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TEACHING NON-EUCLIDEAN GEOMETRIES IN HIGH-SCHOOL: AN EXPERIMENTAL STUDY

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Abstract

In these years it is becoming quite clear that the development of a young student into an adult citizen requires a solid scientific background. Facing the challenges of a quickly changing world where political decisions are not only concerned with economics or ethics, but also with climate sciences, medicine, etc., requires a good education. Citizens are required to exert logical thinking and know the methods of science in order to adapt, to understand and to develop as persons.

At the core of all these required skills sits mathematics, with its formal methods to develop knowledge. Learning the axiomatic method is fundamental to understand how hard sciences work, and helps in consolidating logical thinking, which will be useful for the entire life of a student.

In my experience as a secondary school teacher, I have tried to understand how students perceive mathematics and what difficulties they encounter. One observation I often made was that the axiomatic study of geometry was a problematic topic for students, even for those with an interest for mathematics.

For this reason, I decided to focus my PhD work on the teaching and learning of geometry, focusing explicitly on its axiomatic foundations, in order to concentrate on those aspects that foster the development of a logical thinking, of the ability of proving a thesis, etc., which are necessary, as said, for the growth of a modern citizen.

Axiomatic geometry exposes the students to plural worlds, where the choice of a few base axioms heavily influences the properties of the objects that can be observed. The students, who are used to an intuitive study of geometry in the Euclidean plane, can benefit from the discovery of non-Euclidean geometries in several regards. First, they are shown that different non-Euclidean geometries exist, then they discover how these geometries can be developed by slightly modifying the axioms. Finally, they can be taught how these are useful to model real problems.

The importance of teaching non-Euclidean geometries in high school has been debated for decades and several experiences have been conducted with students in the past. However, all these works were often of qualitative nature, the experimental protocols were poorly documented, and the statistical data was

missing. The main objective of this thesis is to investigate, by means of quantitative experimental protocols, the viability and effects of teaching an introductory course on non-Euclidean geometries to high-school students.

The experimental nature of this study required the classroom work to be concise and limited to a short number of seminars and workshops. Several experiences are described, involving several high school classes and a total of 154 students and 57 teachers. These have been used to evaluate and refine the teaching tools and topics that are covered in this thesis and are reported in detail for use in future experiences. Statistical methods and evaluation questionnaires are discussed to assess the effectiveness of the approach, which will prove necessary in larger-scale experiments.

The outline of the thesis follows. In Chapter 1, a more elaborate motivation and introduction to the topics of the thesis are given. Chapter 2 reports a brief history of the development of Euclidean and non-Euclidean geometries and then discusses whether the birth and the development of non-Euclidean geometries constitute a revolution in mathematics. Chapter 3 discusses the teaching of geometry in secondary schools, with a specific interest in the Italian education system. Chapter 4 gives an informative introduction to the main aspects of the study and open questions; summarise and critiques the studies that have been conducted on the teaching of non-Euclidean geometries; and states the research questions that are investigated in the present thesis:

- *RQ1:What features of a short introductory course in non-Euclidean geometries are effective in engaging high-school students?*
- *RQ2: To what extent do students gain a new perspective on the concept of axiomatic system?*
- *RQ3: How well do students learn the taught concepts of non-Euclidean geometries?*
- *RQ4: To what extent do students' critical thinking and proof skills improve over the duration of the course?*
- *RQ5: Do students' beliefs about mathematics change over the duration of the course?*

Chapter 5 discusses all the details about the experimental phase. Specifically, this chapter describes and justifies the research methods and justifies the choice of adopting an essentially positivist paradigm using quantitative methods; discusses a preliminary experimentation conducted to investigate a suitable methodology, and – in more detail – a second experimentation. In addition, this chapter contains two sections dealing, respectively, with topics related to the experimentations: an experience with high-school teachers; and the description of the necessary adaptations for the distance learning imposed by Covid-19 restrictions. While Covid-19 restrictions impaired the possibility of a large-scale experiment, it allowed me to observe some peculiarities of the distance learning paradigm that must be accounted for when conducting geometry seminars online. Chapter 6 discusses the results of the data analysis, and provide an interpretation of the data shown and many conjectures for future developments. Chapter 7 concludes the doctoral dissertation.

1 Introduction

This research work consists of an investigation on the feasibility and on the effects of teaching a short course on non-Euclidean geometries to high-school students. It stems from my experience as a high-school teacher. As such, I am aware of some difficulties encountered by students in gaining confidence with mathematics and its methods. I strongly believe that teaching mathematics is not just useful to develop their skills in utilitarian terms (e.g., to promote a successful career), but it is essential in shaping them as citizens, whatever their job will be and whatever their role in society will be. For this reason, I have taken the occasion of this PhD studies to concentrate on some aspects of the teaching of mathematics: its role as a medium for passing concepts of logics, its role in shaping a modern scientific view, crafting a rational thinking etc.

1.1 The role of geometry in developing a rational mind

Unfortunately, developing a rational mind requires time and efforts, which students rarely have in a society where multimedia technologies expose them to an excessive stimulation. In Italian, the English term *multitasking* is widely adopted to refer to the ability of responding to multiple contemporary stimuli, which is reputed by many a necessary condition to adapt to current work environments, where emails, text messages, incoming calls and information come from many different channels and electronic devices. In reality, Gui warns that the term *task-switching* would be better suited for this new attitude, inherently suggesting that a real multitasking is not possible, but rather a constant shift of the brain focus to new stimuli, that reduces the quality of the work, since it has been shown that the brain can concentrate deeply on one stimulus at a time (Gui, 2019). He also states, citing multiple studies, i.e. (Calderwood, Ackerman, & Conklin, 2014) (Sana, Weston, & Cepeda, 2013), that in didactical environments, multiple contemporary stimuli reduce significantly the performance of both the student using a device and the nearby class mates. Unfortunately, digital didactic trends tend to adapt to this task-switching attitude, rather than correcting it. Of course, not all digital tools are bad for the students. For example, after evaluating many previous works, Gui states that experimental digital didactic activities are fruitful and yield good results when the didactical goals are clearly stated.

Furthermore, to develop rational skills, subjects such as Mathematics and Physics should be learned well, and this requires some interest from the student. Nowadays a lot of novel didactic research deals with technological tools and approaches, which should purportedly make these subjects more involving. However, it is questionable whether students get engaged with the tool or with the subject, and whether the student gets to know the essence of the subject or its visible surface. Loving, or at least showing an interest in a subject is a rare attitude that must be cultivated with all available means, but to encourage students in this regard, I believe that they have to be shown that they can understand it and manipulate it, rather than gamifying it (especially when this process externalizes certain actions such as computing, drawing or proving, to a software). This does not mean that using modern technologies is a bad practice, however, I believe that the students should gain their reward from the satisfaction of being able to master the subject, observe it from multiple angles and even question its truths. These goals can be achieved by an active practice that relies on discussing and following the students one by one (provided that classes are not too numerous). Indeed, we need to free the students from a passive learning, where truths seem to be immutable, pre-decided and imposed, and turn it into an active involvement, where they can see where these truths stem from, and how they can be questioned. This can be done in several ways, and the one I propose in this work has a very low cost, differently from methods that impose the use of expensive proprietary technologies, and can be therefore applied to schools of all countries and census.

As a case study, in this work I will consider the study of geometry from the axiomatic point of view. An informal study that I conducted, motivated me to work on geometry. I discovered that many students describe the study of geometry as more problematic than algebra. Many students fail to understand the ontological necessity of the theorems and the usefulness of proving them, since Euclidean geometry, which they deal with, seems intuitive or self-evident. I have several clues that could explain this.
(i) High school algebra can be more rewarding, since exercises can be often solved using an algorithmic approach. More specifically, students who cannot (or believe they cannot) motivate their solution steps, apply mechanical rules. Let us consider the solving of an equation of the type $2x - 3 = 0$. Students can learn to

mechanically take the addendum -3 from the left to the right changing its sign, and then move the coefficient 2 "below" 3, to get the correct solution $x = \frac{3}{2}$.
When students fall into believing mathematics as a mechanical subject, thus renouncing to understand it, the reward from feeling capable of solving the problem overcomes the feeling of uncertainty given by the fact of not knowing the reason why.
(ii) One aspect of trouble in geometry is the necessity of conjugating linguistic skills with logical skills. Expressing a statement without reproach requires a lot of effort from the student, and sometimes expressions used in mathematics clash with their use in the everyday language. As an example, the expression "if and only if" in mathematics is used to express a logical biconditional operator, however, it can be mistaken for a reinforcing of an "if" conditional and this may mislead a student in his/her conjectures.
(iii) One last issue that is encountered in geometry, is the self-evidence of some of its properties. This is only true of Euclidean geometry, however, this is the only geometry that students know of, unless they are not engaged with a non-Euclidean geometries (NEG) course.

This last consideration was the pivot of my line of reasonings when I started planning my PhD work. Finding a way to connect geometry to the development of modern science, showing how its truths can be questioned, and how truths hold up in an axiomatic system would be very interesting to develop the students' knowledge, their understanding of mathematics and science in general. In this regard, I find non-Euclidean geometries may suit well my objectives. These represent a topic that intersects the logics of axiomatics systems (which are a methodological foundation of all hard sciences), and provide pointers to other problems (e.g. cartography, relativity in physics). It also clashes with Euclidean geometry, and thus, with our intuitive view of geometry. Non-Euclidean geometries are believed to be a great stimulus for students and is sometimes taught in high school in the form of seminars. Therefore, I decided to undertake a series of introductory courses to non-Euclidean geometries, which will be described in this thesis. The teaching method has been developed through several iterations. My main objective was to assess their impact on students. I decided to study their impact with quantitative tools, a method which to the best of my knowledge has been overlooked in the past.

1.2 Historical perspectives on the axiomatic method

The axiomatic method presents theories using systems of axioms relating to entities. From these axioms, theorems are deduced about these entities. The axiomatic method is a fundamental element of mathematical thinking. The birth of this method in geometry dates back to III century BC and its evolution runs through the centuries to XIX century AD: from Euclid's early axiomatic formalization of geometry, to Hilbert's emancipation of geometry from the reality outside of geometry. While Euclid's idea was to enclose the essence of the reality in a few fundamental properties formulated in a constructive way, Hilbert reformulated the Euclidean axiomatic system in a hypothetical-deductive method without requiring the axioms to relate to the intuition or the appearance of existing objects. In Hilbert's formulation, theorems follow arbitrary hypotheses that satisfy a criterion of coherence.

Federigo Enriques (1871-1946) observed that the new logical imprint given by Hilbert to the axioms is that of relations having an independent meaning from the content of the concepts (Enriques, 1922). The primitive concepts can be known only from the relations that the axioms impose to them. This made it possible to broaden our understanding of axioms and postulates: "*axioms* and *postulates* (the distinction between these tends to cancel in the contemporary vision) will be simply considered arbitrary hypotheses apt to originate a deductive sequence of theories, of which the truth value will have to be validated in its entirety, not by simply judging the primitive prepositions […] The mathematical theory that is conceived as a «*hypothetical-deductive system*», is a fragment of science which the scientist assumes as something completed, isolating it from any other knowledge: the judgement on the coherence of the entire theory is independent from the value of the hypotheses, provided these are not contradictory, and from the meaning of the concepts appearing in this theory"[1] (Enriques, 1938).

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[1] Translated from the Italian "assiomi *e* postulati *(la distinzione stessa tende a cancellarsi nel pensiero contemporaneo) verranno ritenuti semplicemente come ipotesi arbitrarie atte a reggere l'ordine deduttivo delle proprie teorie, il cui valore di verità dovrà essere saggiato nel suo insieme, anziché da un giudizio portato esclusivamente sulle proposizioni primitive […] La teoria matematica concepita nel suo assetto logico di* «sistema ipotetico-deduttivo», *è un frammento di scienza che lo studioso assume come qualcosa di compiuto,*

Guershon Harel observes the differences between the Greek's method of proving and the modern method of proving (Harel, 2007): "While Euclid's *Elements* is restricted to a single interpretation – namely that its content is a presumed description of human spatial realization – Hilbert's *Grundlagen* is open to different possible realizations, such as Euclidean space, the surface of half-sphere, ordered pairs and triples of real numbers, etc. the *Grundlagen* characterizes a structure that fits different models. [...] The transition between these two proof schemes is revolutionary. It marks a monumental conceptual change in humans' mathematical ways of thinking". In his article, Harel adds that understanding this transition may shed light on epistemological obstacles that students encounter upon moving from concrete models of their quantitative or spatial reality to a more abstract setting. Moreover, the author suggests that "a mathematics educator should ask what is the nature of the instructional interventions that can bring students to refine and alter an existing way of thinking to a more desirable one".

During the two millennia that see the development of the axiomatic method, our interpretation of mathematics changes radically. Among the elements that determined such a change of perspective we can undoubtedly include the rise of non-Euclidean geometries and the proof of their logical consistency, guaranteed by the models of Beltrami, Klein and Poincaré. Their discovery was dramatic since they proved something unbelievable before: there is not a unique true geometry (the Euclidean one). There is not a unique truth but more conditionals truths exist, each one depending on the assumed hypothesis. The discovery and development of non-Euclidean geometries "is sometimes claimed to be a revolution in mathematics" (Gillies, 1995). The point whether non-Euclidean geometries have determined a revolution in mathematics, and more in general, whether revolutions in mathematics exist, is largely debated (Gillies, 1995) and in general the outcome depends on the conception one has about mathematics and his/her definition of revolution.

For sure, we can state that the birth of non-Euclidean geometries "was the first opportunity for the diversification of axiomatic theories"[2] (Bachelard, 1978). The

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isolandolo da ogni sapere: il giudizio sulla intera coerenza della teoria è indipendente dal valore delle ipotesi, purché queste non siano contraddittorie, e anche dal significato dei concetti che in essa figurano".

[2] Translated from the Italian *"è stata la prima occasione del diversificarsi delle assiomatiche"*.

unfolding of a plurality of geometries reinforces the strength of abstract thinking. As put by Bachelard (Bachelard, 1978) we are allowed to see "the reality as a special case of the possible"[3] (Palombi, 2017). Even Imre Toth (1921-2010) spends his words on the plurality of the geometries that were revealed by the birth of non-Euclidean geometries: "the great novelty has been the establishment of a plurality of worlds: the universe of geometry is no longer a domain in which there is only one valid truth, two opposing truths are valid for the subject" (Toth, 1991).

Giorgio Gallo, in a position paper (Gallo, 2012), argues that "maths in its entirety, if well taught, can bring a relevant contribution to a culture of peace and nonviolence". By that the author intends peace not only as the absence of war but as a state in which every individual benefits of all rights and means to fully participate to the endogenous development of society. Gallo observes that, in order to build this peace, we need – beyond a strong ethical sense – tools for critical thinking that allow to understand and decipher the reality in which we live and a nonviolent approach in our actions to change reality.

The mathematical culture belongs to this discussion: firstly because it provides analysis, synthesis, abstraction and induction skills. These skills are necessary to build mental models that help us understanding reality and modifying it. Furthermore, in epistemological terms, some mathematical concepts and results could facilitate a nonviolent approach to action to change reality, acknowledging the other as a subject with rights, holding values and carrying truths.

Indeed – and here we return to the main discourse – some concepts (Gallo mentions the passage from Euclidean geometry to non-Euclidean geometries among his examples) allow us to understand that "truth is not something that can be possessed once and for all, but rather a process of continuous discovery that is anything but linear" and that "much of our knowledge, and certainly the most relevant part of it, is conjectural and therefore somehow uncertain and always susceptible to being questioned, modified". The author specifies that he does not intend to affirm that a truth does not exist, nor that there are only interpretations, nor that "something is true in a certain historical context, or given certain presuppositions, or certain social beliefs"[4] (Berto, 2008).

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[3] Translated from the Italian "*il reale come un caso particolare del possibile*".

[4] Translated from the Italian *"qualcosa è vero in un certo contesto storico, o date certe presupposizioni, o certe convinzioni sociali"*.

Instead, he intends "to affirm that the historical context, language, culture and social conventions determine the way in which we formulate the results we have arrived at on our journey towards the truth and that these formulations in any case represent only an approximation that can always be remitted under discussion" (Gallo, 2012). The previous clarification is very important, and we must be careful not to fall into the misunderstanding that all points of view have the same value. The author closes his article by quoting the following words from Munir Fasheh, one of the best-known Palestinian learning theorists and practitioners: "One of the main objectives of teaching mathematics should be to make people understand that there are different points of view and to enforce the right of each individual to choose their own. In other words, mathematics should be used to teach tolerance in an age so full of intolerance (Fasheh, 1997).

1.3 Why teaching non-Euclidean geometries

For what has been said so far, I believe it is important to talk about the path that leads to the birth of non-Euclidean geometries and their basic notions and models. According to several authors, talking about non-Euclidean geometries could help:

- Understand that mathematics is not a rigid discipline and that its evolution is not independent of the historical and social context in which it develops.
- Start reflecting on the concept of truth in mathematics and, in general, in the sciences (such a reflection would also help to clarify the meaning of doing research).
- Better understand Euclidean geometry. In fact, working with non-Euclidean geometries models within the Euclidean one (for example the study of the Poincaré disk) means: interpreting the primitive entities of non-Euclidean geometries in terms of the primitive entities of the Euclidean one; translating the axioms of non-Euclidean geometries into the corresponding Euclidean propositions; prove that the Euclidean statements thus obtained are all valid theorems.
- Take a step forward towards understanding the modern conception of an axiomatic system.
- More specifically: to understand the need not to take for granted those properties which, despite appearing evident, cannot be considered true

because they are not consequences of the assumed axioms. This would allow for perfecting critical skills and proving practice.

- A culture of mutual understanding based on the awareness that different conclusions can be caused not only by logical fallacies but also simply by different starting points.

Later in the thesis I will provide a survey of the available scientific literature, giving space to those works that build experiments in order to prove some of their claims. Considering the importance of supporting a thesis with evidences, I spent my efforts in constructing experiments that are meant to bring new evidences to foster the discussion. In particular, I focused on short introductory courses aimed at high-school students. There are several reasons behind this choice that I will try to resume here.

The first has to do with the axiomatic method itself: this is more easily taught in school, rather than in casual science dissemination TV shows or podcasts. high-school students, in particular, are in the perfect age to start working on abstract concepts, while adults may be less malleable.

Another reason for choosing high-school students is that they are sufficiently autonomous to attend additional courses. Finally, short courses are well adequate to give some preliminary results on the topic and are feasible. Longer experiments (e.g. a 1-year course of non-Euclidean geometries with 1 hour per week) would be equally necessary, however they present steeper difficulties. Special agreements would be necessary between the research institution, the school and its teachers. This would be better suited for a national experimental plan, where the ministry of education takes the conducting role and approves such a long-term experiment on a number of classes.

Actually, there is an interest in teaching the axiomatic approach in high school. In Italy, current lyceum high school national recommendations *Indicazioni Nazionali*[5] introduced in 2010 with the so-called Gelmini-Reform (after the name of the Italian Minister of Education Mariastella Gelmini) represent the "disciplinary declination of the student's educational, cultural, and professional profile at the end of the high school courses"[6]. These recommendations suggest that the student should be able – at the end of his/her studies – to understand the historical context of several

[5] See *Decreto Interministeriale 211 del 7 ottobre 2010 - Indicazioni Nazionali per i Licei*.

[6] See *Regolamenti di Riordino dei Licei, degli istituti tecnici e degli istituti professionali emanati dal Presidente della Repubblica in data 15 marzo 2010*.

mathematical theories and their conceptual meaning. The recommendations also suggest "a clear vision of the axiomatic approach in its modern conception and of its specificity with respect to the classic Euclidean approach". Nonetheless, the recommendations do not report non-Euclidean geometries among the suggested teaching topics. As the Italian mathematician and teacher Walter Maraschini (1949-2017) critically stated in a talk during the round table of the XXIX UMI Conference, this lack weakens the coherence of the recommendation (Maraschini, 2010). In the contemporary historical and methodological asset of mathematical studies, it is hard to understand the modern axiomatic approach without studying non-Euclidean geometries "Non-Euclidean Geometry is not only true, but also necessary. Without it, the development of so-called modern mathematics would be hardly conceivable" (Toth, 2003). By excluding non-Euclidean geometries from the teaching programmes, the recommendations exclude, thus, a large part of the mathematical issues from the 20th century, showing again a contradiction because at the same time they suggest to report results from 20th century mathematical and scientific results in Philosophy courses. To mitigate the importance of some of the lacks and the contradiction hereby discussed, the following quote from *Indicazioni Nazionali* is reported: "these recommendations do not dictate any didactical or pedagogical model. […] The teacher has freedom to enrich what is suggested by these recommendations, with respect to the characteristics of each lyceum curriculum, and also has the freedom to apply the appropriate strategies and methods" (translated from Italian).

The possibility of conducting extra activities with the students determines the proposal of non-Euclidean geometries seminars and workshops in many high schools. Many teachers believe these can be enriching and stimulating for the students, which on one hand is surely true, however, as Lucio Russo warns (Russo, 2016), the introduction of these topics must not just fascinate the students, with the risk of instilling a "reverent admiration" for Science. This will cause the development of an irrational feeling for something that is not understood and must be avoided. Therefore it is crucial that a laboratory on non-Euclidean geometries is designed to show rationally that all those things that appear mysterious and bizarre from the surface, are indeed logical and well understood.

In the development of the experimental work that is the object of this thesis, this point has been taken seriously, and the final formulation of the laboratory came after several design and experimental iterations.

1.4 Summary

In this first chapter I introduced several of the topics that motivate this thesis and inform the reader about the framework that I employed during its development. In Section 1.1 I firstly argued about the role of geometry studied from an axiomatic point of view in the developing of a rational mind. Its logical structure is based on axioms, theorems and proofs, which can be difficult to grasp at first, but allows the evolution of knowledge and science by prediction, abstraction and construction. The axiomatic method also leads to geometric worlds that are different from the Euclidean one and, thus, not always self-evident, yet necessary for the expansion of knowledge or for solving problems in other fields such as physics. The same section also gives some hints about the role of technology in teaching and the differences in learning algebra and geometry.

Then, Section 1.2 highlights the points of view of some important intellectuals on the axiomatic method, spacing from ancient Greece to contemporary authors. Finally, Section 1.3 discusses the benefits of teaching non-Euclidean geometries and motivates some of the design choices of this work. It also briefly introduces the Italian guidelines for teaching mathematics, that will be later discussed in detail in Chapter 3.

2 Euclidean and non-Euclidean Geometries: Historical Context

The present chapter outlines a brief history of the development of Euclidean and non-Euclidean geometries to delineate the historical-epistemological framework of the thesis. One of the paramount aspects is the role of non-Euclidean geometries in the evolution of mathematics: does it constitute a revolution in mathematical thinking or not? Several authors argued this topic and their positions will be outlined.

2.1 Scientific activity during Hellenism

In his essay *La rivoluzione dimenticata (The Forgotten Revolution*) (Russo, 2019), Lucio Russo debunks the myth that science was born in Europe in the 17th century by arguing that it was Hellenistic scientists who gave birth to the scientific method and adds that the scientific revolution of the 17th century was based on the rediscovery of Hellenistic culture. Russo thus departs from other historians of science such as the mathematicians Otto Neugebauer and Thomas Heath who relegate the Hellenistic scientists to the role of pure precursors (Neugebauer, 1969) (Heath, A History of Greek Mathematics, 1921). To clarify his position, he specifies what he means by "Hellenism" and "science".

In the section on Hellenism, he sets the period and some of the protagonists of what he calls the scientific revolution in time and space. The main information is summarised below. The Hellenistic civilisation, according to the terminology introduced by Droysen and accepted by later historiography, begins in 323 BC, with the death of Alexander the Great (although – Russo argues – it would be more logical to choose the date of the beginning of Alexander's expedition, or reign, since the essential novelty of the Hellenistic period consists in the implementation of his programme of Hellenisation of the territories of the ancient empires). The end of Hellenism is usually dated 30 BC, the year of the unification under Roman rule of the whole Mediterranean and the beginning of the imperial period. In truth, it could be said that it endured during the Roman Empire. Autonomous Greek cities distributed throughout the Mediterranean contributed to its development, and not only the Greeks living in the regions that had formed the empire of Alexander the Great (mainly Egypt, the Seleucid state and the Antigonid state).

Science developed from the end of the 4th century BC, mainly in Alexandria, favoured by the policies of Ptolemy I Sotèr and Ptolemy II Philadelphus, and declined rapidly during the 2nd century BC, after the Romans conquered the centres of Hellenism (from 212 BC, the date of the sack of Syracuse and the death of Archimedes). More precisely, scientific activity in Alexandria and, with it, the most flourishing period of Hellenistic science ended tragically in 145-144 BC, with the persecution of the Greek ruling class, which was followed by the almost complete destruction of the Greek ethnic group in the city of Alexandria Alessandria (Polibio, Geographia) (Polibio, Historiae) and the diaspora of intellectuals from the city (Athenaeum). Scientific activity then partially resumed in the imperial era, between the 1st and 2nd centuries AD, during the Pax Romana. However, there are no original documents left of scientific activity in the 4th century AD, only compilations and comments on old works, such as the works of Pappus and Theon. The total decline of ancient science is usually dated to 415 AD, the year of the murder of the mathematician Hypatia for religious reasons (Russo, 2019).

Among the main exponents of the scientific revolution, Russo lists Euclid, Archimedes of Syracuse and Erophilus of Chalcedon but does not fail to mention, among the names of the refined minds that gave birth to science, Ctesbius, Aristarchus of Samos, Eratosthenes, Chrysippus, Philo of Byzantium, Appollonius of Perga and Hipparchus of Nicaea.

In his aforementioned essay, Lucio Russo clarifies what he means by “science”. He starts by defining what – according to him – is not “science”: “science” is not a set of statements that are certainly true. If this were not the case, we could not consider all outdated theories as scientific. Every scientific theory has a limited usefulness: once it has proved inadequate to describe a new phenomenology it will have to be replaced, but it will continue to be a scientific theory and can still be used in its own sphere (Russo, 2019). In order to facilitate the definition of the term “science”, Russo notes that theories that are generally considered scientific (e.g. thermodynamics, Euclidean geometry, calculus of probabilities) share three fundamental characteristics: "their statements do not concern concrete objects but specific theoretical entities"; "the theory has a rigorously deductive structure"; "the applications to the real world are based on rules of correspondence between the entities of the theory and concrete objects". The term "science" can also be used for some theories without rules of correspondence that allow their application to the real world but only applicable to other scientific theories (e.g.: contemporary mathematical theories), theories for which the theory-reality relationship is indirect but still guaranteed by the same mechanism of theory formation. Lucio Russo specifies that by the term "science" he means first and foremost "exact science". By "exact science" he means the set of scientific theories whose usefulness lies in "providing models of the real world within which there is a guaranteed method for distinguishing false statements from true ones". Models make it possible to describe and predict natural phenomena or to modify the existing world by constructing reality corresponding to the model identified theoretically. The rigorously deductive structure of scientific theories means that they can be self-extending. In this way, even if they are born with the aim of describing natural phenomena, they can become models of areas of technological activity. Russo devotes a large part of his essay to the argument that, while at the empirical level technology has existed as long as man has existed, "scientific technology" was born with exact science. I refer to his essay for further discussion of this. I will just point out that Russo returns to the same topic in (Russo, 2015) explaining how in the Hellenistic era we first speak on two different levels: that of scientific theories and that of reality. After having developed the theory as a model of real objects and phenomena, thanks to the proof method it is possible to deduce the behavior of objects existing only in the theory. If they are useful objects, one can choose to move on to scientific design, i.e. to move from the theoretical to the concrete level by actually realizing the objects studied at the theoretical level (see Figure 1).

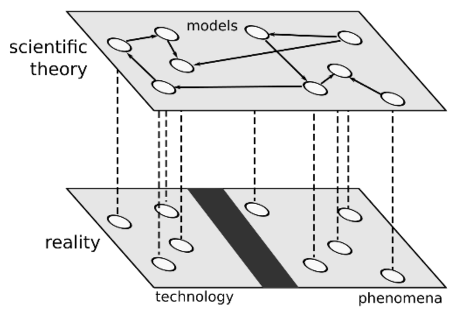

Figure 1. An illustration depicting the relationship among scientific theory, phenomena and technology (adapted from (Russo, 2015)).

Russo lists two essential aspects of exact science: methodological unity and extreme flexibility in considering new objects of study. He also points out that the "scientificity" of the study of a phenomenon does not depend on the type of phenomenon but on the possibility of using scientific theories. Russo also talks about empirical theories, explaining that they differ from the exact sciences because they do not have a rigorously deductive structure. This implies that, since they cannot be extended using the proof method, they can be useful for framing a known reality but cannot be used to design different realities. Moreover, unlike the exact sciences which are differentiated according to theories, the empirical sciences are divided according to the phenomena they study.

In (Russo, 2019), the author criticises Kuhn who, in his essay "The Copernican Revolution" (Kuhn, 1972), using the concept of "paradigm" more extensively than his concept of "scientific theory", includes among the sciences forms of knowledge that do not allow any clear relationship between technology and "science" (e.g. Pythagorean mathematics and Aristotelian physics). Lucio Russo is also aware that adopting his definition of “science” excludes other cognitive methods often referred to as scientific, whose importance he himself does not deny.

Lucio Russo's position has received much praise. We quote, for example, part of Sandro Graffi's review of the essay *La rivoluzione dimenticata (The Forgotten Revolution*): "The scholarly support is unquestionably impressive. It includes a methodological novelty, this time in the examination of the original sources. Thanks to his dual competence in science and philology, Russo does away with a time-honored habit among scholars of antiquity-namely, that humanists only deal with "literary" sources and historians of science with the "scientific" ones" (Graffi, 1998). On the other hand, we also find those who, while praising Russo's great work, criticise some aspects of it (see (Rowan-Robinson, 2004) and (Greene, 2004) for example). Among these, Marcello Cini himself who, writing the preface to the 2001 edition of the essay in question (Russo, 2019), includes himself among those who Russo himself knows consider the definition of "science" he proposed too restrictive.

2.2 Euclid and the *Elements*

As already mentioned, Lucio Russo considers Euclid one of the greatest exponents of the scientific revolution (Russo, 2019). His main work, the *Elements* – writes Attilio Frajese (1902-1986) – "can be described as truly scientific rigour, so that it is not by chance that it has passed through the centuries, obscuring the work of its predecessors and in a certain way determining that of its successors" (Frajese, 1950) (today we do not consider the *Elements* to be an example of true scientific rigour, although with this work the level of rigour rises considerably, however Frajese's sentence is significant in showing the value that Euclid's main treatise had at the time). Despite this, Euclid has been removed from history, as has much of Hellenism (Russo, 2015).
In fact, we know very little about Euclid's life, so much so that the version of his best-known treatise, the *Elements*, edited by Attilio Frajese and Lamberto Maccioni opens with this exclamation: "Euclid, this stranger!" (Frajese & Maccioni, 1970). Even in some biographies, the author of the *Elements* is mistakenly identified with Euclid of Megara and a portrait of the latter often appears in the histories of mathematics. The author of the *Elements* is generally identified with Euclid of Alexandria, so called because he is believed to have been one of the scholars whom King Ptolemy I Sotèr (Eordia, kingdom of Macedonia, 367/366 B.C. - 282 B.C.) called to teach at the academy dedicated to the Muses which was for centuries the highest cultural institution of the Hellenistic world: the Museum, in Alexandria (Boyer, 1990). Other historians, such as Thomas Heath (Heath, 1921), are less certain of this version advocated by Carl Boyer. In addition to the *Elements*, Euclid wrote a dozen treatises on various subjects (e.g. optics, astronomy, music, mechanics, spherical geometry) but unfortunately more than half of his works have been lost (Greenberg, 2008). We refer to the texts by Acerbi (Acerbi, 2007), Boyer (Boyer, 1990), Heath (Heath, 1921), and Russo (Russo, Pirro, & Salciccia, 2017) for further insights into Euclid's life.

The *Elements* is such a well-known treatise that "it has appeared in more editions than any other book besides the Bible. It has been translated into countless languages and has been in print in one country or another almost since the beginning of printing" (Katz, 1993). Boyer claims that this treatise is the most successful book on mathematics ever written (Boyer, 1990). Unfortunately, Euclid's original manuscript has been lost. "The two oldest surviving copies of the

Elements date from the ninth century, one preserved at Oxford University and the other in the Vatican. Only fragments of the book predating these copies have been found. The manuscript preserved at Oxford was copied on parchment in 888 AD by Stephen the Clerk for Aretia of Patras in Constantinople. With the help of the Clay Mathematics Institute, this manuscript has been digitized and can be viewed in high resolution with the corresponding Greek and English text on the Institute's website, www.claymath.org/euclids-elements" (Carroll & Rykken, 2018). The versions of the Elements that we know have been reconstructed from a copy kept in the Vatican Library and from Arabic translations of lost Greek copies. The first printed version appeared in Venice in 1482; it is Campano da Novara's translation made from Arabic. In 1880 Hideberg reconstructed a new Greek version. The English-speaking world mainly refers to the English translation of the latter made in 1908 by Thomas Heath Heath (Greenberg, 2008). In Italian there is the edition translated by Frajese and commented by Maccioni (Frajese & Maccioni, 1970) of the text reconstructed by Heiberg and Monge (Heiberg & Menge, 1895).

The original Greek title (*Στοιχειᾶ*, or Stoicheia) of Euclid's main work has been translated to "Elements", it refers to the rudimentary principles or primary results present in the work (Carroll & Rykken, 2018). Boyer describes the *Elements* not as a compendium of all geometric knowledge of the time but an introductory manual that included all elementary mathematics: arithmetic, synthetic geometry and algebra (understood in geometric terms) (Boyer, 1990). Attilio Frajese and Lamberto Maccioni, on the other hand, state that the Elements "offer an overview of Greek mathematics [...] they represent both a point of arrival and a point of departure. A point of arrival – they explain – of a period of about three centuries of elaboration of mathematics, from the origins of Greek mathematics when Miletus transported geometry from the East to Greece. This period is called pre-Euclidean. The Elements represent, at the same time, a starting point both for scientists who tried to study mathematical questions that were no longer elementary, such as the method of exhaustion (Archimedes) and the theory of conic sections (Apollonius), and for the teaching of elementary mathematics (Frajese & Maccioni, 1970). Lucio Russo also insists on the constructive character present in the *Elements* by emphasizing how they are fundamentally the study of drawings that can be executed with a ruler and compasses. Since drawing was the main tool for finding the results of the other exact sciences (e.g. optics, astronomy and mechanics), the *Elements* constituted the basis for the other exact sciences (which were then part of *Mathematics*).

2.3 Euclid's method

Euclid solved, for the first time, the problem that arose in the Hellenic period of using the proof method without proving any statement (a problem already considered by Aristotle in (Aristotele) (Russo, 2019). Euclid's solution can be seen in the *Elements* and consists in creating mathematics as a scientific theory by "explicitly defining the entities of the theory [...] in terms of a few fundamental entities [...] and listing the statements about these entities that must be accepted without proof". The *Elements* is in fact a treatise organised in 13 volumes containing no less than 465 propositions, all deduced from a few fixed definitions and basic assumptions (common notions and postulates)[7].

Euclid's merit is not that he gave birth to the proof method, which in fact developed in the 4th century BC. For example, Aristotle and Plato already reported proofs of theorems, but their proofs consisted of highly refined forms of logical deduction that, however, chose their starting points each time, making them coincide with statements that seem obvious from the evidence of the design. For example, in Plato's *Menon*, Socrates, in leading a slave to the conclusion that in order to construct a double square of a given square one must construct the square on the diagonal of the given square, takes for granted the existence of the square. Euclid's merit lies in having raised the level of rigour by reducing the theorems he already had to a few fixed basic assumptions from which he could deduce his proofs. The result achieved by Euclid could not have been achieved in classical Greece because at that time culture was predominantly oral so it would have been practically impossible to fix the starting points, no one had the authority to choose the premises from which others would have to start their proofs (Russo, Pirro, & Salciccia, 2017). The editorial production of the Library of Alexandria and the dissemination of the book were decisive for the birth of Euclid's method.

In the *Elements* there are two types of propositions that have a complementary role. Proclus used the terms $\pi\rho o\beta\lambda\acute{\eta}\mu\alpha\tau\alpha$ and $\theta\varepsilon\omega\rho\acute{\eta}\mu\alpha\tau\alpha$ to denote these types of

[7] I reproduce, in Appendix 1, definitions, common notions (or axioms) and postulates found in the version of the *Elements* translated by Thomas Heath.

propositions; we translate them with the terms *problems* and *theorems*. Problems consist of a statement requiring the construction of a figure having certain properties, the indication of how to construct the figure and its construction, the proof – or rather, the presumption of proof – that the constructed figure satisfies the properties indicated in the statement, and finally the closure with the formula *ὅπερ ἔδει ποιῆσαι* (*as it should have been done*). Theorems concern figures whose existence has already been proved and which have therefore been the subject of some previous problem; they consist of an enunciation in the formulation of which a hypothesis and a thesis appear, a proof and they close with the formulation *ὅπερ ἔδει δεῖξαι* (*as it had to be proved*). The same distinction applies to postulates as to propositions: there are constructive and affirmative postulates (e.g. the fifth postulate).

Unfortunately, quite early on – as early as the imperial era – people stopped understanding Euclid's main treatise and betraying its spirit, despite the fact that it was the basis of the scientific method for two millennia (Russo, 2019). The importance of having a fixed set of postulates would only come to be well understood again in the 1830s, with the birth of non-Euclidean geometries.

2.4 Criticisms of Euclid's *Elements*

Although the versions of the *Elements* that have come down to us bear witness to the high level of rigor that existed at the time, we are now aware that this level was perfectible. In particular we can raise the following criticisms of the *Elements* (Heath, 1956): it contains definitions – or attempts at definitions – of fundamental geometrical entities that appear vague or unnecessary because they are never used in the rest of the treatise; there are omissions (Euclid makes implicit assumptions).

Vague or unnecessary definitions

The presence of definitions – or attempts at definitions – of fundamental geometric entities that appear vague or useless because they are never used in the rest of the treatise. As far as the first book is concerned, the contested definitions are only the following: I-VII, XI-XIV, XVII- XXII.
The analysis conducted by Lucio Russo following a historical-philological criterion on Euclid's main treatise supports the thesis that the first definitions present in the versions of the treatise that have come down to us (Heath, 1956) (Heiberg &

Menge, 1895), i.e. the definitions in which an attempt is made to define fundamental geometric entities, were not given by Euclid but were inserted in the imperial era (Russo, Pirro, & Salciccia, 2017).

In the imperial age, a time of cultural decadence, understanding Euclid's text was difficult. With the presumption of helping to overcome this difficulty, the compilers of the *Elements* made modifications by adding comments and results with the presumed aim of making the treatise more complete and more suitable for teaching purposes. As evidence of this, Russo cites the example of the mathematician Theon of Alexandria who in (Teone di Alessandria) explicitly mentions one of his theorems included in his edition of the *Elements* (4th century BC). Euclid's text therefore underwent transformations because it was copied without any philological scruples.

To worsen the situation described so far, there was the fact that the added comments often distorted the original spirit in order to adapt the teaching of mathematics to the neo-Platonic ideas prevailing in the imperial age. In order to clarify this issue, let us take an example of the terminology adopted by Euclid: the case of the term *straight line*. Euclid's terminology was made up of words from ordinary language that had a concrete meaning related to the activity of a draughtsman with a ruler and compass (ungraduated ruler and collapsible compass), these acquired a scientific meaning through the argumentative structure of the text and the choice of postulates. Specifically, by a *straight line* (γϱαμμή ευθεία) Euclid meant a limited straight line that can be extended indefinitely (Euclid's infinity was a potential infinity). He used this fact "only to find a larger finite length" (Kline, 1972) in the proof of propositions 11, 16 and 20. Euclid meant what we today call a *segment*. He started from an entity that corresponded to the lines drawn by a draughtsman and through a process of abstraction imagined that he could extend the line. In the imperial era, on the other hand, priority was given to the abstract infinite entity because it was believed that the concepts that have true reality are the ideas that are in the Hyperuranium, and so a Neoplatonic philosopher started from the concept of an infinite line (actual infinity). The segment was then defined from the line. Thus Euclid's idea was distorted in his commentary on the Elements. Our terminology today is affected by this transformation: we start from the abstract rather than the concrete.

The definition of straight line that we know – as well as other definitions – is said to have been extracted from a preparatory treatise to the *Elements* written by Hero of Alexandria in which the author explained terms left undefined by Euclid (e.g.

the definition of *point* and that of *straight line*). According to Russo's thesis, therefore, parts of Hero's passages have been inserted – not always faithfully – into the later editions of the *Elements* and passed down to the present day as if they were definitions belonging to Euclid's original text. As far as the straight line is concerned, a passage from Hero contains the following expression:
“Εὐθεῖα μὲν οὖν γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐπ’ αὐτῆς σημείοις κεῖται ὀρθὴ οὖσα καὶ οἷον ἐπ’ἄκρον τεταμένη ἐπὶ τὰ πέρατα”
(“a straight line is [that] which with respect to [all] its points lies straight and taut at the maximum between the ends”).
Russo notes that the earlier description of a straight line given by Hero can be reasonably traced back to Archimedes. In fact Archimedes had postulated that the straight line between two ends was the shortest line of events at those ends (Mugler, 1970–1972). Hero might have added *“in the same way with respect to [all] its points”* because, in transforming Archimedes' postulate into a characterization of the line, he could not limit himself to considering only two of its points. The person who wrote the Elements in the form that has come down to us was not a mathematician but a simple copyist and, most probably, having decided to insert the preceding passage of Hero as a definition of a straight line at the beginning of the Elements, he will have cut the sentence to form the following one which, although syntactically correct, is meaningless:
“A straight line is a line which lies evenly with the points on itself” (Heath, 1956).
It is possible to elaborate on Russo's thesis in (Russo, 1992) (Russo, 1998) (Russo, 2019).

The position we read in the comments to the edition of the *Elements* edited by Frajese and Maccioni is different from the one exposed by Russo: they do not doubt that it was Euclid who gave the definition of a straight line, but they insist on the different conception of definition by Euclid and, more generally, by the Greeks with respect to the one we have today. In particular, it is argued that for the Greeks, defining a concept did not mean constructing it, almost creating in the spirit those geometric entities that are considered: it was only a matter of describing them, so that they could be easily recognised through a satisfactory nomenclature. "Those geometrical entities, that is, already exist: the definition has for Euclid only the sense of identifying them. This is why in the Elements we find at the beginning, in the first place, precisely that "definition" of point which modern arrangement, starting from other views, has renounced" (Frajese & Maccioni, 1970).

Implicit assumptions

When analysing the first book of the *Elements*, the following omissions can be observed:

- the implicit assumption that the circles drawn in the proof of the first proposition (construction of an equilateral triangle from one of its sides) cut one another and, similarly, the implicit assumption that the given line and the circle drawn in the proof of the twelfth proposition (construction of a perpendicular to a given line passing through a point outside the line) intersect. These omissions prevent – in fact – the proof of the existence of the equilateral triangle and the perpendicular. Hilbert resolved this shortcoming by using the theory of real numbers and, therefore, the axiom of continuity, which was beyond the reach and thought of Euclid.

  On this point, Frajese takes a different, more nuanced view: Euclid would not have been completely silent on the existence of the above-mentioned intersections between circles and between line and circumference: "even if Euclid did not explicitly state the postulates concerning the intersections of circle and line and of circle and circle, he nevertheless fixed the conditions necessary for the intersections to exist" (Frajese, 1968). See his article for more details.
- The implicit assumption that one can employ the technique of superposition to prove the congruence of figures (proposition 4 and proposition 8: congruence schemes SAS and SSS) thus taking it for granted that, in moving a figure, it does not change its properties. Not of this opinion is Heath, who, commenting on Common Notion 4, states "It seems clear that the Common Notion, as formulated here [*Things which coincide with each other are equal to each other*], is intended to assert that superposition is a legitimate way of proving the equality of two figure which have the necessary parts respectively equal, or – in other words – to serve as an *axiom of congruence*" and "The phraseology of propositions, e.g. I. 4 and I. 8, in which, Euclid employs the method indicated, leaves no room for doubt that he [Euclid] regarded one figure as actually *moved* and *placed upon* the other" (Heath, 1956). Hilbert resolved this shortcoming by formulating his fifth Axiom of Congruence ("If for two triangles $\triangle ABC$ and $\triangle A'B'C'$ we have $AB \cong A'B'$, $AC \cong A'C'$, $\measuredangle BAC \cong \measuredangle B'A'C'$, then we also have $\measuredangle ABC \cong \measuredangle A'B'C'$").

