.7) implies that for each $x\in U$, $\mathcal{L}(x,\cdot)$ is injective with inverse which is continuous on the image of $\mathcal{L}(x,\cdot)$. Using again (A.7) together with the continuity of $\mathcal{L}$ on $U\times\mathbb{R}^m$, we get that the map $\mathcal{P}$ is injective with continuous inverse as well. Hence,by (A.7),

$$\big(x(s),\dot x(s)\big) = \mathcal{P}^{-1}\left(x(s), A + H(s) + \frac{2C_z - d(x)}{\beta(x(s))}\omega_{x(s)}\right) \tag{A.8}$$

and so x is of class C^1. By (A.5), even $\dot t$ is continuous, so z is of class C^1 in the coordinate system where the stationary product type local structure (A.3) of L holds and then in any other coordinate system. Hence, the function h is actually a continuous function, and the right-hand side of (A.7) is of class C^1. Since ω is a C^1 one-form, (A.7) shows that also $\partial_\nu L_0(x,\dot{x})$ is of class C^1. Being $\partial_\nu L(z,\dot{z})$ identifiable with

$$\Big(\partial_{nu} L_0(x,\dot{x}) + \Big(2\omega \dot{t} d(x) - 2\beta(x)\dot{t}\Big)\mathrm{d}t\Big),$$

we deduce that the function $s \in [0,1] \mapsto \partial_\nu L\big(z(s),\dot{z}(s)\big)$ is C^1 as well, and then (A.1) holds. Moreover, by standard arguments, z satisfies the conservation law (A.2) for some constant $E \in \mathbb{R}$ (see, e.g., Proposition 1.16 of [14]). □

Remark A.2 We notice that, if L_0 admits positive definite vertical Hessian at some vector in $\nu \in TU$, then $\mathcal{L}$ admits a bijective fiberwise derivative, so it is a local C^1-diffeomorphism in a neighborhood $\mathcal{V}$ of ν in TU. Hence, $\mathcal{P}$ has a C^1 inverse on $\mathcal{P}(\mathcal{V})$ and then from (A.8) we get that $\dot{x}$ is C^1 on an open interval J containing the instant s_0 such that $\dot{x}(s_0) = \nu$. From (A.5), $\dot{t}$ is C^1 as well on J and then $z \in C^2(J, M)$. We observe that this holds in particular when L is the quadratic form associated with C^1 stationary Lorentzian metric g_L (see Example 3.8), hence its critical curves are C^2 on the interval where they are defined and then they are classical geodesics.

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