- The implicit assumption that the prolongation of the median drawn in the proof of the sixteenth proposition (*"In any triangle, if one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles"*), is all contained by the external angle taken into consideration (see Section 5.5.7.3).
- The implicit assumption that a closed polygon divides the plane into two parts (one finite and the other not) and the characterisation of the interior and exterior of a polygon. This does not escape Hilbert's attention, he is aware that this is not an elementary fact and he deals with this issue in Theorem 9 of "The Foundations of Geometry" (Hilbert, Foundations of geometry, 1971).

2.5 The fifth postulate: from flaw to necessary element

Another major flaw was visible, in the past, when studying Euclid's *Elements*: the formulation of the fifth postulate. This formulation clashed with the demand for the intuitive evidence of the postulates and, therefore, their universal acceptability without proof. Today we know that what was once considered a flaw on Euclid's treatise is instead a necessary element of it. In (Villani, 2006), Vinicio Villani observes that "Euclid must have been well aware of the lack of intuitive evidence for his fifth postulate. This is testified by the fact that the Elements are organised in such a way as to postpone, as far as possible (precisely until proposition 29), recourse to this postulate. But in order to proceed further, Euclid was forced to make a choice between three possible alternatives: (I) To consider the statement of the fifth postulate as a theorem and to succeed in giving a "pseudo-proof" of it (for example analogous to the one of Saccheri) based on his other postulates. (II) To omit the fifth postulate altogether, i.e. to limit oneself to the development of absolute geometry. (III) Insert it as a postulate (in one of its equivalent versions)". Villani then observes that the first alternative "would have been irretrievably invalidated by the construction of hyperbolic geometry, the second would have been reductive since "the most important theorems of Euclidean geometry [...] are not valid in absolute geometry" (e.g., Pythagoras' theorem), the only objection that can be made to the third alternative – which we have already raised: its lack of intuitive evidence – has "become irrelevant from the mathematical point of view since the researches of Bolyai, Lobacecskij, Riemann and Hilbert collapsed the ancient belief of the identification of Euclidean geometry with physical reality".

Villani concludes that "if one wants to insist on seeing a "stain" or "imperfection" or "flaw" in Euclidean geometry, this does not depend on Euclid's choices, but on the theory one wishes to develop". Bertrand Russell observes that, although the theory of parallels and the theory of proportions were long thought to be the only two weak points in Euclid, they have proved to be almost the only points where Euclid does not offer any criticism (Russell, 1965).

2.6 Development of non-Euclidean geometries: a revolution in mathematics?

For over two thousand years after the writing of Euclid's *Elements*, Euclid's postulates were held to be self-evident truths concerning physical space. The self-evidence of the postulates of geometry guaranteed the truth of the propositions they deduced. The certainty in the self-evident truths underlying Euclid's geometry persisted even as errors in the foundations of the other sciences began to be observed (Moise, 1990). This certainty began to waver only with the development of hyperbolic geometry and definitively collapsed once models of hyperbolic geometry became known, demonstrating its consistency (a consistency relative to the consistency of Euclidean geometry). "We now think not of a unique, physically "true" geometry, but of a number of mathematical geometries, each of which may be a good approximation of physical space, and each of which may be useful in various physical investigations. Thus, we have lost our faith not only in the idea that simple and fundamental truths can be relied upon to be self-evident, but also in the idea that geometry is an aspect of physics" (Moise, 1990). Moise observes a "philosophical revolution", a shift in philosophy that "had been developing independently of the mathematicians", and that "helped to give mathematicians the courage to undertake non-Euclidean investigations and publish the results" (Moise, 1990).

In Chapter 1, I have already observed that the discovery and development of non-Euclidean geometries is sometimes claimed to be a revolution in mathematics. Nevertheless, there is a debate on the following topics: have non-Euclidean geometries determined a revolution in mathematics? Are there revolutions in mathematics?

Many authoritative scientists and philosophers have addressed these questions. In the following I will try to report different views that have been expressed around the concept of revolution in mathematics.

Donald Gillies, in the introduction of his book (Gillies, 1995), discusses on the debate begun in the USA in the mid-1970's on the concept of revolution in mathematics. This debate involved, among others, Michael Crowe and Joseph Dauben. They both were pushed to reflect whether the general pattern of Thomas Kuhn's theory of the structure of scientific revolutions is applicable to mathematics. In "The structure of scientific revolutions" (Kuhn, 1962), Thomas Kuhn (1922-1996) discusses his conception of how science grows: the growth of science consists of non-revolutionary periods (periods of "normal sciences") interrupted by periods of revolutionary (or "extraordinary") science characterized by a paradigm[8] shift.

In his chapter published in 1975 in *Historia Mathematica* (Crowe, 1975)[9], Michael Crowe expresses his famous *Law 10*: "Revolutions never occur in mathematics". He stresses on the proposition "in" in his *Law 10* explaining that revolutions never occur "in" mathematics, even if "revolutions may occur in mathematical nomenclature, symbolism, metamathematics (e.g., the metaphysics of mathematics), methodology (e.g. standard of rigour), and perhaps even in the historiography of mathematics". Crowe justifies his law by arguing the following condition: "a necessary characteristic of a revolution is that some previously existing entity (be it a king, constitution, or theory) must be overthrown and irrevocably discarded". Therefore, Craw argues that, for example, the Copernican revolution satisfies his condition while the same condition excludes the possibility that revolutions exist in mathematics. In his chapter, Crowe observes that "Euclid was not deposed by, but reigns along with, the various non-Euclidean geometries". Therefore, the discovery and the development of non-Euclidean geometries was not – according to Crowe – a revolution in mathematics. However, he admits that non-Euclidean geometries led to a revolutionary change in views as to the nature of mathematics.

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[8] "A paradigm is what the members of a scientific community share, and, conversely, a scientific community consists of men who share a paradigm" (Kuhn, The Structure of Scientific Revolutions, 1970).

[9] Reprinted in (Gillies, 1995)

Joseph Dauben agrees on the fact that older theories in mathematics are not discarded in the way that has happened to some scientific theories. At the same time, he argues that "there are certain critical moments, even in mathematics, that suggest that revolutions do occur – that new orders are brought about and eventually serve to supplant an older mathematics" (Dauben, 1984)[10]. Dauben observes that Crowe adopts an "unnecessarily restrictive" definition of the term *revolution*. He adds: "To say that mathematics grows by the successive accumulation of knowledge, rather than by the displacement of discredited past theory by new theory is not the same as to deny revolutionary advance". Discussing about the nature of scientific advances reflected in the development of the history of mathematics, Dauben juxtaposes the term *resolution* to the term *revolution*. He writes: "Like microscopist, moving from lower to higher levels of resolution, successive generations of mathematicians can claim to understand more, with a greater stockpile of results and increasingly refined techniques at their disposal. […] Discoveries accumulate, and some inevitably lead to revolutionary new theories uniting entire branches of study, producing new points of view, sometimes wholly new disciplines that would have been impossible to produce within the bounds of previous theory" (Dauben, 1984). He includes the discovery of non-Euclidean geometries among inventions and discoveries that did not involve crisis or the rejection of earlier mathematics although they represented a response to the failures and limitations of prevailing theory, among inventions and discoveries that have transformed mathematics, one of the case in which "the old mathematics is no longer what it seemed to be, perhaps no longer even of much interest when compared with the new and revolutionary ideas that supplant it (Dauben, 1984).

In (Dunmore, 1995), Caroline Dunmore supports the idea that revolutions do occur in mathematics. She points out that they are confined entirely to the mathematical component of the community's shared background. For this reason, developments in mathematics appear to be cumulative. But it should be more correct to state the following: mathematics evolution is conservative on the object-level but revolutionary on the meta-level. Dunmore takes non-Euclidean geometries as an example of revolution that constituted inclusive advances on the object-level but that demanded the replacement of metamathematical beliefs.

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[10] Chapter reprinted in (Gillies, 1995)

Dunmore argues that "the hard-core of the research programme within which mathematicians are working at any time […] induces in them certain expectations about the results they are to produce. Any anomalous result that violates these expectations is resisted and must struggle to the surface, the strength of the community's reaction to an anomaly being in direct proportion to the strength of the beliefs it violates". Dunmore observes what Dauben already pointed out (Dauben, 1984): the community's resistance to change is a good indication of revolutionary developments.
Moreover, Dunmore underlines a difference between social sciences and mathematics. She argues that in empirical sciences "there are two different types of revolution, those in which some concepts are totally discarded and those in which they are retained as special cases of some more general concepts". Unlike, in mathematics – she suggests – there is only one kind of revolution and "they all exhibit two characteristics: exclusiveness on the meta-level and inclusiveness on the object-level". In conclusion, Dunmore states: "In mathematical revolution, concepts are always conserved, but what are irrevocably discarded are metamathematical principles; so that, while concepts are not actually discarded, their scope and meaning can be altered by reinterpretation in the metamathematical component of the mathematical word".

In (Boi, 1995), Luciano Boi argues why he rejects the use of all sociological concepts to describe and to explain the nature of mathematical knowledge and the intrinsic reason for its development and changes. He explicitly rejects the concept of *revolution*, the concept of *scientific community* and the concept of *paradigm*. Boi argues that "mathematical, and particularly geometrical, knowledge is neither 'revolutionary' nor 'cumulative'. He clarifies his position arguing that mathematics develops through a subtle internal dialectic, and that "it is possible (and very interesting) to outline a 'genealogy' of their [the theoretical sciences] ideal forms and to understand their historicity, that is, the development of their [the theoretical sciences] concepts". According to Boi, important conceptual transformations beset geometrical thought in the nineteenth century; they produced a conceptual turning-point in all mathematical field and in theoretical physics. Specifically, he remarks the discovery and the development of non-Euclidean geometries. He argues that the work of Lobachevsky and Bolyai "consisted in showing the existence, alongside Euclidean geometry, of other geometries equally logically justified and well founded, and also the possibility of constructing these geometries independently of Euclid's fifth postulate. However, they did not call into question the Euclidean conception of space and geometry". Boi cites: "There is an essential difference between Lobachevsky and Riemann: while Lobachevsky carried on the work of Euclid and wanted to become, as it were, a second Euclid, Riemann introduced a completely different approach by conceiving of space as a number-manifold (*Zahlnmannigfaltigkeit*), and by applying this to the whole of analysis" (Lie, 1893). Boi observes that Gauss pushes beyond Lobachevsky. Indeed, Gauss not only criticized the age-old conviction that Euclidean geometry could claim to be the only true description of physical phenomena. He also "acknowledges explicitly that the nature of the space is not determined once and for all, and that it is not prescribed *a priori* by one system of geometry. While admitting that geometry is a mathematical science, he asserts that from the point of view of physics the principle or laws (*Gesetze*) of geometry cannot be completely determined *a priori*, since they depend on a (constant) magnitude (*Grösse*) whose value must be sought experimentally". Therefore, Gauss makes explicit the distinction between geometric space (which is a mathematical construction) and physical space. According to Boi, two reasons permit us to state that the discovery and the development of non-Euclidean geometries represent a rupture within the history of mathematics and in scientific thought in general. The first is the fact that the existence of "a plurality of geometries" becomes evident, "and that each of them can be investigated as an autonomous mathematical theory, thus removing geometry and the concept of space from the realm of the absolute". The second reason concerns the fact that – thanks to Gauss – "The problem of constructing geometry as a mathematical theory was thus completely separated from the question of finding the type of geometry most capable of explaining the phenomena of our physical space".

In his book, *The Non-Euclidean Revolution*, Richard J. Trudeau compares the Euclidean geometry with the hyperbolic one, talks about the invention of hyperbolic geometry as a revolution that brought the scientists asking themselves whether science wasn't a very different thing that they had always thought. Moreover, the author talks about his considerations in his book as striking evidence – though not proof – of the impossibility of a significant and certain knowledge about the world. He addresses the readers who are still not aware of the existence of non-Euclidean geometries and who are going to deal with his book

The Non-Euclidean Revolution "like a 16-th century astronomer hearing of Copernicanism for the first time".

As we have seen in the above paragraphs, the concept of revolution in geometry has been largely discussed by intellectuals in the past. Their opinions are far from converging and other novel perspectives will be probably expressed in the future. What we can conclude is that the discovery of non-Euclidean geometries, if not revolutionary, will be at least an eternal source of debate, inspiration and scientific advancement. After all, this is one of the reasons why non-Euclidean geometries are being considered for the schools, to stir up interest, to change the static and merely technical vision that students have of mathematics, and to help students move to an abstract reasoning.

2.7 Summary

The chapter starts with a discussion about science as a conquer of the Hellenistic era, quoting the studies of Lucio Russo and outlining some of its aspects, that are still at the core of modern science.

Sections 2.2 and 2.3 discuss philological aspects of Euclid's *Elements* and introduce its method. The revolutionary relevance of the book is discussed, since it represents the first historical attempt of providing rigorous proofs in a scientific discipline starting from a minimal kernel of assumptions and building on top of these. Despite its novelty and the high level of rigor attained by Euclid's *Elements*, there are some pitfalls in the book, discussed in Section 2.4. Euclidean geometry has been, thus, revised by Hilbert and other modern authors. An adaptation of this theory is what it is usually taught in schools. The evolution of Euclidean geometry is a useful didactical example of how scientific theories evolve.

Another point to discuss in Euclid's *Elements* is the fifth postulate (Section 2.5). For a long time – even though its validity was not questioned – it was seen as a flaw in the theory: it either needed to be proved (Euclid and Saccheri tried to write a pseudo-proof for it), or to be taken as a postulate. A third option is to omit the postulate, but it was overseen or rejected for centuries. Such an option leads to the development of absolute geometry. With the evolution of modern thinking, it became clear that the fifth postulate can be considered valid or not, leading to several geometries that can represent different geometrical spaces, all licit in their respect. The development of non-Euclidean geometries is discussed in Section 2.6 with a focus on its role as a “revolution”.

3 The teaching of Euclidean geometry in the Italian secondary school

This chapter deals with fundamental topics related to teaching and learning of Euclidean and non-Euclidean geometries in school, with a specific focus on the Italian high school. We start off with a historical perspective on the Italian school system, from the late XIX century to the current days. The role of the school as an institution, the political reforms and the changes in the didactic curricula are analysed and related to the various points of view that the ministers proposed in the teaching of geometry.

3.1 From the unification of Italy to the Coppino reform

In 1859, two years before the Unification of Italy, in the territory of the old Kingdom of Sardinia and in Lombardia freed from Austrian occupation and annexed to Piemonte, the Casati Law came into force, named after the then Minister of Education Gabrio Casati. This law modifies the school legislation established by Bon Compagni and Lanza respectively in 1848 and 1857. It was soon extended to the whole Kingdom of Italy (Regno d'Italia) and, with the exception of a few modifications, it constituted the school system in force until the Gentile law of 1923 which, however, took up its fundamental characteristics. The Casati Law is particularly important for various aspects. Among the positive ones, we recall the recognition of the right-duty of the State to intervene in school matters, replacing and flanking the Church, holder of the monopoly of education; the introduction of free elementary school of four years open to all; the mention of an obligation to attend the first two years of elementary school (in reality the obligation was entrusted to the discretion of the municipalities); the creation of the normal school for the preparation of teachers, the definition of the figure of the public teacher; the priority assigned to public schools over private ones; the overcoming of the distinction, for the purposes of education, between males and females. Among the negative aspects introduced by the law are: the rigidity of the curricula; the limits to the freedom of teaching; the absence of professional education. The Casati Law organizes secondary school education by dividing it into three addresses: the classical address, the technical address and the magistral address. The first begins with three years of lower gymnasium, continues with two years of upper gymnasium and ends with three years of high school. The second is developed in three years of Technical School followed by four years of Technical Institute. The last, which trains the teachers, consists of three years of Normal School which can be accessed after the three years of lower gymnasium or technical schools. This subdivision remains unchanged until, in 1940, the Bottai reform establishes the Scuola Media unifying the lower Ginnasio and the Technical School.

As far as the teaching of geometry is concerned, the most popular book is the Italian translation of "Elémentes de géométrie" by Adrien-Marie Legendre (of 1784), a book that makes up for the lack of texts by Italian authors. Unlike Euclid, Legendre separates the problems from the theorems and places the former at the end of the various chapters of his work because he does not believe that the constructions of the various geometric entities are indispensable for the demonstration of the theorems in which they intervene (Giacardi, 2004). Legendre's text is often judged as not very rigorous, so much so that in the preface of the first Italian edition, the publisher wrote: "The Elements of Geometry are accused of being not very rigorous: although several works of this kind have particular advantages, and satisfy very well the purpose for which they were composed, there is none that comes to demonstrate all the propositions in a way that satisfies the spirit completely" (Legendre, 1802)[11].

3.2 From the Coppino reform to the Gentile reform

In 1867, the so-called Coppino Reform (Michele Coppino at the Ministry of Education) is approved. The Minister of Education calls the mathematicians Luigi Cremona, Enrico Betti and Francesco Brioschi to formulate the mathematics curricula. The reform includes the study of Euclid's *Elements*: the study of the first book in the last year of the Gymnasium and the study of books II to VI, XI and XII

[11] Translated from the Italian *"Si dà taccia agli elementi di geometria d'esser poco rigorosi: benché diverse opere di tale sorte abbiano dei vantaggi particolari, e soddisfacciano assai bene al fine per cui sono state composte, non ve n'è alcuna che giunga a dimostrare tutte le proposizioni in un modo da appagare completamente lo spirito"*.

at the Lyceum[12]. In the Official Gazette of the Kingdom of Italy of October 24, 1867, which contains the instructions and curricula for teaching in the public schools of the Kingdom, we read the spirit to which the teaching of mathematics must adhere: "Mathematics in classical secondary schools is not to be looked upon merely as a complex of knowledge or theories, useful in themselves, of which young people must acquire knowledge in order to apply it later to the needs of life; but principally as a means of intellectual cultivation, as a gymnastics of thought, directed to develop the faculty of reasoning, and to help that just and healthy criterion which serves as a light to distinguish the true from that which has only the appearance of it"[13]. Specifically, with regard to geometry we read: "In geometry, in order to give maximum educational effectiveness to the teaching, and to reduce the subject within modest boundaries, it is sufficient to apply to ours the example of the English schools, returning to the elements of Euclid, which by universal consensus are the most perfect model of geometric rigor. The method of teaching must be but one, that is, all the individual parts must be closely linked together and carried out with rational order and a rigorously scientific process. Euclid is the unsurpassed master of this method"[14]. And again, after observing that "taught with the method of the ancients, geometry is easier and more attractive than the science of numbers", it is recommended "to the teacher that he stick to the Euclidean method, because this is the most appropriate to create in young minds the habit of inflexible rigor in reasoning. Above all, do not cloud the purity of ancient geometry, transforming geometric theorems into algebraic formulas, that is, substituting concrete quantities (lines, angles, surfaces, volumes) with their measures: but accustom your students to always reasoning on the former, even

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[12] See *Istruzioni e programmi, l'insegnamento secondario classico e tecnico, normale e magistrale, ed elementare nelle pubbliche scuole del Regno.*

[13] Translated from the Italian *"La matematica nelle scuole secondarie classiche non è da guardarsi solo come un complesso di cognizioni o di teorie, utili in sé, delle quali i giovani devono acquisire conoscenze per applicarle poi ai bisogni della vita; ma principalmente come un mezzo di coltura intellettuale, come una ginnastica del pensiero, diretta a svolgere la facoltà del raziocinio, e ad aiutare quel giusto e sano criterio che serve di lume per distinguere il vero da ciò che ne ha soltanto l'apparenza"*.

[14] Translated from the Italian *"Nella geometria, per dare all'insegnamento la massima efficacia educativa, e per ridurre a un tempo la materia entro modesti confini, basta applicare alle nostre l'esempio delle scuole inglesi, facendo ritorno agli elementi di Euclide, che per consenso universale sono il più perfetto modello di rigore geometrico. Il metodo d'insegnamento non deve essere che uno, cioè che tutte le singole parti sieno strettamente collegate fra loro e svolte con ordine razionale e con processo rigorosamente scientifico. Di questo metodo è appunto Euclide insuperabile maestro"*.

when considering their relationships"[15]. It is clear that he distanced himself from the tradition inaugurated by Légéndre, who had put in the background the geometric "purity" in the algebraic-geometric tradition (Bottazzini, 1998).
While the Coppino reform did not provide for the teaching of geometry at the lower secondary school, in the parallel Technical Schools geometry was to be taught with the graphical-intuitive method. Also in the Official Gazette of the Kingdom of Italy (translated from the Italian "Gazzetta Officiale del Regno d'Italia") of October 24, 1867 we read: "the teacher will ensure that the students draw on paper with precision the figures that he outlines on the table, and will accustom them to follow on the drawing reasoning that he considers appropriate to do. The reasoning of which will be reduced to obtaining from the drawn figure the intuitive proof of the properties that pertain to it. In this way the construction taught for the solution of a problem (as would be that of conducting the perpendicular to a line from a given point outside of it) can lead intuitively to the discovery of other truths (place of the points equidistant from two given points, properties of the isosceles triangle, etc.). It is not important that the way taken to prove a proposition be rigorously scientific; it is important that the students acquire the knowledge of that proposition and the persuasion of its truth"[16].
Even less scientific rigor is required in the teaching of geometry for normal schools; rather, it is recommended that drawing be used to help students understand definitions and properties of geometric figures.

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[15] Translated from the Italian "*insegnata col metodo degli antichi, la geometria è più facile e più attraente che non la scienza dei numeri*", si raccomanda "*al docente che si attenga al metodo euclideo, perché questo è il più proprio a creare nelle menti giovanili la abitudine al rigore inflessibile nel raziocinio. Sopratutto non intorbidi la purezza della geometria antica, trasformando teoremi geometrici in formole algebriche, cioè sostituendo alle grandezze concrete (linee, angoli, superficie, volumi) le loro misure: ma avvezzi i suoi scolari a ragionare sempre sulle prime, anche là dove se ne considerano i rapporti*"

[16] Translated from the Italian"*il maestro farà sì che gli scolari disegnino sulla carta con precisione le figure che egli delinea sulla tavola, e li abituerà a seguire sul disegno ragionamenti che egli stima opportuno di fare. I quali ragionamenti del resto si ridurranno a ricavare dalla figura disegnata la prova intuitiva delle proprietà che le competono. Per tal modo la costruzione insegnata per la soluzione di un problema (come sarebbe quello di condurre la perpendicolare ad una retta da un punto dato fuori di essa) può condurre intuitivamente allo scoprimento di altre verità (luogo dei punti equidistanti da due date, proprietà del triangolo isoscele, ecc.). Non importa che la via battuta per dimostrare una proposizione sia rigorosamente scientifica: importa bensì che gli scolari acquistino la cognizione di quella proposizione e la persuasione della sua verità*".

In 1868, Luigi Cremona, in order to implement the Coppino Reform, published an edition of the *Elements* edited by Betti and Brioschi. In the new text, the two editors, then members of the Superior Council of Education, comment: "Euclid's supreme accuracy is no longer appreciated in our schools, and inexact demonstrations of properties, which can only be revealed by the senses, are preferred to those axioms and postulates which Galileo judged to be 'such honest and «admissible questions that if the factory of geometry was raised on such foundations, it could only be very strong and very stable»'"[17] (Maracchia, 1998).
In 1869, in a letter to the editor of the *Journal of Mathematics for Italian University Students*[18], after attacking the mathematician J. M. Wilson for describing the *Elements* as *"antiquated, artificial, unscientific and ill-adapted for a textbook"* and criticizing the mnemonic study of the *Elements* in English schools, Broschi and Cremona wrote: "English schools are all classical and everyone must pass through them: instead our gymnasiums and high schools are intended to give a high, exceptional culture. In them, the aim is not to teach geometric drawing, nor does it matter that young people learn this or that proposition, nor that they study many things in a short time. What matters is that they learn to reason, to demonstrate, to deduce"[19] (Brioschi & Cremona, 1869). They also add that they introduced Euclid's *Elements* mainly to free the school from bad "booklets" even though they are aware of the flaws of Euclid's text, they would in fact accept a revised Euclid as long as it is not a "disfigured Euclid".
Criticism of Euclid's *Elements* persisted, however, as a result of which, with a Ministerial Circular of 1870, this text was made compulsory only in books of plane geometry, leaving teachers free to teach solid geometry (Mammana, 2000). A few years later, the obligation was permanently removed.

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[17] Translated from the Italian *"la suprema accuratezza di Euclide non è più apprezzata nelle nostre scuole, e vi si preferiscono dimostrazioni inesatte di proprietà, le quali non ponno essere rivelate che dai sensi, a quegli assiomi e postulati che il Galileo giudicava «domande così oneste e concedibili che se la fabbrica della geometria veniva inalzata sopra tali fondamenti, non poteva essere che fortissima e stabilissima»"*.
[18] Translated from the Italian "*Giornale di matematica ad uso degli studenti universitari italiani*".
[19] Translated from the Italian "*Le scuole inglesi sono tutte classiche e tutti devono passare per esse: invece i nostri ginnasi e i nostri licei sono destinati a dare una coltura elevata, eccezionale. In essi non si mira a insegnare il disegno geometrico, né importa che i giovani vi apprendano la tale o tal'altra proposizione, né che studino molte cose in poco tempo. Importa invece che apprendano a ragionare, a dimostrare, a dedurre*".

In 1881, "Recognized as necessary and useful to diminish the excessive amount of teaching to which students must be accustomed"[20], the Minister of Education, Guido Baccelli, approved a decree that modified the existing curricula. One of the changes made consists in instituting for the first time in the world the inclusion of so-called intuitive geometry (also known as experimental or constructive geometry) and geometric design. He instituted it at the gymnasium, while at the high school he let plane geometry be dealt with by the "Euclidean method"[21]. The purpose of this choice is to provide the first and most important notions of geometry and to make students desire the rational study of geometry itself, which is reserved for high school. This measure also contrasts the gap created between the teaching of geometry in elementary schools and high school.

Almost a century later, Emma Castelnuovo, after observing that the principles expressed by Cremona, Betti and Brioschi did not fit the age of pre-adolescence, wrote these words: "We owe to a great doctor, then Minister of Education, Guido Baccelli, the idea of premising to the logical-deductive course a course of geometry of experimental character" and, further on, "If it is maintained that the geometric entity is a construction of the human mind, independently of the consideration of real objects, it does not evidently make sense to premise to the course of rational geometry a study of experimental, sensory character. It is therefore understandable that countries such as France, which supported the rationalist thesis, did not consider appropriate the introduction of a course of intuitive geometry. But if we start from the hypothesis that the geometric entity is formed in the human mind «by abstraction», starting from observations of real objects and from experiences on these, we will have, on the didactic level, to make the deductive course proceed from a study of experimental character, where the axioms find their natural roots"[22] (Castelnuovo E., 1962).

[20] Translated from the Italian "*Riconosciuto necessario e utile il diminuire la soverchia mole degli insegnamenti cui debbono accingersi gli studenti*".

[21] See *Regio Decreto del 16 giugno 1881. Gazzetta Ufficiale del Regno d'Italia (Mercoledì 31 agosto 1881). Roma*.

[22] Translated from the Italian "*Si deve a un grande medico, allora Ministro della Pubblica Istruzione, Guido Baccelli, l'idea di premettere al corso logico-deduttivo un corso di geometria a carattere sperimentale*" e, più avanti, "*Se si sostiene che l'ente geometrico è una costruzione della mente umana, indipendentemente dalla considerazione di oggetti reali, non ha evidentemente senso premettere al corso di geometria razionale uno studio a carattere sperimentale, sensoriale. Si comprende perciò come quei paesi come la Francia, che sostenevano la tesi razionalista non abbiano ritenuto opportuna l'introduzione di un corso di geometria intuitiva. Ma se si parte dall'ipotesi che*

In 1884, Minister Coppino returned to the government and again changed the structure of the teaching of geometry: he brought forward the study of rational geometry to the fourth grade and eliminated intuitive geometry. More than a century later, the ministerial inspector Vincenzo Vita will observe in (Vita, 1990) that this suppression is generated by the inability to write textbooks adherent to the spirit of the new teaching but too conditioned by the Euclidean model (Maracchia, 1998). The adoption of the Elements as a textbook was no longer recommended, but "modern texts that would follow its spirit and order"[23] (Vita, 1986).

Between 1884 and 1910, *fusionism*, an orientation in the teaching of elementary geometry, was introduced: the simultaneous treatment of related topics in plane geometry and geometry of space, and the use of the latter also for demonstrations of plane geometry (Borgato, 2006). This orientation had already developed in Europe since the '40s of the nineteenth century following the publication in France of the treatise of Alcippe Mahistre (Mahistre, 1844) and in Germany of the treatise of Carl Anton Bretschneider (Bretschneider, 1844). In Italy, this project is taken up in the treatise of Riccardo De Paolis (De Paolis, 1884) and in that of Giulio Lazzeri and Anselmo Bassani (Lazzeri & Bassani, 1891). Fusionism ignites many debates but ends up being abandoned and leaving room again for traditional teaching, despite the minister Niccolò Gallo including it in the 1900 curricula as an optional choice (Maracchia, 1998).

In 1900, Minister Gallo restored intuitive geometry in the curricula for the first classes of high school. In that same year, Federico Enriques, in line with what David Hilbert and Stefan Cohn-Vossen will affirm ("the intuitive tendency [...] proposes to reach a clear perception of the objects considered and a concrete representation of their reciprocal relations"[24] (Hilbert & Cohn-Vossen, 1972)) emphasizes that intuition has a complementary role to rational understanding, affirming that geometric certainty is represented by the empirical basis from which

l'ente geometrico si formi nella mente umana «per astrazione», a partire da osservazioni di oggetti reali e da esperienze su questi, dovremo, sul piano didattico far procedere il corso deduttivo da uno studio a carattere sperimentale, dove gli assiomi trovino le loro radici naturali".

[23] Translated from the Italian "testi moderni che ne seguissero lo spirito e l'ordine".

[24] Translated from the Italian *"la tendenza intuitiva […] si propone di giungere ad una chiara percezione degli oggetti considerati e a una rappresentazione concreta delle loro relazione reciproche".*

postulates are drawn, which, however, have a different character than simple physical observations, a character that resides in the feeling of necessity that accompanies geometric evidence and almost gives the illusion of a logical necessity (Merenghi, 2010). The curricula no longer impose any topic or text for the teaching of rational geometry, it is sufficient that these follow the "Euclidean method" (Maraschini & Menghini, 1992).
In 1903 was published the first edition of the text of Federigo Enriques and Ugo Amaldi "Elementi di Geometria", which, although with subsequent editions and reprints, is the main reference book until the seventies and remains on the market throughout the twentieth century. This text, "born in the wake of Hilbertian axiomatic arrangement, can be considered the culminating point of a restructuring of the teaching of geometry that began immediately after the birth of the Italian State"[25] (Furinghetti,1996).

Following official reports that judged the teaching of mathematics to be inadequate (for example: the VIII and IX official bulletins, respectively of 1882 and 1883, of the Ministry of Public Education) or that showed flaws in the lyceum when compared to German schools (for example, a criticized characteristic was the purely rational teaching method that left no room for practical applications), an attempt was made to revitalize a school system that no longer seemed to reflect the needs of Italian society. To this end, "to free congenitally incompetent students from a useless burden"[26], the minister Vittorio Emanuele Orlando, in 1904, allowed students in the second and third year of high school to choose between Greek and mathematics (Giacardi, 2006). Thus, we are left in the wake of legislative measures which, since 1881, have progressively weakened the teaching of mathematics, both because the content of the curricula decreased and because less time was dedicated to it.

Between 1904 and 1923 there were other proposals for reform, but none of them were realized. Among these was the Boselli Project, abandoned in 1909. Giovanni Vailati (1863-1909) also participated in this project. Among Vailati's proposals there is the one to make a gradual transition between the teaching of *experimental*

[25] Translated by the Italian "nato sulla scia della sistemazione assiomatica hilbertiana, si può considerare il punto culminante di una ristrutturazione dell'insegnamento della geometria iniziata subito dopo la nascita dello Stato italiano".
[26] Translated from the Italian, "*per liberare gli studenti congenitamente incapaci da un inutile fardello*"

geometry (or *operational*, adjectives that he prefers to the term *intuitive*) and the teaching of *rational* geometry, "applying first of all deductive reasoning not to prove propositions that already appear quite obvious to the students, or of the truth of which they have already been convinced by direct observation, but rather to derive, precisely from the latter, other propositions that they do not yet know and that they can then easily verify by resorting to the same means"[27] (Vailati, 1909). Vailati believes that by doing so, deduction will present itself to the students as a way to "economize" experiences and to arrive without them at "predicting" the result, and thus the demonstrative procedure will also appear to have utility as a means of discovery. Another way, according to Vailati, to cultivate in the students the confidence in the deductive method is not to limit themselves to a single demonstration, but to show, for the most important propositions, how the same conclusion can be reached assuming different starting points. Even a couple of years before, Vailati had suggested that guiding and pushing the student to obtain, "by way of experiment and, in particular, with the use of drawing instruments, the greatest possible number of factual knowledge on how to construct figures and their properties, especially not "intuitive"" was "the best way to give birth in him the desire and the need to understand "how" and "why" these properties exist, and to predispose him to consider as interesting the learning, or the search for deductive connections between them, and reasoning that lead to recognize them as consequences of each other"[28] (Vailati, 1907). The article just cited opens an epistolary discussion with Beppo Levi (1875-1961) and a discussion with Giuseppe Veronese (1854-1917). Both disagree with Vailati. In particular, Levi argues that experimental teachings are exclusively informational teachings and can present "a very serious danger with respect to the education of the mind"[29] (Giacardi, 1999)

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[27] Translated from the Italian "*applicando anzitutto il ragionamento deduttivo non già a dimostrare proposizioni che agli alunni appaiano già abbastanza evidenti, o della cui verità essi si siano già convinti per via di diretta constatazione, ma piuttosto a ricavare, appunto da queste ultime, altre proposizioni che essi ancora non conoscano e che essi possano poi facilmente verificare ricorrendo agli stessi mezzi*".

[28] Translated from the Italian "*per via di esperimento e, in particolare, col ricorso agli strumenti di disegno, il più gran numero possibile di cognizioni di fatto sul modo di costruire le figure e sulle loro proprietà, soprattutto non «intuitive»*" fosse " *il miglior mezzo di far nascere in lui il desiderio e il bisogno di rendersi ragione del «come» e del «perché» tali proprietà sussistano, e di predisporlo a riguardare come interessante l'apprendimento, o la ricerca, di connessioni deduttive tra esse, e di ragionamenti che conducano a riconoscerle come conseguenze le une delle altre*".

[29] Translated from the Italian "*un pericolo gravissimo rispetto all'educazione della mente*".

and writes: "I believe that the foundation of geometry is primarily intuitive: external experience is of great importance as an occasion for the manifestation and determination of the geometric forms of our spirit, but it is not what constitutes them"[30] (Levi, 1907).

In 1913, in the Curricula for the Modern Gymnasium-Lycée of Minister Credaro, some of the methodological indications suggested by Vailati in the Boselli Project of 1909 were included. In fact, it is suggested that the teacher beware "of two opposing dangers that would make his work ineffective: the danger of falling into a gross empiricism and that, no less serious, of suffering the flattery of an exaggerated criticism. The empirical method, by hiding the links that pass between the facts suggested by experience and keeping silent about the theories that refer to them, would deprive mathematics of its formative value for the mind and would obscure the fascination that it must exercise on those students in whom the logical faculties prevail. On the other hand, a teaching in which the subtleties of modern criticism would penetrate would be accessible to few and would give these same students a one-sided, and therefore false, idea of what science is"[31] (Castelnuovo G., 1909).

The thought of Federigo Enriques influences the teaching of mathematics giving it a very different imprint with respect to the usual one in which the student had a passive role of admiration and contemplation. Reading (Enriques, 1921) we see that according to the author, the student must no longer move in a pre-established, perfect and well-ordered whole, but must understand that the bases of one theory or another are arbitrary and that the whole mathematical construction is a continuous evolution and expansion in order to solve problems for which the elements and premises that were sufficient to solve simpler questions are

[30] Translated from the Italian *"Io credo che il fondamento della geometria sia principalmente intuitivo: l'esperienza esterna ha grande importanza come occasione al manifestarsi e al determinarsi delle forme geometriche del nostro spirito, ma non è essa a costituirle".*

[31] Translated from the Italian *"da due opposti pericoli che renderebbero inefficace la sua opera: il pericolo di cadere in un grossolano empirismo e quello, non meno grave, di subire le lusinghe di un esagerato criticismo. Il metodo empirico, nascondendo i legami che passano fra i fatti suggeriti dall'esperienza e tacendo delle teorie che ad essi si riferiscono, toglierebbe alla matematica il valore formativo della mente e oscurerebbe il fascino che essa deve esercitare su quegli allievi nei quali le facoltà logiche prevalgono. D'altra parte un insegnamento dove penetrassero le sottigliezze della critica moderna riuscirebbe accessibile a pochi ed a questi stessi darebbe un'idea unilaterale, e quindi falsa, di ciò che è scienza.*

insufficient (Castelnuovo E., 1957). This was Enriques' thought at the gates of the Gentile Reform.

3.3 From the Gentile reform to Piano Nazionale Informatica (PNI) and Progetto Brocca

The Gentile Reform, an organic school reform, takes its name from the neo-idealist philosopher who inspired it: Giovanni Gentile. This is one of the first acts approved by the fascist regime and was passed in 1923. Among the various measures, the physical-mathematical section of the technical institute was cancelled and the scientific high school was instituted. This choice was considered by Lucio Russo "the worst product of the Gentile reform because it lowered the level of scientific skills provided by the Italian school"[32].

The reform unified the teaching of mathematics with that of physics and reduced their timetable. The mathematics and physics curricula are called *Exam Curricula* (translated from the Italian "*Programmi d'esame*") and the teaching of mathematics (and geometry in particular) is given a more utilitarian value than a formative one. As far as the gymnasium is concerned, mathematics should be taught from the fourth gymnasium using the rational method. At the examination of classical maturity, the "ability of the candidate to understand and make his own a rigorous deductive system"[33] are also tested; however, there is no mention of a reworking of the subject studied and a historical-critical study of the foundations that would allow a cultural vision of mathematics (however, the most motivated teachers show the subject taught in all its cultural depth, helped in this by very good texts) (Maracchia, 1998). The guidelines regarding the scientific baccalaureate exam take up those of the classical high school, but require candidates to be more confident in their exposition and broaden the topics in which they must be prepared.

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[32] Translated from the Italian*"il prodotto peggiore della riforma Gentile perché ha abbassato nettamente il livello di competenze scientifiche fornito dalla scuola italiana".* Retrived from www.senato.it/4800?newsletter_item=1973&newsletter_numero=190.

[33] *See Regio Decreto 31 dicembre 1925 n 2473. Programmi di esami di ammissione, di licenza, di maturità e di abilitazione per gli istituti medi d'istruzione. Gazzetta ufficiale del Regno d'Italia (25 gennaio 1926)(19). Roma.*

In open opposition to the Gentile reform there are many mathematicians (e.g.: Federigo Enriques, Guido Castelnuovo, Vito Volterra), scientific associations (e.g.: Mathesis, Academia dei Lincei) and Science Faculties (Giacardi, 2006).
As noted in (Baresi, 2011), actually Gentile's thinking was broader than what passed into the reform that bears his name. He saw mathematics and science as an "ever new attitude of the spirit in the face of reality"[34] (Gentile, 1959) and supported their teachings because he believed they conformed to the idea of a school that truly teaches and forms living and fruitful souls (Gentile, 1959).

Until 1945 there are no major changes to the Gentile reform (Villani, 2011). The only attempt to radically reform the school system is advanced by Giuseppe Bottai in 1939 with *La carta della scuola* but remains largely unimplemented due to the outbreak of World War II. What remains is the unified middle school: no longer the lower gymnasium, a single three-year period common to all.

From 1945, immediately after the end of the war, a commission appointed by the governments of the victorious powers formulated new curricula intended to temporarily replace those of the Gentile era (Villani, 2011). In (Maracchia, 1998) we read that this commission "tries to mitigate a static rigorism that had gradually formed with a dynamic vision of rigor accompanied by the historical development of theories, also giving ample space to an initial intuition"[35]. In 1950 Frajese wrote in the Bulletin of the Italian Mathematical Union (Unione Matematica Italiana) a paper entitled "Storia della matematica e insegnamento medio" in which he invited, rather than giving isolated historical notions, to try "to repeat in teaching the same historical development that has presided over the development of science" (Frajese, 1950).

The development of elementary geometry is one of the chapters of scientific development which he considers suitable for using this method and he examines "in what organic way a true and profound relationship can be established between the history of mathematics and teaching"[36]. Frajese proposes to follow the historical

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[34] Translated from the Italian *"atteggiamento sempre nuovo dello spirito di fronte alla realtà"*.
[35] Translated from the Italian *"cerca di mitigare uno statico rigorismo che si era via via formato con una visione dinamica del rigore accompagnato allo sviluppo storico delle teorie, dando anche largo spazio ad una iniziale intuizione"*.
[36] Translated from the Italian *"in quale modo organico possa stabilirsi una vera e profonda relazione tra storia della matematica ed insegnamento"*.

development of elementary geometry in four phases, perfecting – with respect to what is already done, at least in the broad outlines of teaching – the historical-didactic correspondence. The four phases are as follows: “1) Pre-Hellenic geometry (Egyptian, etc.): essentially linked to materiality; 2) Geometry of the Hellenic period (600-B00 B.C.): formation of the Elements system, i.e. passage from intuition to reasoning; 3) Elements of Euclid (300 B.C.): Euclidean rigor, perpetuated, albeit with minor refinements, until the end of the XYIII century; 4) Nineteenth century: organic revision of the "Euclidean rigor" and establishment of a "perfect rigor"[37] (Frajese, 1950). In the same article, Frajese also observes that the repetition of the phases should not be carried out in a literal sense, but according to the spirit of the scientific development of the various periods. And he adds that, at least in the broad outlines, what he proposes is already happening, but it is necessary that we acquire full awareness and perfect the historical-didactic correspondence. In fact – he adds – in the elementary school, it is already taught a kind of Egyptian geometry in the sense of a measurable and calculating geometry that starts from matter, and remains at least partially linked to "matter"; in the middle school the student begins to form in his mind the system of the Elements (corresponding to the period from 600 BC to 300 BC) through an insensitive, gradual transition from intuition to reasoning; after middle school the student is ripe for initiation into rational geometry, the development of which will last throughout the course of study; organic revision of "Euclidean rigor" is suggested for the last year of high school, as part of a general organic repetition, which for many students would be a true orienting revelation (Frajese, 1950).

The curricula on which they worked in 1950 took into account the experiences gained abroad, particularly in France, Belgium and England (Villani, 2011). One of the elements that has touched the Italian scene is the spread of Bourbakism that originated in France in the 1930s. Bourbakism is the movement linked to the collective of French mathematicians known by the pseudonym Bourbaki. According to the Bourbakist project, in its original version, all mathematics was to be completely permeated by the axiomatic and deductive method and reduced

[37] Translated from the Italian *“1) Geometria preellenica (egiziana, ecc.): essenzialmente legata alla materialità; 2) Geometria del periodo ellenico (600-B00 a. C): formazione del sistema degli Elementi, cioè passaggio dall’intuizione al ragionamento; 3) Elementi di Euclide (300 a. C.): rigore euclideo, perpetuatosi, sia pure con perfezionamenti di poco conto, fino a tutto il secolo XYIII; 4) Secolo XIX: revisione organica del «rigore euclideo» ed instaurazione di un «rigore perfetto”*

and unified on the basis of the great abstract structures of modern mathematics: the algebraic, topological and order structures. Bourbakism also advocates for overcoming the old traditions in mathematics education by inciting revolutionary changes under the label of "Mathématique Nouvelle" (Villani, 2011). According to the Bourbakist position, one must proceed from the real numbers, establishing rules for operations on a set of undefined objects, so as to create a vector space structure. Euclidean geometry can be treated in three lectures, in which a system of axioms is presented; the properties of triangles play no role in this path. Conditioned by the new wind of bourbakism, the geometry textbooks that spread in France in the 1960s propose a concatenation of definitions, postulates and theorems without reference to geometric intuition, there are very few graphic representations, and heuristic methods and applications to empirical reality are absent. The problem of the relationship mathematics-reality is completely removed, the discipline is proposed as a pure formal game. Bourbakist positions were expressed in a desecrating way – to the cry "À bas Euclid! Mort aux triangles!" (in English, "Down with Euclid! Death to triangles!") – by the French mathematician Jean Alexandre Eugène Dieudonné (1906-1992) during the conference of CIEAEM (Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques - International Commission for the Study and Improvement of Mathematics Teaching) held in 1959 in Royaumont (Paris). CIEAEM is an international commission founded in 1950 with the aim of finding new approaches to mathematics education suitable to the changed mathematical and social contexts (Furinghetti, Menghini, Arzarello, & Giacardi, 2008). The founder is the mathematician, pedagogue and philosopher Caleb Gattegno (University of London). His main collaborators in the first ten years are the French mathematician Gustave Choquet and the Swiss psychologist and theorist of cognition Jean Piaget. "Choquet brought into the discussion the ideas of a reform guided by the new restructured "architecture" of mathematics, Piaget presented his famous research results on cognition and conveyed new insights into the relationships between mental-cognitive operational structures and the scientific development of mathematics, Gattegno attempted to link the new mathematical meta-theory to psychological research through a philosophical and pedagogical synthesis and to create and establish relationships with mathematics education as an important part of general education"[38]. Among the members we find the Italians Filippo Spagnolo and Emma Castelnuovo who, between 1979 and 1981 will be president of the commission.

Middle school curricula change in 1979, under Minister Mario Pedini. Although this school is now unified and attended by the vast majority of people, its curricula become increasingly challenging. Maracchia describes the method indicated for mathematics: a gradual passage from an initial "operativity", to a "mathematical interpretation" of real phenomena, until reaching a "systematicity" in view of a tendency towards increasingly abstract situations. Furthermore, especially in the mathematics curricula, there is more explicit mention than elsewhere of the need for interdisciplinarity among the various subjects of study (Maracchia, 1998).

3.4 From Piano Nazionale Informatica (PNI) and Progetto Brocca to the Gelmini reform

The curricula of the upper secondary school also undergo a series of variations by specially appointed commissions. The most important of these are the Piano Nazionale Informatica (PNI) and the Progetto Brocca. In the addresses that adhere to these two experiments, the number of weekly hours of mathematics is greater than in the other addresses.

The Piano Nazionale Informatica (PNI) experimentation was introduced in 1985, under Minister Falcucci (we note that computer science had already officially entered Italian schools in the years 1965- (Barozzi & Ciarrapico, 2003) in technical education). The PNI curricula[39] recognize Mathematics as "a relevant part of human thought and a driving element of philosophical thought itself"[40] and indicate, for the two-year periods of all secondary schools, the aim of determining "in young people the habit of sobriety and precision of language, care for

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[38] See *Regolamenti di Riordino dei Licei, degli istituti tecnici e degli istituti professionali emanati dal Presidente della Repubblica in data 15 marzo 2010.*

[39] See *Circolare ministeriale 6 febbraio 1991, n. 24, Piano Nazionale per l'introduzione dell'Informatica nelle scuole secondarie superiori*.

[40] Translated from the Italian "*parte rilevante del pensiero umano ed elemento motore dello stesso pensiero filosofico*".

argumentative coherence, and a taste for the search for truth"[41]. The learning objectives related to geometry that students must achieve at the end of the first two years are: to be able to identify invariant properties for simple transformations and to be able to demonstrate properties of geometric figures. We observe that it is required that the student, at the end of the two-year period, is also able to historically frame some significant moment in the evolution of mathematical thought. The purpose of the teaching of geometry in the two-year period is primarily "that of progressively leading the student from the intuition and discovery of geometric properties to their rational description, and as such represents a privileged guide to argumentative awareness"[42]. The teacher is invited to pursue these aims by adopting a method which, "making use of the intuitive knowledge learned by the student in middle school, proceeds to the rational development of limited chains of deductions"[43] and to clearly recognize and explicitly formulate each hypothesis or admission to which recourse will be made. It is further explained that the teacher can "lead the student to familiarize himself with the hypothetical-deductive method on circumscribed parts of geometry, without the concern of arriving at the construction of a global system of axioms"[44]. And in this perspective, he can plan "an organic frame of reference, a choice of properties (theorems) of plane figures to demonstrate, using the geometry of transformations or following a more traditional path"[45]. It is left to the teacher to present first the plane geometry and then the space geometry, or merge the two expositions. In order to deal with the need for a rational construction of mathematical knowledge, it is suggested to follow its historical evolution.

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[41] Translated from the Italian *"nei giovani abitudine alla sobrietà e precisione del linguaggio, cura della coerenza argomentativa, gusto per la ricerca della verità"*. Retrieved from https://www.cieaem.org/index.php/en/about-us-en/some-famous-names-of-the-cieaem-and-their-testimonies-en

[42] Translated from the Italian *"quella di condurre progressivamente l'allievo dalla intuizione e scoperta di proprietà geometriche alla loro descrizione razionale, e rappresenta come tale una guida privilegiata alla consapevolezza argomentativa"*.

[43] Translated from the Italian *"facendo leva sulle conoscenze intuitive apprese dall'allievo nella scuola media, proceda allo sviluppo razionale di limitate catene di deduzioni"*.

[44] Translated from the Italian *"condurre l'allievo a familiarizzarsi con il metodo ipotetico-deduttivo su parti circoscritte della geometria, senza la preoccupazione di pervenire alla costruzione di un sistema globale di assiomi"*.

[45] Translated from the Italian *"un quadro di riferimento organico, una scelta della proprietà (teoremi) delle figure piane da dimostrare, utilizzando la geometria delle trasformazioni oppure seguendo un percorso più tradizionale"*.

As far as the three-year course is concerned, it is specified that the teaching of mathematics cannot be conducted autonomously and detached from the other subjects and must be integrated in the individual institutes with their specific singularities in relation to the characteristics of the course. Focusing on the classical and scientific high schools, we read that, at the end of the three-year period, students must have achieved the following objectives: to have assimilated the deductive method and understood the meaning of axiomatic systems; to have understood the relationship between philosophical thought and mathematical thought; to be able to historically frame the evolution of fundamental mathematical ideas. Among the listed contents we find the non-Euclidean geometries from the elementary point of view and the axiomatic arrangement of Euclidean geometry. It is commented that "The presentation of the non-Euclidean geometries will not be an end in itself, but it will serve to better clarify the concepts of axiom and of axiomatic-deductive system; it can be conducted also through the illustration of the most significant attempts of demonstration of the V postulate of Euclid. The acquisition of these concepts will allow the critical re-examination and the logical chaining of the Euclidean geometry topics already studied, as well as the enucleation of the relative system of axioms"[46]. Among the methodological indications to be adopted in the three-year course, those suggested in the two-year program are repeated. Moreover, the teacher is invited to "illustrate, and possibly deepen, with the help of the colleague of philosophy and through the reading of historical pages, some issues of epistemology of the discipline"[47].

In 1988, a commission, the Brocca Commission ("Commissione Brocca"), was established with the mandate of "revising" the curricula of the first two years of upper secondary education, in view of the extension of compulsory education to the sixteenth year of age. The Brocca Commission concluded its work of revising the Italian public education system in 1992. "Mathematics and computer science"

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[46] Translated from the Italian "*La presentazione delle geometrie non euclidee non sarà fine a se stessa, ma servirà a chiarire meglio i concetti di assioma e di sistema assiomatico- deduttivo; essa potrà essere condotta anche attraverso l'illustrazione dei più significativi tentativi di dimostrazione del V postulato di Euclide. L'acquisizione di questi concetti consentirà il riesame critico ed il concatenamento logico degli argomenti di geometria euclidea già studiati, nonché la enucleazione del relativo sistema di assiomi*".

[47] Translated from the Italian "*illustrare, ed eventualmente approfondire, con il concorso del collega di filosofia ed attraverso la lettura di pagine a carattere storico, alcune questioni di epistemologia della disciplina*".

is a common discipline for all courses of studies. Regarding geometry, the objectives that the student must achieve at the end of the two-year period are the same as those set by the PNI experiment, including the ability to historically frame some significant moment in the evolution of mathematical thought. Similarly to the PNI experiment, the aim is to progressively lead the student from the intuition and discovery of geometric properties to their rational description, making explicit every hypothesis or admission used. Furthermore, it underlines the importance of taking care of the constructive processes that concern the discipline and to start from didactic situations that favor the emergence of mathematical problems, the practice of heuristic procedures to solve them, the genesis of concepts and theories, the approach to axiomatic and formal systems taking as sources the real world, mathematics itself and all the other sciences, thus also favoring interdisciplinarity. It is also stated: "There are exemplary historical models of the processes of mathematization that can also illustrate the intrinsic difficulties: one thinks of the pre-Euclidean mathematization in geometry and of its difficult and rigorous Euclidean-Hilbertian approach, of the formal system of arithmetic, of the theories regarding real numbers, of logic, of probability, etc.". In this sense, the reflection on the role of models and mathematical language in physics and in the complex systems of biology and sociology makes us understand the importance of this reference also for the teaching of mathematics. The central didactic problem that is posed to the teacher in the implementation of the curricula resides in the choice of situations particularly suitable for conjectures, hypotheses and problems to arise in a natural way. For such a didactic practice, the results of research in the historical field offer priority inspiration"[48]. Another analogy with PNI experimentation is the faculty given to the teacher to present first plane geometry and then space geometry, or to merge [...] the two expositions.

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[48] Translated from the Italian "*Dei processi di matematizzazione esistono modelli storici esemplari in grado di illustrarne anche le intrinseche difficoltà: si pensi alla matematizzazione pre-euclidea in ambito geometrico e al suo difficile rigoroso approdo euclideo-hilbertiano, al sistema formale dell'aritmetica, delle teorie riguardanti i numeri reali, alla logica, alla probabilità ecc.. In tal senso proprio la riflessione sul ruolo dei modelli e del linguaggio matematico in fisica e nei sistemi complessi della biologia e della sociologia fa cogliere la portata di questo riferimento anche per la didattica della matematica. Il problema didattico centrale che si pone al docente nell'attuazione dei programmi risiede nella scelta di situazioni particolarmente idonee a far insorgere in modo naturale congetture, ipotesi, problemi. Per una pratica didattica così finalizzata, offrono prioritaria ispirazione i risultati delle ricerche in campo storico*".

Among the objectives that the student must achieve at the end of the three-year course we read: to be able to develop proof within axiomatic systems; to solve geometric problems in the plane synthetically or analytically; to intuitively interpret geometric space situations; to recognize the contribution of mathematics to the development of experimental sciences; to historically frame the evolution of fundamental mathematical ideas; to understand the interactions between philosophical and mathematical thought. The teaching indications (common to all addresses) are then reiterated: "it is insisted on the opportunity that the teaching is conducted by problems [...] It should be remembered that the term problem should be understood in its broadest sense, that is, also referred to internal issues of mathematics itself; in this hypothesis, it could be didactically profitable to historicize the issue by presenting it as a succession of attempts brought to higher and higher levels of rigor and abstraction; in this regard, the process that led to non-Euclidean geometry, and the one which led to the integral field have been recalled"[49] (among the contents of the three-year course are in fact non-Euclidean geometry from the elementary point of view; the hypothetical-deductive method: primitive concepts, axioms, definitions, theorems: consistency and independence of a system of axioms; formal systems and models).

Among the subsequent reforms, the most relevant is the one promoted by Minister Moratti in 2003[50]. This reform is still in force today, even if its implementing decrees have been profoundly modified by the regulatory acts issued between 2008 and 2010 by the Gelmini reform, reform named by the Italian Minister of Education Mariastella Gelmini.

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[49] Translated from the Italian "*si insiste sulla opportunità che l'insegnamento sia condotto per problemi […] Si ricorda che il termine problema va inteso nella sua accezione più ampia, riferito cioè anche a questioni interne alla stessa matematica; in questa ipotesi potrà risultare didatticamente proficuo storicizzare la questione presentandola come una successione di tentativi portati a livelli di rigore e di astrazione sempre più spinti; sono stati a riguardo ricordati il processo che portò alle geometrie non euclidee e quello che sfociò nel campo integrale*".

[50] See *Legge n. 53 del 2003. Gazzetta Ufficiale della Repubblica Italiana (Mercoledì 2 aprile 2003). Roma.*

3.5 From the Gelmini reform to present day

The Gelmini Reform was implemented in school year 2010/2011 but became fully operational in school year 2014/2015. It introduces high school national recommendations regarding the specific learning objectives for high schools: *Indicazioni Nazionali*[51]. These represent the disciplinary declination of the educational, cultural and professional profile of the student at the end of high school courses"[52]. For each discipline, general guidelines have been drawn up that include a description of the competences expected at the end of the course; this is followed by the specific learning objectives divided into subject areas for each two-year period and for the fifth year. The disciplinary cores are the following Arithmetic and Algebra (both two years); Geometry (all years); Relations and Functions (all years); Data and Predictions (both two years); Elements of Computer Science (first two years only).

Among other suggestions, these guidelines observe that the student should be able – at the end of his/her studies – to understand the historical context of several mathematical theories and their conceptual meaning. Among the concepts that must be the object of study we read: the elements of Euclidean geometry of the plane and of space within which the characteristic procedures of mathematical thought take shape: definitions, proofs, generalizations, axiomatizations (it is suggested not to reduce the Euclidean approach to the two-year course to a purely axiomatic formulation, but no alternative modalities are suggested); a clear vision of the axiomatic approach in its modern conception and of its specificity with respect to the classical Euclidean approach. However, the recommendations do not list non-Euclidean geometries among the suggested teaching topics. The Italian mathematician and lecturer Walter Maraschini (1949-2017) critically stated in a speech during the round table of the XXIX UMI Conference, that this lack weakens the coherence of the recommendations because in the contemporary historical and methodological framework of mathematical studies, it is difficult to understand the modern axiomatic approach without studying non-Euclidean geometries,

[51] See *Decreto Interministeriale 211 del 7 ottobre 2010 - Indicazioni Nazionali per i Licei*.
[52] See *Regolamenti di Riordino dei Licei, degli istituti tecnici e degli istituti professionali emanati dal Presidente della Repubblica in data 15 marzo 2010*.

which unlike the Euclidean "descriptive" method, is a hypothetical-deductive method (Maraschini, 2010).

3.6 Summary

This chapter discussed the teaching of Euclidean geometry in Italian school from the unification of Italy to the current days, with the intent of leading to current guidelines and the role of non-Euclidean geometries have in it.

At the beginning (see Section 3.1), there was no Italian textbook for teaching geometry, therefore one popular book was from Legendre, and used to divide the problems from the theorems, which at the time was not considered very rigorous. With the first years of the Italian kingdom (Regno d'Italia), many reforms were written (Section 3.2) to build and reform a national school system, often leading to radical changes in the guidelines and objectives in the span of a few years. For example, Euclid's *Elements* were described as either perfectly fit for teaching rationality or as aged and "classical". In these years, intuitive geometry was introduced not only in technical schools but also in the Gymnasium, and various forms of syncretism between the rational Euclidean geometry and the more practical-oriented drawing were proposed. Another discussed topic was the bridging of the way geometry was taught in elementary schools and in high school, the former more intuitive and based on experience. In the attempt to established proper ways to teach geometry in school, Italian authors published their textbooks in Italian, expressing their take on the topic.

Important changes in the Italian school system were introduced by the so-called Gentile reform, in 1923, described in Section 3.3. This was inspired by the fascist intellectual Gentile's ideology. In his view, the role of arts and humanities was much higher than that of science and technique, therefore he gave emphasis to these in the Gymnasium curriculum. He instituted the *Liceo Scientifico*, which is still very present in the Italian school system and the school was mandatory up to the third year of middle school. Some authors argue that the reform gave a utilitarian role to mathematics. In the years after the constitution of the Italian Republic, several reforms were done and more rigor was introduced back in the teaching of mathematics, following French cognitivism and Bourbakism.

An important turning point in upper secondary education was the introduction of the PNI in 1985 and the indications of the Brocca Commision in 1988 (Section 3.4). In these years a great weight is given to mathematics and computer sciences. In particular, the indications of PNI and Brocca Commision give high relevance to the axiomatic method and to axiomatic geometry as a tool to develop logical skills, critical thinking and a historical perspective of mathematics. Non-Euclidean geometries are also mentioned: they do not need to be studied per-sé, but as a way to show the importance of an axiomatic system. The intersection between disciplines such as mathematics, physics and philosophy is also emphasized as a way to empower students, with interdisciplinary classes.

Finally, in Section 3.5 the Gelmini reform (2010) is introduced. The reform produced new indications where the role of the axiomatic method is still important but with a reduced emphasis. For example, they indicate that Euclidean geometry should not be taught only from an axiomatic point of view. These new indications completely omit non-Euclidean geometries, significantly reducing the ability to clarify to the student the properties and importance of an axiomatic method.

4 An experimental study

In the present chapter I will first give a motivation for my interest in non-Euclidean geometries (Section 4.1) and then investigate some open questions in their teaching (Section 4.2). Then, in Section 4.3, I will provide a broad overview of experimental studies related to the teaching of non-Euclidean geometries that will be taken as a starting point for my experimental study. I will then formulate the research questions for the work in Section 4.4 and state how an answer as been sought.

4.1 Rigor and proof: non-Euclidean geometries as a testbench

It is well known that many students have a problematic relationship with mathematics. The problem is exacerbated and becomes even more prevalent if we focus on the relationship between students and the concept of *proof* [53] and, more generally, of *axiomatic system*.

The difficulties that students encounter in proving are not only attributed to inability to reason logically, but also depend on social and metamathematical aspects, particularly on their perception of the meaning of *proof*. Indeed, students struggle to attribute a clear function to *proving* in terms of meaning, purpose, and

[53] The concept of *proof* is a complicated one and an epistemological knot. In this context – that is, in the context of the teaching practice in high school – by the term *proof* I mean a logical and coherent chain of reasoning that guarantees that something follows from what we have assumed to be valid. One might wonder what is meant by "guaranteeing". Asking when it is guaranteed that something follows from what we have assumed to be valid is related to asking when a proof is rigorous. Villani et al., in the context of their book, state that a proof is rigorous when it is judged to be so by mathematicians; or, perhaps better, when there are reasons to believe that, if one wanted, it would be possible to write in detail that list of elementary steps mentioned in the logical proof (Villani, Bernardi, Zoccante, & Porcaro, 2012). I could go on to ask how is it possible to – and, in particular, how can a student – realize that a proof is rigorous? With what degree of confidence is it possible to state this? When to assess that the level of rigor achieved by a high-school student is adequate? In this context, I assess that a student has achieved a good level of rigor when: he/she does not activate recursive reasoning; he/she does not draw inferences from graphs; he/she does not take statements for granted when it is not possible (specifically, I would focus attention on statements that - following focused didactic intervention - the student could disprove by showing appropriate counterexamples).

utility (Freudenthal, 1958) (De Villiers, 1990) (Healy & Hoyles, 2000) (Baccaglini-Frank, Di Martino, Natalini & Rossolini, 2018). Gonobolin argues that "the students […] do not […] recognize the necessity of the logical proof of geometric theorem, especially when these proofs are of a visual obvious character or can easily be established empirically" (Gonobolin, 1954).
Schoenfeld (Schoenfeld, 1991) – exploring the ways that mathematics is understood and used in our culture and the role that schooling plays in shaping those mathematical understandings – notice that, despite the best intention to the contrary, often classroom behavior and teacher's comments (too much focused on form) reinforce the message that "proof is a ritual activity in which they [the students] confirm results that are already known to be true and that were intuitively obvious to begin with". Indeed, in students' perspective, to prove that, e.g. *two straight lines perpendicular to a given straight line are parallel to each other*, or *the base angles of an isosceles triangle are congruent*, is a ritual. Therefore, students do not grasp the idea that mathematics is a vehicle for sense making. As stated by Fischbein, "Being apparently self-evident, intuitively accepted cognitions have a coercive impact on our interpretation and reasoning strategies. […] Intuition may become obstacle[54] in the learning, solving, or invention process" (Fischbein, 1994).

We could attempt to solve the above problem in, at least, two different ways: not requiring to prove statements whose truth is generally accepted by students on an intuitive level; challenging "truths" that students usually take for granted.
The first option, in my opinion, is more about avoiding a problem than solving it. Indeed – as Villani et al. also argued – taking all intuitively obvious facts for granted is very dangerous, not only because of the risk that too many properties, perhaps conflicting with each other, will be accepted. Proving only the least intuitive facts risks losing the typically mathematical taste for the arrangement and organization of a theory (Villani, Bernardi, Zoccante, & Porcaro, 2012). Without doubt, the goal of not taking for granted what seems obvious is not easy to achieve. It suffices to recall that even "Euclid often uses arguments that are not logical consequences of his initial assumptions but are rooted in human's intuitive physical experience" (Harel, 2007). This reflection – along with a discussion of the development of non-Euclidean geometries – can be done also with students and could help show mathematics as a human activity (Fischbein, 1994), a human

[54] Here Fischbein refers to the concept of "epistemological obstacles" emerged in (Bachelard, 1938).

construction subject to improvement. In addition, showing students that some of the obstacles they encounter in the process of learning mathematics are also found at a more macro level in the history of mathematics (Kelley, 2000) (Sfard, 1991) (Swetz, 1989), could help them suspend any negative judgments they have about their ability to succeed in mathematics. Consequently – according to (Di Martino, 2001) (Di Martino & Zan, 2001) (Zan, 2007), that have found the dependence of students' performance on their beliefs – their performance could improve.

The second option is the one that led me to conceive the experimental work described in this thesis, i.e. the implementation of courses on non-Euclidean geometries, in order to challenge intuition. Indeed, "the entire conception of mathematics had to be changed in order to feel free to accept, as axioms, statements that contradict intuition" (Fischbein, 1994). As also argued in (Arzarello, et al., 2012), working with spaces that have properties defined by different systems of axioms and for which not even theorems among the most famous ones are valid, force the students to give importance to those valid theorems and axioms on the Euclidean surface that serve to define Euclidean geometry among other geometries. The exercise of understanding and accepting what goes against our intuition, could have a great benefit for high school students. Non-Euclidean geometries are counterintuitive. For example, students do not feel the need to prove that two straight lines perpendicular to the same straight line are parallel to each other because they think it is obvious, and they feel self-evident that through a point not on a given straight line, there exist one and only one straight line that are parallel to the given line. Dealing with spherical geometry, in which that is not true, could let them realize that a proof is needed and that they always must have clear in mind the hypothesis (postulates) that they are assuming. Dealing with hyperbolic geometries, in which the proposition "Through a point not on a given straight line, there exist at least two straight lines that are parallel to the given line" is true and is a postulate of this geometry, could let them realize that propositions that do not appear self-evident can be postulated. Therefore, the study of non-Euclidean geometries can play a fundamental role in conquering the idea of an axiomatic foundation of geometry (Iacomella, 1992). Moreover, working on non-Euclidean surfaces allows one to initiate critical reflection on definitions and on what it means to define in mathematics. E.g., is defining a square as an equilateral and equiangular quadrilateral equivalent to defining it as an equilateral quadrilateral with all right angles?

4.2 Investigating open questions in non-Euclidean geometries teaching

Following the considerations of the previous section, I decided to focus my work on the teaching of non-Euclidean geometries. As I have shown, they can be considered useful for their role in supporting the development of a more abstract and logical thinking. However, is this always true? Are students successfully engaged in such a development process when exposed to non-Euclidean geometries?

While dealing with Euclidean geometries, students may learn (given the proper stimuli) how an axiomatic system is built, however, they are rarely pushed to understand this deeply, since Euclidean geometries build on axioms that seems self-evident and reflect the reality of the material world. Axiomatic systems are fundamental in the construction of scientific knowledge: can we help students in understanding science more deeply?

Of course, teaching non-Euclidean geometries requires time, which is not allocated for this topic on the current Italian guidelines for the high schools, as seen in Chapter 3. However, if the goal is to stimulate reflection and propose different axiomatic systems (generating doubts, and useful discussions), a short introductory course or a laboratorial experience, may be sufficient. Laboratory experiences have been proposed in the past, with such an intent. In the words of (Arzarello, et al., 2012) "A teaching path [on non-Euclidean geometries] such as the one presented, which forces us to ask ourselves what it means to "go straight" or to experiment concretely what it means to "draw a straight line between two points", in a context other than the Euclidean plane, can undermine "truths" considered obvious, forces us to "deal" with spaces that have properties defined by different systems of axioms and for which not even the most "famous" theorems are valid and therefore, on the other hand, forces us to give importance to those theorems or axioms valid on the flat surfaces that serve to define Euclidean geometry among the many geometries"[55].

Can a short introduction to non-Euclidean geometries with some laboratory activities, help students gain a new perspective on mathematics? Can they see mathematics under a new light, see it as a creative and stimulating activity? Can they look at mathematics as a topic that stirs discussion? Can they understand how mathematical process of growing and consolidating was achieved through scientific debate, and not through a mere affirmation of dogmas? It would be very interesting to see whether a first contact with non-Euclidean geometries can break the dogmatic view of hard sciences that is often found in students (and later, in adult citizens).

Comparing a known axiomatic system (provided that they learned enough from their regular Mathematics classes) such as the Euclidean one, with a non-Euclidean geometry, requires to sharpen one's own logical thinking abilities. Can the students improve their critical thinking and proof skills avoiding taking propositions for granted when they cannot? Can students correctly formulate counterexamples?

I believe these questions have been formulated many times before. As an early example, in (Krauss & Okolica, 1977) the authors argue that teaching non-Euclidean geometries should emphasize some of the properties of a postulation system and of the deductive reasoning, including the fact that "the choice of postulates is arbitrary, though they must be consistent; that once a set of consistent postulates is chosen, a certain set of valid conclusions can be drawn from them; that changing a set of postulates may allow us to arrive at a different set of equally valid conclusions". Furthermore the authors stress "the difference between truth and validity".

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[55] Translated from the Italian "*Un percorso [sulle geometrie non Euclidee] come quello presentato, che costringe a interrogarsi su cosa voglia dire "andare dritto" o a sperimentare concretamente cosa significhi "tirare una linea dritta tra due punti", in un contesto diverso dall'ordinario piano Euclideo, può mettere in crisi "verità" ritenute scontate, costringe a "fare i conti" con spazi che hanno proprietà definite da diversi sistemi di assiomi e per i quali non valgono nemmeno i teoremi più "famosi" e quindi, per contro, costringere a ridare importanza proprio a quei teoremi o assiomi validi sulla superficie piana che servono a definire la geometria Euclidea tra le tante geometrie*"

More recently, I have seen many laboratory[56] experiences and courses proposed to high-school students with these objectives. Up to my knowledge, the impacts of these teaching experiences have been evaluated, at most, in the short term.
It would be intriguing to know their impact also in the long term: do students remember the learned lesson, and are able to use critical thinking in their lives or in their university studies? Of course, a long-term evaluation is very hard to conduct, but it would be very important. As stated in Section 3.4, for several years non-Euclidean geometries could be introduced in high school lessons. Therefore, in principle, a comparison between students (nowadays adults) who were exposed to the fundamental concepts of non-Euclidean geometries and those who were not, is possible. However, the many confounding variables would impair such a research question or at least, would make it harder to reduce uncertainty in the obtained results. This triggers an additional question: why were non-Euclidean geometries removed from the national guidelines? Was this an evidence-based decision?
Moreover, up to my knowledge, the laboratory experiences and courses proposed to high-school students with the objectives mentioned above have been evaluated, at most, by adopting qualitative approaches. Therefore, these works are more relevant to school operators than to policy-makers (Vannini, 2009). Given that, generally, such experiences have been evaluated positively, would it not be worthwhile to conduct quantitative research aimed at policy-makers, with the purpose of investigating whether or not to include basic elements of non-Euclidean geometries within school curricula? Most of the questions raised above are too complex to find an answer in the restricted scope of my research work alone.

4.3 Previous experimental works and other useful references

In this section I first discuss those works that deal with investigations on the teaching of Euclidean geometry by mixing it with non-Euclidean geometries to

[56] With the term "laboratory" (or "workshop") I mean a didactic activity in which students have an active role in it, and in which teachers became expert guides (Dedò & Di Sieno, 2013).

high-school or university students or to teachers. Then I present some textbooks useful to build a didactic path with Euclidean and non-Euclidean geometries.

There is a strain of reasoning about the role of teaching multiple geometries altogether to school students that dates back at least at the 1960s. Up to my knowledge the first seminal ideas were provided by the following works: (Adler, 1968), (Kárteszi, 1972), (Krauss & Okolica, 1977). In the first work, Adler observes that "if the student knows of only one set of conclusions, deduced from the commonly used assumptions, he can easily fall into the trap of thinking that the conclusions are necessarily true rather than being merely consequences of the assumptions". Therefore, the author suggests teaching the axioms of hyperbolic geometry. The second work is a paper in Hungarian language, however its contents have been briefly introduced in English in (Bosnyák & Nagy-Kondor, 2008) and (Makara & Lénárt, 2004). From (Kárteszi, 1972), (Krauss & Okolica, 1977) it emerges that the original idea of Kárteszi was to compare and contrast the plane geometry with another geometry, in order to create a continuous confrontation with the students and a deeper understanding of the plane geometry. He also suggests that the use of manipulation improves the understanding of the geometrical concepts.
In (Krauss & Okolica, 1977), the authors report "a promising classroom-tested alternative in high school geometry". The study is conducted with students that have completed at least the 10^{th} grade and who voluntarily engaged in a course dealing with Neutral geometry (Hilbert's refinements of Euclid's postulates excluding the parallel postulate), Lobachevskian (hyperbolic) geometry and Riemannian geometry (single and double elliptic). The article reports interesting activities and observations that I took into account when designing my course (in particular, also during my course, the proof of the existence of a parallel line to a given straight line was analysed and it was pointed out that this does not depend on the fifth postulate). However, among the various activities carried out with the students, only those related to Neutral geometry were described in detail (possibility of proving SSA criterion and observation about the fact that, before proving it, students believed that it did not apply-in Neutral geometry, since generally such a theorem is proved from the sum of the interior angles of a triangle equal to 180°; proof of the existence of a parallel; non-existence of rectangles; existence of parallelograms in Neutral geometry). Those carried out in the area of

hyperbolic and elliptic geometry are not equally detailed (e.g., it is explained that consequences of Lobachevsky's postulate have been treated and proved but it is not stated which consequences). The article does not refer to how many students joined the course or when it was conducted. Moreover, the authors do not make explicit whether they refer to a particular theoretical paradigm. Their investigation is exploratory in nature and the main conclusion drawn by the authors is as follows: investigating several postulation systems "seemed" to help the students to improve their understanding of geometry with respect to students who did not attend the course. The authors also observed that students showed much enthusiasm toward the course and that before they have never encountered such degree of enthusiasm "from students ranging in mathematical ability from moderate to above average".

Following these early works, the Hungarian mathematician and teacher trainer István Lénárt has been working on a Comparative Geometry for decades. Lénárt calls Comparative Geometry the teaching of two or three different geometries at the same time, mainly through direct experimentation with hands-on tools, and intensive use of discussion between classmates (Lénárt, 2007) (Lénárt, 2021). The idea behind the project is explained in (Rybak & Lénárt, 2017). Lénárt suggests the teaching of Comparative Geometry "for all level of general education with adequate educational material and teaching aids", he adds that "It fits traditional classroom environment and e-learning environment as well" (Lénárt, 2007) and discusses some examples. The mathematician argues that it is worth introducing Comparative Geometry – specifically, spherical geometry – within Euclidean geometry teaching practices (Lénárt, 1993). Lénárt invented the so called Lénárt sphere in the early 1990s. He describes its use and teaching activities to compare Euclidean and spherical geometry in (Lénárt, 1996). The activities described are specifically designed for middle and high-school students. His work was also translated in Italian (Lénárt, 2012).
In (Lénárt, 2021) the mathematician observes that "in the past two millennia Euclidean geometry has maintained absolute dominance in the European cultural and educational tradition. Spherical geometry had been involved in secondary education until the middle of the twentieth century, because of its applications in geography, navigation, astronomy and art, but then was gradually pushed out of the school curriculum. Only in recent years it has returned in a completely

subordinate role to plane geometry and in science popularization books such as Glen Van Brummelen's work" (here, Lénárt is referring to (Van Brummelen, 2013)). In the same paper, Lénárt describes how he tried to adapt a university course to an unforeseen emergency situation, a course at the time of Covid-19 pandemic.

In (Makara & Lénárt, 2004), the authors describe experiences in teaching comparative geometry for prospective teachers of primary schools. They discuss examples that refer to changes in their students' thinking, in their mathematical knowledge and their learning and teaching attitudes. The authors' aim was to give students "self-confidence, clearer understanding of geometric concepts, direct experience in mathematical discovery, and joy and satisfaction in their mathematical studies". They – inspired from (Kárteszi, 1972) – based on the following three postulates: "Comparing and contrasting properties of the plane with properties of another well-known surface leads to a deeper understanding of the concept of the plane"; "The concerted process of learning and teaching requires continuous comparison and contrast. Without these activities, comprehensive understanding and operative knowledge remain unattainable for the student"; "Illustration and manipulation are instrumental in the teaching of geometry, because they give way to a quicker, deeper and more effective understanding of the concept and its consequences". The authors compared spherical geometry with Euclidean geometry by means of the Lénárt sphere and some accessories. They also adventured into 3-D space and gave a brief outlook of the Bolyai-Lobachevskian hyperbolic geometry, based on the hemispherical Poincaré model. The authors state that comparing different geometries can be appealing for the students because it offers different ways of thinking, showing different approaches to the same problem (i.e., parallelism, angle, area, sum angles). They conclude their paper venturing to think that "comparative geometry may be part of a living curriculum, from middle school and up, for the mathematics classroom of the twenty-first century".

In (Lénárt, 2021, 2), the author discusses the possibility of adding the hyperbolic geometry on the hemisphere to the plane and the sphere by means of hands-on experimentation and ICT. In this paper, Lénárt describes a comparative geometry syllabus, the main topics, wordings and illustrations by which he tried to make the basic concepts accessible to his students at ELTE University, Faculty of Preschool and Primary Education. In the syllabus, he gives explanation on the following topics: "Surface and basic elements"; "*Lines through two points*"; "*Common points of two lines*"; "*Pencils of straight lines*"; "*Measuring hyperbolic distance*" but he states that he usually omits this topic; "*Measuring hyperbolic angle*"; "*Sum of interior angles in a hyperbolic triangle*"; "*Khayyam-Saccheri quadrilateral*"; "*Lambert quadrilateral*"; "*Napier shape*"). He concludes his paper observing that "There are many other mathematically and historically interesting and important concepts that can be added to this material, such as the equidistant line (the set of points equidistant from a line), the classification of circles and cycles, or the measurement of area".

In (Guven & Karatas, 2009), the authors describe how student mathematics teachers (a university level course) explore new conjectures in spherical geometry and how their conjectures lead them to find proofs by means of dynamic geometry software. The authors first introduced the basic concepts of spherical geometry, such as great spherical lines, spherical angles, polar points, spherical triangles and polar triangles, to the students by using Lénárt spheres. Then, they give place to their students to explore activities by means of Spherical Easel, a program that allows to explore the geometry of the sphere. The activities explored were the following ones: "The sum of the three sides of a spherical triangle is less than 360°"; "The sides and angles of a polar triangle are respectively the supplements of the angles and sides of the primitive triangle". In the conclusions of their paper, the authors assert that working with a dynamic geometry software such as Spherical Easel allowed the course to be transformed into a laboratory in which students could explore relationships and conjectures. In particular, they state that the software allowed students to validate their intuitions and prompted them to seek proof of their conjectures. However, the authors clarify that they do not consider working with a dynamic geometry software more suitable than working on real objects.

In (Gambini, 2021) an eight-year didactic experiment is described, dealing with Euclidean plane geometry and spherical geometry in primary school. The didactic activities were conducted for five years, then the long-term effects have been evaluated after three years from the end of the didactic experiences, using questionnaires. The author states that this work on comparative geometry has several benefits: the students are more engaged with geometry, more motivated and self-confident with mathematics; the acquired knowledge remains over time; the results obtained by (Lénárt, 1993) are confirmed also for elementary school. Another interesting work (Gambini & Lénárt, 2021) collects a large number of experiences with teachers over the span of several years. The authors discuss their activities in the training of Italian and Hungarian primary-school pre-service and in-service teachers using Lénárt sphere. The paper describes several useful anecdotes and the issues with conducting the training remotely.

In (Liguori & Capone, 2017), the authors present a geometry activity aimed at second year students of high school. The activity introduces non-Euclidean geometries in an intuitive way, avoiding the geometric-mathematical formalism. The goal is to make students understand that geometry is around us. By using materials that are easy to find such as spheres, cards, bottles, leaves, the students were not only able to check the validity of known geometric properties, but also to generate new knowledge. The introduction of hyperbolic geometry was inspired by the work of Daina Taimina (Henderson & Taimina, 2005). The teaching methodology adopted was Inquiry-Based Science Education (IBSE), specifically the researchers adopted the 5E model with the variant of the European project Teaching Inquiry with Mysteries Incorporated (TEMI).

In (Bini, 2017), the author presents a teaching experience on non-Euclidean geometries involving the use of artefacts and physical experiences. The main purpose of this experience was to encourage the "process of translation from reality to mathematics and vice versa" (Jablonka & Gellert, 2007). The activities conducted involved 25 high-school students. They have been inspired by (Lénárt, 2009) and (Lamb, 2015) but, in addition, used artefacts and physical experiences. After an introduction to non-Euclidean geometries (discussion on the role of the fifth postulate; validity of the theorems about parallels and transversals; validity of the theorems about the sum of a triangle internal angles), students working on spherical geometry by the use of an orange, elastic bands and a protractor. An example of task given to the student was deriving the formula for the surface area of a spherical triangle. The author states that this task encouraged critical reflection. This task requires that the students switch from the real-world model (the triangle on the orange) to the mathematical model (the formula for the surface of the triangle). The author also states that the use of artefacts and physical experiences let the students support their learning.

In (Arcara, 2005), the author debates the efficacy of a short course on non-Euclidean geometries that she conducted with Italian high-school students. Specifically, she conducted three one-hour classes with students of a first two years of an Italian *liceo scientifico* high school. The work was carried on during the scholastic year 2004-2005, when the non-Euclidean geometries were a subject of study at the last year (the fifth one) of *liceo scientifico* (in the frame of the national PNI project already discussed in Section 3.4). The classes have been conducted in form of frontal-dialogue lessons, moreover the author used some plastic balls when dealing with the Riemann model. The classes were on the following topics: axioms and postulates of the Euclidean geometry; the fifth postulate; attempts of proving the fifth postulate; non-Euclidean geometries, specifically elliptic and hyperbolic geometries; the Riemann model; the Klein model.

Schiano reports results of an extensive experimentation conducted with high-school students (Schiano, 2011). The class activities are only based on 2-hours workshops and no details on their content are reported. However, this work shows data that seem to support that teaching non-Euclidean geometries lets students change their view of mathematics.

Other international research papers that highlight interesting methods for teaching non-Euclidean geometries follow. In 1993 a research group reported the exploration of hyperbolic geometries employing software tools such as NonEuclid (Castellanos, Dan Austin, & Darnell, 1993). A recent book chapter (Kotarinou & Stathopoulou, 2017), reports on progresses in teaching hyperbolic geometries by means of the same software, NonEuclid, which indeed is still actual and has been updated in the years to work on current Internet browsers. Relating to the proposed activities the chapter states that "The students' contact with a non-Euclidean geometry was an opportunity for them to renegotiate the basic concepts of Euclidean geometry – as a geometry and not as 'the' Geometry". It is worth noting that besides the ICT aspects of this experimental work, other modern pedagogical resources are employed, such as the Drama in Education method.

In the frame of the national PLS project (*Progetto Lauree Scientifiche*)[57], many laboratories focused on non-Euclidean geometries have been conducted in different university locations, by university referents who collaborate with high-school teachers of mathematics.

Among these, the short course entitled "Mathematical Revolutions: non-Euclidean geometries" described in (D'Agostino, et al., 2015). This course was carried out over the years 2011-2012/2012-2013 (when non-Euclidean geometries was a topic

[57] It is a national project for the training of teachers on scientific subjects and for the orientation of students of the last years of high-schools towards scientific degrees.

included in the lyceum high-school curriculum). The purpose of these course was trying to change the static, rigid and purely technical vision which too often is associated with mathematics by students. The course was mainly developed in a workshop mode; the workshops were preceded by an introductory lesson and a final lesson with a university teacher. During the introductory lesson, the teacher outlined the origins and historical developments of non-Euclidean geometries. Instead, during the final lesson, the teacher dealt with the links between geometric and physical space. The laboratory activities were conducted using the Geogebra software and working with the Poincaré half-plane model for hyperbolic geometry. D'Agostino et al. do not report quantitative data on the effect of the course on the students. However, they report the description of the course by the high-school teachers involved in first person to its realization. All teachers gave positive reviews. Among the positive aspect appreciated by the teachers, there are the following ones: the course gave the students the opportunity to tackle a curricular topic through laboratory activities, using experimentation for understanding abstract concepts; working in groups, students had to use communication skills. A teacher observes that during the workshops the students were directly involved in the discovery of the figures and properties in hyperbolic geometry, this led the students to compare the new results with the correspondents in Euclidean geometry and to detect the results independent from the fifth postulate.
Arzarello et al. offers a rich variety of educational activities suitable for secondary school students (Arzarello, et al., 2012). These activities were conducted – in the frame of the PLS project – with the use of software or of other didactic material. The book provides supplementary material for download from Springer's online platform. The authors argue that their work is intended to be an introduction to the concept of geodesics as a basic concept in geometry. To help understand the concept of geodesics, the authors propose activities on various surfaces: the sphere, the cone, the cylinder, the plane and the pseudosphere. The authors do not report quantitative data on the effect of the course on the students. Nevertheless, they argue that the method they proposed allows the students to understand the mathematical concepts underlying the activities carried out and to arrive at an advanced symbolic reconstruction. The basic ideas of the authors follow. "To strengthen and re-evaluate the geometry of space; to encourage exploration and discovery activities of geometric properties; to pay attention to the links between

the study of geometry and the real world; to seek historical insights as an opportunity for philosophical reflection"[58].

In (Benvenuti & Cardinali, 2018), the authors report a general positive judgment on the PLS activity on non-Euclidean geometries organized by the University of Camerino since the academic year 2007. The proposed activities proved to be very successful, therefore the number of involved schools and the number and age of the students has been increasing with time. The research work object of the present thesis merge with the PLS project of the University of Camerino in order to expand the research work, specifically on the evaluation of the methods.

To conclude the survey of the related works. I must mention some of the textbooks that are available to build a didactic path with Euclidean and non-Euclidean geometries. Some of these have been inspiring for my experimental study.

In (Benvenuti, 2008), the author deals with Euclidean geometries in a rigorous and, at the same time, simple way. Benvenuti addresses the subject not only from the theoretical point of view but also from the historical and application point of view and shows the influences that the birth and the development of non-Euclidean geometries had in other fields of knowledge (e.g. art and literature).

Even the reading of (Agazzi & Palladino, 1978) does not require previous knowledge of mathematics from the reader and is aimed not only at lovers of philosophy and mathematics, but also at anyone who wants to deepen their knowledge of one of the fundamental stages of scientific thought.

Villani instead targets primarily teachers and future teachers of mathematics (Villani, 2006). Among the topics dealt with, the following were of particular interest: the synthesis of the characteristics of the axiomatics of Euclid, of Hilbert, of Birkhoff, of Diedounné and of Choquet and the reflections on the relative mathematical and didactic aspects. The author reflects on whether introducing a geometry other than the Euclidean one into secondary education is convenient. He recommends proceeding with caution and gradualness in presenting the axiomatic structure of Euclidean geometry to high-school students. He also advises against teaching hyperbolic and elliptical geometry due to their difficulty. The author

[58] Translated from the Italian *"Rafforzare e rivalutare la geometria dello spazio; favorire attività di esplorazione e di scoperta di proprietà geometriche; porre attenzione ai collegamenti tra lo studio della geometria e il mondo reale; ricercare spunti storici come occasione di riflessione filosofica"* (Arzarello, et al., 2012).

indicates as the main difficulty in the study of elliptical and hyperbolic geometry the following one: "the impossibility of referring to global models made in Euclidean physical space without changing the 'rules of the game'". Another difficulty lies in the lack of applications to problems of concrete interest for our daily life. For the aforementioned reasons, the figural intuitive component and the application aspect are missing, leaving only the conceptual aspect. On the other hand, Villani's position towards teaching spherical geometry is different. This – according to the author – has the educational advantage of being viewable on familiar objects and of finding applications in our life. Villani recommends the use of concrete spherical objects rather than resorting to the use of two-dimensional figures (including those created by software).

Also Dedò recommends great caution in showing geometry as an axiomatic method at school, even at high-school (Dedò, 2016). This does not mean banning proofs. But – argues Dedò – by "proof" we must not understand the "mathematical proof" as understood during university studies. This is in order to not stop the curiosity of the students and, at the same time, continuing to pursue the goal of accustoming the students to have a critical attitude and not to take anything for granted. Dedò mentions the following examples: the proof of the congruence of the two angles of an isosceles triangle; the proof that two lines perpendicular to a given line are parallel to each other. She clarifies that if we really want, for example, to show that two lines perpendicular to a given line are parallel to each other, then it is appropriate to show that this is not true on the spherical surface. Moreover, Dedò notes that students are more likely to feel the need to demonstrate when we propose difficult problems.

Bertolini et al. describes three possible paths of workshops on spherical surfaces to be conducted with students (Bertolini, Bini, Cereda, & Locatelli, 2012). The authors clarify that they almost always introduce concepts of spherical geometry starting form analogous concepts of Euclidean geometry.

Carroll and Rykken give an introduction to Euclidean and non-Euclidean geometries designed for an upper-level college geometry course (Carroll & Rykken, 2018). It grew out of authors' previous experience teaching junior/senior level advanced geometry and history of mathematics. I found Carroll and Rykken' work interesting mostly for its first five chapter. In the first one, the authors introduce the "protagonists" of their book: the *straight line* and the *circle*. The second chapter deals with definitions and axioms present in the Heath's translation of the Euclid's *Element*. In the third chapter, they introduce the Neutral geometry. In the fourth chapter the authors come back to the concept of "straight line" working on the sphere. The fifth chapter deals with Taxicab geometry. The authors state that the exposure of the hidden flaw in Euclid's reasoning "sheds light on the importance of the axiomatic development of mathematics, and creates an avenue to discuss the differences between axiomatic systems and their models, as well as the desirable properties of such systems".

Russo et al. reinterpret of the Euclidean elements made mainly following a historical-philosophical criterion (Russo, Pirro, & Salciccia, 2017). The authors reconstruct the I book of the Euclid's *Elements* by removing some passages. The passages removed are the ones considered spurious because – according to the authors – their mathematical content is extraneous to the logical nature of the work. At the same time, the authors change the order of the exposition of some definitions and postulates. In this work, the *segment* – and not the *straight line* – is assumed as a primitive term, which is what we know Euclid referred to when he used the term *straight line*. This book is born from a project conducted students of an Italian high school. The book also contains the Greek version of the Euclid's work, translated by the students thanks to the help of their teacher.

4.4 Research questions

By reading the works discussed above, the following recurring features emerges:

- the usefulness of teaching basic concepts of non-Euclidean geometries and their historical development to let students change their vision on mathematics (generally static, rigid and purely technical), making the students more engaged with mathematics, more motivated and self-confident;
- the usefulness of teaching different geometries – specifically, spherical geometry – within Euclidean geometry teaching practices to explore relationships and conjectures, to renegotiate the basic concepts of Euclidean geometry, and to see Euclidean geometry as "a" geometry and not as "the" geometry;
- the call to conduct a course on non-Euclidean geometries in which students can work in a laboratory manner by means of hands-on experimentation to encourage critical reflection; it is generally considered more convenient to

work on concrete objects rather than resorting to the use of two-dimensional figures; it is convenient to use ICT especially when concrete object cannot be used.

Nevertheless, not all the authors agree on the idea that teaching Comparative Geometry is suitable for all level of general education. For example, Villani advises against teaching hyperbolic and elliptical geometry to high-school students due to their difficulty, while he sees spherical geometry as advantageous because it is simpler, viewable on familiar objects and of finding applications in our lives. (Villani, 2006).

A critical point of view is also the one of Lucio Russo, who – disapproving the tendency to give up on conveying the scientific method in secondary schools and criticizing those who have polemicised against his arguments on the educational value of Euclidean geometry – argues that dissemination on non-Euclidean geometries and their introduction in schools risk having a counterproductive effect. Indeed, Russo argues that, with the intention of using "mystery to fascinate young people, and more generally lay people, by attracting them to science"[59], the mainly conveyed notion is that "'familiar' geometry, the one that can be guessed, is false, for reasons that cannot be understood"[60]. Such dissemination – Russo adds – would induce "an attitude of reverent admiration for science precisely because it is deemed incomprehensible"[61], and would foster the spread of irrationalism (Russo, 2016).

The bibliographic research I conducted reinforced my initial idea: teaching basic concepts of non-Euclidean geometries to high-school students is feasible and it could help the students to develop a more mature understanding of the nature of mathematics and also abstract mathematical thinking. At the same time, Russo's critique warned me from designing a course that appear only mysterious and from developing an irrational feeling for something that is not understood.

Moreover, the divergence of opinions of some authors on teaching hyperbolic geometry to high-school students (see, (Villani, 2006) and, on the other side,

[59] Translated from the Italian *"il mistero per affascinare i giovani, e più in generale i profani, attirandoli verso la scienza"*.

[60] Translated from the Italian *"la geometria 'familiare', quella che si può intuire, è falsa, per ragioni che non si possono capire"*.

[61] Translated from the Italian *"un atteggiamento di reverente ammirazione per la scienza proprio in quanto è ritenuta incomprensibile"*.

(Krauss & Okolica, 1977) or the experiences described in (Arcara, 2005), (D'Agostino, et al., 2015), and (Liguori & Capone, 2017)) prompted me to research in this direction as well.

The main lacks I observed are the following ones: materials and methods are often not available or not adequately detailed; to the best of my knowledge, there are no quantitative studies conducted with high-school students.

As early as 1979, Keeves (Keeves, 1979), stated that qualitative and quantitative studies can lead to generalizable conclusions, but "qualitative studies generally require such high expenditures that they must be limited to small, unrepresentative samples. Their role then is to offer the rich observed detail that suggests explanations of the more cursory effects recorded in quantitative *surveys* or investigations. Recommendations regarding educational policy, based on the limited qualitative research, can only be of a very tentative nature. On the contrary, and although the measurements made by them are rather coarse, the results of nomothetic research are generalizable and we believe that they better enable us to predict or measure the effects of policies developed on this basis"[62] (cited in (De Landsheere, 1999) and in (Vannini, 2009)). Therefore, taking into account the lack of quantitative studies and the fact that quantitative studies seem more likely to inform the policy-makers, I felt it was important to conduct a quantitative study.

From the previous discussion on the research literature, I can condense the most interesting points in the following research questions:

RQ1. What features of a short introductory course in non-Euclidean geometries are effective in engaging high-school students?

RQ2. To what extent do students gain a new perspective on the concept of axiomatic system?

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[62] Translated from the Italian *"gli studi qualitativi e quantitativi debbano completarsi in un programma di ricerca. I due modi d'indagine possono condurre a conclusioni generalizzabili, ma gli studi qualitativi richiedono in generale spese così elevate che essi debbono limitarsi a piccoli campioni non rappresentativi. Il loro ruolo consiste allora nell'offrire il ricco particolare osservato che suggerisce spiegazioni degli effetti più sommari registrati nelle indagini o surveys di tipo quantitativo. Raccomandazioni relative alla politica educativa, basate sulle ricerche qualitative limitate, non possono che rivestire un carattere assai provvisorio. Al contrario, e benché le misurazioni da esse effettuate siano piuttosto grossolane, i risultati delle ricerche nomotetiche sono generalizzabili e noi crediamo che essi permettano meglio di predire o di misurare gli effetti di politiche elaborate su questa base".*

RQ3. How well do students learn the taught concepts of non-Euclidean geometries?

RQ4. To what extent do students' critical thinking and proof skills improve over the duration of the course?

RQ5. Do students' beliefs about mathematics change over the duration of the course?

To answer to *RQ1* I will present considerations arising from my experience in conducting courses in non-Euclidean geometries with high-school students. The experimental phase consisted of several stages and some of the features have been added or removed, based on the feedback obtained during the classes. An answer to *RQ2*, *RQ3*, *RQ4*, and *RQ5*, will be given by analysing four questionnaires specifically designed: the *VHL test* (shown in Appendix 3), the *NEG questionnaire* (shown in Appendix 4), the *PROOF questionnaire* (shown in Appendix 5), and the *BELIEFS questionnaire* (shown in Appendix 6). A summary of the results obtained in regard to each *RQ* can be found in Section 6.5.

Before closing the section, it is worth to provide some additional clarifications on the research questions:

- In answering *RQ2* I focused on investigating the following:
  - *RQ2.1*: *Do students understand that in formulating an axiomatic system, some terms must be left undefined and it is necessary to assume that some statements are valid (postulates or axioms)?*
  - *RQ2.2: Do students understand that the postulates underlying a theory are not required to be self-evident but not to generate contradictions?*
- Answering to *RQ4* is far from trivial. In this work, I have limited myself to investigating whether knowing the existence of geometries other than Euclidean geometry and working on these geometries, teach students not taking for granted statements just because they seem obvious. Specifically, I wanted to check if students can evaluate whether a sequence of logical steps is a valid proof and, if not, show at least a wrong step justifying their answers (e.g., providing a counterexample).
- *RQ5* takes pace from (Schiano, 2011). Specifically, the beliefs I addressed are the following:
  - mathematics is discovered or invented;
  - mathematical concepts are subject to historical revisions;
  - socio-cultural factors influence mathematical knowledge;

- revolutionary changes exist also within the development of mathematical knowledge.

4.5 Summary

The main focus of this chapter is to extract useful research questions to inform the whole experimental work. To do so, I have first discussed in Section 4.1 the concept of proof, which may not only be hard to conduct for a student, but also hard to accept and understand fully. Proving is part of the practice of rigor that mathematics needs, but to fully understand its value, the student may require to see when it is most useful, i.e., when mathematical properties are not self-evident, and this can be easily shown by recurring to non-Euclidean geometries. Teaching non-Euclidean geometries is far from trivial and is a topic yet to be fully discovered. For this reason, in Section 4.2 I discuss some of the open issues and in Section 4.3 I further dive into the scientific literature dealing with experimental studies in the teaching of non-Euclidean geometries. Some reference textbooks are also highlighted which are very useful to teach non-Euclidean geometries. From all the discussions above I could extract some research questions for the experimental phase of the work, detailed in Section 4.4.

5 Experimental phase

The main phases of my experimentations are divided into two sections dealing, respectively, with the pilot study, i.e. a preliminary experimentation conducted initially to investigate a suitable methodology (Section 5.3) and with a second experimentation that draws conclusions on it (Section 5.5).

In addition, this chapter contains two sections dealing with topics related to the experimentations. Section 5.1 describes and justifies the research methods that were used; Section 5.3 describes an experience with high-school teachers; and Section 5.4 describes the necessary adaptations for the distance learning imposed by Covid-19 restrictions.

5.1 Research methods

In developing my work, I initially had no preference on which paradigm and method to adopt for the research. I agree with Pring who – after having contrasted two different paradigms that are in accord with Hammersley's summary of quantitative research and qualitative research[63] – argues his contrariety to the *a priori* adoption of either a quantitative or qualitative view of the world (Cohen, Manion, & Morrison, 2018) (Pring, 2015).

In the words of Vannini, qualitative and quantitative approaches to research have different purposes, and the contexts in which the results are applied determine the lesser or greater appropriateness of one methodological paradigm rather than another. It is the context of educational and school policies that largely calls for the prevalence of quantitative methodologies (Vannini, 2009). Husén also states that these two approaches have different purposes. In his opinion, much of the research oriented to school policy in its broad outlines must be conducted according to the 'positivist' paradigm oriented toward the quantitative, while research relevant to the school practitioner must be more hermeneutic and qualitative in its approach: every pupil and every classroom is unique (Husén, 1993).

[63] Topics discussed in (Hammersley, 2013).

Once the study of the state of the art had been carried out, it was evident that – to the best of my knowledge – so far, the teaching non-Euclidean geometries to high-school students had been approached mainly with qualitative approaches. The results of these studies give us excellent motivation to explore the subject further and – why not? – to orient the investigation to policy-makers.

Therefore, in order to explore the subject further and to expand the range of approaches, but with a view that orients the study to policy making and, possibly, to discover new aspects, I decided to adopt an essentially positivist paradigm using quantitative methods. Quantitative results obtained with reproducible and reliable methods can inform policy-makers. Unfortunately, the inconveniences due to the Covid-19 pandemic intruded into the study and prevented the work from being conducted on a large scale. The originally thought-out framework (repurposed out of necessity for distance learning) remains, though.

Having adopted, in the context of this thesis, a positivist approach, I choose to conduct surveys with students. I focused on quantitative research and I formulated the research work according to a teaching protocol and a rigorous assessment.

My idea was to gather some knowledge through these steps:

- Selecting a sample of students;
- Assessing their abilities and beliefs;
- Formulating a repeatable teaching protocol and addressing this to all the students;
- Assessing their abilities and beliefs again;
- Testing for changes.

Of course, in the field of teaching and learning we need to be careful in the interpretation of this data. Yet, it can provide us novel insight. Additionally, we can draw qualitative data from the students by discussing with them, interviewing them one by one, analysing their thinking with open questions and so on. All this work will lead to draw some specific conclusions in response to the questions raised above, however, most of these are too complex to find an answer in the restricted scope of my own research work alone.

5.2 An experience with high-school teachers

In September 2018, an intensive training course for high-school teachers was organized and conducted by Silvia Benvenuti (who is conducting – in first person or as project supervisor – workshops on non-Euclidean geometries since the academic year 2007) and me, in the frame of the PLS (*Piano Lauree Scientifiche*) (see footnote at page 70) project of the University of Bologna.

The course aimed to present the teaching of non-Euclidean geometries as a tool to promote the understanding of the modern axiomatic method in mathematics, to solicit the students' attitude to logical thinking and to consolidate, developing it critically, the knowledge of Euclidean geometry – as indicated by the national guidelines of high school. To reach this goal, I proposed several workshops that can be implemented at school with the students. Along the course I conducted the teachers exploring non-Euclidean-geometries using inexpensive materials (i.e., balls, markers, rubber bands) and referring to world maps, paintings and woodcuts (i.e., "Boy fascinated by the flight of a non-Euclidean fly" by Max Ernst; the series of "Circle Limit" by M.C. Escher).

The course lasted three days, involved more than 50 high-school teachers and some lower-secondary-school teachers. Teachers involved in the course responded with great enthusiasm to the activities that I proposed. Many teachers – included some lower-secondary-school teachers – expressed interest in conducting activities with their students under our supervision.

5.3 Pilot study

I carried out a pilot study based on class activities on non-Euclidean geometries. The pilot study was aimed to understand how to improve the class activities and how to better design useful questionnaires to collect data for the second experimental study.

5.3.1 Involved subjecs

The pilot experimental phase was conducted with three sets of high-school students involved in the *Piano Lauree Scientifiche* (PLS) of the University of

Camerino (UNICAM) and with a set of students involved in the *Piano Lauree Scientifiche* (PLS) of the University of Bologna (UNIBO).
The activities with Set 1 and Set 2 were conducted during the school year 2018-2019 and the the ones with the Set 3 and Set 4 were conducted during the school year 2019-2020. The subjects involved in the study came from *liceo classico* (classical lyceum) or from *liceo scientifico* (scientific lyceum). *Liceo classico* school offers a wide selection of subjects, but the core ones are those related to Literature; Latin, Ancient Greek, Italian, History and Philosophy. *Liceo scientifico* shares a part of its program with *liceo classico* in teaching Italian, Latin, History and Philosophy, but is more oriented towards Mathematics, Physics, Chemistry, Biology, Earth Science and Computer Science. Axiomatic Euclidean geometry is taught during the first two years of both *liceo classico* and *liceo scientifico*.
Table 1 summarizes the four sets and the involved subjects. A whole class is taken for each one of the first three sets, while the fourth set was assembled collecting voluntary students from different classes and with a strong interest for mathematics.

	Set 1	**Set 2**	**Set 3**	**Set 4**
Number of students	10	18	20	29
Volunteer students	No	No	No	Yes
Class	Students from the same IV class	Students from the same IV class	Students from the same IV class	Students from III-IV-V classes of different schools
School	*Liceo scientifico*	*Liceo scientifico*	*Liceo classico*	*Liceo scientifico - Liceo classico*
School year	2018/2019	2018/2019	2019/2020	2019/2020
University PLS plan	UNICAM	UNICAM	UNICAM	UNIBO

Table 1. Subjects involved in the pilot study.

I trained the mathematics teacher of the class to conduct the first four meetings autonomously. As prescribed by the PLS plan of the University of Camerino, a professor from the university held the last meeting.

5.3.2 Material

In order to facilitate comprehension, I have collected daily life objects that represent a specific geometrical surface. However, I also recurred to 3D-printing to manufacture hemispheres and pseudospheres. These objects can be written with a pencil and reused for several workshops. The 3D printing allows these objects to be replicated for future workshops, if a regular 3D printer is available, or they can be generated from the rendering of simple equations. In addition to these tools, a flexible 3D-printed ruler was devised, with a thin cut in the middle allowing to trace straight lines on a curved surface with a pencil.

Other collected objects emulating non-Euclidean surfaces were: polystyrene spheres that can be written with markers or pinned; a globe; a rugby ball; a paper saddle; plastic bottles and a swim ring.

To draw straight lines or show how to follow them on a curved surface we used the aforementioned 3D-printed flexible ruler, a tiny toy car with no steering (also shown in Fig. 1), sewing thread and pin to be used with the polystyrene spheres.

All the material used in the workshops is shown in Figure 2.

Figure 2. Some material used for the class activities.

5.3.3 Class activities

The activities proposed to the students, were planned as shown in Table 2: five meetings of two hours each, beginning with a workshop on spherical geometry. This ended with an interactive session, with the aim of reflecting on the differences between spherical and Euclidean geometry, focusing on the five postulates of Euclidean geometry and whether they hold on spherical surfaces. In the second meeting the concept of curvature is introduced after discovering the formula for computing the area of a spherical triangle. In the third meeting the concept of curvature is generalized, exploring negative curvature surfaces. During the fourth meeting, I focused on the Poincaré disk model with the use of the software NonEuclid, and I also provided a brief overview of the Beltrami-Klein model and the Poincaré half-plane model. This shows the students that there can be more than an interpretation model for a geometry and to avoid the misconception of identifying a geometry with one of its models. Finally, the last meeting revolved around a frontal-dialogue lesson resuming all the previous meetings and contextualizing non-Euclidean geometries historically and from an application point of view (linking e.g. to relativity in physics or the global positioning system in engineering).

Activity	Topic	Working format
I (2 hours)	Spherical surfaces	Workshop
II (2 hours)	Curvature of a line in one of its points. Gauss curvature of a surfaces in one of its points (non-negative case)	Frontal-dialogue lesson
III (2 hours)	Geometry on the pseudosphere. Hyperbolic geometry	Workshop
	Gauss curvature of a surfaces in one of its points (negative case and general case). Elliptical/parabolic/flat/hyperbolic points Theorema Egregium	Frontal-dialogue lesson
IV (2 hours)	Poincaré disk model	Workshop with the NonEuclid software
V (2 hours)	Resume of previous meetings. Historically contextualization of non-Euclidean geometries	Frontal-dialogue lesson

Table 2. Plan of the class activities.

It is important to note that, for sets 3 and 4, workshops were conducted with slight differences from what reported above:

- For Sets 3 and 4, the workshop on a pseudosphere was replaced with a workshop on a surface with a saddle point. This latter surface is easier to work with in a practical setting and is more readily available (as less expensive) for schools (Figure 3 and Figure 4 show the material used for the workshops).
- I conducted all the activities with Set 3 in first person. Almost two extra hours had to be dedicated to the part on spherical geometry. The work with the NonEuclid software was cut and the introduction of the models of hyperbolic surfaces were integrated in the last meeting.
- The class activity with Set 4 was conducted by Silvia Benvenuti and me, and it consisted of only 6 hours. The frontal-dialogue lessons of the second

and third activities were cut as well as the work with the NonEuclid software.

Figure 3. 3-D printed pseusphere models used for the workshop on hyperbolic geometry with sets of students 1, 2, and 3.

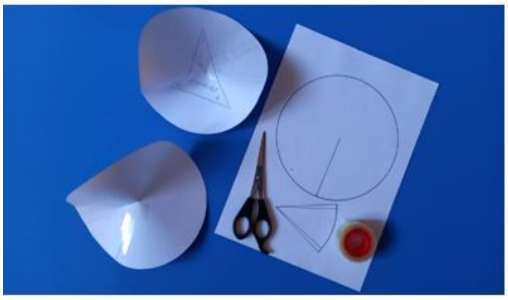

Figure 4. Material used for the workshop on hyperbolic on hyperbolic. geometry with Set 4.

5.3.4 Data collection

To collect data, I administrated questionnaires to the students the week before the beginning of the class activities and the week after the end of the class activities. In the following I describe the three pairs of questionnaires administrated to the students.

QB questionnaire: to collect data on students' beliefs on mathematics, I used two questionnaires, one before the activities and one after the activities (these will be named in the following QBI and QBF). These questionnaires take pace from (Schiano, 2011). They consist in two sets of five questions and – specifically - were used to asses potential changes on students' beliefs on mathematics. QBI and QBF were used only with Set 3. The questionnaires are made up of closed questions, two answers for every question; four of them require the justification of the answer. One of the two possible answers is related to the prescriptive (or normative) account of mathematics that is, as explained by Ernest in (Ernest, 1994), the assumption that "a) the nature of mathematical knowledge is an absolute secure objective knowledge, the cornerstone of all human knowledge and rationality (the assumption of absolutism), and b) the mathematical objects such as numbers, sets and geometric object all exist in some objective superhuman realm (the assumption of Platonism)". The other answer is related to the descriptive (or naturalistic) account of mathematics that, as exposed in (Ernest, 1994), rejects "the epistemological and ontological assumption of prescriptive philosophy of mathematic" and "concern to broaden the scope of a philosophy of mathematics to that of a giving an account of mathematics acknowledging the centrality of mathematical practice and social process". Therefore, an answer is related to the assumption of an objective mathematics while the other one to the view of mathematics as the outcome of a social process, fallible and eternally open to revision (Ernest, 1994).

QK questionnaire: I designed a questionnaire (QK, in the following), to be filled in by the students before and after the class activities, to assess whether they have some prior knowledge on non-Euclidean geometries, to monitor the learning of its basics and of the meaning of axiomatic deductive system, to understand if students are able to not take properties for granted when it is not possible, to detect possible misconceptions. This questionnaire also allowed me to improve the data collection in future and establish a protocol for teaching non-Euclidean geometries with this workshop format. To reach these objectives, I formulated a questionnaire of 10 questions dealing with the following topics: definition of the term *postulate*, negation of the parallel postulate, notion of spherical geometry, intuitive recognition of curvature from example figures, etc.

Van Hiele geometry test: I administrated – before and after the class activities – the van Hiele geometry test as formulated in (Usiskin Z., 1982) (translated to Italian) to get experimental data on the levels of geometric understanding, according to the van Hiele theory. I used this with Set 1, Set 2, and Set 3.

5.3.5 Preliminary results from the pilot study

In this section I present results from the QBI and QBF questionnaires (see Section 5.3.5.1). Then I discuss students' difficulties emerging from the QK questionnaire (see Section 5.3.4). Students also showed many difficulties with practical tasks, such as finding their own resolutive strategy, and sometimes in understanding and following the instructions described in the forms for group-tasks.

5.3.5.1 Students’ beliefs on mathematics

As stated above, the questionnaires QBI and QBF focused on aspects related to beliefs in mathematics. Specifically, the five questions aimed to test whether in the student’s opinion:

Q1. mathematics is discovered or invented;

Q2. in mathematics it is more appropriate stating that an axiomatic system is consistent/non consistent rather than true/false;

Q3. mathematical concepts are subject to historical revisions;

Q4. socio-cultural factors influence mathematical knowledge;

Q5. the human physical-biological nature constitutes a limit to mathematical knowledge.

To monitor potential changes in students’ beliefs, I assigned a score of 0 for every answer related to the assumption of an objective mathematics and 1 point otherwise. Please note that the assignment of a positive score to the latter does not reflect my inclination toward that belief, but is merely for the sake of a quantitative evaluation. There were no empty answers in these questionnaires that I got from the students. Figure 5 shows the results collected from QBI and QBF. As can be seen, the scores for QBF seems to be higher than QBI, on average. I conducted a statistical significance test to reject the following hypothesis h_0: the median score before and after the workshop was identical. The p-value was computed according to the Wilcoxon test and allowed us to reject the null hypothesis with a high confidence ($p < 0.02$). Furthermore, I performed an *effect size* test. Given the low number of subjects (n=20) I employed the Hedge’s *g* coefficient, which corrects Cohen’s *d* for experiments with $n \leq 20$. The computed value is 0.78, which is a rather large value in didactic research (Pellegrini, Vivanet, & Trinchero, 2018).

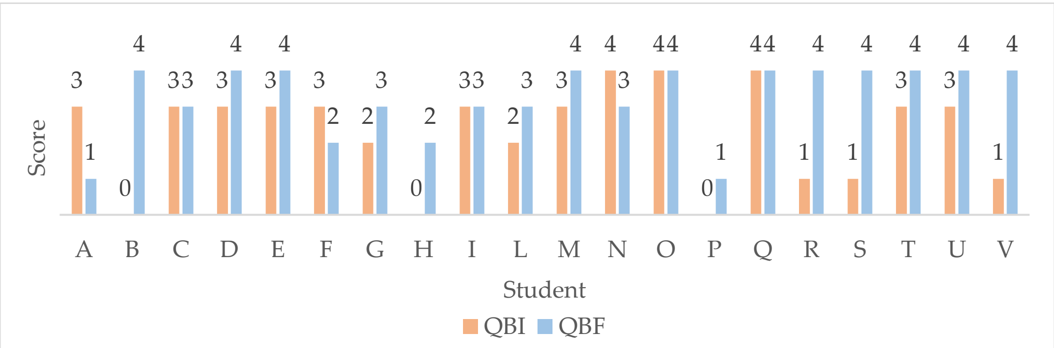

Figure 5. Results of the QBI and QBFfor each student (pilot sudy). Each student is denoted by a letter.

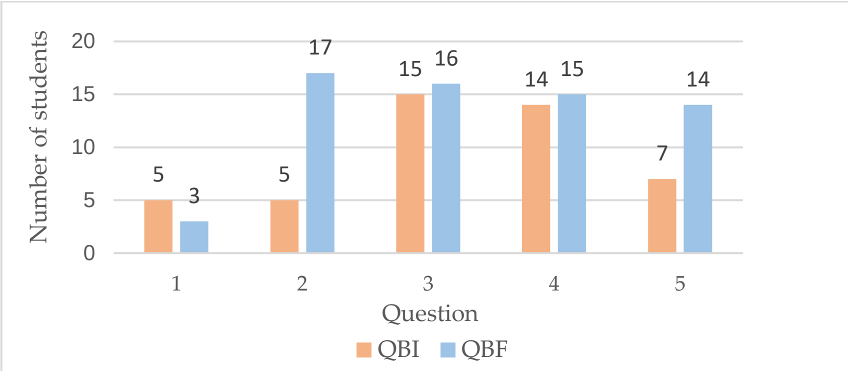

Figure 6. Number of students with positive score for each of the five questions in QBI and QBF (pilot study).

By taking a closer look into the individual questions (Figure 6), it is possible to observe an increase of the score in all four questions 2-5 but a slight reduction in question 1, that previously had a very low score. This suggests that the belief that mathematics is a discovery is resistant to change. This is a debated question among mathematicians and there is no general consensus, therefore there is no ground truth for the evaluation of the students' answers.

5.3.5.2 Students' difficulties

Data obtained from QK (Section 5.3.4) show that the understanding of non-Euclidean geometries improves but some relevant answers are still wrong. In the following, I will discuss some of the questions and the answers provided by the students. It is worth investigating the wrong answers, as they hint to some of the students' difficulties.

Difficulties due to attention issues and low ability to interpret the text

Figure 7 shows one of the questions from QK. The aim of this question is to detect previous knowledge of the concept of *shortest path between two points* in spherical geometry.

With reference to the spherical surface in a three-dimensional space represented in Figure A that approximates that of the Earth, observe the highlighted parallel arcs AB and CD. Which of the two arcs represents the shortest path between its extremes (considering only the surface of the sphere)?

- neither of the two arcs;
- only the arc AB;
- only the arc CD;
- both arcs.

Justify your answer: ....
....
....
....
....

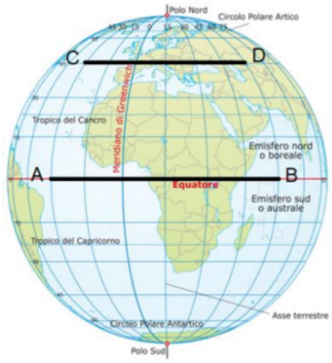

Figure A

Figure 7. Questions on the shortest path between two points in spherical geometry.

Table 1 classifies students' answers. It is possible to see that, before the class activities, no one of those who gave the correct answer could provide the right justification. Furthermore, 64,3% of the subjects (that is 18 of the 28 students involved) compared the length of the path AB with the one of the path CD. Once the time available to answer the questionnaires has expired, I asked to the students how they interpreted the question shown in shown in Figure 7. Their answers confirmed my hypothesis: they gave the wrong answer due to attention issues and low ability to interpret the text, even if it is well formulated. After observing these issues with the first two sets of students, I decided to modify the question for the next two sets, in order to avoid their misunderstanding to impair the ability to detect previous knowledge of the concept of shortest path between two points in spherical geometry.

		Pre			Post		
		Set 1	**Set 2**	**TOT**	**Set 1**	**Set 2**	**TOT**
	Number of subjects	10	18	28	10	18	28
Non correct answer	**Comparison of the two paths**	70,0 %	61,1 %	64,3 %	10,0 %	16,7 %	14,3 %
	Other	20,0%	38,9 %	32,1 %	30,0 %	27,8 %	28,6 %
Correct answer	**No justification**	10,0 %	0,0 %	3,6 %	10,0 %	0,0 %	3,6 %
	Correct justification	0,0 %	0,0 %	0,0 %	50,0 %	55,6 %	53,6 %

Table 3. Students' answers to the question about shortest path in spherical geometry.

Difficulties in negating a statement

Item 2 of QK (shown in Figure 7) asked the students to negate the parallel postulate of Euclidean geometry.

Question translated in English	Original question in Italian
The *parallel postulate* states: "In a plane α, there exists one and only one straight line parallel to a given straight line r and passing through a point P not on r". Write the negation of the parallel postulate: ……………………..	Il *postulato delle parallele* afferma: "in un piano α, esiste una e una sola retta parallela ad una retta data r e passante per un punto P esterno ad r". Scrivi la negazione del postulato delle parallele: ………………………………..

Frame 1. Item 2 of QK questionnaire

Table 4 shows data related to students' answers. I classified an answer as "totally wrong or hard to classify" if formulated in an ambiguous way or if it does not express a negation. I classified three kinds of answers as not completely correct as consisting in only a part of the fifth postulate's negation (*"There are no […]", "There are more than one […]", "There are infinite […]"*).

Before the class activities, only 1 over 77 students gave the correct answer and 61,1% gave no answer or gave an answer that is totally wrong or hard to classify. Some examples of these answers are provided in Table 5. These make me suppose that some students believe that negation of a postulate cannot even be expressed (see, e.g., Student 4).

Students' answer	**Pre**					**Post**				
	Set 1	**Set 2**	**Set 3**	**Set 4**	**TOT**	**Set 1**	**Set 2**	**Set 3**	**Set 4**	**TOT**
Correct	10,0%	-	-	-	1,3%	-	38,9%	35,0%	79,2%	48,1%
"There are no […]"	10,0%	11,1%	5,0%	6,9%	7,8%	10,0%	27,8%	20,0%	3,4%	14,3%
"There are more than one […]"	20,0%	11,1%	35,0%	20,7%	22,1%	40,0%	-	5,0%	3,4%	7,8%
"There are infinite […]"	10,0%	5,6%	-	13,8%	7,8%	10,0%	-	5,0%	3,4%	3,9%
Totally wrong or hard to classified	40,0%	38,9%	60,0%	44,8%	46,8%	30,0%	22,2%	35,0%	10,3%	22,1%
No answer	10,0%	33,3%	-	13,8%	14,3%	10,0%	11,1%	-	-	3,9%

Table 4. Students' answers to the question about negation of Euclid's fifth postulate. Zeros are replaced by dashes with improved readability

	Set	QK	Answer translated in English	Original answer in Italian
Student 1	1	Pre	"In a plane α there is no more than a straight line passing through a point P and parallel to r"	*"In un piano α non esiste più di una retta passante per un punto P e parallela a r"*
		Post	"There is no more than a straight line parallel to a straight line r lying on the α plane, passing through the point P"	*"Non esiste più di una retta parallela a una retta r che giace sul piano α, passante per il punto P"*
Student 2	1	Post	"Since there exist coincident parallels, for P external to r pass infinite straight lines, parallel to r"	*"Esistendo le parallele coincidenti, per P esterno a r passano infinite rette, parallele a r"*
Student 3	2	Pre	"Non-parallel lines are incident so they meet at one point"	*"Le rette non parallele sono incidenti quindi si incontrano in un punto"*
		Post	"More straight lines can pass but they will all lie on the first one"	*"Possono passare più rette ma saranno tutte giacenti sulla prima"*
Student 4	2	Pre	"There is no denial, it is a postulate so it is true a priori"	*"Non esiste una negazione, è un postulato quindi è vero a priori"*
Student 5	3	Pre	"In a plane α there are several straight lines parallel to a given straight line as long as they have no points in common with it"	*"In un piano α esistono più rette parallele a una retta data purché non abbiano punti in comune con questa"*
		Post	"In spherical geometry there are more parallels to a given straight line or none if the straight parallels are not considered"	*"In geometria sferica esistono più parallele ad una retta data oppure nessuna se non si considerano i paralleli rette"*
Student 6	3	Pre	"Given a point P external to a straight line r exists a *unique* parallel to r and passing through P. There cannot be two straight lines parallel to r and passing through P since one would not be coplanar"	*"Dato un punto P esterno a una retta r esiste unica la parallela a r e passante per P. Non possono essere due le rette parallele a r e passanti per P poiché una non sarebbe complanare"*
Student 7	3	Post	"In a sphere there is no straight line parallel to a given straight line r and passing through an external point to r"	*"In una sfera non esiste alcuna retta parallela ad una retta data r e passante per un punto esterno ad r"*

Table 5. Examples of the answers that do not express a negation of Euclid's fifth postulate.

5.3.6 Discussion

Non-Euclidean geometries are counterintuitive. Indeed, "the entire conception of mathematics had to be changed in order to feel free to accept, as axioms, statements that contradict intuition" (Fischbein, 1994). The exercise of understanding and accepting what goes against our intuition, has a great benefit for high-school students.

The pilot study reinforced my opinion that teaching non-Euclidean geometries triggers a change in the students' beliefs on mathematics. The data collection suggests that conducting individual interviews with students could be useful, adding insights to the quantitative method. These have been planned for the next phase. The data collection also showed the need to devise new questionnaires. Specifically, wrong answers given by the students and observed during the pilot study enabled me to develop more appropriate tests for my objective research.
One aspect which worth mentioning is the fact that most students find it difficult, especially at the beginning, to work on the given assignments related to practical and manipulation tasks. The teachers, who conducted some of the activities with their students, referred that their students often ask for suggestions and they need hints to start. At the same time, it must be noted that students, even if they do not excel in mathematics, they involved in the group work and in the practical experiences.

5.3.7 Towards the second experimental phase

The outcomes of the pilot study are positive: the students learnt that geometries other than the Euclidean one exist, they experiment that intuition should not be trusted blindly. However, my goal is not only to let the students experience other geometries, but is also to focus on the axiomatic method. The teaching activities of the pilot study were centered around the concept of curvature and the examples of geometries that arise from surfaces with different curvature. If the courses cannot be arranged to last longer, the time to dedicate to the axiomatic method is too short to concentrate on this. Rather, in the pilot study, the axiomatic method can be seen as the arrival point. After experiencing concepts that go against intuition, the axiomatic method is proposed as the only way to stitch together the pieces of knowledge that the students gather during the course. However, the axiomatic method – per se – is not discussed in detail and is left as an activity for the regular teaching activities with the class to be done after the course.

Another observation stemming from the course, is that more work is required on the practical activities, trying to make the students more autonomous.

5.4 Pandemics, distance teaching and their impact on the experimentations

The beginning of the second experimental phase of the project happened to occur at the inception of the Covid-19 pandemics. This fact had a deep impact on the whole PhD project. First of all, this unexpected event changed tremendously all the activities in the Italian school. When the spread of the pandemics was getting fast, the Italian government issued a lockdown of all citizens and students except for a minor part of essential workers. This lockdown started in March 2020 and the situation was constantly monitored to try to cope with an unpredictable adversity. In the meanwhile, the teachers had to switch to Distance Teaching and try to adapt to it, reformulate their didactical activities, try to find new ways to gather the attention of their students given the issues (connectivity, internet access, lack of physicality, etc.). The return to school was delayed several times, and with it all the planned workshop that I organized together with the schools. With the end of the year getting closer the teachers must be sure they complete the didactic program by the end of the year and the students must improve their marks if they are not sufficient. Furthermore, an extra amount of work was weighing on the students and the teachers imposed by the distance learning paradigm. For these reasons I was unable to reschedule my activities with the schools for several months. The school year ended with distance learning, therefore the students returned to school in presence only at the start of the next school year, i.e. on September 2020. However, very soon the activities returned to be conducted remotely in total or at least in part. A part of the teachers that originally applied to my experimental activities accepted to start the activities notwithstanding these troubles. However other teachers declared to be unable to participate at the last moment. The experimental activities were conducted in the Fall of 2020. At that time, it was clear that I could not conduct the activities in presence, therefore my experimental courses must be modified accordingly.

My method was adapted to synchronous online teaching due to the restrictions imposed by the anti-pandemic plan. I also taught all the courses personally, since the teachers preferred not to conduct it personally to avoid taking responsibilities they could not bear in case of new lockdowns. The online platform we used was Webex. This tool allows to create sub-groups with their own chatroom where smaller groups of students could work separately. This allowed to conduct some of the practical activities, where the student could discuss and find their own responses to my questions. As a teacher, I could also join any of the sub-groups to observe their activities and give some feedback or help them figure out their work. The questionnaires were adapted as well, specifically they were prepared as online Google forms.

The second and final stage of my experimental work has been conducted in this framework, with the classes being concluded by the end of 2020 and some of the interviews being done in March 2021.

5.5 Second experimental phase

The pilot study served the goal of gathering data and feedback about my proposal of experimental workshops on geometry. From these, it was possible to adjust the methods, the objectives and the didactical transposition to begin a new experimental phase. In this second phase the focus is shifted towards a more rigorous data collection approach. For this reason, an entire chapter will be dedicated to the analysis of the results. Unfortunately, as explained in the previous section, the second experimental phase was delayed and adjusted to the troubles and constraints of the Covid-19 pandemics. For the time being, it was not possible to conduct this experimentation in a in-presence workshop, which still remains the targeted scenario.

The following paragraphs provide all the details related to the methods, the materials and the questionnaires (the pre-questionnaires were administered to students the week before the start of the course, while the post-questionnaire were administrated the week after the end of the course). The results are analysed in Chapter 6.

5.5.1 Involved subjects

Currently, non-Euclidean geometries are not considered by Italian high-school guidelines, therefore their teaching is not compulsory. To ensure a collaboration with teachers, I proposed the course to high-school teachers that had already expressed their interest in conducting a course in non-Euclidean geometries.
The high schools involved were two *liceo scientifico* high school type. Several curricula of *liceo scientifico* exist. One of the four classes that took part in the study

follows the traditional curriculum of the *liceo scientifico* already discussed in 5.3.1, two classes are involved in the *Scienze Applicate* (Applied Sciences) curriculum, while another one is involved in the *Cambridge International* curriculum. *Scienze Applicate* curriculum renounces some of the aspects of humanistic culture, those linked to the study of Latin classicism, in favor of more scientific oriented programmes. The *Cambridge International* curriculum allows to learn the English language at high levels of competence by supporting the English teacher with a native speaker, and teaching two disciplines, generally of a scientific nature, in two languages. In their first two years all the curricula of *liceo scientifico* deal with the study of Euclidean geometry from the axiomatic point of view.
Students from two classes for each school took part in the project. Specifically, 18 students from a second-grade class, 25 students from a third-grade class, and two sets of students from two fifth-grade classes (a set of 10 students and the other set made of 24 students). No one of the students had learning disabilities identified.

The inconveniences created by the Covid-19 pandemic reduced the number of classes involved in the study and forced us to conduct the course outside school hours. To avoid dispersion, the teachers strongly encouraged their students to attend the course and demanded from them to justify their possible inability to participate (i.e. students who practice sport at a competitive level have mandatory afternoon workouts and have therefore been justified).

Table 6 shows data on the subjects involved in the study. Note that the number of subjects involved is minor that the total number of students who attended the course. This because I consider as subject of my study only those students who attended the course and answered to all the questionnaires planned for my research.

I distributed the questionnaires (pre-questionnaires and post-questionnaires) via Google Form, at two different days, depending on the school to which the students belonged. Students filled out the initial *questionnaires* the week before the beginning of the course and the final *questionnaires* the weak after the end of the course. I clarified with the students that: only the researchers involved in the study would see their answers; there would be no evaluation; the researchers involved in the study could contact them to discuss their answers.

	Set 1 (II SA class)	**Set 2 (III CI class)**	**Set 3 (V SA class)**	**Set 4 (V LS class)**
High school	*Liceo scientifico* School A	*Liceo scientifico* School A	*Liceo scientifico* School B	*Liceo scientifico* School B
Curriculum	*Scienze Applicate*	*Cambridge International*	*Scienze Applicate*	No special curriculum
Students	Students from a single II grade class	Students from a single III grade class	Students from a single V grade class	Students from a single V grade class
Number of students who attended the course	18	25	10	24
Number of subjects *	14	20	8	14

* Students who attended the course and answered all the questionnaires

Table 6. Subjects involved in the study.

5.5.2 Material

Before starting the course, I ensured that all students had the necessary materials for the workshops. Among these, the following material: polystyrene spheres (that can be split into two hemispheres) that can be written with markers or pinned; sewing thread and pins to draw straight lines on the polystyrene hemispheres; rulers; protractor; and compass. I was provided with the same materials as the students and more: a globe; a tiny toy car with no steering; 3D-printed hemispheres, pseudospheres and flexible ruler. All the material used in the workshops is shown in Figure 8

Figure 8. Some material used for class activities.

To collect data, I administrated questionnaires to the students the week before the beginning of the class activities and the week after the end of the class activities as shown in Table 7. In sections 5.5.3-5.5.6 I describe the four pairs of questionnaires administrated to the students: the *VHL test*, the *NEG questionnaire*, the *PROOF questionnaire*, and the *BELIEFS questionnaire*.

Week before the class activities	
Day 1	*VHL test* *BELIEFS questionnaire*
Day 2	*NEG questionnaire* *PROOF questionnaire*

Class activities →

Week after the class activities	
Day 1	*VHL test* *BELIEFS questionnaire*
Day 2	*NEG questionnaire* *PROOF questionnaire*

Table 7. Questionnaires administration.

5.5.3 *VHL test*

The van Hiele model is one of the most popular pedagogical theories. This theory expresses the teaching and learning of geometry through several levels. The van Hiele levels describe how students reason when solving geometrical problems or working with geometrical elements (objects, definitions, classifications, etc.). The choice of using the van Hiele model is not arbitrary or casual: as stated by Vinicio Villani in (Villani, 2006), the Italian school system has features in common with what was modeled in the van Hiele theory. Villani observes: "Although I am aware of the risk of an excessive trivialization, I believe it can be said that, at least as a first approximation, the five levels of *van Hiele* correspond quite closely to the teaching-learning of geometry, as it was already developed in Italy in the first half of the twentieth century, that is, even before the development of *van Hiele*'s theory, and how it is (or how it should be) still developed, respectively in middle school, upper secondary school or university courses"[64].

A husband-and-wife team of educators, Pierre van Hiele and Dina van Hiele-Geldof, developed it in their thesis at the University of Utrecht in 1957 (Usiskin Z., 1982). They postulated five levels of thought in geometry, each level indicates how individuals think over geometrical concepts. Hoffer summarizes – and Usiskin proposes again – general descriptions of the van Hiele's levels as follows (Usiskin Z., 1982) (Hoffer, Geometry (Teacher's Edition), 1979) (Hoffer, 1981):

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[64] Translated from the Italian *"Pur essendo consapevole del rischio di una banalizzazione eccessiva, credo si poter affermare che, almeno in prima approssimazione, i cinque livelli di van Hiele corrispondono abbastanza da vicino all'insegnamento-apprendimento della geometria, come veniva sviluppato in Italia già nella prima metà del Novecento, ossia ancor prima dell'elaborazione della teoria di van Hiele, e come viene (o come dovrebbe venire) sviluppato tuttora, rispettivamente nella scuola media, nelle scuole secondarie superiori o nei corsi universitari"*.

- Level 1 (recognition): the student can learn names of figures and recognizes a shape as a whole (e.g.: squares and rectangles seem to be different).
- Level 2 (analysis): the student can identify properties of figures (e.g.: rectangles have four right angles).
- Level 3 (order): the student can logically order figures and relationships but does not operate within a mathematical system (e.g.: simple deduction can be followed, but proof is not understood).
- Level 4 (deduction): the student understands the significance of deduction and the roles of postulates, theorems, and proof (e.g.: proofs can be written with understanding).
- Level 5 (rigor): the student understands the necessity for rigor and can make abstract deductions (e.g.: non-Euclidean geometries can be understood).

Pierre M. van Hiele identifies four properties of the levels (Van Hiele, 1958-59), to which Usiskin assigned names (Usiskin Z., 1982):

- Property 1 (fixed sequence): a student cannot be at van Hiele level n without having gone through level n-1.
- Property 2 (adjacency): at each level of thought what was intrinsic in the preceding level becomes extrinsic in the current level.
- Property 3 (distinction): each level has its own linguistic symbols and its own network of relationships connecting those symbols.
- Property 4 (separation): two persons who reason at different levels cannot understand each other.

Usiskin designed a test ("van Hiele test") to detect the level of thought in geometry according to the van Hiele theory (Usiskin, 1982). There are 25 questions, 5 questions for each level.

There are two criteria to assess if a student satisfies a certain level ("fits"): the "3 of 5 criterion" and the "4 of 5 criterion". The first one considers the level as passed if the student answers correctly to at least 3 of the 5 questions of that level. The second one, called "strict criterion", considers the level passed only if the student answers correctly to at least 4 of the 5 questions of that level. Usiskin suggests that the choice of the criterion is done based on the wish to reduce Type I (false positive) or Type II error (false negative).

Usiskin observed that sometimes level 5 items turned out to be easier for students than items at levels 4 or even 3, and that the reliability of the test for the fifth level is discussed. For these reasons, Usiskin considers two different theories: the *classical* one and the *modified* one. The so-called modified theory differs from the classical one for the fact that level 5 is not considered. The assigning of levels in either the classical or modified case requires that the student at level *n* satisfy the criterion not only at that level but also at all preceding levels (Usiskin Z., 1982). For example, if a student fits – according to a certain criterion – level 1, 2, 3, and 5 (but not level 4), he/she is classifiable only under the modified theory. Specifically, we say that the student fits level 3 of the modified theory.

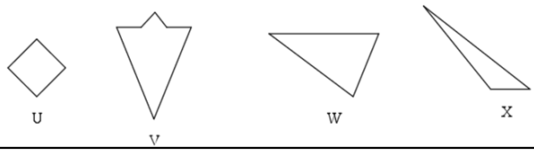
The *VHL test* (shown in Appendix 3) is the test I used to detect the students' levels of geometric thinking according to the van Hiele theory. Moreover, item 19 of the *VHL test* – combined with items 1 and 2 of the *NEG questionnaire* and item 2 of the *BELIEFS questionnaire* – addresses the research question *RQ2 (To what extent do students gain a new perspective on the concept of axiomatic system?)*.

I adapted the van Hiele test formulated by Usiskin (Usiskin Z., 1982) to the need of my research. Specifically, the Usiskin test was translated to Italian and, as a slight modification to the original test, I clarified in the items, when necessary, that the question was referred to a Euclidean plane. The test was provided before (pre-test) and after (pre-test) the course, to assess the initial van Hiele geometry levels of the students, and to compare these with the final ones. Frame 2-6 shows samples of items related to van Hiele levels 1-5, respectively.

I administered the *VHL test* with the *BELIEFS questionnaire* the week before the beginning of the class activities, and with the *BELIEFS questionnaire* the week after the end of the class activities (see Table 7). The time allowed to fill out the *VHL test*, before and after the class activities, was 45 minutes. This is 10 minutes more than what prescribed by Usiskin and is meant to accommodate for the difficulties given by the computer screen in answering and reviewing the question with respect to paper. All the tests and the questionnaires were administrated via Google Form.

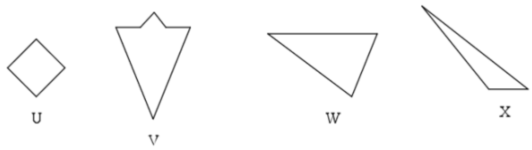
2) Nel piano euclideo, quali di questi sono triangoli?

a) Nessuno.
b) Solo V.
c) Solo W.
d) Solo W e X.
e) Solo V e W.



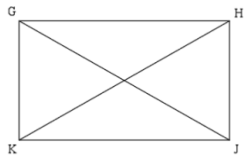
2) In the Euclidean plane, which of the following are triangles?

a) None.
b) V only.
c) W only.
d) W and X only.
e) V and W only.



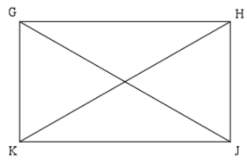
Frame 2. Item 2 of the NEG questionnaire, original version (above) and its english translation (below). Sample of item related to van Hiele level 1.

7) GHJK è un generico rettangolo nel piano euclideo, GJ e HK sono le sue diagonali. Tra le prime quattro seguenti affermazioni, quale NON è vera? [Indica la quinta opzione se credi che le precedenti siano tutte vere.]



a) In ogni rettangolo del piano euclideo ci sono quattro angoli retti.
b) In ogni rettangolo del piano euclideo ci sono quattro lati.
c) In ogni rettangolo del piano euclideo le diagonali hanno la stessa lunghezza.
d) In ogni rettangolo del piano euclideo i lati opposti hanno la stessa lunghezza.
e) Le precedenti affermazioni sono tutte vere.

7) GHJK a generic rectangle in the Euclidean plane, GJ and HK are its diagonals. Of the first four of the following statements, which one is NOT true? [Select the fifth option if you believe that the previous ones are all true.]



a) In every rectangle in the Euclidean plane there are four right angles.
b) In each rectangle in the Euclidean plane there are four sides.
c) In every rectangle in the Euclidean plane, the diagonals have the the same length.
d) In every rectangle in the Euclidean plane, the opposite sides have the same length.
e) The previous statements are all true.

Frame 3. Item 7 of the NEG questionnaire, original version (above) and its english translation (below). Sample of item related to van Hiele level 2.

12) Leggi i due enunciati scritti qui sotto (Enunciato S ed Enunciato T) e indica quale delle affermazioni di seguito riportate è corretta.

Enunciato S: “Il triangolo ABC del piano euclideo ha tre lati della stessa lunghezza”.

Enunciato T: “Nel triangolo ABC del piano euclideo gli angoli $\widehat{B}$ e $\widehat{C}$ hanno la stessa misura”.

a) Gli enunciati S e T non possono essere entrambi veri.
b) Se S è vero allora T è vero.
c) Se T è vero allora S è vero.
d) Se S è falso allora T è falso.
e) Nessuna delle precedenti affermazioni è corretta.

12) Read the two statements written below (Statement S and Statement T) and select which of the statements below is correct.

Statement S: “Triangle ABC of the Euclidean plane has three sides of the same length.”

Statement T: “In triangle ABC of the Euclidean plane, angles $\widehat{B}$ and $\widehat{C}$ have the same measure.”

a) Statements S and T cannot both be true.
b) If S is true then T is true.
c) If T is false then S is false.
d) If S is false then T is false.
e) None of the previous statements are correct.

Frame 4. Item 12 of the NEG questionnaire, original version (above) and its english translation (below). Sample of item related to van Hiele level 3.

17) Leggi i due enunciati scritti qui sotto (Enunciato S ed Enunciato T) e indica quale delle affermazioni di seguito riportate è corretta.

Proprietà D: “Ha diagonali della stessa lunghezza”.

Proprietà Q: “È un quadrato”.

Proprietà R: “È un rettangolo”.

Quale tra le seguenti affermazioni è vera?

a) D implica Q che implica R.
b) D implica R che implica Q.
c) Q implica R che implica D.
d) R implica D che implica Q.
e) R implica Q che implica D.

17) Read the three statements, referred to a figure, written below (Statement D, Statement, and Statement R):

Statement D: "It has diagonals of the same length."

Statement Q: "It is a square."

Statement R: "It is a rectangle."

Which of the following statements is true?

a) D implies Q which implies R.
b) D implies R which implies Q.
c) Q implies R which implies D.
d) R implies D which implies Q.
e) R implies Q which implies D.

Frame 5. Item 17 of the NEG questionnaire, original version (above) and its english translation (below). Sample of item related to van Hiele level 4.

22) TRISECARE un angolo significa dividerlo in tre parti di uguale misura. Nel 1847, P. L. Wantzel dimostrò che, in generale, è impossibile trisecare angoli utilizzando soltanto un compasso e un righello NON graduato. Quale delle seguenti affermazioni si può dedurre dalla sua dimostrazione? [Leggi con attenzione.]

a) In generale, è impossibile BISECARE angoli usando solamente un compasso e un righello non graduato.
b) In generale, è impossibile TRISECARE angoli usando solamente un compasso e un righello graduato.
c) In generale, è impossibile TRISECARE angoli usando qualsiasi strumento da disegno.
d) È ancora possibile che in futuro qualcuno possa trovare un modo generale per TRISECARE angoli usando solamente un compasso e un righello non graduato.
e) Nessuno potrà mai trovare un metodo generale per TRISECARE angoli usando solamente un compasso e un righello non graduato.

22) To TRISECT an angle means to divide it into three parts of equal measure. In 1847, P. L. Wantzel proved that, in general, it is impossible to trisect angle using only a compass and an UNMARKED ruler. Which of the following statements can be deduced from his proof? [Read carefully.]

a) In general, it is impossible to BISECT angles using only a compass and an unmarked ruler.
b) In general, it is impossible to TRISECT angles using only a compass and an marked ruler.
c) In general, it is impossible to TRISECT angles using any drawing instruments.
d) It is possible that in the future someone may find a general way to trisect angles using only a compass and an unmarked ruler.
e) No one will ever be able to find a general method for trisecting angles using only a compass and an unmarked ruler.

Frame 6. Item 22 of the NEG questionnaire, original version (above) and its english translation (below). Sample of item related to van Hiele level 5.

5.5.4 *NEG questionnaire*

The *NEG questionnaire* addresses to the research question *RQ3* (*How well do students learn the taught concepts of non-Euclidean geometries?*) and – combined with the *PROOF questionnaire* – to *RQ4* (*To what extent do students' critical thinking and proof skills improve over the duration of the course?*). Indeed, the *NEG questionnaire* is one of the two questionnaires that aim at evaluating students' ability to apply critical and logical thinking. It is more focused on the knowledge of Euclidean geometry and non-Euclidean geometries and the application of what was learned during the lessons.
Moreover, items 1 and 2 of the *NEG questionnaire* – combined with item 19 of the *VHL test* and item 2 of the *BELIEFS questionnaire* – addresses to the research question *RQ2* (*To what extent do students gain a new perspective on the concept of axiomatic system?).*

The questionnaire contains seven items articulated in several sub-items that vary according to the choices given by the individual students. The items start with a closed question and, based on the answer given by the student, proposes other questions aimed at investigating whether the given answer is well motivated or not. In the following, I analyse each of the seven items.

I administered the *NEG questionnaire* with the *PROOF questionnaire* the week before the beginning of the class activities, and the *NEG questionnaire* with the *PROOF questionnaire* the week after the end of the class activities (see Table 7). The time allowed to fill out the *NEG questionnaire*, before and after the class activities, was 50 minutes. All the questionnaires were administrated via Google Form.

Items 1 and 2

Items 1 and 2 of the *NEG questionnaire*, whose English translation is shown in Frame 7, investigate the students' view of the concepts of *axiom*. Specifically, we want to find out whether students have a "modern" view of the concepts of *axiom* (any statement taken as starting point of a theory, provided that it is consistent with the other assumed axioms).

We consider that a student has a modern view of the concepts of *axiom* if she/he gives all the following answers:
- in item 1: No;
- the justification in item 1 must be correct (e.g: "any statement taken as starting point of a theory, provided that it is consistent with the other assumed axioms, is an axiom", "a statement can be an axiom if it is not contradictory to the other axioms of the theory"; "the fifth Euclidean geometry's axiom is not evident");
- in item 2: "Consistency".

Item 1. *An axiom (also called a postulate) is a statement that, without being proven in advance, is assumed to be the foundation of an axiomatic theory (also called an axiomatic system). Does such a statement, in order to be called an "axiom" (or "postulate"), also have to be self-evident?* o *Yes.* o *No.* o *I cannot answer.* o *Other: ....* *Justify the previous answer: ....*
Item 2. *What is the fundamental property that an axiomatic system should satisfy?* o *Consistency.* o *Completeness.* o *Evidence.* o *Independence.* o *Truthfulness.* o *I cannot answer.* o *Other: ....*

Frame 7. Item 1 and item 2 of the NEG questionnaire, traslated to English.

Items from 3 to 7, sub-items a and b

Items from 3 to 7, sub-items *a* and *b,* of the *NEG questionnaire* aim to assess if, respectively before and after the course on non-Euclidean geometries:

1. students have knowledge regarding geometries other than the Euclidean one;
2. students know that some given propositions are assumptions (see item 3.a shown in Frame 8) or merely consequences of a specific axiomatic system (the Euclidean one), or if they think that these propositions are necessarily true.

At the same time, students' answers to the above cited items before the course on non-Euclidean geometries allows us to detect possible students which have very low levels of knowledge in Euclidean geometry.

Items from 3 to 7, sub-items *a* and *b,* are concerned with specific propositions that are true at least in Euclidean geometry but are false at least in spherical geometry or in hyperbolic geometry. To clarify how these items are formulated, Frame 8 shows the first case: item 3 (sub-items 3.a and 3.b).

Item 3.a. *Read the following proposition: "Given a line r and a point P outside r, there exists one and only one line through P and parallel to r".* *From the following statements, select the correct one:* o *The preceding proposition is true.* o *The preceding proposition is false.* o *The preceding proposition is true in some circumstances and false in others.* o *I cannot answer.*
Item 3.b. (This question is activated only if the third statement was selected in sub-item "a") *- Show at least one case in which the previous proposition ("Given a line r and a point P outside r, there exists one and only one line through P and parallel to r") would be true: ……* *- Show at least one case in which the previous proposition ("Given a line r and a point P outside r, there exists one and only one line through P and parallel to r") would be false: …...*

Frame 8. Item 3.a and 3.b of the NEG questionnaire, traslated to English.

We have seen in Frame 8 that sub-items 3.a and 3.b deal with the proposition *"Given a line r and a point P outside r, there exists one and only one line through P and parallel to r"* (in the pre-questionnaire and in the post-questionnaire). The other ones (from 4 to 6, sub-item "a" and "b") – structured in the same way of sub-items 3.a and 3.b – deal with the following propositions:

- *"Let r, s and t be two distinct lines belonging to the same surface. If r is parallel to s and s is parallel to t, then r is parallel to t (i.e. the transitive property for parallelism between lines applies)"* (item 4.a of the pre-questionnaire and of the post-questionnaire).
- *"Let r and s be two lines belonging to the same surface. If r and s are parallel to each other, then all points on r have the same distance from s and, vice versa, all points on s have the same distance from r"* (item 5.a of the pre-questionnaire and of the post-questionnaire).
- *"Taking any two triangles ABC and DEF belonging to the same surface and not congruent with each other, the sum of the interior angles of triangle ABC is equal to the sum of the interior angles of triangle DEF"* (item 6.a of the pre-questionnaire and of the post-questionnaire).

Note that questions from 3 to 6 in the post-questionnaire are identical to questions 3 to 6 of the pre-questionnaire. To understand if possible improvements of the students depend on this factor, I choose to change question in item 7: item 7.a-b of the post-questionnaire is different from item 7.a-b of the pre-questionnaire. They are structured in the same way as item 3.a-b, but the proposition used in the pre-questionnaire is different from the one used in the post. The proposition used in the pre-questionnaire is *"Given any triangle, any of its external angles is congruent to the sum of the internal angles not adjacent to it"*, while the one in the post-questionnaire is *"If A, B, C and D are four two-by-two different points on the same surface such that BD is perpendicular to AB and the angle BAC is acute, then the line through A and C and through B and D intersect"*. Since the item 7 in the pre-questionnaire is not the same of the one in the post-questionnaire, I only used items from 3 to 6 to evaluate possible students' improvements.

We observe that, at the beginning of each *NEG questionnaires*, I wrote the following note:

> *In order to avoid misunderstandings, read well the definition of "PARALLEL STRAIGHT LINES" and the definition of "INCIDENT STRAIGHT LINES" that we assume to be valid (you will find them repeated before each question, even if the terms will not be part of the text).*
> *We say that "two lines are parallel to each other" if they belong to the same surface and have no points in common.*
> *We say that "two lines are incident to each other" if they have one and only one point in common.*

To monitor potential changes in students' understanding I classify their answers and I give them scores. Since sub-items "b" are necessary to classify students' understanding in case they select that a given proposition is *true in some circumstances and false in others*, I classify the pairs of sub-items "a-b" as it is a unique question and I assign it a unique score. Table 8 shows how I classify students' answers and how I assign the scores.

We observe that classifying and scoring the selection *"I cannot answer"* deserves particular attention. A student who selects this statement in the post-questionnaire is a student who, despite having followed the course on non-Euclidean geometries, shows no sufficient knowledge in either Euclidean or non-Euclidean geometries. Then I would assign 0 points to this student. Instead, a student who selects *"I cannot answer"* in the pre-questionnaire could be a student who does not know the truth value of the given proposition even in Euclidean geometry or, on the other hand, she/he could be a student who knows the truth value of the given proposition in Euclidean geometry but who, without being sure, admits the possibility that this proposition is not valid in other geometries. I think correct to assign 0 points to the selection in the first case, while a positive score to the selection in the latter case. We also observe that a student who fits the latter case should never select *"The preceding proposition is true"* and *"The preceding proposition is false"*. Since, regarding my experimentation, every student who selects at least one time *"I cannot answer"* also selects at least one time *"The preceding proposition is true"* or *"The preceding proposition is false"*, I assign 0 points to all these students.

As I have already explained, to evaluate possible students' improvements I compare their answers to item 3.a-b, 4.a-b, 5.a-b, and 6.a-b of the pre and of the post-questionnaire. I do not use item 7.a-b because the one in the pre-questionnaire is different from the one in the post. I used it only to understand if possible improvements of the students depend on the fact that the other items are identical in the two questionnaires.

In the following part of the present section, I give more details on each item from 3 to 7.

Student's answers	**Score**
The student • selects *"The preceding proposition is true in some circumstances and false in others"*; • shows correct understanding in Euclidean geometry making at least a correct example in this case; • shows correct understanding in non-Euclidean geometries making at least a correct example in this case.	1
The student • selects *"The preceding proposition is true in some circumstances and false in others"*; • shows correct understanding in Euclidean geometry making at least a correct example in this case; • shows not very correct understanding making at least a correct example in this case but adding a wrong example. E.g.: the "transitivity" of the parallelism holds in Euclidean geometry, but it does not hold in hyperbolic geometry and in spherical geometry. It does not hold in hyperbolic geometry because there can be three straight lines *a*, *b*, and *c* such that *a* and *b* are parallel, *a* and *c* are parallel, but *b* and *c* are not parallel. It does not hold in spherical geometry because there are no parallel straight lines.	0.75
The student • selects *"The preceding proposition is true in some circumstances and false in others"*; • shows correct understanding in Euclidean geometry making at least a correct example in this case; • shows awareness of the existence of non-Euclidean geometries but not correct understanding: he/she does not show correct examples in non-Euclidean geometries. E.g.: the "transitivity" of the parallelism holds in Euclidean geometry, but it does not hold in spherical geometry since there are no parallel lines in spherical geometry.	0.5
The student • shows correct understanding in Euclidean geometry • does not show awareness of the existence of non-Euclidean geometries. This student selects: "*The preceding proposition is true*" or *"The preceding proposition is true in some circumstances and false in others"* but she/he does not mention example in non-Euclidean geometries. E.g.: the proposition "Given a line r and a point P outside r, there exists one and only one line through P and parallel to r" is true in case of a Cartesian plane with x and y axis is taken into consideration, while it is false if the point and the straight line are in the Cartesian plane with x, y and z axes.	0.25
All the other cases (the student does not show knowledge even in Euclidean geometry).	0

Table 8. Classification of students' answer and scoring to the pair of sub-items 3.a-3.b, 4.a-4.b, 5.a-5.b, 6.a-6.b, and 7.a-7.b.

Item 3 (a, b)

Item 3 (Frame 8) concerns the modern formulation of the V Euclidean postulate. Indeed, it deals with the following proposition P_3: *"Given a line r and a point P outside r, there exists one and only one straight line through P and parallel to r"*. A completely correct answer is one that cites the Euclidean plane as a case in which P_3 is true and the spherical surface or the hyperbolic surface as a case in which P_3 is false. Throughout the non-Euclidean course, I have shown several times that on a spherical surface P_3 is false because there is no straight line through *P* and parallel to the given straight lines *r*, while on a hyperbolic surface P_3 is false because there exists more than one (there exist infinite) straight line through *P* and parallel to the given straight lines r. Moreover, proposition P_3 was object of one assignment I proposed through the laboratory activities with the software *NonEuclid* (see Figure 9).

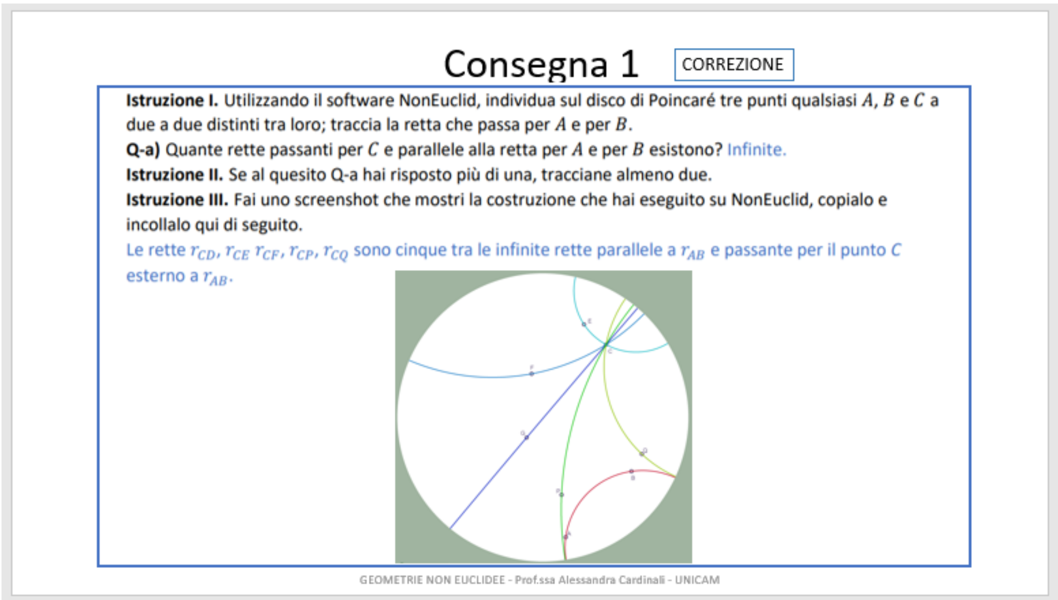


Figure 9. Slide on the correction of the assignment on the modern formulation of the V Euclidean postulate I proposed through the laboratory activities. I ask the students how many straight lines passing trought a point C are parallel to a given straight line not passing through C. Moreover, in case they answer "more than one", I ask them to draw at least two of these (using the software NonEuclid), and to show a screenshot of their drawing.

Item 4 (a, b)

Item 4 (shown in Appendix 4) concerns the following proposition P_4: *"Let r, s and t be two-by-two distinct lines belonging to the same surface. If r is parallel to s and s is parallel to t, then r is parallel to t (i.e., the transitive property for parallelism between lines* *holds)"*. A completely correct answer is one that cites the Euclidean plane as a case in which P_4 is true and the hyperbolic surface as a case in which P_4 is false. Instead, the spherical surface is not a correct counterexample to refuse P_4 since it does not verify P_4's hypothesis: there are no parallel straight line on spherical surfaces. Throughout the non-Euclidean course, I observed that proposition P_4 holds on the Euclidean plane. Moreover, I noted that P_4 does not hold on a hyperbolic surface showing a counterexample on a model of pseudosphere and another one on the Poincaré disk model (see Figure 10). To monitor whether students can autonomously produce logically correct counterexamples to refute a statement, I never observed along the course that a spherical surface is not a counterexample of proposition P_4. In the post-questionnaire, 45 students select that proposition P_5 is true in some circumstances and false in others; among these, 15 students show not having the above skill: they explicitly refer to spherical surface.

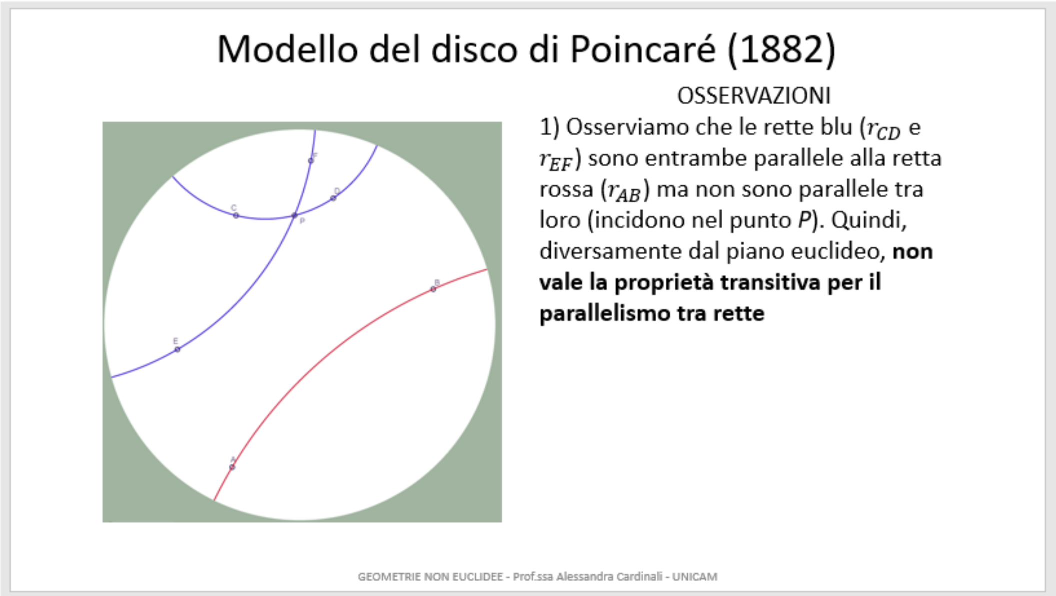


Figure 10. Slide used to show that the proposition P4 does not hold on the Poincaré disk model.

Item 5 (a, b)

Item 5 (shown in Appendix 4) concerns the following proposition P_5: *"Let r and s be two lines belonging to the same surface. If r and s are parallel to each other, then all points on r have the same distance from s and, vice versa, all points on s have the same distance from r"*. As in the previous case of item 4, completely correct answer is one that cites the Euclidean plane as a case in which P_4 is true and the hyperbolic surface as a case in which P_4 is false. In this case too, the spherical surface is not a correct counterexample to refuse P_5 since it does not verify P_5's hypothesis: there are no parallel straight line on spherical surfaces. Throughout the non-Euclidean course, I have observed that P_5 holds on the Euclidean plane but does not hold on a hyperbolic surface, I showed a counterexample on a model of pseudosphere (it is enough to consider two) and another one on the Poincaré disk model (see Figure 11 and Figure 12). To monitor whether students can autonomously produce counterexamples to refute a statement, I never observed along the course that a spherical surface is not a counterexample of proposition P_5. In the post-questionnaire, 44 students select that proposition P_5 is true in some circumstances and false in others; among these, 12 students show not having the above skill.

Figure 11. Straight lines p and s (generatrix straight lines of the given model of pseudosphere) are parallel but not all points on p have a same distance from s and, vice versa.

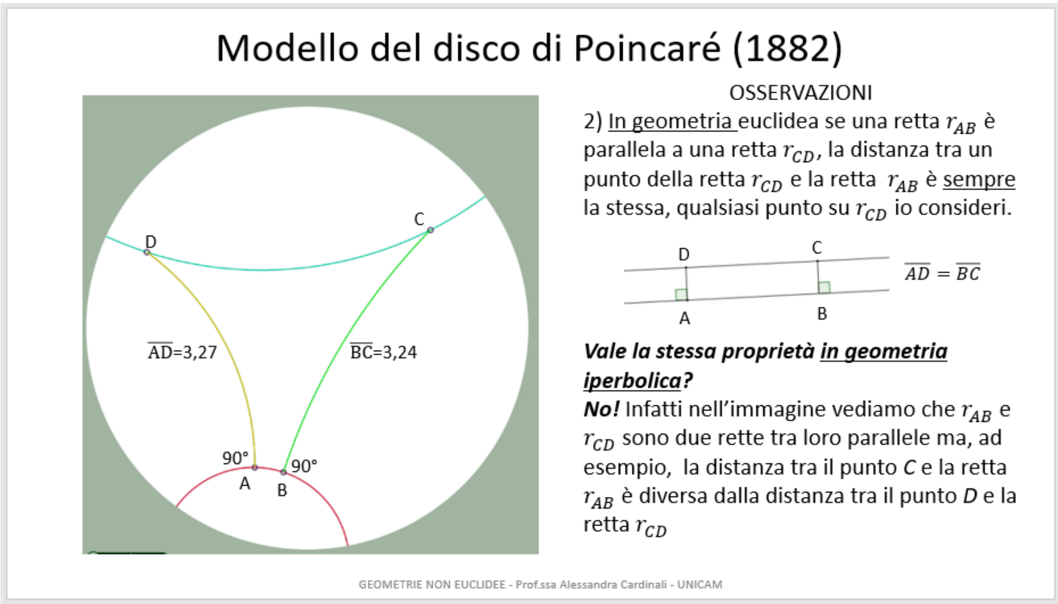


Figure 12. Slide used to show that the proposition P4 does not hold on the Poincaré disk model.

Item 6 (a, b)

Item 6 (shown in Appendix 4) concerns the following proposition P_6: *"Taking any two triangles ABC and DEF belonging to the same surface and not congruent with each other, the sum of the interior angles of triangle ABC is equal to the sum of the interior angles of triangle DEF"*. A completely correct answer is one that cites the Euclidean plane as a case in which P_6 is true and a hyperbolic surface or a spherical suface as a case in which P_6 is false. Throughout the non-Euclidean course, I observed that proposition P_6 holds on the Euclidean plane. Moreover, I noted that P_6 does not hold on a hyperbolic surface and on a spherical suface showing counterexamples on a model of pseudosphere, on the Poincaré disk model and on the surface of a shere. Moreover, proposition P_6 was object of one assignment I proposed through the laboratory activities on spherical surfaces and the one with the software NonEuclid (see Figure 13 and Figure 14).

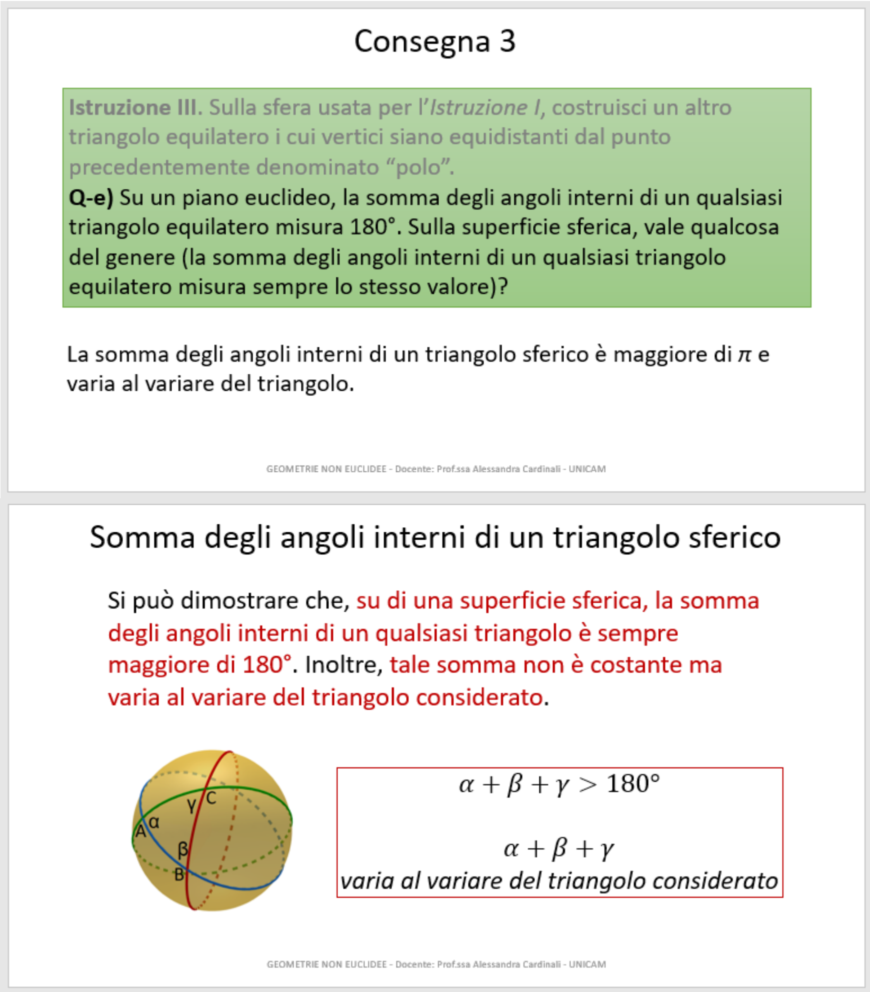


Figure 13. Slides on the correction of the assignment on the sum of internal angles of triangles. After the workshop on spherical geometry, I confirm the students' conjecture that the sum of the interior angles of any triangle is greater than 180° and it varies as the triangle considered varies.

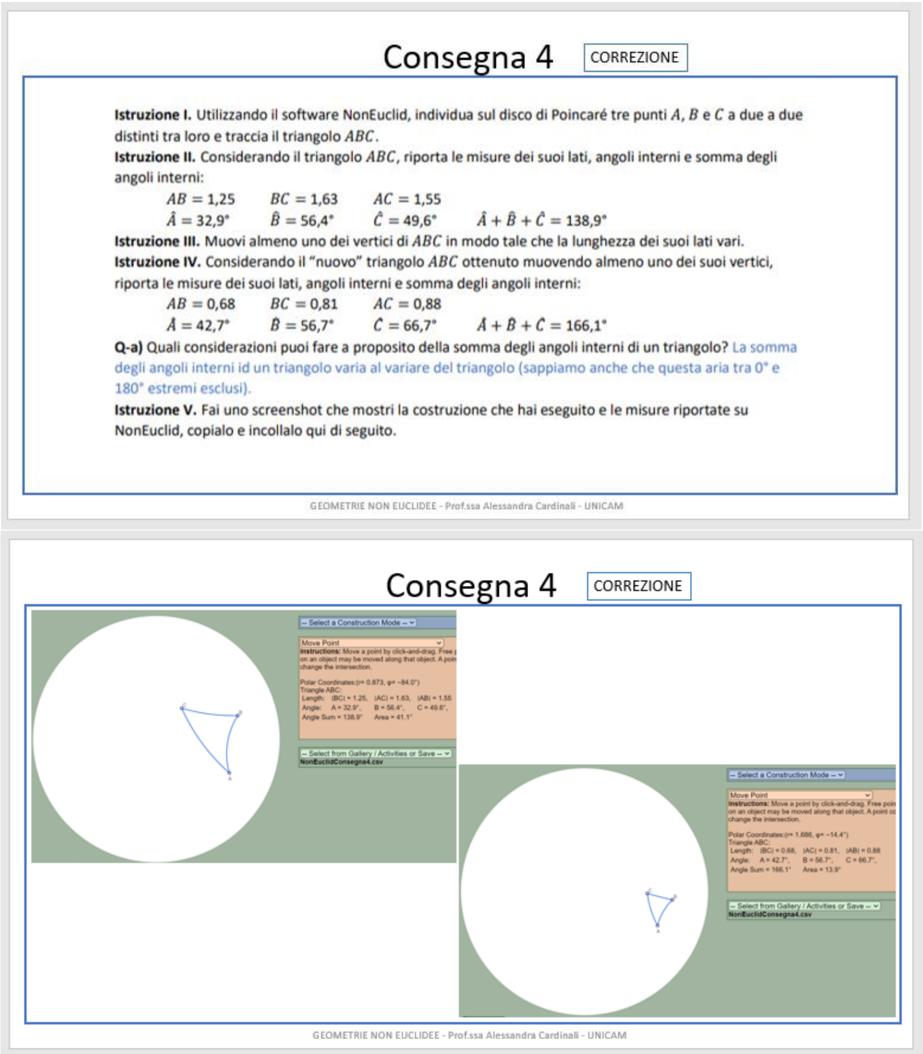


Figure 14. Slides on the correction of the assignment on the sum of internal angles of triangles. Students have to draw a hyperbolic triangle, then move their vertices and observe the sum of interior angles misure.

Item 7 (a,b)

Questions from 3 to 6 in the post-questionnaire are identical to questions 3 to 6 of the pre-questionnaire. To understand if potential improvements of the students depend on this factor, I choose to change question in item 7: item 7.a-b of the post-questionnaire is different from item 7.a-b of the pre-questionnaire (shown in Appendix 4).

Item 3, sub-items c, d and e

Discussing the data collected from the pre-questionnaire and in the post-questionnaire used in the previous pilot study (see Section 5.3.5.2), I stated that, probably, some students believe it impossible to express the negation of a postulate. Trying to investigate this aspect more thoroughly, I decided to formulate the question by first asking the students if it is possible to negate the postulate. More details follow. In the second study, I divide the question into several parts as shown in Frame 9. I first asked, in sub-item 3.c, if it is possible to state the negation of the fifth postulate of the Euclidean geometry expressed as follow: *"Given a line r and a point P outside r, there exists one and only one line through P and parallel to r"*. Then, I diversified the subsequent questions (sub-items 3.d and 3.e) on the basis of the students' answers in sub-item 3.c. To better understand, see sub-items 3.c, 3.d, and 3.e in Frame 9.

Item 3.c. *Is it possible to state the negation of the previous proposition? ("Given a line r and a point P outside r, there is one and only one line through P and parallel to r")?*

- *Yes.*
- *No.*
- *I cannot answer.*
- *Other: ………………………………*

Item 3.d. (This question is activated only if "Yes" was selected in sub-item 3c)

Carefully read statements A, B, C, D and E in the box and select the option that would correspond to the negation of the previous proposition ("Given a line r and a point P outside r, there is one and only one line through P and parallel to r") from the list under the box.

> *A: "Given a line r and a point P outside r, there are no lines passing through P and parallel to r ".*
> *B: "Given a line r and a point P outside r, there is at least one line passing through P and parallel to r ".*
> *C: "Given a line r and a point P outside r, there is more than one line through P parallel to r ".*
> *D: "Given a line r and a point P outside r, there are many lines passing through P and parallel to r ".*
> *E: "Given a line r and a point P outside r, there are an infinite number of lines through P parallel to r ".*

- *The statement A.*
- *The statement B.*
- *The statement C.*
- *The statement D.*
- *The statement E.*
- *The union of statement A with statement B*
- *The union of statement A with statement C.*
- *The union of statement A with statement D.*
- *The union of statement A with statement E.*
- *The union of statement B with statement C.*
- *The union of statement B with statement D.*
- *The union of statement B with statement E.*
- *The union of statement C with statement D.*
- *The union of statement C with statement E.*
- *The union of statement D with statement E.*
- *None of the above options.*
- *I cannot answer.*

Item 3.e. (This question is activated only if "No" was selected in sub-item 3.c)

Explain why it would not be possible to state the negation of the previous proposition ("Given a line r and a point P outside r, there is one and only one line through P and parallel to r"): ….

Frame 9. Sub-items 3.c, 3.d, and 3.e.

5.5.5 *PROOF questionnaire*

The *PROOF questionnaire* (shown in Appendix 5) is focused on analytical reading of the text, exerting logical skills, understanding the importance of hypotheses, and exerting the ability to show counterexamples when they are needed. This questionnaire – combined with the *NEG questionnaire* – addresses to the research question *RQ4* (*To what extent do students' critical thinking and proof skills improve over the duration of the course?*).

The *PROOF questionnaire* consists of two items. Each item contains a proposition and a possible proof for it. The proof, however, always contains a mistake that the student should find out. The mistake concerns the use of properties that are valid only in some circumstances among those admitted by the hypotheses, and which therefore cannot be taken for granted. The *PROOF questionnaire* requires that the student identifies which step of the proof is not valid and justifies why showing a correct counterexample. If the student is unable to do that, then he/she may not know how to apply critical and logical thinking to geometry.

To identify the mistake, the student must critically confront each step of the possible proof and ask himself if this step is allowed. The steps that are not always valid under the hypothesis specified in the statement regard the following properties:

- "Each external angle of a triangle is congruent to the sum of the internal angles not adjacent to it".
- Original formulation of Euclid's fifth postulate.
- "Each external angle of a triangle is greater than any internal angle not adjacent to it".
- "If *l*, *m*, and *k* are three straight lines belonging to the same surface such that *m* is parallel to *l*, and *k* is parallel to *l*, then *m* is parallel to *k*".

Table 9 shows where each of the previous properties exactly appears (in the pre-questionnaire or in the post-questionnaire). Note that for each questionnaire there is one property in which the term *parallel* is not in the statement (neither in its potential proof) and another one in which it does not compare.

These properties are all valid on a Euclidean surface but not always on a non-Euclidean one. In fact, the first property does not hold both on a spherical surface and on a hyperbolic one, the second and the fourth ones do not hold on a hyperbolic surface, the third does not hold on a spherical surface.

Obviously, I would expect that a student who does not yet have knowledge of non-Euclidean geometries is able to recognize the validity of the previous properties in Euclidean geometry (uncertainties are admitted only on the validity of the fifth postulate of Euclid stated in its original formulation) but cannot explicitly limit this validity to Euclidean geometry.

Each item is divided into three parts. In the following subsection I give more details on items' structure and its purpose.

	Pre-questionnaire	**Post-questionnaire**
Properties in item 1 (does not contain the term *parallel*)	"Each external angle of a triangle is congruent to the sum of the internal angles not adjacent to it"	"Each external angle of a triangle is greater than any internal angle not adjacent to it"
Properties in item 2 (contains the term *parallel*)	Original formulation of Euclid's fifth postulate	"If l, m, and k are three straight lines belonging to the same surface such that m is parallel to l, and k is parallel to l, then m is parallel to k"

Table 9. Properties that are not always valid under the statement's hypothesis.

I administered the *PROOF questionnaire* with the *NEG questionnaire* the week before the beginning of the class activities, and the *PROOF questionnaire* with the *NEG questionnaire* the week after the end of the class activities (see Table 7). The time allowed to fill out the *PROOF questionnaire*, before and after the class activities, was 60 minutes. All the questionnaires were administrated via Google Form.

5.5.5.1 Items' structure

The previous pilot study showed us numerous cases of students who, instead of showing a counterexample (when it is required) to prove that a statement is not true, they show a circumstance in which the hypotheses of the given conjecture are not all verified. For example, they mention the spherical geometry despite the hypothesis of the statement deal with parallel straight lines. These kinds of mistakes can be due to a lack of attention on the part of the students. Or the students just do not know that the existence of circumstances that violate the hypothesis has no implications for the truth value of a proposition. To investigate this aspect, I focused the first part of each item on the following aspect: does the existence of circumstances that do not verify the hypotheses of a statement determine its truth value? If yes, which one?

Frame 10 shows the English translation of first part of item 1 in the post-questionnaire.

Question 1a. After carefully reading the proposition P1 below, select the correct statement from the ones below.

> Proposition P1. *Let ABC be a triangle belonging to a surface α such that the side AC is greater than the side AB. Under the previous hypothesis, the angle $\widehat{ABC}$ is greater than the angle $\widehat{BCA}$.*

- The possible existence of triangles which do not satisfy the hypothesis of proposition P1 would imply that proposition P1 is true.
- The possible existence of triangles which do not satisfy the hypothesis of proposition P1 would imply that proposition P1 is false.
- The possible existence of triangles which do not satisfy the hypothesis of proposition P1 would have no implications for the truth value of proposition P1.
- The three statements above are all wrong.
- I do not know which of the previous four statements is correct.

Frame 10. English translation of Question 1a – Item 1 – Post-questionnaire.

The second part of each item is the main one. I give a statement and a possible proof of it. I require that the student identifies which step of that proof is not valid. Frame 11 shows the English translation of part b of item 1 in the post-questionnaire.

The only correct answer is "The proof is incorrect because it contains at least one step (a, b, c, d, e, f, g or h) which is not always valid under the hypothesis stated in proposition P1". Indeed the hypothesis does not specify which kind of surface α is. In case α is a spherical surface, step c) is not correct (just think of a spherical triangle equal to a quarter of the spherical surface, with all angles equal to 90°). All the other possible answers are wrong.
The first two (wrong) answers are based on the most common answers given in open questions of the pilot study. If a students selects the first wrong answer ("The proof is correct because all its steps are always valid under the hypothesis of proposition P1") we can deduce that probably she/he is still anchored to a

Euclidean view of geometry. If a students selects the second wrong answer ("The proof is incorrect because there are triangles which do not satisfy the hypotheses of proposition P1") we can deduce that her/his logical skills are not developed enough. Indeed, this student thinks that the existence of a circumstance that violate the hypothesis invalidate the truth of the statement, therefore he/she could not be able to build correct counterexamples.

Question 1b. After carefully reading again the previous proposition P1 and the proof written below, select the correct statement from the ones below.

> *Proposition* P1. *Let ABC be a triangle belonging to a surface α such that the side AC is greater than the side AB. Under the previous hypothesis, the angle $\widehat{ABC}$ is greater than the angle $\widehat{BCA}$.*
>
> *Proof.*
>
> *a) By hypothesis, the side AC is greater than the side AB so* ***there exists a point D on the side AC such that that AD is congruent to side AB.***
>
> *b) Thanks to Euclid's first postulate,* ***we can join B with D with a segment****.*
>
> *c) Since angle $\widehat{ADB}$ is an exterior angle of triangle BCD, it is greater than the interior angle and not adjacent $\widehat{BCD}$: $\boldsymbol{\widehat{ADB} > \widehat{BCD}}$.*
>
> *d) Since, by construction, the side AD is congruent to the side AB, then the angles $\widehat{ABD}$ and $\widehat{ADB}$ are congruent: $\boldsymbol{\widehat{ABD} \cong \widehat{ADB}}$.*
>
> *e) From the deductions in c) and d) it follows that $\boldsymbol{\widehat{ABD} > \widehat{BCD}}$.*
>
> *f) Since the whole is greater than the part, the angle $\widehat{ABC}$ is greater than the angle $\widehat{ABD}$: $\boldsymbol{\widehat{ABC} > \widehat{ABD}}$.*
>
> *g) From the deductions in e) and f) it follows that $\boldsymbol{\widehat{ABC} > \widehat{BCD}}$.*
>
> *h) Since, by construction, the angles $\widehat{BCD}$ and $\widehat{BCA}$ coincide, it follows that $\boldsymbol{\widehat{ABC} > \widehat{BCA}}$.* $\blacksquare$

- The proof is correct because all its steps are always valid under the hypothesis of proposition P1.
- The proof is incorrect because there are triangles which do not satisfy the hypotheses of proposition P1.
- The proof is incorrect because it contains at least one step (a, b, c, d, e, f, g or h) which is not always valid under the hypothesis stated in proposition P1.
- The three previous statements are all wrong.
- I do not know which of the previous four statements is correct.

Frame 11. English traslation of Question 1b – Item 1 – Post-questionnaire.

The last part of each item does activate only if the student has correctly answered the previous question 1b. I ask the student which one of the steps is not correct (step c) and why. To correctly answer to Question 1c, a student must both select step c and give a right counterexample.

Item 2 (shown in Appendix 5) is analogous to item 1: they have the same structure. The proposition mentioned in item 2 is the following: "If *l*, *m*, and *k* are three straight lines belonging to the same surface such that *m* is parallel to *l*, and *k* is parallel to *l*, then *m* is parallel to *k*".

5.5.6 *BELIEFS questionnaire*

Similarly to what was done in the pilot study, I used two questionnaires to collect quantitative data on student's beliefs on mathematics, one before the activities and one after the activities. These questionnaires (shown in Appendix 6), named QBI and QBF, are substantially identical to the ones in the pilot study, except for the fifth question.

The *BELIEFS questionnaire* addresses to the research question *RQ5* (*Do students' beliefs about mathematics change over the duration of the course?*). Specifically, the five questions aimed to test whether in the student's opinion:

- Q1. mathematics is discovered or invented;
- Q2. in mathematics it is more appropriate stating that an axiomatic system is consistent/non consistent rather than true/false;
- Q3. mathematical concepts are subject to historical revisions;
- Q4. socio-cultural factors influence mathematical knowledge;
- Q5. revolutionary changes exist also within the development of mathematical knowledge.

Students can select only one answer between five given options. Selecting the first answer, the student agrees with the statement in the question; selecting the second answer the student does not agree with the statement in the question; selecting the third answer the student state he/she cannot answer; selecting the fourth answer ("Other: …") the student can state a different opinion from "I agree/I do not agree". Specifically, the first answer is related to the prescriptive (or normative) account of mathematics while the second answer is related to the descriptive (or naturalistic) account of mathematics (see Section 5.3.4).

I administered the *BELIEFS questionnaire* the week before the beginning of the class activities and the *BELIEFS questionnaire* the week after the end of the class activities, both with the *VHL test* (see Table 7). The time allowed to fill out the

BELIEFS questionnaire. before and after the class activities, was 35 minutes. All the tests and the questionnaires were administrated via Google Form.

5.5.7 Class activities

The activities proposed to the students were planned as shown in Table 10. Five meetings of two hours each, beginning with an interactive session whose main objective was to understand what a circle and a straight line look like on a spherical surface, during this session we also deal with the definitions of segment, angle, polygon, and triangle on a spherical surface. At the beginning of the second session, I assigned the students to groups based on their results at the pre-test, in such a way as to minimize the chance that high ability students will huddle together leaving others out. Each group was of four students, exception for some groups of three students. I also created a virtual room for each group on the Cisco Webex platform, virtual rooms in which I could log into to monitor the work of the groups. During the second session, the students, divided into groups, tried their hand at tasks to be carried out on polystyrene spheres. These tasks allowed each student to explore the spherical surface and to observe that there exist geometric figures' properties that hold on a plane surface while they do not hold on a spherical surface. They deduced that we were dealing with a geometry different from the one we already knew (the Euclidean geometry). The third session revolved around the following question: "Why, are there geometric figures' properties that hold on a plane surface and that do not hold on a spherical surface, and vice versa?". We refreshed the basic elements of the Euclidean geometry and discussed on the possible validity of the five postulates of Euclid on a spherical surface. We observed that there are interpretations that allow us to consider the five postulates, formulated by Euclid, also valid in spherical geometry (Carroll & Rykken, 2018). Afterward, we analysed the Proposition 31 of the first Book of the Euclid's *Elements* ("Through a given point to draw a straight line parallel to a given straight line"), and its proof that relies on Proposition 16 (*Exterior Angle Theorem*). We understood that there is a flaw in the proof of Proposition 31. This led me to mention the Hilbert formalization of Euclidean geometry (specifically, the third axiom of order and the axiom of parallel), and the meaning of consistency, completeness, and independence of an axiomatic system. Connecting to the concept of independence of an axiomatic system, the fourth session focused on the controversy surrounding Euclid's fifth postulate, on the birth of the hyperbolic geometry, and on the importance of having models for an axiomatic system. I used a 3D-printing models of pseudospheres to show geometric figures' properties that hold on a plane surface but that do not hold on a pseudosphere and vice versa. The fourth session ended with a discussion on the loss of meaning of the question "Which geometry is the true one?", and contextualizing non-Euclidean geometries from an application point of view (linking e.g. to relativity in physics or the global positioning system in engineering). Finally, the last meeting consisted of a workshop on the Poincaré disk model and a final discussion to resume the whole course. The aim of the workshop on the Poincaré disk model was to let the students become more familiar with hyperbolic geometry, understand that there can be more than a model for a geometry, and avoid the misconception of identifying a geometry with one of its models.

<table>
<tr><th>Activity</th><th>Topic</th><th>Working format (online)</th></tr>
<tr><td>I
(2 hours)</td><td>Circumference, straight lines, segments, triangles, polygons on a spherical surface</td><td>Frontal-dialogue lesson/Workshop</td></tr>
<tr><td>II
(2 hours)</td><td>Constructions on a spherical surface</td><td>Group work</td></tr>
<tr><td rowspan="3">III
(2 hours)</td><td>Euclidean geometry and the potential validity on a spherical surface of the five postulates formulated by Euclid</td><td rowspan="3">Frontal-dialogue lesson</td></tr>
<tr><td>Euclid's flaw on Proposition I.16 and mention to the Hilbert formalization of Euclidean geometry</td></tr>
<tr><td>Introduction to the meaning of consistency, completeness, and independence of an axiomatic system</td></tr>
<tr><td rowspan="4">IV
(2 hours)</td><td>The independence of the fifth postulate of Euclid</td><td rowspan="4">Frontal-dialogue lesson</td></tr>
<tr><td>Hyperbolic geometry</td></tr>
<tr><td>Models for an axiomatic system</td></tr>
<tr><td>"Which geometry is the true one?"</td></tr>
<tr><td rowspan="2">V
(2 hours)</td><td>Poincaré disk model</td><td>Workshop with "NonEuclid" software</td></tr>
<tr><td>Final discussion</td><td>Frontal-dialogue lesson</td></tr>
</table>

Table 10. Plan of the class activities.

5.5.7.1 I session

The main objective of the first interactive session was to understand what a circle and a straight line look like on a spherical surface.

Circumferences on a spherical surface

I show the students the spheres represented in Figure 15 on whose surfaces a maximum circumference and a non-maximum circumference are drawn. I ask which geometric figures are those drawn in green. Students generally answer correctly that the pictures drawn are circles.

Figure 15. Circles on spherical surfaces.

I ask for the definition of circumference. We agree that the circumference is the set of equidistant points (i.e. they have the same distance) from a fixed point called the centre (skipping the definition of distance). I ask how, practically, we can mark all points that are at the same distance from the chosen centre. We conclude that, on a plane, it is possible to use a compass or, in the absence of this, a pin, a sewing thread and a pencil to trace a circumference. We observe that this second method can also be used by a being who lives immersed in the surface of the sphere, making sure that the sewing thread remains adherent and taut, for its entire length, on the spherical surface (Figure 16). It is good to insist on the fact that the radius of the traced circumference is the length of the sewing thread used.

Figure 16. Tracing a circumference on the surface of a plane and on that of a sphere using a pin, sewing thread and pencil.

I confirm that the plotted figures are circles and, showing how they were drawn, I highlight the centre and radius. We notice that the circumferences on the surface look very much like those on the plane. There is, however, a big difference: it concerns the lengths of the circumferences that can be drawn: on a plane there is no theoretical limit to the size of a circle, while on the sphere this limit exists. On the surface of a sphere, the largest possible circumferences are those that split the sphere into two hemispheres. A pair of antipodal points is defined as any pair of points belonging to the same maximum circumference and diametrically opposed to each other (from the extrinsic point of view, however, as a pair of points corresponding to the intersection of the spherical surface with any line passing through the centre of the sphere). It is convenient to work with a spherical model of Earth: the equator and the meridians of the globe are examples of maximal circumferences, the terrestrial parallels other than the equator are examples of non-maximal circumferences, the pair North Pole-South Pole is a pair of points diametrically opposed to each other. In demonstrating the above analogy, we must prevent students from creating the following misconception: only meridians and the equator are maximal circumferences; only parallels are non-maximal circumferences. Similarly, if we leave the example of the globe, we must avoid the following misconception: if we fix two antipodal points, only the maximum circumferences passing through the two given points and the maximum circumference orthogonal to them are the only maximum circumferences of a spherical surface (Figure 17). We observe that each maximum circumference contains infinite pairs of antipodal points and that infinite maximum circumferences pass through each pair of antipodal points.

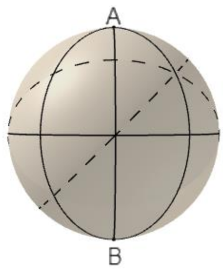


Figure 17. Possible misconception: given two antipolar points A and B, the solid lines are straight lines on a spherical surface while the dashed ones are not.

It is possible to open a parenthesis to make the students think about other ways to obtain a circumference on the surface of a sphere: by intersecting the spherical surface with a plane; if this plane passes through the centre of the sphere, a maximum circumference is obtained (see Figure 18). We observe that the point of view adopted here is different from the previous one: here we stand from an extrinsic point of view, whereas before it was intrinsic.

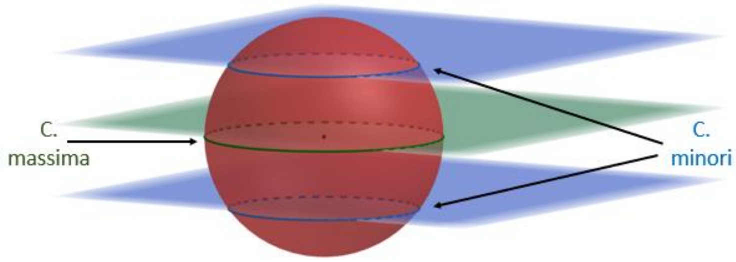


Figure 18. Spherical circle as the intersection of a spherical surface with a plane. The intersection of a spherical surface with a plane passing through the centre of the sphere gives a maximum circle.

Since the object of the workshop is the spherical surface, it is appropriate to clarify that – unless otherwise specified – we will work at the intrinsic level and, in particular, it is appropriate to remember that when we use the term *spherical radius* we will refer to the radius of a circumference as previously specified and not to the radius of the sphere. To let the students become familiar with the spherical surface, circumferences and their radii – referring to a sphere of which the length of the maximum circumferences $l_{M.}$ is known – I asked some questions. For example: "How large is the radius of the maximum circumferences (r_M)?"; "Where is the circumference having centre in P [I indicate a point on the surface] and a radius equal to half of $l_{M.}$?"; "Where is the circumference having centre in P [I indicate a point on the surface] and a radius between r_M and the length of a maximum semicircle? (see *Figure 19*). We also observe that the same circumference can be constructed from different centres and radii and that two different centres are antipodal points.

Let us summarise the most important steps of the activity carried out so far. We have defined the circumference as a set of points equidistant from a point called the centre. We have observed the spherical surface from two different points of view: the intrinsic and the extrinsic. In both cases we have seen how operationally it is possible to locate a circumference on a spherical surface. We have observed that on a spherical surface there are maximum circumferences and that these divide the spherical surface into two equal parts.

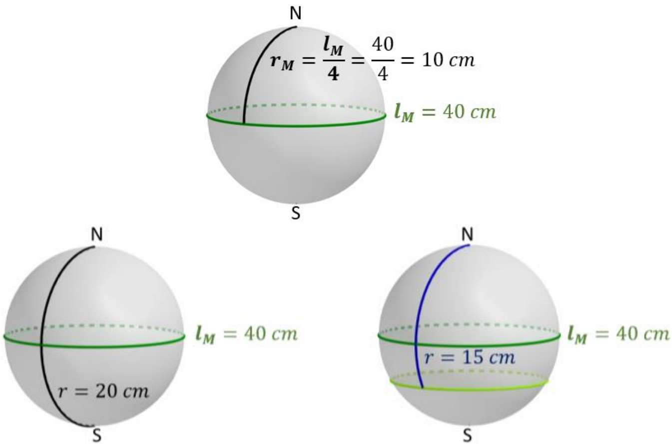


Figure 19. (a) Relationship between the length of a maximum circumference (l_M) and its radius (r_M). (b) Circumference with centre N and radius equal to half of l_M: degenerate circumference coinciding with the antipodal point at N. (3) Position of a circumference with radius between r_M and the length of a maximum semicircle.

Triangles, segments and straight lines on a spherical surface

Figure 20. Triangle and figures that resemble triangles on spherical surfaces.

I show the students the spheres represented in Figure 20. On the surfaces of these spheres I had previously drawn triangles or figures that may look like triangles. I ask which geometric figures are those drawn in red. Students generally answer that all figures are triangles or that only some are. I ask for the definition of triangle. Generally the debate evolves in the following way: initially some students try to define the triangle but they express the definition in an imprecise way (eg, "geometric figure", "flat figure", "three-sided figure"). After a while someone specifies the statements stating that *a triangle is a three-sided polygon*. We remember that a polygon is a figure made of a polygonal and of the points inside it. It should also be remembered that a finite sequence of segments $A_1A_2 \ldots A_{n-1}A_n$ *is called polygonal if* A_1A_2, A_2A_3, … $A_{n-1}A_n$ *are consecutive but not adjacent segments and each end of the segments is common to a maximum of two of them*. We call *sides of the polygon* the segments A_iA_{i+1} $(i = 0, \ldots, n)$, and *vertices of the polygon* the ends of the sides.

Then I ask what a segment is. Generally, some students answer inaccurately, often affirming that "a segment is a line that connects two points". Often, only after showing, on a Euclidean plane, a line that connects two data points but that is not a segment, some student specify the statement in one of the following ways:

A1. "a segment is the shortest line connecting two points".

A2. "a segment is the part of a straight line bounded by two points";

A3. "a segment is a straight line connecting two points".

If the students generally answer do not express the three previous definitions, I ask if they are familiar with those in *A1* and in *A2*[65]. Then I invite them to discuss the given statements. Below I indicate how, in general, the discussion evolves based on the answers given by the students.

Comments on the statement *A1*: I note that this statement corresponds to Archimedes' idea (Russo, Pirro, & Salciccia, 2017). I ask how, given two points, we could identify the shortest line connecting them, and be certain that this is the shortest. If the possible lines are infinite, we should be able to measure them all to be sure which is the shortest. Typically, some students comment that we could do this by drawing the "straight line" or the "straight line through the two points". I anticipate that we will come back to this topic.

I observe that this is indeed the definition usually given of a segment. Then I ask for the definition of a straight line. The students generally try answers such as the following: "it is the extension of a segment"; "it is a straight line". We deconstruct the first argument by noting that we cannot use the term segment because we have used the term *straight line* to define the term *segment*. On the other hand, the second case rejoins statement *A3*.

Comments on the statement *A3*: I ask what *to go straight* means. Generally, the students feel uneasy and do not answer. At the most they answer that it means not

[65] Working on the spherical surface it is possible to show that the two definitions coincide in the Euclidean plane but do not coincide in general: in general, going straight on a surface (i.e. following a geodesic) does not correspond to running the shortest path; these two definitions coincide only locally. Following the shortest path implies going straight, but the vice versa does not apply. Please note that the term geodesic will not be used during the course due to a research requirement: in the questionnaires it is more appropriate to use only the term *straight line* whatever the geometry considered, in order to avoid giving clues as to how to answer some questions.

to go crooked. I ask them what it means *to go crooked* and they point out that it leads to a vicious circle.

I reassure the students by pointing out that their difficulty in defining what a segment is, what a straight line is, and what *to go straight*, means is understandable. Indeed, it is not possible to proceed backwards by defining every term we use. Therefore, we have to assume some terms without defining them. These, called *primitive terms*, are fundamental to formulate all the other terms. In plane Euclidean geometry - the geometry on which the students have been working so far - we consider the following as primitive terms: *point*, *line* (in the sense of a *straight line*) and *plane*. If it is true that primitive terms are not explicitly defined, it is also true that it is possible to give a sort of implicit definition by specifying their mutual relations (*axioms* or *postulates*). E.g.: *for two points it is possible to draw a straight line*.

It is possible to open a historical digression on the definition of straight line (see Section 2.4) that face the students with three aspects rarely discussed in school

- There is a debate on historical sources also in mathematics.
- Mathematical knowledge is also influenced by the culture of the time in which it is developed.
- the difficulty students had in trying to define the meaning of *going straight* was also found at a historical level. The same can be found for other difficulties common to many students.

I recall that our goal is to determine whether the figures drawn on the spheres (Figure 20) are triangles. To do this, we must estabilish whether the edges of these figures are segments, i.e., parts of straight lines between two points. We then pose the problem of establish the correspondence, on spherical surfaces, of straight lines on plane surfaces. We must therefore understand which path we follow if we go straight ahead on a spherical surface. In order to understand this, we use a toy car that cannot steer and that, therefore, can only go straight. We observe that the toy car, while moving on the sphere without being subject to any force, can only follow the maximum circumferences (Figure 21).

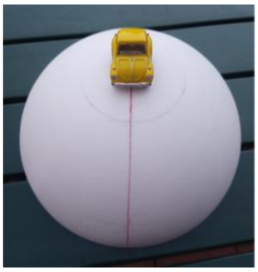


Figure 21. A small car that can only move by going straight on the surface of a sphere.

Therefore, maximum circumferences on a spherical surface correspond to straight lines on a plane surface. To reinforce this idea, we repeated on the surface of a sphere what we have already seen on the plane by stretching a sewing thread: we stretch the sewing thread between two points randomly chosen on a path covered by the toy car (in such a way that the sewing thread remains adherent to the surface of the sphere), and we observe that the sewing thread lies on the path.

We "translate", on the spherical surface, the primitive terms assumed in plane geometry (see Table 11).

	Plane surface	**Spherical surface**
Primitive terms	Plane	Surface of the sphere
	Point	Point on the spherical surface
	Straight line	Spherical straight line ≡ Maximum circumference

Table 11. Correspondence between the primitive entities of plane geometry and those of spherical geometry.

We go back to the definition of the term *segment* as part of a straight line bounded by two points and we observe that, if we fix two points on a spherical straight line, the straight line is divided into two parts having the two given points as extremes (Figure 22). We wonder whether only one of the two parts has to be considered a segment, and if so which one. We observe that if the two points are antipodal, then the both parts into which the line is divided correspond to half of a maximum circumference, therefore they are congruent. On the other hand, if the two points are not antipodal, then two non-congruent arcs of the circumference will be formed (Figure 22).

Figure 22. Two points on a spherical line divide the line into two parts (in the figure: the dotted line and the solid line).

We observe two peculiarities that would occur if we let both parts into which the line is divided be considered segments. Follow the two peculiarities

- The existence of triangles (one concave and one convex) having the same vertices but not congruent with each other. For example, in the sphere in Figure 23, both the yellow figure (in which the side *AB* corresponds to the smallest arc of maximum circumference passing through *A* and *B*) and the green one (in which the side *AB* corresponds to the longest arc of maximum circumference passing through *A* and *B*) would be triangles with vertices *A*, *B* and *C*, but the two figures are evidently not congruent.
- The existence of triangles as the one shown in Figure 24.

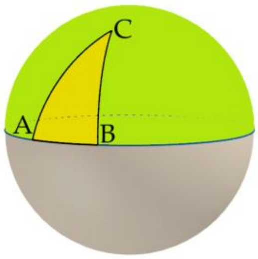


Figure 23. The yellow and green surfaces have the same points A, B and C as vertices but are not congruent

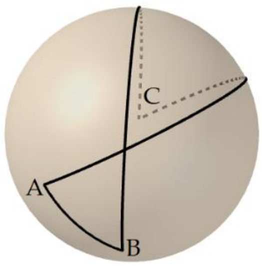


Figure 24. Figure having three vertices (A, B, C) and such that the lines AC and BC have lengths greater than a maximum semicircle.

We agree on the following definition of a *spherical segment* having two given endpoints: the shortest part of the spherical line between the two endpoint. Thus, a spherical segment is the shortest arc of the maximum circumference passing through two points (e.g. the dashed line in Figure 22). In the case of two antipodal points, both parts of the line can be called segments. Moreover, in the latter case, there are an infinite number of segments with the two points as extremes. This is because, as we have already seen, for each pair of antipodal points there are infinite maximal circumferences. We observe that by defining a *spherical segment* as the shortest part of a spherical line between two points, we repeat Archimedes' idea of associating to the segment the shortest line between two points. At the same time, we observe that going straight between two points does not imply travelling the shortest path.

I mention the routes of aircraft as an application of what we have said. Indeed, these tend to follow maximum circumferences in order to minimise distances (if we approximate the globe with a sphere).
The Google Maps application also reinforces the idea that in order to minimise distances, one must follow maximum circumferences. In fact, Google Maps can show the route in line of sight connecting any two cities (the route whose length corresponds to the distance), and its length. If we choose two cities that lie on the same terrestrial parallel (e.g. Rome and New York) as start and end points, and display the line of sight between them, we observe that the points on the route are not on the same parallel, in fact they do not have the same terrestrial latitude. This allows us to dispel the following common misconception among students: the shortest route connecting two cities that have the same latitude is to follow the parallel to which they belong.

I resume what we have seen so far by dealing with the concepts of triangle, segment and line. We started by asking ourselves whether the figures drawn on the spheres are triangles, we recalled the definition of a triangle (polygon with three sides) and, going backwards by trying to define the terms used, we asked for the definition of a segment. We came to understand the need to assume primitive terms. On the sphere, we identified the correspondents of the primitive terms in the plane (point, line and plane). We agreed to define the spherical segment with two points given as extremes as the smallest part of the line between the two points (i.e. the spherical segment corresponds to the smallest arc of the maximum circumference passing through the two points).

We conclude the discussion we have had so far. We go back to the figures drawn on the sphere and we observe that, in order to estabilish whether some of them are triangles, we have to check, for each figure, that all drawn line sections are segments. I ask the students how we could determine this. Generally, some students propose an appropriate strategy: stretching a sewing thread between the ends of each line and checking whether the sewing thread lies exactly along the drawn path, or observing whether the toy car can move along the three lines that make up the drawn closed line. It turns out that only some of the figures shown are triangles.

I point out that, since we have defined the triangle as a polygon, and since a polygon is a part of the surface bounded by a polygonal, the line we see is the polygon bounding the triangle. We realise that the definition of a triangle needs to be made more precise. We agree on the following definition: *a spherical triangle is a three-sided polygon in which each angle is less than a flat angle*.
So if we look at, for example, the sphere in Figure 25, we conclude that the spherical triangle with vertices *A*, *B,* and *C* is the dark figure.

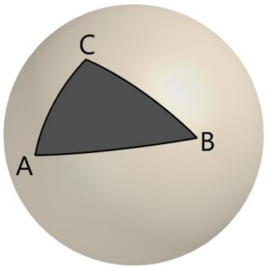


Figure 25. The dark figure is a spherical triangle while the light figure is not a spherical triangle.

We also observe that if we had let both surfaces delimited by the polygonal *ABC* be triangles, then we would have had two triangles (one concave and one convex) having the same sides but not congruent with each other.

Quadrilaterals on spherical surface

We hand out spheres on which figures are drawn that may look like quadrilaterals (Figure 26). I ask what figures are those drawn on the surface of the spheres. Having understood, thanks to the previous activity on spherical triangles, that it is appropriate to determine which of the two parts of the plane delimited by the line marked on the spheres is a quadrilateral, we agree on the following definition of *quadrilateral*: *polygon of four sides having each angle smaller than a flat angle*. After appropriate practical verification with sewing threads and pins, we conclude that only on some spheres there are quadrilaterals drawn.

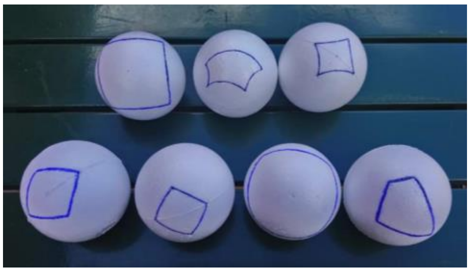


Figure 26. Quadrilaterals and figures that resemble quadrilaterals on spherical surfaces.

Angles on a spherical surface

So far, we have referred to the concept of angle on spherical surfaces only in an intuitive way. During the next workshop, the students will need to measure angles on spherical surfaces in order to perform some tasks. Therefore, I make some clarifications on this issue.

We observe that if we consider a small portion of a spherical surface this can be well approximated by a flat surface. Students have no difficulty in accepting this fact, indeed, sometimes they themselves comment that the same approximation applies when considering some portions of the Earth's surface.

This clarification is sufficient for the purposes of our course. We use the following tools to measure angles on the surface of a sphere: a sheet of tracing paper, a protractor, a ruler, three pins and a pencil. To measure an angle we pin the sheet of tracing paper at the vertex of the spherical angle. We copy on the sheet of tracing paper a point A on one side of the angle and a point B on the other side of the angle (close to the vertex to reduce as much as possible the error due to approximation of a portion of spherical surface with a plane). This work, illustrated in Figure 27, allows us to reconstruct the angle $A\hat{P}B$ on the tracing paper and measure its amplitude.

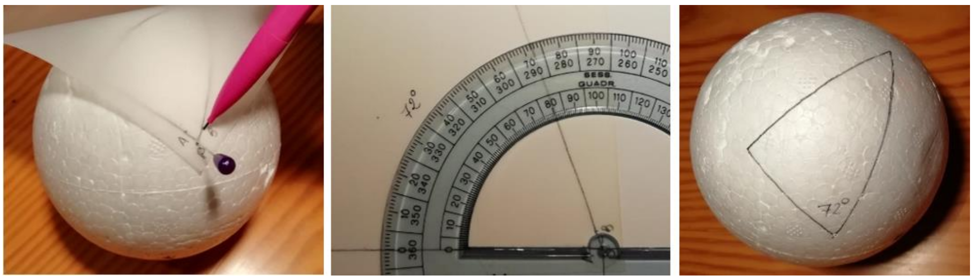


Figure 27. How to measure (approximately) a spherical angle.

If we consider it appropriate – sometimes students themselves stimulate further clarification – we can give an accurate definition of spherical angle using the concept of *dihedral angle*. The measure of the dihedral angle corresponds to that of the plane angle obtained by sectioning the dihedral angle with a plane perpendicular to the corner. Once defined the dihedral angle, we can define the angle between two straight lines on a spherical surface.

5.5.7.2 II session

The second session revolves around a workshop on spherical geometries. The tasks assigned allow the students exploring the spherical surface and observing that there exist geometric figures' properties that hold on a plane surface while they do not hold on a spherical surface. The students deduce that we are dealing with a geometry different from the one we already knew (the Euclidean geometry). The students, divided into groups, follow instructions given by me, that let them construct figures on the surface of a polystyrene sphere (to facilitate the work, the students can split the sphere into two hemispheres and work only on one of them).

The deliverables assigned to the groups are discussed below.

Task 1. Ratio of a circle's circumference to its radius

Task 1 revolves around the ratio of a circle's circumference to its radius. Frame 12 and Frame 13 show, respectively, the original Italian version of Task 1 and its translation into English.

CONSEGNA 1

Q-a) Considerando solo la superficie di una sfera, quanto vale il rapporto tra una circonferenza massima e il suo raggio?

Istruzione I. Sulla superficie di una sfera traccia una circonferenza che non sia massima.

Q-b) Come hai proceduto per tracciare la circonferenza richiesta nella precedente istruzione?

Q-c) Considerando solo la superficie della sfera utilizzata per eseguire l'Istruzione I, quanto vale il rapporto tra la lunghezza della circonferenza tracciata e quella del suo raggio?

Q-d) Su un piano euclideo, il rapporto tra una qualsiasi circonferenza e il suo raggio è costante (vale 2π). Sulla superficie sferica, vale qualcosa del genere (il rapporto tra una qualsiasi circonferenza e il suo raggio corrisponde sempre a uno stesso valore)?

Istruzione II. Fotografa la sfera in maniera tale che si veda bene ciò che hai tracciato e copia e incolla l'immagine qui sotto.

Frame 12. Task 1 on spherical surfaces (original verision).

TASK 1
Q-a) Considering only the surface of a sphere, what is the ratio between a maximum circumference and its radius? **Instruction I.** On the surface of a sphere, draw a circumference that is not maximum. **Q-b)** How did you proceed to trace the circumference required in the previous instruction? **Q-c)** Considering only the surface of the sphere used to perform Instruction I, what is the ratio between the length of the traced circumference and that of its radius? **Q-d)** On a Euclidean plane, the ratio between any circumference and its radius is constant 2π. On the spherical surface, does something like this hold true (does the ratio of any circumference to its radius always correspond to the same value)? **Instruction II.** Photograph the sphere in such a way that you can clearly see what you have drawn and copy and paste the image below.

Frame 13. Task 1 on spherical surfaces (english translation).

Generally, students have no problem understanding this delivery. Similar consideration was faced with the teacher during the I session (see Circumferences on a spherical surface in Section 5.5.7.1). Thus, the most problematic aspect was the practical one. In particular, students pose the problem of how to plot and how to measure the length of circumference and radius.

Finding a maximum circumference is easy because they can just split the sphere into two hemispheres and consider the bound of the surface of the hemisphere. In contrast, identifying a non-maximum circumference is more problematic. Students implement the following strategies to plot a non- maximum circumference:

- use the procedure discussed in *Circumferences on a spherical surface* (in Section 5.5.7.1), i.e. using a compass constructed from a pin, a sewing thread and a pencil;
- use a traditional compass. Some students who use traditional compasses initially make the mistake of assuming the radius of the circle to be the compass opening.

Students implement the following strategies to measure a circumference:

- measure the length of a sewing thread as long as the circumference;
- rotate a ruler on the circumference trying to make the ruler stay on the plane of the circumference;
- use a tape measure.

The students verify what was previously observed: the ratio of a circle's circumference to its radius is not constant. After a discussion I led, we conclude that, on a given sphere, the ratio of a circle's circumference to its radius and increases as the radius decreases and can vary between the value 4 (excluded) for a maximum circumference and the value 2π (excluded) for circumferences of radius tending to 0 and, therefore, on portions of the sphere assimilated to the plane.

If deemed appropriate – particularly for students who already have a rudiments of goniometry and analysis – find out the previous limits by means of extrinsic observations. Referring to the circumference γ in Figure 28, having the arc r_s as radius, it is possible to derive that γ/r_s depends only on the angle α, that cannot be evaluated by those immersed in the surface of the sphere. This ratio is equal to $2\pi\frac{sin\,\alpha}{\alpha}$, thus varies between 4 and 2π for α in the range $0, \frac{\pi}{2}$.

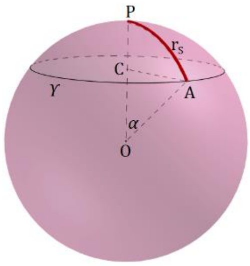


Figure 28. The image used to show how to calculate the ratio circumference/radius on a spherical surface.

Task 2. Squares

Task 2 revolves around the concept of square on a spherical surface. Frame 14 and Frame 15 show, respectively, the original Italian version of Task 2 and its translation into English.

CONSEGNA 2

Istruzione I. Su un piano euclideo, esegui la seguente costruzione:
- traccia un segmento AB, poi un segmento BC congruente e perpendicolare ad AB;
- traccia un segmento CD congruente e perpendicolare a BC e sullo stesso semipiano di AB rispetto a BC;
- traccia un segmento DE congruente e perpendicolare a DC e sullo stesso semipiano di BC rispetto a DC.

Q-a) Che cosa si può osservare a proposito dei punti A ed E?
Q-b) Che cosa si può osservare a proposito dei segmenti AB e DE?
Istruzione II. Sulla superficie di una sfera, ripeti la costruzione indicata in *Istruzione I*.
Q-c) Che cosa si può osservare a proposito dei punti A ed E?
Q-d) Che cosa si può osservare a proposito dei segmenti AB e DE?
Istruzione III. Fotografa il piano e la sfera in maniera tale che si veda bene ciò che hai tracciato e copia e incolla l'immagine qui sotto.

Frame 14. Task 2 on spherical surfaces (original verision).

TASK 2

Instruction I. On a Euclidean plane, perform the following construction:

– draw a segment AB, then a segment BC congruent and perpendicular to AB;

– draw a segment CD congruent and perpendicular to BC and on the same half-plane as AB with respect to BC;

– draw a segment DE congruent and perpendicular to DC and on the same half-plane as BC with respect to DC.

Q-a) What can be observed about points A and E?

Q-b) What can be observed about the segments AB and DE?

Instruction II. On the surface of a sphere, repeat the construction given in Instruction I.

Q-c) What can be observed about the points A and E?

Q-d) What can be observed about the segments AB and DE?

Instruction III. Photograph the plane and the sphere so that you can clearly see what you have traced and copy and paste the image below.

Frame 15. Task 2 on spherical surfaces (english translation).

Following the instruction given in the delivery, you draw a square on a Euclidean plane (Figure 29) and – depending on the length of the side *AB* – either an open braided broken line or a tri-rectangular triangle (the latter case is obtained in case of the side *AB* is a quarter of the maximum circumference as shown in Figure 30).

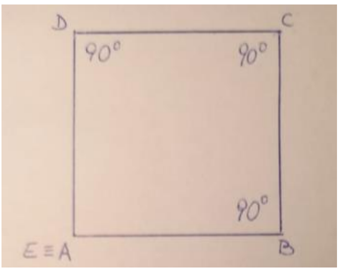


Figure 29. Figure drawn on a Euclidean plane following the instructions given in Task 2.

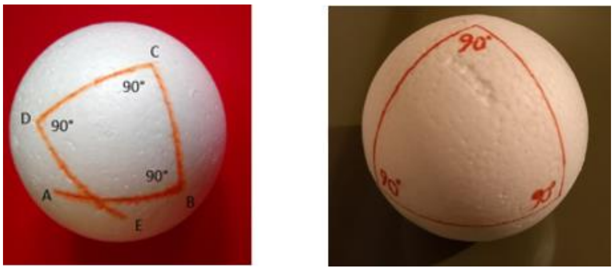


Figure 30. Figure drawn on a spherical surface following the instructions given in Task 2.

Some observations made during the workshop follow:

O1. All students perform the construction on the plane without any difficulty.

O2. Many students who correctly perform the construction on the spherical surface believe they make a mistake, moreover they force their construction by trying to get a figure they believe to be a square.

O3. Some students misunderstand the task and try to draw a figure that they think is a square. E.g., they cut out the square drawn on the plane and try in vain to adhere it to the surface of the sphere in order to trace its side.

The implementettion of the erroneous technique described in O3 lead us to an argument mentioned talking about the angles on a spherical surface (see *Angles on a spherical surface* Section 5.5.7.1). With my help, the students understand that the technique used is incorrect because it is impossible for a portion of a plane to adhere perfectly to a spherical surface, and vice versa. Indeed, an approximation between plane and spherical surface is possible only if we consider "small" portions of surfaces.

We also observe the existence of a figure that does not exist on the plane, the *equilateral triangle trirectangle*. Such a triangle allows us to show that on the sphere does not apply the Pythagorean theorem.

Discussing the Task 2, we wonder if it is possible to draw squares on the surface of a sphere. The students divided in two groups: a group who believe that it is possible to draw squares on the surface of a sphere, and a group who believe that it is not possible. I remind you that taking positions, it is suitable to agree on the same definitions of the terms we are considering (now, *square*). I exhort to define the *square*. The answers suggested by students – when not completely wrong – are generally the following

A1. "quadrilateral with all sides and all angles congruent."
A2. "quadrilateral with all sides congruent and all angles right."
A3. "quadrilateral with sides two by two parallel and congruent, and with congruent angles."

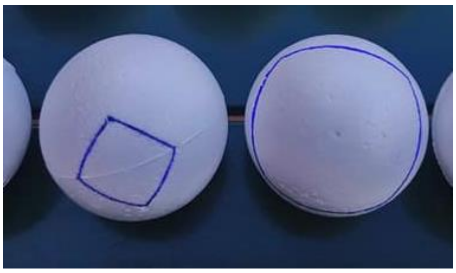


Figure 31. Regular quadrilaterals on spherical surfaces.

I show some spheres on which I had previously traced some figures that resemble squares (e.g., the ones in *Figure 31*). We verify that these figures are actually quadrilaterals. Working with these figures allows us to conclude that, on the surface of a sphere, the three answers given are not equivalent. In fact, the figures that satisfy the characterization given in A1, satisfy neither the one given in A2 nor the one given in A3. Moreover we prove that there cannot exist quadrilaterals with the sides two by two parallel and congruent and the angles congruent (A3) because there are no parallel straight lines. Indeed all the spherical straight lines intersect in two antipodal points.

Task 2 leads the students to conjecture that, on a spherical surface, quadrilaterals with all congruent sides and all right angles do not even exist. I confirm that there are no quadrilaterals that satisfy the characterization given in A2.

We conclude that, among the previous definitions of *square*, the only one that can also be assumed on spherical surfaces is the following one: *quadrilateral with all sides and all angles congruent* (A1).

Task 3. Angles of an equilateral triangle

Task 3 revolves around the measure of angles in spherical triangles. Frame 16 and Frame 17 show, respectively, the original Italian version of Task 3 and its translation into English.

CONSEGNA 3

Istruzione I. Sulla superficie di una sfera scegli un punto che chiamerai "polo".

Istruzione II. Costruisci un triangolo equilatero i cui vertici siano equidistanti dal polo.

Q-a) Come hai proceduto per tracciare il triangolo richiesto nella precedente istruzione?

Q-b) Sia α la misura di ogni angolo interno del triangolo equilatero costruito. Quanto vale α?

Istruzione III. Sulla sfera usata per l'*Istruzione I*, costruisci un altro triangolo equilatero i cui vertici siano equidistanti dal punto precedentemente denominato "polo".

Q-c) Sia β la misura di ogni angolo interno del secondo triangolo equilatero costruito. Quanto vale β?

Q-d) Su un piano euclideo, ogni angolo di un qualsiasi triangolo equilatero misura 60°. Sulla superficie sferica, vale qualcosa del genere (ogni angolo di un qualsiasi triangolo equilatero misura sempre lo stesso valore)?

Q-e) Su un piano euclideo, la somma degli angoli interni di un qualsiasi triangolo equilatero misura 180°. Sulla superficie sferica, vale qualcosa del genere (la somma degli angoli interni di un qualsiasi triangolo equilatero misura sempre lo stesso valore)?

Istruzione IV. Fotografa la sfera in maniera che si veda bene ciò che hai tracciato e copia e incolla l'immagine qui sotto.

Frame 16. Task 3 on spherical surfaces (original verision).

TASK 3
Instruction I. On the surface of a sphere, choose a point that you will call "pole".
Instruction II. Construct an equilateral triangle whose vertices are equidistant from the pole.
Q-a) How did you go about drawing the triangle required in the previous instruction?
Q-b) Let α be the measure of each interior angle of the constructed equilateral triangle. What is the value of α?
Instruction III. On the sphere used for Instruction I, construct another equilateral triangle whose vertices are equidistant from the point previously called the "pole."
Q-c) Let β be the measure of each interior angle of the second constructed equilateral triangle. What is the value of β?
Q-d) On a Euclidean plane, each angle of any equilateral triangle measures 60°. On the spherical surface, is something like this true (each angle of any equilateral triangle always measures the same value)?
Q-e) On a Euclidean plane, the sum of the interior angles of any equilateral triangle measures 180°. On the spherical surface, does something like this hold true (the sum of the interior angles of any equilateral triangle always measures the same value)?
Instruction IV. Photograph the sphere in such a way that you can clearly see what you have drawn and copy and paste the image below.

Frame 17. Task 3 on spherical surfaces (english translation).

Some observations made during the workshop follow:

O1. Many students draw an equilateral triangle and identify the point called the pole with a vertex of the triangle. These students misinterpret the term *equidistant* [66].

O2. Many students draw an equilateral triangle by attempts, i.e., by placing pins on some points of the sphere and then checking if these points can be considered vertices of an equilateral triangle. Once this is done, they try to identify the point that is equidistant from such vertices.

O3. Some students locate the so-called pole, draw a circle with that point as the pole, and try to divide the circle into three congruent arcs to locate the vertices of a triangle that meets the requirements in Task 3. To divide the circumference, students first try to measure it.

O4. A few students divide into three equal parts the round angle having vertex in the pole.

O5. No student came up with alternative strategies.

O6. Some students estimate the measure of the angles of the drawn triangle but do not measure it. Especially in such cases, and particularly if the triangles drawn have similar dimensions, some students answer that the angles of the triangles drawn have equal measure.

[66] Translated from the Italian "*equidistante*".

O7. Most students infer that, on a spherical surface, equilateral triangles do not have a fixed value for angles (varying triangle can vary the value of angles), unlike the Euclidean case where equilateral triangles have angles equal to 60°. They also deduce that the sum of the interior angles of a spherical triangle varies as the triangle varies.

To conclude the discussion on this task, I add that – although we will not prove it during our course – on a spherical surface, the sum of the interior angles of any triangle is always greater than 180°.

Task 4. Polygons with only two angles

Unlike the students involved in the pilot study described in Section 5.3, the ones involved in the second experimental phase, here discussed, are unable to carry out Task 4 due to lack of time. This task revolves around the existence of polygons having only two angles and is preparatory to Task 5 which aims to make students find the formula to calculate the area of a spherical triangle. Frame 18 and Frame 19 show, respectively, the original Italian version of Task 4 and its translation into English.

CONSEGNA 4

Istruzione I. Sulla superficie di una sfera, traccia due rette tra loro perpendicolari.
Q-a) In quante parti viene suddivisa la superficie della sfera?
Q-b) Le varie parti in cui vengono suddivisa la superficie possono essere considerate poligoni? Se sì, calcola le loro aree.
Istruzione II. Sulla superficie di una sfera, traccia due rette che non siano tra loro perpendicolari.
Q-c) In quante parti viene suddivisa la superficie della sfera?
Q-d) Le varie parti in cui vengono suddivisa la superficie possono essere considerate poligoni? Se sì, calcola le loro aree.

Frame 18. Task 4 on spherical surfaces (original verision).

TASK 4

Instruction I. On the surface of a sphere, draw two straight lines that are perpendicular to each other.
Q-a) How many parts is the surface of the sphere divided into?
Q-b) Can the various parts into which the surface is divided be considered polygons? If yes, calculate their areas.
Instruction II. On the surface of a sphere, draw two straight lines that are not perpendicular to each other.
Q-c) How many parts is the surface of the sphere divided into?
Q-d) Can the various parts into which the surface is divided be considered polygons? If yes, calculate their areas.

Frame 19. Task 4 on spherical surfaces (english translation).

Students should be guided to establish that the surface of a spherical polygon having only two angles (e.g., one of the two congruent green parts into which we see the sphere divided in Figure 32) equal to α is $2\alpha R^2$.

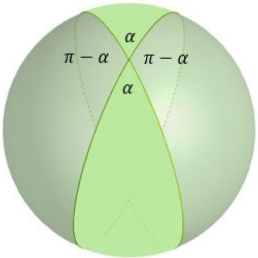


Figure 32. Spherical surface divided into four parts two by two congruent with each other.

Task 5. Sum of the interior angles of a triangle

Task 5 aims to lead the students to find the formula to calculate the area of a spherical triangle. Frame 20 and Frame 21 show, respectively, the original Italian version of Task 5 and its translation into English.

CONSEGNA 5

Istruzione I. Sulla superficie di una sfera, costruisci un triangolo trirettangolo.
Q-a) Come si può calcolare l'area del triangolo trirettangolo?
Istruzione II. Sulla superficie di una sfera, costruisci un triangolo che non sia trirettangolo.
Q-b) Come si può calcolare l'area del triangolo costruito?

Frame 20. Task 5 on spherical surfaces (original verision).

TASK 5

Instruction I. On the surface of a sphere, construct a trirectangular triangle.
Q-a) How can the area of the tri-right triangle be calculated?
Instruction II. On the surface of a sphere, construct a triangle that is not trirectangular.
Q-b) How can the area of the constructed triangle be calculated?

Frame 21. Task 5 on spherical surfaces (english translation).

As already observed, unlike the students involved in the pilot study described in Section 5.3 (that revolved around the concept of curvature), the ones involved in the second experimental phase, are unable to carry out this task due to lack of time. I only remark that, on a spherical surface, the sum of the interior angles of any triangle is always greater than 180° (see *Task 3. Angles of an equilateral triangle*).

5.5.7.3 III session

During the previous session, the students tried their hand at tasks that allowed them to explore the spherical surface and to observe that there exist geometric figures' properties that hold on a plane surface while do not hold on a spherical surface. The object of the III session is the comparison between Euclidean and spherical geometry. The aim is to understand what changes, from the axiomatic point of view, in the passage from Euclidean to spherical geometry.

The main steps of the II session follow

- Remind that, during the previous session, we experimented that on the surface of the sphere are valid results different from the ones known by our studies on Euclidean geometry.
- Introduce Euclid and his main work, the *Elements*. Point out that Euclid's merit lies in having unified and systematized mathematics by deducing propositions then known from a few postulates and definitions. Note that his work has survived to us, but not as an original manuscript written by Euclid himself.
- Introduce the first book of the Elements. In discussing definitions, point out that some may not even have been written by Euclid. Also remember that – as already anticipated in 5.5.7.1 – "definitions" that introduce the primitive terms (*point*, *line*, and *plane*) are not real definitions but useful descriptions to visualize the concepts. Present Euclid's postulates as statements that do not need to be proved but are assumed to be valid, statements that constitute, together with the common definitions and rules, our starting point for all our deductions.
- Analyse Euclid's postulates and reflect on their possible validity in spherical geometry. Conclude that the five postulates of plane geometry expressed by Euclid can be considered valid – through appropriate interpretations – even on the surface of a sphere. Figure 33 show a slide used to present while discussing the potential validity in spherical geometry of the Euclid's postulates; Figure 34 show its translation to English.

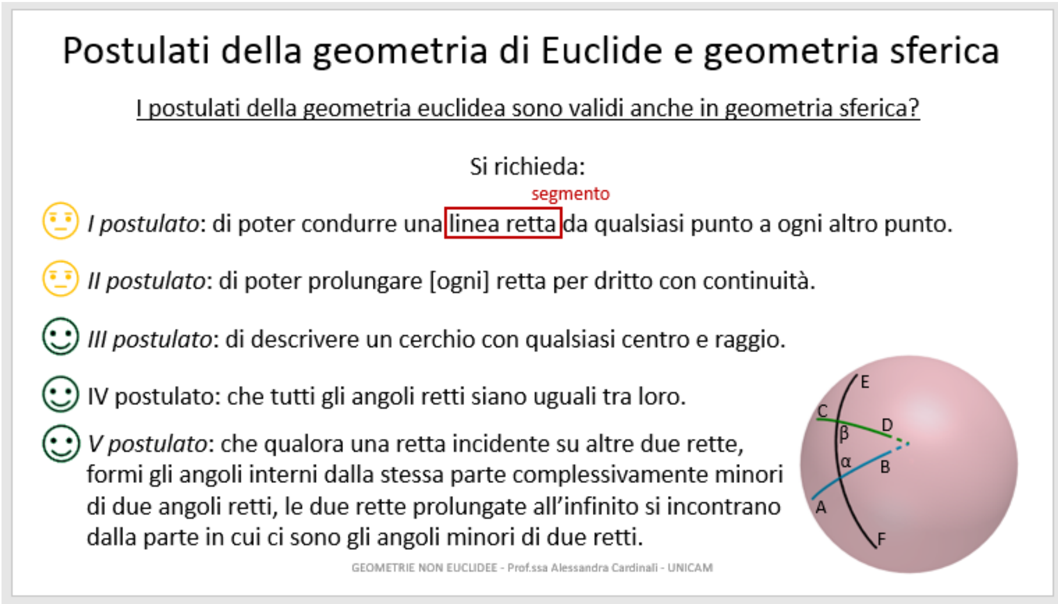


Figure 33. Slide used to analyse the potential validity of the postulate of Euclid in spherical geometry (original version).

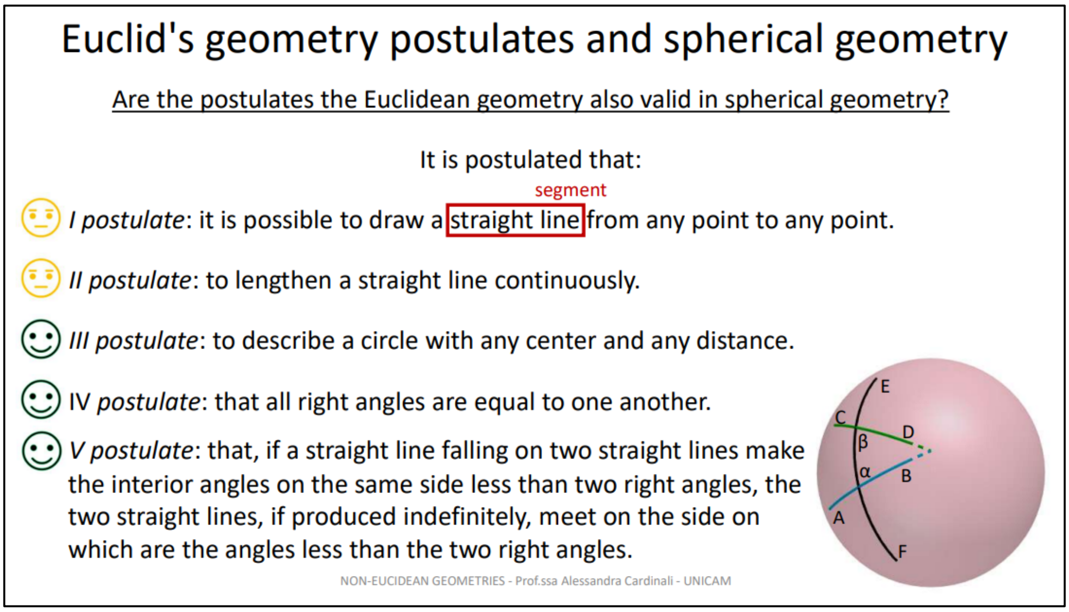


Figure 34. Slide used to analyse the potential validity of the postulate of Euclide in sferical geometry (english traslation).

We discuss the interpretations of Euclid's postulates that allow them to be valid also in spherical geometry. In the following the interpretation are explained. The I postulate can be considered valid because it does not postulate the uniqueness of a line passing through any two points. The I postulate of Euclid only postulate the existence. In addition – as I already mentioned during the I session – Euclid, for *straight line*, means a bounded straight-line (a *segment*). The II postulate can be considered if we distinguish between the notions of *infinite* and *boundaryless*. "While a great circle is certainly finite in length, it is also free of any boundaries, in that we can continuously produce the line, never reaching an end" (Carroll & Rykken, 2018). There are no particular problems in accepting that Euclid's other postulates are also valid on spherical geometry. I note that I give the Euclid's formulation of the fifth postulate and I disclose that today we state it in a different way.

I invite students to reflect on the fact that although all of Euclid's postulates are valid in spherical geometry, not all of the consequences that can be deduced from them are valid. For example, on the surface of a sphere there are no lines parallel to a given line. In contrast, on the plane, it is always possible to conduct a line parallel to a given line (Proposition 31, I book, Euclid's *Elements*). This means that there is a gap in the axiomatic system formulated by Euclid. Identifying this gap would allow us to understand why there are geometric figures' properties that hold on a plane surface and that does not hold on a spherical surface.

Our goal now focuses on understanding where is the gap in Euclid's reasoning. We start our investigation from the Proposition 31 of the I book of the *Elements*. I recall its statement (*"To a given point it is possible to draw a straight line parallel to a given straight line")* and its proof. Proceeding backwards through the logical steps of the proof, we try to understand which step are invalid on the surface of a sphere.

We observe that Proposition I.27 (*"If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another")*) is not valid on the surface of a sphere.

We go on in our backward progression. Euclid proves Proposition I.27 through Proposition I.16 (*"In any triangle, if one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles"*). Proposition I.16 does not hold on a spherical surface (e.g., see *triangle trirectangle* in Section *5.5.7.2*). In analysing the proof of Proposition 16 we observe that what would seem obvious to us, cannot be deduced from Euclid's postulates. In the following, I refer to Figure 35. Proving that the angle $B\hat{C}A$ is greater than the angle $A\hat{B}C$, Euclid extends *BM* (median of *BC*) by a segment *MD* congruent to *AM*. If we repeat the same construction on the surface of a sphere, we observe that the prolongation does not always be completely included in the external angle taken into consideration (see Figure 36). This is a omission in Euclid's reasoning.

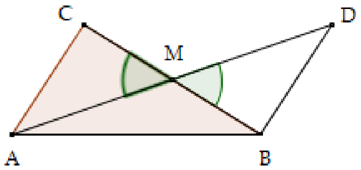


Figure 35. Diagram used to prove Proposition I.16.

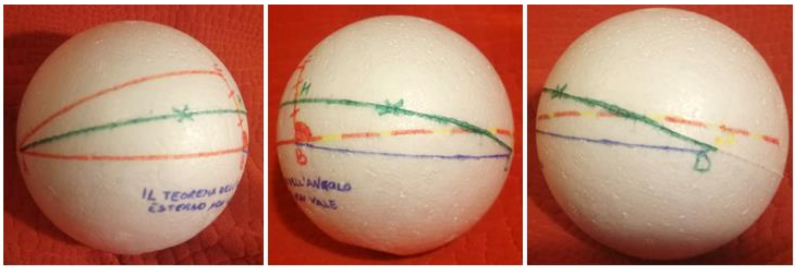


Figure 36. Diagram on a spherical surface used to show that the the median of BC may not be totally included in a single half-plane bounded by the straight line through B and C.

The previous omission gives the opportunity to introduce Hilbert's reformulation of the Euclidean axiomatic system. I observe that – besides correcting some "oversights" present in Euclid's work – Hilbert also postulates the following (Hilbert, 1971):

- *"For every two points A, B there exist no more than one straight line that contains each of the points A, B"*. This axiom makes explicit that the straight line passing through two given points is unique.
- *"Of any three points on a straight line there exists no more than one that lies between the other two"*. This axiom does not allow considering a circumference a straight line.
- *"Let a be any line and A a point not on it. Then there is at most one straight line in the plane, determined by a and A, that passes through A and does not intersect a"*. This is the modern version of the fifth postulate of Euclidean geometry.

I highlight that, while Euclid's axiomatic approach was aimed to codify the intuition of a physical space, Hilbert's axiomatic approach aimed to build an abstract mathematical object. Hilbert's goal was to axiomatize all mathematics, that is, to formalize the various branches of mathematics by developing the "best" axiomatic systems for them. And, by "best" axiomatic systems, Hilbert meant an axiomatic system that was consistent, complete, and independent. I give an explanation of what it means for an axiomatic system to be consistent, complete, and independent by focusing on consistency and independence.

I anticipate students that the concept of independence will be central to our next meeting. Indeed, for two millennia, mathematicians tried to prove that Euclid's fifth postulate was not independent of the other postulates. All the attempts failed but led to the birth of a geometry different from both Euclidean and spherical geometry, they led to the formulation of hyperbolic geometry.

The III session described here is conducted in frontal-dialogue mode. As expected, many students never intervene in the discussion and – also because of the distance mode – it is difficult to realize how much attentive they were. However, among the students who intervene the most, there are some who – according to their teachers – are usually more reserved and less cooperative.

It is worth noting that – before introducing the postulates of Euclidean geometry – generally, the students cannot state any postulate of Euclidean geometry. Only few students postulate the existence of a straight line passing through two given points. Between these students, some also postulate the uniqueness of this straight line. As for the other postulates, I have observed much confusion.

Several students intervene in the explanation of the consistency, independence, and completeness of an axiomatic system.

5.5.7.4 IV session

In hyperbolic geometry, Euclid's fifth postulate is replaced by the following postulate: *"for a point not belonging to a given straight line, there exists more than one straight line parallel to the given straight line"*. Other properties, quite out of the ordinary, hold in hyperbolic geometry. For example: the sum of the interior angles of a triangle is less than 180° and can vary as the triangle varies; parallel straight lines are not everywhere equidistant from each other (by *equidistant lines* we mean two lines r and s such that if a point R of r has distance d from s, then all points of r have distance d from s, as well as all points of s have distance d from r); the rectangles (if we pretend that they are quadrilaterals with all right angles) do not exist (and therefore do not exist even the squares); the Pythagorean theorem does not hold.

I reassure students by noting that if they cannot imagine a world in which such results would hold, it is well understandable. Along the history of Mathematics, many mathematicians run into the same difficulty. I follow course that led from doubting the independence of the fifth postulate to the acceptance of the hyoerbolic geometry. In the following, the principal steps:

- As the validity of the fifth postulate was not in doubt, why do mathematicians tried to prove the indiendence of the fifth postulate of Euclid from the other postulates?
- The equivalence of Euclid's fifth postulate – in case the first four postulates of Euclid's postulates are assumed – to the formulation we adopt today.
- The drama of some episodes in which some mathematicians were involved. For example, the drama with which Girolamo Saccheri, expresses his rejection of the results he arrives at by assuming that there are no straight lines parallel to a given straight line passing through a point outside it (Saccheri states *"The hypothesis of the acute angle is absolutely false, because repugnant to the nature of the straight line"* (Carroll & Rykken, 2018)). The dramatic event involving János Bolyai, his father Farkas Bolyai, and Carl Friedrich Gauss.
- Gauss's reluctance to publish his results.
- The difficulties that the publications of Saccheri, Bolyai, and Lobachevsky encountered in being accepted by the mathematical community (because of the obscurity of some of the publications, the languages in which they were written, and – most of all – the difficulty on the part of the mathematical community in developing a general understanding of this strange new world).
- The development of Euclidean models for hyperbolic geometry was the key element that aided understanding and led to accept hyperbolic geometry. By a Euclidean *model* for hyperbolic geometry was meant an interpretation of the primitive terms of hyperbolic geometry made in such a way that all the axioms of hyperbolic geometry make sense in Euclidean geometry. Thus, such a model let imagine the results of hyperbolic geometry. It also allows us to claim that the new geometry is as consistent as the Euclidean geometry.

A mere theoretical explanation of what a Euclidean model of hyperbolic geometry is cannot but be incomprehensible or, at least, evanescent. Therefore, I introduce models of pseudospheres (it would be better to say a portion of pseudospheres) printed with the 3-d printer. On these models I have previously drawn several figures: straight lines (or, better to say, parts of straight lines);

circumferences; triangles; a figure drawn following the instructions given in in Frame 14. These figures allow us to make the following observations

- Euclid's fifth postulate is not valid.
- The ratio of the length of a circle to its diameter is not constant.
- The sum of the interior angles of a triangle varies as the triangle considered varies.

I ask students how I could have drawn the straight lines. Some students state that I have stretched a wire between two ends of the pseudosphere (as we did in spherical geometry). Unfortunately, this experience could only be done at a distance so the students could not get their hands on the models. However, observing me, they understood that it is not possible to repeat the same strategy adopted on the surface of a sphere because the stretched wire would not always adhere to the surface. There is not always adherence because of the shape of the pseudosphere. Then, I show a kind of soft ruler, printed with the 3-d printer, used by me to draw the straight lines (see Figure 40).

Figure 37. Three straight line on a model of pseudosphere.

Figure 39. Poligonal drawn following the instructions given in Frame 14.

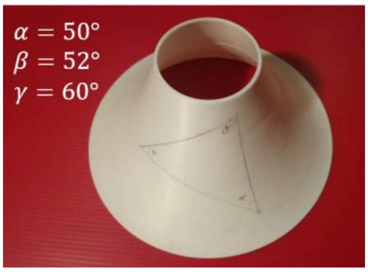


Figure 38. Triangle drawn on a model of pseudosphere.

Figure 40. 3D-printed flexible rules used to drawn straight lines on a pseudosphere.

We interpret the primitive terms of Euclidean geometry. Then, we again review Euclid's five postulates and we comprehend that the first four postulates hold also on the surface of the pseudosphere. In contrast, the fifth postulate does not hold.

Interpreting the primitive terms of Euclidean geometry is more difficult than interpreting them on the surface of a sphere. Indeed, while dealing with spherical geometry models it is possible to see representations of entire spherical straight lines, dealing with the pseudosphere this is not possible. Although not easy to perform manually, it is still suitable to refer to figures that can be visualized in their entirety in the available model (e.g., the ones in Figure 38 and in Figure 39).

I open a discussion asking what is the truth, is it true that the sum of the interior angles of a triangle is less than, equal to, or greater than 180°? Is it true that through an external point to a given straight line pass infinite number, only one or no parallel to the given straight line? We conclude that my question is wrong: there is no absolute truth but there are statements that can be true – in the sense of *provable* – depending on the premises that we assume. I specify that, in the modern conception of mathematics, postulates can be freely chosen. There is no longer the will to describe an existing object but what is required of an axiomatic system is consistency. Then, mathematicians can also create an axiomatic system according to their aesthetic sense, but they must create it in such a way that it is consistent (it is important to not lead the students understand that every theory is possible or that all theories are equally important.). If a mathematical theory is consistent, it could arrive a moment in which this theory find some application. This moment has arrived also for non-Euclidean geometries; in fact today we can move on Earth (with GPS) and in the cosmos (general relativity) also thanks to non-Euclidean geometries that were born as abstract inventions.

5.5.7.5 V session

I presented another model of hyperbolic geometries, the Poincaré disk model. I present this model using the NonEuclid software and referring to the illustration in Figure 41.

Figure 41. M.C. Escher, Cerchio limite IV (1960).

As usual, we first interpret the primitive terms of Euclidean geometry (see Table 12). Then I observe that if the geometry of the hyperbolic world contains two propositions that contradict each other, the inner geometry of the Euclidean circle, and by implication the geometry of the entire Euclidean plane, should also be inconsistent. Since this is not the case, we can consider the hyperbolic geometry as consistent as the Euclidean geometry.

	Plane surface	**Poincaré disk model for hyperbolic geometry**
Primitive terms	Plane	Open disk
	Point	Point inside the disk
	Straight line	Hyperbolic straight ≡ open diameter of the boundary circumference or an open arc of a circle that is orthogonal to the boundary circumference of the disk

Table 12. Correspondence between the primitive entities of plane geometry and those of Poincaré disk model for hyperbolic geometry.

After an introduction to the NonEuclid software, students perform Task 1 and Task 2 shown in Frame 22-25.

CONSEGNA 1.

Istruzione I. Utilizzando il software NonEuclid, individua sul disco di Poincaré tre punti qualsiasi A, B e C a due a due distinti tra loro; traccia la retta che passa per A e per B.
Q-a) Quante rette passanti per C e parallele alla retta per A e per B esistono?
Istruzione II. Se al quesito Q-a hai risposto più di una, tracciane almeno due.

Frame 22. Task 1 on the Poincaré disk model (original verision).

TASK 1.

Instruction I. Using the NonEuclid software, locate on the Poincaré disk any three points A, B and C two by two distinct from each other. Draw the line that passes through A and B.
Q-a) how many lines passing through C, and parallel to the line through A and B, exist?
Instruction II. If you answered “more than one” to question Q-a, draw at least two of them.

Frame 23. Task 1 on the Poincaré disk model (english translation).

CONSEGNA 2.

Istruzione I. Utilizzando il software NonEuclid, individua sul disco di Poincaré due punti qualsiasi A e B distinti tra loro.

Istruzione II. Traccia una circonferenza di centro A e di raggio AB.

Istruzione III. Individua almeno tre differenti punti della circonferenza tracciata e uniscili al centro A.

Istruzione IV. Misura le lunghezze dei segmenti (raggi) tracciati.

Q-a) Come sono tra loro le lunghezze dei segmenti (raggi) tracciati?

Q-b) Come appaiono, a noi osservatori esterni, le lunghezze dei segmenti (raggi) tracciati?

Istruzione V. Controlla le lunghezze dei segmenti tracciati mentre muovi A (il centro della circonferenza tracciata) verso il centro del disco.

Q-c) Come appaiono, a noi osservatori esterni, le lunghezze dei segmenti tracciati se muovi A (centro della circonferenza tracciata) verso il centro del disco?

Frame 24. Task 2 on the Poincaré disk model (original verision).

TASK 2.

Instruction I. Using the NonEuclid software, locate any two points A and B on the Poincaré disk that are distinct from each other.

Instruction II. Draw a circle with center A and radius AB.

Instruction III. Identify at least three different points on the traced circle and join them at the centre A.

Instruction IV. Measure the lengths of the drawn segments (radii).

Q-a) Compare the lengths of the traced segments (radii) between them?

Q-b) How do the lengths of the plotted segments (radii) appear to us outside observers?

Instruction V. Check the lengths of the plotted segments as you move A (the center of the plotted circle) toward the center of the disk.

Q-c) How do the lengths of the plotted segments appear to us, outside observers, if you move A (center of the drawn circumference) toward the center of the disk?

Frame 25. Task 2 on the Poincaré disk model (english translation).

Previous deliveries on Poincaré disk model help students become familiar with Poincaré model. Many of the students' questions focus on measurements of lengths. I explain that, as outside observers, we perceive lengths in a distorted way. To make the problem clearer, I note that it is impossible to represent the Earth's surface on a plane in such a way that both lengths and angles are faithfully represented. This implies that all the cartography distorts lengths or angles.

We discuss the validity of Euclid's first four postulates and the invalidity of the fifth postulate of Euclidean geometry (both Euclid's and modern versions).

Students perform Task 4 and Task 5 shown in Frame 26-29. Then we discuss these tasks. We conclude that the sum of the interior angles of a triangle varies as the triangle varies; we conjecture that this sum is always less than 180° (conjecture confirmed by me); we observe that, as had happened on the pseudosphere, Instruction III in Task 4 leads to the construction of a simple polygonal chain (see Figure 42).

CONSEGNA 3.

Istruzione I. Utilizzando il software NonEuclid, individua sul disco di Poincaré tre punti A, B e C a due a due distinti tra loro e traccia il triangolo ABC.

Istruzione II. Considerando il triangolo ABC, riporta le misure dei suoi lati, angoli interni e somma degli angoli interni:

$AB =$ .... $BC =$ .... $AC =$ ....

$\hat{A} =$ .... $\hat{B} =$ .... $\hat{C} =$ .... $\hat{A} + \hat{B} + \hat{C} =$ ....

Istruzione III. Muovi almeno uno dei vertici di ABC in modo tale che la lunghezza dei suoi lati vari.

Istruzione IV. Considerando il "nuovo" triangolo ABC ottenuto muovendo almeno uno dei suoi vertici, riporta le misure dei suoi lati, angoli interni e somma degli angoli interni:

$AB =$ .... $BC =$ .... $AC =$ ....

$\hat{A} =$ .... $\hat{B} =$ .... $\hat{C} =$ .... $\hat{A} + \hat{B} + \hat{C} =$ ....

Q-a) Quali considerazioni puoi fare a proposito della somma degli angoli interni di un triangolo?

Frame 26. Task 3 on the Poincaré disk model (original verision).

TASK 3.

Instruction I. Using the NonEuclid software, locate on the Poincaré disk any three points *A*, *B* and *C* two by two distinct from each other and draw the triangle ABC.

Instruction II. Considering triangle ABC, report the measurements of its sides, interior angles, and sum of interior angles:

$AB =$ .... $BC =$ .... $AC =$ ....

$\hat{A} =$ .... $\hat{B} =$ .... $\hat{C} =$ .... $\hat{A} + \hat{B} + \hat{C} =$ ....

Instruction III. Move at least one of the vertices of *ABC* such that the length of its sides varies.

Instruction V. Consider the "new" triangle *ABC* obtained by moving at least one of the ends of AB, and write the measurements of the sides and angles and the sum of the interior angles:

Instruction IV. $AB =$ .... $BC =$ .... $AC =$ ....

$\hat{A} =$ .... $\hat{B} =$ .... $\hat{C} =$ .... $\hat{A} + \hat{B} + \hat{C} =$ ....

Q-a) What considerations can you make about the interior angles of an equilateral triangle?

Frame 27. Task 3 on the Poincaré disk model (original verision).

CONSEGNA 4.

Istruzione I. Ripassa la costruzione con riga e compasso che, data una retta r e un punto *P* appartenente ad essa, permette di tracciare una retta *s* perpendicolare a *r* e passante per *P*.

Q-a) La precedente costruzione dipende dal quinto postulato di Euclide?

Istruzione II. Ripeti l'Istruzione I sul disco di Poincaré.

Istruzione III. Sul disco di Poincaré, esegui la seguente costruzione:

- traccia un segmento AB, poi un segmento BC congruente e perpendicolare ad AB;
- traccia un segmento CD congruente e perpendicolare a BC e sullo stesso semipiano di AB rispetto a BC;
- traccia un segmento DE congruente e perpendicolare a DC e sullo stesso semipiano di BC rispetto a DC.

Q-b) Che cosa si può osservare a proposito dei punti A ed E?

Q-c) Che cosa si può osservare a proposito dei segmenti AB e DE?

Frame 28. Task 4 on the Poincaré disk model (original verision).

TASK 4.

Istruzione I. Review the construction with ruler and compass which, given a line *r* and a point *P* belonging to it, allows you to draw a line *s* perpendicular to *r* and passing through *P*.

Q-a) Does the previous construction depend on Euclid's fifth postulate?

Istruzione II. Repeat Instruction I on the Poincaré disk.

Istruzione III. On the Poincaré disk, perform the following construction:

– draw a segment *AB*, then a segment BC congruent and perpendicular to *AB*;

– draw a segment *CD* congruent and perpendicular to *BC* and on the same half-plane as *AB* with respect to *BC*;

– draw a segment *DE* congruent and perpendicular to *DC* and on the same half-plane as *BC* with respect to *DC*.

Q-b) What can be observed about points *A* and *E*?

Q-c) What can be observed about the segments *AB* and *DE*?

Frame 29. Task 4 on the Poincaré disk model (original verision).

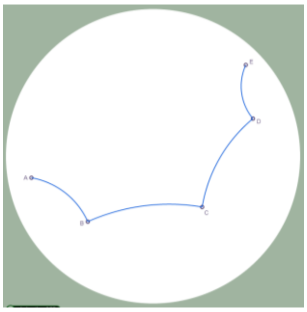


Figure 42. Poligonal drawn following the instructions given in Frame 28 (p. 154).

We also observe that two parallel straight lines do not keep the same distance all over their length (see Figure 43).

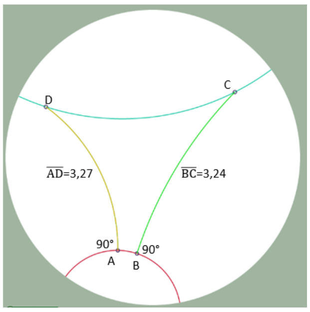


Figure 43. An illustration depicting two parallel straight lines on the Poincaré disk model and the distance from two points of one straight line to the other.

We conclude the session recapping all the activities we conducted.

5.6 Summary

In this chapter all the details about the experimental phase have been discussed. The research method has been presented in Section 5.1, then three stages have been discussed in detail: an experience with high-school teachers, a pilot study with high-school students and a second experimental round with students. Unfortunately, between the pilot study and the second round, the Covid-19 pandemics arose, requiring a large number of adjustments to the didactical method and impairing the research by reducing the number of available classes and the possibility of conducting one-to-one interviews with students after the workshops. In any case, the choices and the materials for teaching non-Euclidean geometries and evaluating the students' abilities and attention remain as a useful resource for future experiments.

6 Results

6.1 *VHL test*

I used the *VHL test* to assess the students' levels of geometric thinking according to the van Hiele theory. Moreover, item 19 of the *VHL test* – combined with items 1 and 2 of the *NEG questionnaire* and item 2 of the *BELIEFS questionnaire* – addresses the research question *RQ2* (*To what extent do students gain a new perspective on the concept of axiomatic system?).*

In the present section, I report results regarding the van Hiele levels detected by the *VHL test* before and after the class activities on non-Euclidean geometries, and I analyse students' answers to particular *VHL test* questions. In addition to providing statistical tests on the *VHL test*, I also conducted interviews with some subjects, who have been selected based on their answers to the questionnaires. I conducted the interviews about three months after the end of the course. This chapter includes excerpts from the interviews that are of interest to assess students' understanding of geometry and how students approach solving specific problems. Since, to the best of my knowledge, there are no studies similar to mine, I do not have a comparative term for the effect sizes of my interventions. Nevertheless, I will report the effect sizes because it could be useful for future studies. Indeed, like observed in (Bakker, Cai, English, Kaiser, & Mesa, 2019), findings on effect size should be related to "comparable studies with similar characteristics (research design, sample size, type of measurement, type of variable influenced, etc.)" in terms of "smaller/larger than typical under such conditions," or "comparable with other studies with similar characteristics (research design, alignment between intervention and assessment, sample size, type of variable influenced etc.)".

6.1.1 Van Hiele levels

In the present subsection I first report results that answer the following question:
Q1. How are students distributed before the class activities on non-Euclidean geometries with respect to the levels detected by the *VHL test*?
Q2. How are students distributed after the class activities on non-Euclidean geometries with respect to the levels detected by the *VHL test*?

I answer the two previous questions considering: case 1) all the 56 students who answered to all the four questionnaires involved by the experimentation, before and after the course; case 2) only the students who fit the classical van Hiele theory both in the pre-test and in the post-test; and case 3) only the students who fit the modified van Hiele theory both in the pre-test and in the post-test. Each of the previous three cases are divided in two subcases: the 3 of 5 criterion and the 4 of 5 criterion. For cases 2) and 3) I state if the differences between the post-test and the pre-test are significant and I report the effect sizes of the non-Euclidean activities on the levels detected by the *VHL test*. For case 1 I cannot report the effect size or whether the difference is significant because there are students that do not fit any van Hiele level in the pre-test or in the post-test or in both the tests.
As stated in (Usiskin Z., 1969) regarding his van Hiele test, for what concern the reliability, the *VHL test* is considered as 5-item tests. The computed Cronbach's α for the five parts in the pre-test are 0.44, 0.54, 0.56, -0.13, and 0.67, while in the post-test the computed Cronbach's α are 0.58, 0.61, 0.78, 0.52, and 0.39. I observe, as done by Usiskin, that one reason for the low reliabilities is the small number of items; similar tests at each level 20 items long would have the following Cronbach's α: 0.89, 0.91, 0.92, 0.79, and 0.94 in the pre-test, while 0.92, 0.92, 0.96, 0.90, and 0.91 in the post-test.

6.1.2 Results regarding students who answered to all the questionnaires included in the experimentation (case 1)

The graphs in Figure 44 show the students' distribution with respect to the levels detected by the *VHL test*. All the 56 students who answered to all the questionnaires included in the experimentation are included. Analysing the pre-test, I see that – according to the 3 of 5 criterion and the 4 of 5 criterion, respectively – roughly 23% and 27% of students do not fit the classical theory, while roughly 11% and 12% of students do not fit the modified theory. Analysing the post-test, I see that – according to the 3 of 5 criterion and the 4 of 5 criterion, respectively – roughly 23% and 27% of students do not fit the classical van Hiele level, while roughly the 21% and 4% of students do not fit the modified theory.

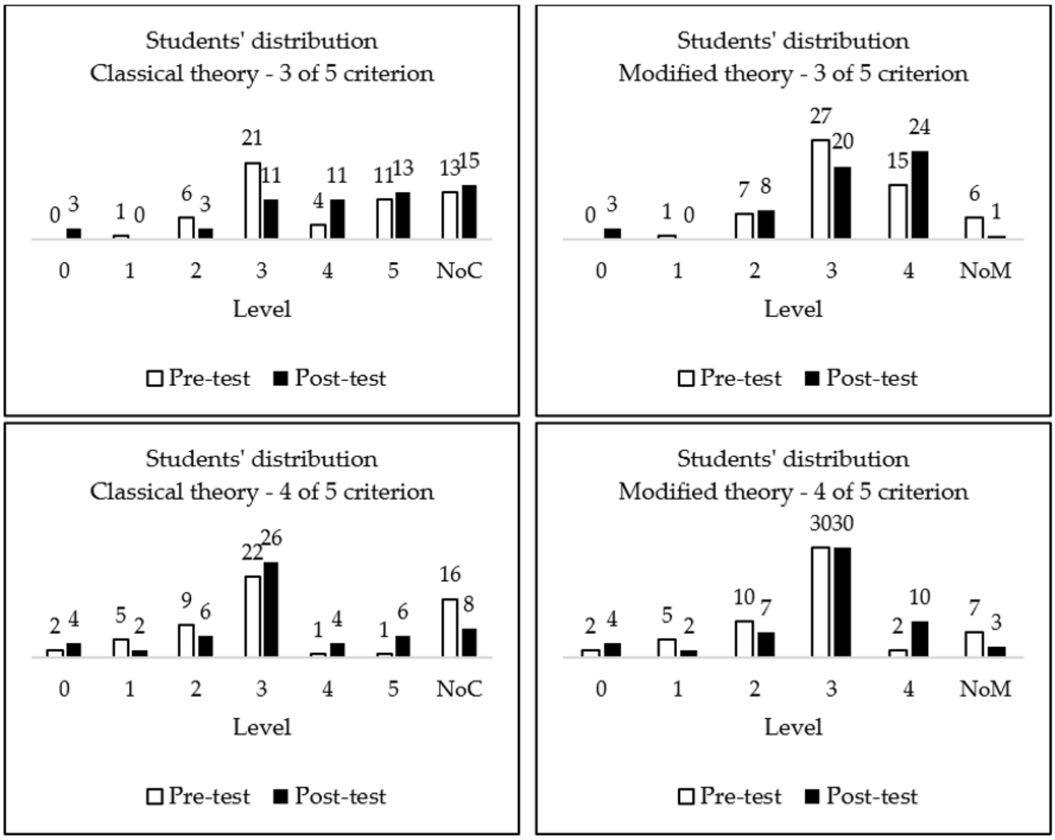


Figure 44. Distribution of the 56 students with respect to the levels detected by the van Hiele. (a) Classical theory - 3 of 5 criterion; (b) Classical theory - 4 of 5 criterion; (c) Modified theory - 3 of 5 criterion; (d) Modified theory - 4 of 5 criterion.

6.1.3 Results regarding students who fit the classical theory both in the pre-test and in the post-test (case 2)

I answer question Q1 and question Q2 written at the beginning of the present section considering the van Hiele levels of thought in geometry according to the classical theory (i.e., the level if the entire theory is considered). Please observe that here I consider only students who, according to Usiskin (Usiskin Z., 1982), fit the classical theory, both in the pre-test and in the post-test.
Table 13 shows the distributions of the students with respect to the levels detected by the *VHL test* according to the classical theory.

In both cases (3 of 5 criterion and 4 of 5 criterion), the levels for the post-test seem to be, on average, slightly higher than the ones for the pre-test but the difference is not statistically significant. Indeed, I conducted the Wilcoxon Signed-Rank test to understand whether reject the following hypothesis h_0: "the median level before and after the workshop is identical" and its results does not allow us to reject the null hypothesis with a high confidence (p-value > 0.05).

	The classical theory							
	3 of 5 criterion				4 of 5 criterion			
	Pre-test		Post-test		Pre-test		Post-test	
Level	**N. students**	**%**	**N. students**	**%**	**N. students**	**%**	**N. students**	**%**
0	0	0	2	6	2	6	4	11
1	1	3	0	0	4	11	1	3
2	5	15	2	6	8	23	3	9
3	15	45	8	24	20	57	23	66
4	2	6	9	27	0	00	1	3
5	10	30	12	36	1	03	3	9
Tot	33	100	33	100	35	100	35	100
Mean	3.45		3.76		2.43		2.71	
Std. dev.	1.18		1.35		1.01		1.25	

Table 13. Distribution of the students with respect to the levels detected by the VHL test – The classical theory.

I also calculate, for each case, the Cohen's d effect size as follow:

$$d = \frac{\overline{x_{post}} - \overline{x_{pre}}}{s},$$

where s is the pooled standard deviation:

$$s = \sqrt{\frac{(n_{post}-1)s_{post}^2 + (n_{pre}-1)s_{pre}^2}{n_{post}+n_{pre}+2}},$$

And

$$s_{pre}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{pre,i} - \bar{x}_{pre})^2, \qquad s_{post}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{post,i} - \bar{x}_{post})^2.$$

The Cohen's d computed value is 0.24 in the case of the 3 of 5 criterion, while it is 0.25 in the case of the 4 of 5 criterion.

6.1.4 Results regarding students who fit the modified theory both in the pre-test and in in the post-test (case 3)

I answer question Q1 and question Q2 written at the beginning of the present section considering the van Hiele levels of thought in geometry according to the modified theory (i.e., level 5 is removed from consideration). Please note that here I consider only students who, according to Usiskin (Usiskin Z., 1982), fit the modified theory, in both the pre-test and the post-test.

Table 14 shows the distributions of the students with respect to the levels detected by the *VHL test* according to the modified theory.

	The modified theory							
	3 of 5 criterion				4 of 5 criterion			
	Pre-test		Post-test		Pre-test		Post-test	
Level	**N. students**	**%**	**N. students**	**%**	**N. students**	**%**	**N. students**	**%**
0	0	0	2	4	2	0.04	4	9
1	1	2	0	0	4	0.09	1	2
2	7	14	7	4	9	0.20	5	11
3	26	53	16	33	29	0.63	26	57
4	15	31	24	49	2	0.04	10	22
Tot	49	100	49	100	46	1.00	46	100
Mean	3.12		3.22		2.54		2.80	
Std. dev.	0.73		0.98		0.89		1.09	

Table 14. Distribution of the students with respect to the levels detected by the VHL test – The modified theory.

For the 3 of 5 criterion, the levels for the post-test seem to be, on average, slightly higher than the one for the pre-test but the difference is not statistically significant: the Wilcoxon Signed-Rank test I conducted does not allow us to reject the null hypothesis h_0 (h_0: "the median level before and after the workshop is identical") with a high confidence as p-value < 0.05 (p-value = 0.36). The Cohen's *d effect size* I computed is 0.12.

For the 4 of 5 criterion, the level for the post-test is, on average, higher than the one for the pre-test and the Wilcoxon Signed-Rank test I conducted let us conclude the difference is statistically significant (p-value < 0.05) so I can reject the null hypothesis h_0 (h_0: "the median level before and after the workshop is identical") with a high confidence as p-value < 0.05: p-value = 0.03. In this case, the Cohen's *d effect size* is 0.26.

Since the changes in last case (modified theory - 4 of 5 criterion) are the ones with the highest effect size and since changes are statistically significant, I give some more quantitative details on it.

I show in Figure 45 the changes between the average levels resulted from the pre-test and the ones from the post-test for each set of students (II SA: 9 students; III CI: 18 students; V SA: 7 students; V SC: 12 students). Changes are positive only for those classes whose starting level is roughly 3, these classes are V SC and III CI. The Wilcoxon Signed-Rank test I conducted let us conclude the change regarding set III C is statistically significant (p-value < 0.01). Set V SA worsen its result while group II SA does not change its average. I can conclude that changes do not depend on the grade but on the starting level of thought in geometry.

Figure 46 shows in detail how many students improve, worsen, or do not change their level of thought in geometry (according to the modified theory – 4 of 5 criterion). About the 34,8% of students improves, about the 8.7% of students worsens and about the 56.5% of students does not change their level.

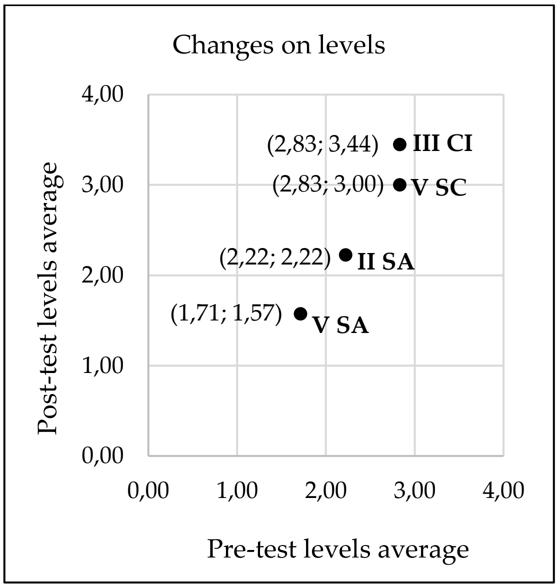


Figure 45. Changes on the average levels for each group of students.

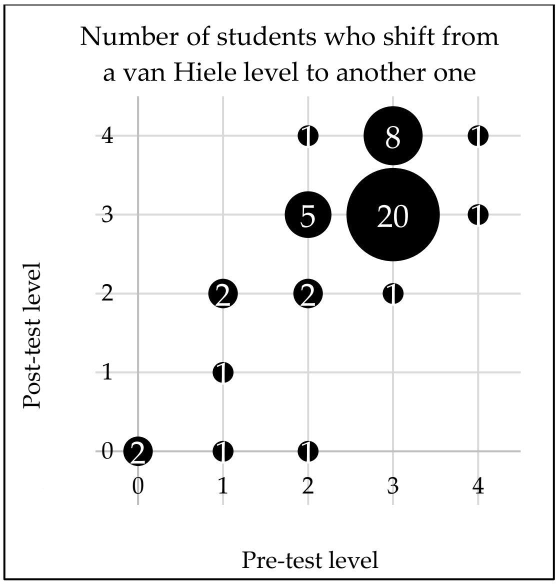
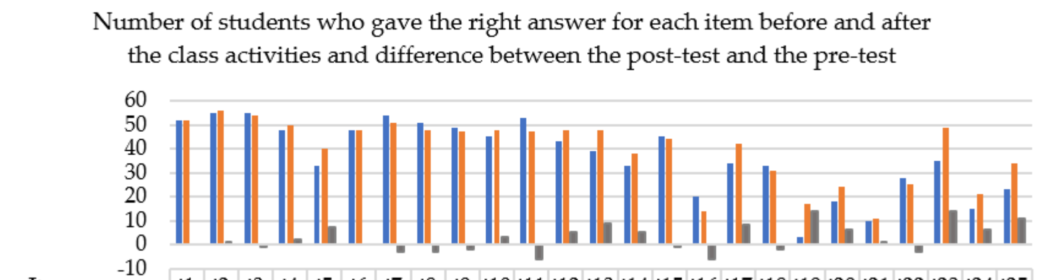


Figure 46. Number of students who shift from a van Hiele level in the pre-test to another one in the post- test (modified theory - 4 of 5 criterion).

6.1.5 Students' answers to specific *VHL test* items of interest

Figure 47 shows, for each item, the number of students who answer correctly in the pre-test, the number of students who answer correctly in the post-test and the difference between the post-test and the pre-test.



Item	i1	i2	i3	i4	i5	i6	i7	i8	i9	i10	i11	i12	i13	i14	i15	i16	i17	i18	i19	i20	i21	i22	i23	i24	i25
Pre	52	55	55	48	33	48	54	51	49	45	53	43	39	33	45	20	34	33	3	18	10	28	35	15	23
Post	52	56	54	50	40	48	51	48	47	48	47	48	48	38	44	14	42	31	17	24	11	25	49	21	34
Δ (Post - Pre)	0	1	-1	2	7	0	-3	-3	-2	3	-6	5	9	5	-1	-6	8	-2	14	6	1	-3	14	6	11

Figure 47. Number of students who gave the right answer for each item before and after the classes activities and difference between the post-test and the pre-test.

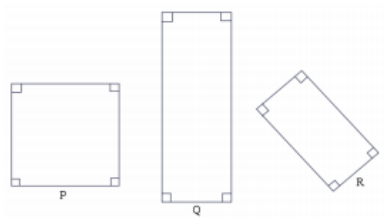
A Wilcoxon Signed-Rank test showed that statistically significant changes are the ones regarding items 13 (p-value < 0.05), 19 (p-value < 0.01), and 23 (p-value < 0.01). The Cohen's d effect size I computed for item 13 is 0.39, the one for item 19 is 0.68 while the one I computed for item 23 is 0.60. The improvements regarding item 19 and item 23 (shown in Figure 50 and in Figure 51, respectively) were predictable since the non-Euclidean course has dealt with the birth of hyperbolic geometry and with the concept of an axiomatic system. Since item 19 and item 23 deal with the need of undefined terms and assumed statements (item 19) and with inventions in a mathematical system (item 23). The fact that the improvement regarding item 13 is statistically significant is understandable because I dealt with the definition of a square during the class activities.

In the following subsections, I analyse student's answers to items 13, 19, and 23. Additionally, I discuss items 21, 22 which were expected to be affected by the course. Item 21 and 22 relate, respectively, to the ability to conduct correct logical deductions in a four-points finite geometry, the meaning of mathematical impossibility, and the meaning of definition. None of these items was correctly answered by more than half the students neither in the pre-test nor in the post-test, and the changes regarding their answers are not significant.

Item 13

Item 13 deals with squares and rectangles. I have slightly changed the wording of item 13 compared to as proposed by (Usiskin Z., 1982) by adding "in the Euclidean plane" in the question. Before the class activities, about the 30% of the students think a square is not a rectangle, while this percentage decrease to about 14% in the post-test. Figure 48 shows item 13 and, in details, how students' answers change.

I.13. *Which of these can be called rectangle in the Euclidean plane?*



1. *All can.*
2. *Q only.*
3. *R only.*
4. *P and Q only.*
5. *Q and R only.*
6. No answer

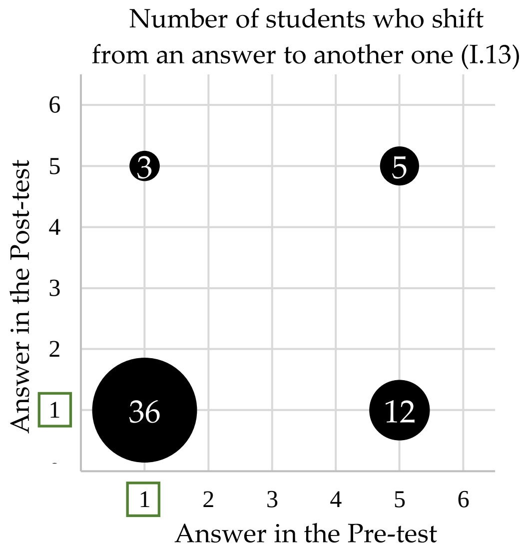


Figure 48. Item 13 and number of students who shift from an answer to another one.

I have conducted interviews with students. Table 15 shows parts of students' interviews[67].

[67] Note that I have assigned a code to each student: the first character ("S") stands for "student", the second ("F" or "M") stands for "female" or "male", the next two indicate the number I associated to the student, the last ones indicate the student's class.

Student	Interview
SF09VLS	Researcher: *Please, read item 13 and the related answers and choose the one you think correct.* SF09VLS: *First, we are on the Euclidean plane. All three have right angles. Q and R definitely. P in doubt because it is also a square. The difficulty is to understand if a square is also a rectangle. I would say no because if there is a particular name to indicate the square then this will not be a rectangle. If there is a name that describes it, I don't understand why it could be a rectangle.*
SF27IIICI	R: *Please, read item 13 and the related answers and choose the one you think correct.* SF27IIICI: *Only Q and R.* R: *How do you define the term "rectangle"?* SF27IIICI: *Actually, all three. The square also has congruent sides. The rectangle is a figure that has parallel sides in pairs, and 90° angles. In theory, also P is a rectangle but we call it a square. Nevertheless, if the square were a rectangle then it would also be a rhombus, but it cannot be that it is rhombus and rectangle at the same time.* – Student reflects – *I want to be consistent with my initial idea, therefore I select "Only Q and R".*
SM29IIICI	R: *Please, read item 13 and the related answers and choose the one you think correct.* SM29IIICI: *If I correctly remember, "All" if I'm not mistaken. Because they have all angles of 90° and – in the Euclidean plane – they can be represented as such.* R: *How do you define a rectangle?* SM29IIICI: *A quadrilateral having four right angles… with two sides… with… yes… two congruent sides and the other two congruent too. Congruent and parallel.*
SM38IIICI	R: *Please, read item 13 and the related answers and choose the one you think correct.* SM38IIICI: *Only Q and R.* R: *How do you define the term "rectangle"?* SM38IIICI: *A four-right-angle figure, with congruent opposite sides… It is a "trick question" because it could deceive the fact that one can consider the square a rectangle.*

Table 15. Interviews with students on item 13.

All the causes of error relate with the idea of "concept image" (Vinner, 2002). Vinner wrote: "A concept name when seen or when heard is a stimulus to our memory. Something is evoked by the concept name in our memory. Usually, it is not the concept definition, even in the case the concept does have a definition. It is what I call 'concept image' (Tall & Vinner, 1981) (Vinner, 1983), while others (Davis, 1984) call it 'concept frame'". The case of student SM29IIICI is an example. He explicitly refers to a representation of the rectangles ("they can be represented as such") and seems describing his concept image of rectangle in as much detail as possible.

As noted in (Antonini, 2019), a concept image is subjective, it can be very dynamic and can constantly evolve over time. While student SM29IIICI's concept image of rectangle is dynamic enough to include the square, the same cannot be said for

students SF09VLS, SM27IIICI, and SF38IIICI. They might have mistakenly extended properties of their prototype of rectangle (e.g. a non-regular quadrilateral) to all rectangles. Their prototype of rectangle might be so firm to be predominant on the definition. Here, I referred to the *Prototype Theory* formulated by Eleanor Rosch under which some of the elements belonging to the same category are more typical than others (Rosh, 1975) (Lakoff, 1987) (Langacker, 1987) (Gärdenfors, 2000). Teachers and textbooks' author can prevent students from stabilizing too rigid prototypes of a concept image, see the discussion on avoidable misconceptions ("misconcezioni evitabili") in (Santi & Sbaragli, 2007).
I cite again (Antonini, 2019) to observe that different parts of the concept image can be in contradiction with each other and can be the cause of latent conflicts that become evident only when they are evoked at the same time[68]. For example, the student SF27IIICI has a conflict on the fact that a certain figure (the square, in this case) can be categorized at the same time as rectangle and rhombus. She gives a definition of rectangle that contradicts her way to classify a figure. Indeed, according to her definition of rectangle, all the figures should be recognized as rectangles but, according to her way of classify a figure – of course, by *exclusive classification* – there are only two rectangles in the frame. It was not the aim of my interview, but it would have been profitable to take advantage of this conflict to perfect the student's knowledge.

The interviews also highlight that students have an erroneous conception of what "to define a term" means. Herbst, Gonzalez, and Macke – in an article where they "discuss how a teacher can prepare the terrain for students to understand what it means to define a figure" – observe that students do not understand why to prefer succinct definitions (to provide necessary and sufficient conditions) since, in their prior knowledge, defining a word means spelling out as much as can be said about the new word to foster understanding and proper usage. Moreover they observe that students come to high school with "a sense of familiar with geometric figures that can conspire against our desire to develop in the students the sense that definitions are needed" (Herbst, Gonzalez, & Macke, 2005). Above, I have observed that the student SM29IIICI seems describing his concept image of rectangle in as much detail as possible. Indeed, to the request to define the term

[68] Translated from the Italian *"parti diverse della concept image possono essere in contraddizione tra loro e possono essere la causa di conflitti latenti che diventano attuali solo nel momento in cui sono evocate contemporaneamente"*.

“rectangle”, he first answers “A quadrilateral having four right angles” (a right definition), but then he continues by listing other characteristics of the rectangle (congruent and parallel side).

Item 19

Item 19 (shown in Figure 49) – combined with items 1 and 2 of the *NEG questionnaire* and item 2 of the *BELIEFS questionnaire* – is addressed to the research question *RQ2* (*To what extent do students gain a new perspective on the concept of axiomatic system?*). Specifically, this item deals with the need to let some terms undefined and the necessity to have some statements which are assumed true.

I_19. In geometry: (A) Every term can be defined and every true statement can be proved true. (B) Every term can be defined but it is necessary to assume that certain statements are true. (C) Some terms must be left undefined but every true statement can be proved true. (D) Some terms must be left undefined and it is necessary to have some statements which are assumed true. (E) None of (A)-(D) is correct.

Figure 49. Item 19.

As already shown in Figure 47 (p. 163), item 19 is one of the two items with the highest difference in the number of students who answer correctly: about the 5% of the students (3 of 56) in the pre-test and about the 30% (17 of 56) in the post-test. The present item created difficulties for most of the students, both in the pre-test and in the post-test. A pilot study based on the van Hiele theory of geometric thinking at Czech secondary schools on a sample of 215 students from three types of schools (secondary general school 112, secondary technical school 55, secondary business school 48) shows item 19 was the most problematic question (Haviger & Vojkůvková, 2015). Moreover, also Usiskin felt discouraging the results relative to item 19 in his study (Usiskin Z., 1982). Figure 50 shows item 19 and, in details, how students’ answers change.

I.19. *In geometry, which of the following statements is correct:*

1. *Every term can be defined and every true statement can be proved true.*
2. *Every term can be defined but it is necessary to assume that certain statements are true.*
3. *Some terms must be left undefined but every true statement can be proved true.*
4. *Some terms must be left undefined and it is necessary to have some statements which are assumed true.*
5. *None of the above statements are correct.*
6. No answer

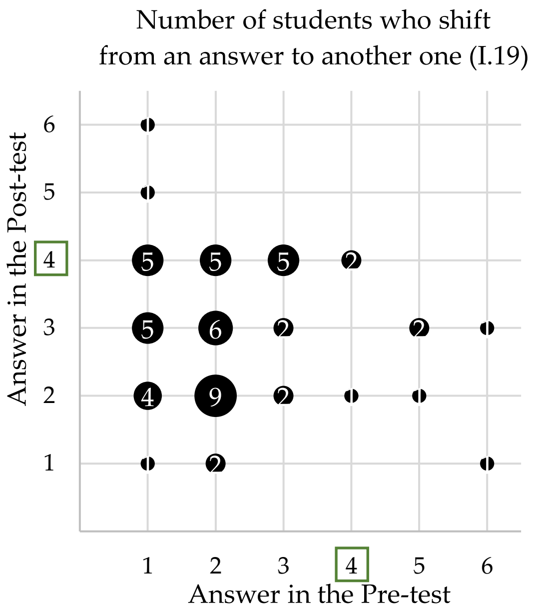


Figure 50. Item 19 and number of students who shift from an answer to another one.

15 students of the 17 that answered correctly in the post-test were wrong in the pre-test. Specifically: 5 of these because they argued that “it is possible” or that “we must be able to prove the truth of every true statement”; 5 because they argued that any “term can be defined”; 5 because they argue the both incorrect claims.

In the pre-test, about the 21% of the students answer that some terms must be left undefined, while in the final questionnaire this percentage rises to 59%. In the pre-test, about the 45% of the students answer that “it is possible” or that “we must be able to prove the truth of every true statement”, while in the final questionnaire this percentage rises to 61%.

Three students pass from correct to incorrect regarding the primitive terms (from "some term must be left undefined" to "every term can be defined"). One of them (S10VLS) is the only one that answered correctly (“Some terms must be left undefined and it is necessary to have some statements which are assumed true”) in the pre-test. I thought possible that the interpretations of primitive terms on the surface of a sphere and on the hyperbolic models may have been misunderstood. I interviewed them and my hypothesis was not confirmed since no one of them refer to the primitive terms’ interpretations on non-Euclidean geometries models. Specifically, here is how student SM10VLS answers:

R: Please, read the question and the relatives answers and choose the one you think correct.

SM10VLS: "2" ["Any term can be defined but it is necessary to assume that some sentences are true"].
R: How do you define the term "square"?
S10VLS: A particular rectangle with congruent sides. Or even a particular rhombus with right angles.
R: How do you define "side"?
SM10VLS: Considered a polygon, one side is a segment and part of a polygon [suspended tone as if to complete his sentence but he stops, he is thoughtful].
R: However, would you use the term "polygon" to define "side"?
SM10VLS: It is certainly a segment.
R: How do you define "segment"?
SM10VLS: A limited portion of the straight line.
R: How do you define "straight line"?
SM10VLS: I've always been told it's an infinite set of points. But the segment or circle is also an infinite set of points.
R: How do you define "point".
SM10VLS: The point is a convention. A choice, a necessity. It is a concept, a rule.
Then he comments that maybe it is not possible to define every term or maybe it is possible but he cannot. He remains in doubt as to whether it is possible to define each term.

Item 23

Item 23 is the second item with the highest difference in the number of students who answer correctly: 35 students (about the 63%) in the pre-test, 49 (about the 88%) in the post-test. Figure 51 shows item 23 and, in details, how students' answers change.

I interviewed the only student that answered well at the pre-test but wrong at the post-test (student SM48IISA) to understand if some concept covered during the non-Euclidean course has affected his answer. He did not remember why he selected "None of the above claim can be deduced". Moreover, during the interview, he stated the correct option. So, I can assume that his mistake was random.

I.23. *There is a geometry invented by a mathematician J in which the following is true: "The sum of the measures of the angles of a triangles is less than 180°". Which of the first four following statements can be deduced? [Mention the fifth option if you believe that none of the above statements can be deduced.]*

1. *J made a mistake in measuring the angles of the triangle.*
2. *J made a mistake in the logical reasoning.*
3. *J has a wrong idea of what is meant by "true".*
4. *J started with different assumption than those in the usual geometry.*
5. *None of the above claim can be deduced.*
6. No answer

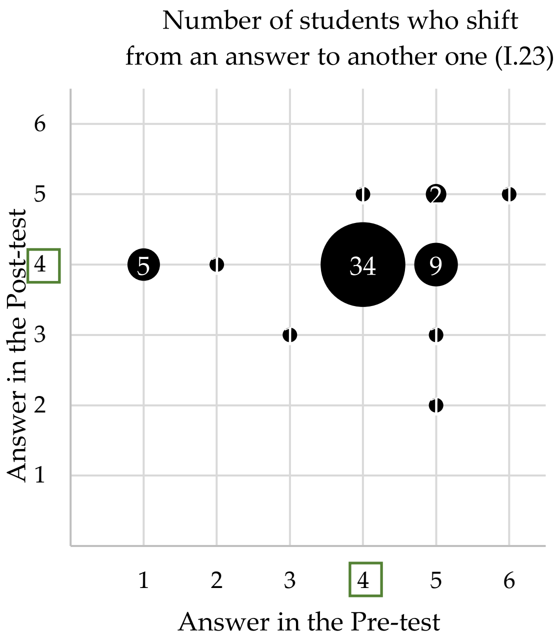


Figure 51. Item 23 and number of students who shift from an answer to another one.

Item 21

As mentioned above, I would expect that my non-Euclidean geometries course would affect answers to items 21 and 22. This motivate my special interest for these items.

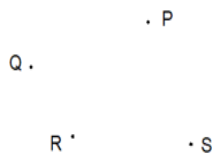
For the first time, during the course, the students involved in my study faced axiomatic systems other than the Euclidean one, and mental representations of the term "straight line" different from the one usual in Euclidean geometry. Therefore, I think it is interesting to monitor students' answer to item 21. Indeed, this item requires students to lead correct deduction in an axiomatic system never introduced before: a four-points finite geometry. Figure 52 shows item 21 and, in details, how students' answers change

I.21. *In an F-geometry, a geometry other than the Euclidean one, there exist exactly four points and six straight lines. Each straight lines contains exactly two points. If the points are P, Q, R and S, the straight lines are {P, Q}, {P, R}, {P, S}, {Q, R}, {Q, S}. Below it is written as the terms "intersect" and "parallel" are used in F-geometry.*

"The straight lines {P, Q} and {P, R} intersect in P because {P, Q} and {P, R} have the point P in common"

"The straight lines {P, Q} and {R, S} are parallel because they have no points in common".

Based on the information provided here, which of the following statements is correct?



1. *{P, R} and {Q, S} intersect.*
2. *{P, R} and {Q, S} are parallel.*
3. *{Q, R} and {R, S} are parallel.*
4. *{P, S} and {Q, R} intersect.*
5. *None of the above statements are correct.*
6. No answer

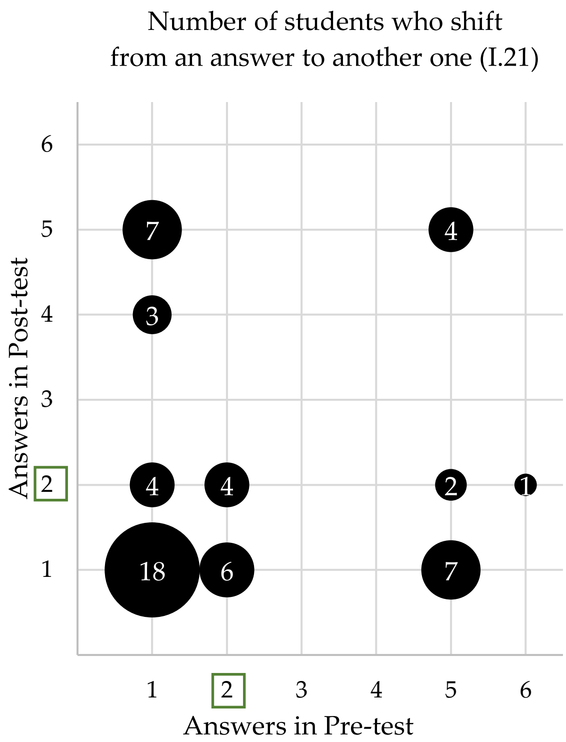


Figure 52. Item 21 and number of students who shift from an answer to another one.

About the 7% of students give the correct answer both in the pre-test and in the post-test. About the 11% of the students answer well in the pre-test but give the same wrong answer ("{P, R} and {Q, S} intersect") in the post-test. Vice versa, about the 13% of students wrong in the pre-test but answer correctly in the post-test.

Table 16 shows parts of students' interview.

Student SM10VLS selected the first option in the pre-test while gave the right answer in the post-test. During his interview, he showed a positive performance in resolving item 21, and clarified how the course made him correctly deal with this item in the four-point geometry. If the non-Euclidean geometries course positively influenced student SM10VLS, I cannot say the same for all the students which improved their results on item 21. As an example, I consider the interview to student SF34IIICI. She did not answer in the pre-test, later she selected the right option in the post-test, but she got wrong during the interview selecting the first option. During the interview, she showed a functional fixedness (Duncker, 1945) regarding the term "straight line". As Mayer remembered, Duncker used the term *functional fixedness* to refer to a situation in which a problem solver cannot think of using an object in a new function that is required to solve the problem (Mayer, 2012). In the present case, she continued to link the term "straight line" to her mental representation of straight line in Euclidean geometry, despite the item explains how to interpret this term (as a set of two points). She overcame her rigidity and resolved item 21 only after I presented her item M21 (modified item 21, Figure 53), showing successful transfer by analogy. Item M21 requires the same formal deduction of item 21. Nevertheless, student SM10VLS was able to autonomously answer only item M21. The mental representation of the term "straight line" is so rooted that it has been an obstacle to a correct formal deduction.

Student	**Interview**
SM10VLS	R: *Please, read item 21 and the related answers and choose the one you think correct.* SM10VLS: *I answer 2: {P, R} and {Q, S} are parallel.* R: *Why?* SM10VLS: *Because the course let me understand that I can detach myself from the usual concept of segment. Above all, thanks to the activity on the disk of the French mathematician... I do not remember the name.*
SF34IIICI	R: *Please, read item 21 and the related answers and choose the one you think correct.* After reading the question very carefully, she thinks about it for a long time and answers: SF34IIICI: *"1". They intersect because we have that the straight lines intersect at a point whose name we do not know.* R: Something change if I delate the diagram with the four points? SF34IIICI: No. R shows the modified item: item I.M21 (Figure 53) and asks to the students to answer. The student reads the new item and seems illuminated and amused. SF34IIICI: *Option "1" no. Option "2" yes. Option "3" no. Option "4" no. "5" no because I said "2" is true. Then, "2".* R proposes the original question again. SF34IIICI: *At this point, based on the reasoning just made, I must conclude that the correct answer is "2". Before, I had answered "1" because I had not considered that the points are only P, Q, R, and S.* R: *It seemed to me that the modified item amused you.* SF34IIICI: *Yes, because with those terms it was easier for me answering. I had associated another meaning to the word "point". From there, it was all a decline.*

Table 16. Interviews with students on item 21.

I.M21. In *a world different from our world*, there exist exactly four *qqq* and six *www*. Each *www* contains exactly two *qqq*. If the *qqq* are P, Q, R and S, the *www* are {P, Q}, {P, R}, {P, S}, {Q, R}, {Q, S}. Below it is written as the terms "*asdf* " and "*zxcv*" are used in F-geometry.

"The *www* {P, Q} and {P, R} *asdf* in P because {P, Q} and {P, R} have the *qqq* P in common"

"The *www* {P, Q} and {R, S} *zxcv* because they have no *qqq* in common".

Based on the information provided here, which of the following statements is correct?

1. {P, R} and {Q, S} *asdf*.
2. {P, R} and {Q, S} *zxcv*.
3. {Q, R} and {R, S} *zxcv*.
4. {P, S} and {Q, R} *asdf*.
5. None of the above statements are correct.
6. No answer

Figure 53. Modified item 21.

Item 22

Item 22 dealt with the meaning of mathematical impossibility. Figure 52 shows item 22 in and, in details, how students' answers change.

The percentage of correct answers decreases from 50% to about 45% of the total answers. Moreover, 69% of students who gave the right answer in the pre-test, got wrong in the post test because they selected option 4 ("It is still possible that in the future someone may find a general way to TRISECECT angles using only a compass and a NON-graduated ruler"). It is possible that this error is due to a misunderstanding that originated during the course. Indeed, a superficial understanding of the creation and evolution of non-Euclidean geometries can have the harmful consequence of questioning any certainty in mathematics and, among these, the everlasting validity of any theorem.

It is also worth mentioning an observation made by a student during his interview. Student SM43IIICI discarded option 1 ("In general, it is impossible to BISECT angles using only a compass and a NOT graduated ruler") because, in his opinion, the possibility of bisecting an angle is a necessary condition to trisect an angle (as the possibility of trisecting an angle is a necessary condition to divide an angle in four equal parts, and so on). Student SM43IIICI clearly justifies his decision explaining that if P. L. Wantzel is dealing with the proof of a trisection, it means that he has already proved that the bisection is possible.

I.22. *TRISECT an angle means to divide it into three parts of equal size. In 1847, P. L. Wantzel proved that, in general, it is impossible to trisect angles using only a compass and a NOT graduated ruler. Which of the following statements can be deduced from his proof? [Read carefully.]*

1. *In general, it is impossible to BISECT angles using only a compass and a NOT graduated ruler.*
2. *In general, it is impossible to TRISECECT angles using only a compass and a graduated ruler.*
3. *In general, it is impossible to TRISECT angles using any drawing tool.*
4. *It is still possible that in the future someone may find a general way to TRISECECT angles using only a compass and a NON-graduated ruler.*
5. *None of the above claim can be deduced.*
6. No answer

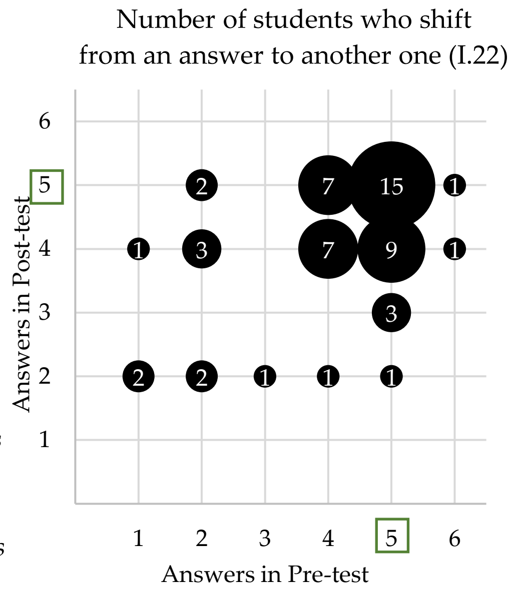


Figure 54. Item 22 and number of students who shift from an answer to another one.

6.1.6 Discussion

In the previous sections I reported results from an experimental evaluation of the impact of the non-Euclidean geometries course for students of different classes and starting with different knowledge levels. I detected changes analysing the *VHL test* filled out by the students before and after the non-Euclidean geometries course. I can observe (Figure 45, p. 162), that changes do not depend on the grade whereas they depend on the starting level of thought in geometry. This should let us conclude that the class activities on non-Euclidean geometries – at least in the way I designed them – should be conducted after having tested students' level of thought in geometry. Non-Euclidean geometries seminars in Italy are often conducted with V grade students (the last grade before university), assuming that their level of thought is high enough to learn non-Euclidean geometries. However, this seems to be not necessarily the case. On the other hand, I have seen that the students from III grade class involved in my study have sufficient abstraction level to learn basic concepts of non-Euclidean geometries and correctly express concepts of axiomatic geometry.

In the previous sections I also analysed the results for items in which students shown statistically significant changes between the pre-test and the post-test, and for items of special interest since I would expect that the non-Euclidean geometries course would affect answers to these items. I reported in detail relevant interviews with students that highlight some of their beliefs.

6.2 *NEG questionnaire*

The *NEG questionnaire* addresses to the research question *RQ3* (*How well do students learn the taught concepts of non-Euclidean geometries?*) and – combined with the *PROOF questionnaire* – to *RQ4* (*To what extent do students' critical thinking and proof skills improve over the duration of the course?*). Moreover, items 1 and 2 of the *NEG questionnaire* – combined with item 19 of the *VHL test* and item 2 of the *BELIEFS questionnaire* – addresses to the research question *RQ2* (*To what extent do students gain a new perspective on the concept of axiomatic system?*).

In the present section I present in detail the data obtained by analysing the students' answers to the *NEG questionnaires* (shown in Appendix 4) and I compare the students' answers given in the *pre-NEG questionnaire* with the ones given in the *post-NEG questionnaire*.

Since, to the best of my knowledge, there are no studies similar to mine, I do not have a comparative term for the effect sizes of my interventions. Nevertheless, I will report the effect sizes because it could be useful for future studies.

Item 1 and 2

Figure 55 shows the students' answers to questions 1 and 2 (shown in Frame 7, p. 104). I divided their answers in six groups:

- The students who answer correctly: in item 1 they select "No", and correctly justify their selection, moreover they select "Consistency" in item

2. I consider this group as the one who have a modern view of the concept of axiom.

- The students who correctly select "No" and "Consistency, but who does not justify.
- The students who correctly select "No" and "Consistency", but who give a wrong justification. Table 17 shows the wrong or not clear justifications given by the students that selected "No" in item 1 and "Consistency" in item 2. Observe that some students seem to have misunderstood the question, indeed they discuss why an axiom has not to be proved (see SM50IISA, SM25IIICI, and SM29IIICI' answers).
- The students who correctly select "No" in item 1, but do not select "Consistency" in item 2.
- The students who select "Yes" in item 1.
- The students who select "I cannot answer" in item 1.
- The students who select "Other" in item 1.

We can conclude that the number of students who have a modern view of the concept of *axiom* rises after the NEG course. Indeed, it goes from 0 in the pre-questionnaire to 7 (12,5%) in the post-questionnaire. Of these 7 students, 2 belongs to *set 1 - II SA class* (14,3% of student in set 1), 3 to *set 2 - III C*I class (15% of student in set 2), 1 to *set 3 - V SA class* (12,5% of student in set 3), and 1 to *set 4 - V SC class* (7,1% of student in set 4).
At the same time, the number of students who believe that an axiom must be self-evident is almost constant: 22 students (39,3%) in pre-questionnaire, 24 (42,9%) in the post-questionnaire. The biggest change is in the number of students who have no idea whether an axiom should be self-evident: 24 (42,9%) in the pre-questionnaire, 2 (3,6%) in the post-questionnaire.

		Wrong or no clear justification	
	Students	**Original answer**	**Answer's English translation**
Pre-questionnaire	SM21VSA	*Non c'è bisogno che sia evidente, basta che si riveli corretto e pertinente*	It does not need to be evident, as long as it is correct and relevant
	SM50IISA	*Non deve essere dimostrato perché è già un fondamento della teoria e non servono dunque dimostrazioni*	It does not have to be proved because it is already a foundation of the theory and therefore no proofs are needed
Post-questionnaire	SM19VSA	*Un assioma non necessariamente deve essere evidente, l'importante è che la sua definizione si verifichi sempre*	An axiom does not necessarily have to be evident, the important thing is that its definition always occurs
	SM21VSC	*Può anche non essere evidente, l'evidenza non è una caratteristica principale per definire se un enunciato è corretto o meno*	It may not even be evident, the evidence is not a main characteristic for defining whether a statement is correct or not
	SM22VSC	*L'assioma non è evidente in tutte le geometrie*	The axiom is not evident in all geometries
	SM25IIICI	*Perché può essere comunque vero senza essere dimostrabile*	Because it can still be true without being provable
	SM26IIICI	*Non ha bisogno di essere dimostrato*	It doesn't need to be proven
	SM29IIICI	*Perché l'assioma non ha bisogno di essere dimostrato*	Because the axiom does not need to be proved
	SM39IIICI	*Il concetto di evidenza in sé è molto vago e in un modo o nell'altro soggettivo*	The concept of evidence itself is very vague and, in one way or another, subjective

Table 17. Wrong or no clear justifications given by the students that selected "No" in item 1 and "Consistency" in item 2.

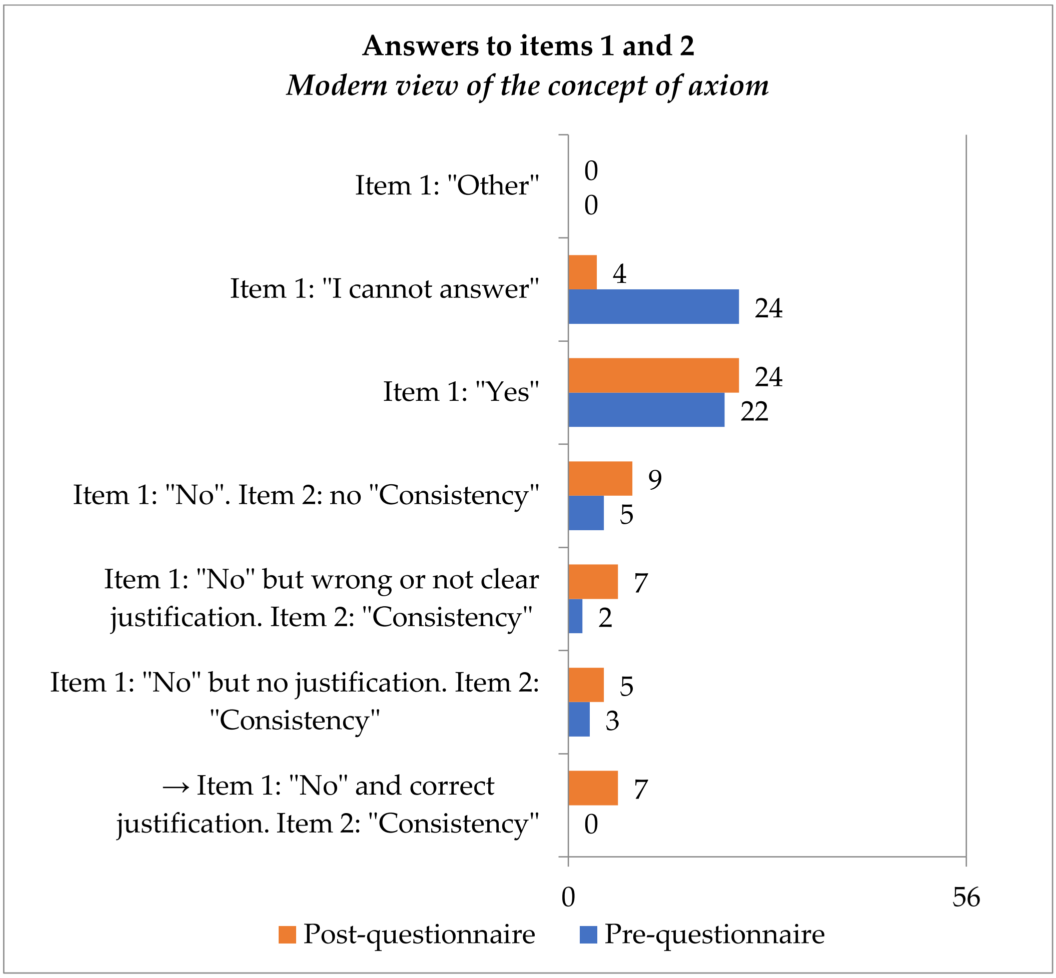


Figure 55. Students' answers to item 1 and 2 of the NEG questionnaire.

Items from 3 to 7, sub-items a and b

If we denote by X the 56x1 vector composed by the score assigned to each student for her/his answer to question 7 in the post-questionnaire, and by Y the 56x1 vector composed by the average of the score assigned to each student for her/his answer to questions from 3 to 6, the Pearson correlation coefficient performed between X and Y is 0.51. This is positive despite no laboratory activities were carried out on the proposition object of question 7, unlike what concern proposition in item from 3 to 6. Moreover, answering question 7 requires greater attention. Indeed, students also must draw an appropriate graph before answering. I give more details discussing item 7 in Appendix 2.

For what concern the reliability of the questionnaire, if we consider the four items from 3 to 6, the computed Cronbach's α is 0.85.

Table 18 shows the average and the standard deviation for all sets of students, before and after the course. If we consider all the 56 students, the average in the pre-questionnaire is 0.24 and the standard deviation is 0.08, while they are respectively 0.77 and 0.28 in the post-questionnaire.

	Average Score		**Standard deviation**	
	Pre	***Post***	**Pre**	**Post**
All students	*0.24*	*0.77*	0.08	0.28
Set 1 (II SA)	*0.20*	*0.62*	0.06	0.30
Set 2 (III CI)	*0.28*	*0.93*	0.10	0.11
Set 3 (V SA)	*0.23*	*0.52*	0.03	0.36
Set 4 (V SC)	*0.24*	*0.85*	0.02	0.20

Table 18. Average score and standard deviation for each set of students.

To understand if there is a significant difference between the two groups of scores (post-pre), I computed the Wilcoxon Signed-Rank test because the prerequisites for a dependent samples t-test are not met. Indeed, the sample data is not symmetric around the average. Moreover, the Wilcoxon Signed-Rank test is robust to the presence of outliers. The Wilcoxon Signed-Rank test I conducted let me conclude that the difference is statistically significant (p-value < 0.01) so it is possible to reject the null hypothesis h_0 (h_0: "the median after the non-Euclidean course is identical") with a high confidence since p-value < 0.01: p-value = 0.00164.

I also calculate the Cohen's d effect size as shown in Section 6.1.3. The observed effect size d is 2,62. This indicate that the result is strong.
Figure 56 shows, for each student, her/his average score in the pre-questionnaire and the one in the post-questionnaire. For all but two students, the average score in the post-questionnaire is higher than the one in the pre-questionnaire.

Figure 57 and Figure 58 show, for each set of students, the average scores calculated considering items from 3 to 6 respectively before and after the class activities. The 36% of students, after the NEG course, correctly answer to all questions and give correct examples and counterexamples. Set 2 (III CI class) is the one that reach better results both before and after the course. The 60% of their students correctly answer to all questions and give correct examples and counterexamples.

We can conclude that:

1. All the students but two (SM23IIICI; SM39IIICI) have no knowledge of non-Euclidean geometries before the NEG course.
2. The two students who have knowledge, before the course, of the existence of non-Euclidean geometries are just aware of the existence of these geometries. Indeed, they wrong some answers and some justifications in the pre-questionnaire. The same students improve their understanding of non-Euclidean geometries: they always select the correct option in the post-questionnaire. Nevertheless, they give at least one not completely correct justification to their selections.
3. The 36% of students, after the NEG course, correctly understand that the propositions I give in the questionnaire are assumptions or merely consequences of a specific axiomatic system (the Euclidean one) and not necessarily true proposition. These students correctly answer to all questions and give correct examples and counterexamples in the post-questionnaire.
4. Set 2 (III CI class) is the one that reach better results both before and after the course. The 60% of their students correctly answer to all questions and give correct examples and counterexamples.

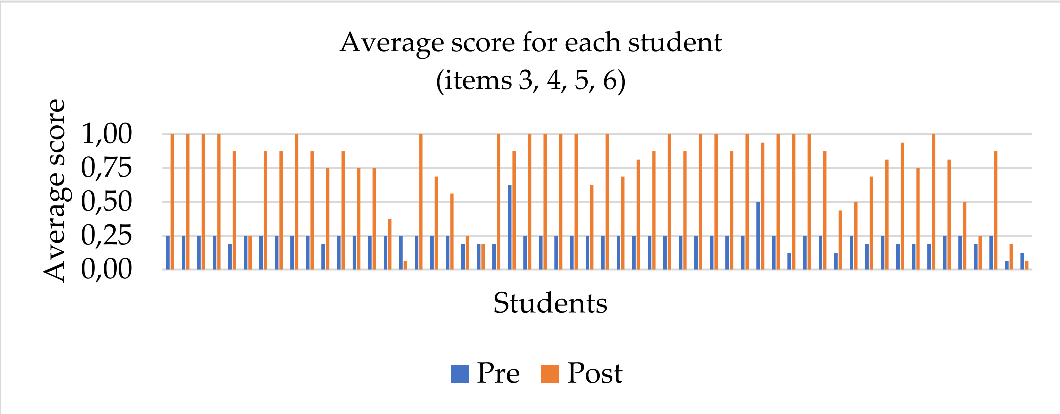


Figure 56. Average score for each student (items 3, 4, 5, 6)

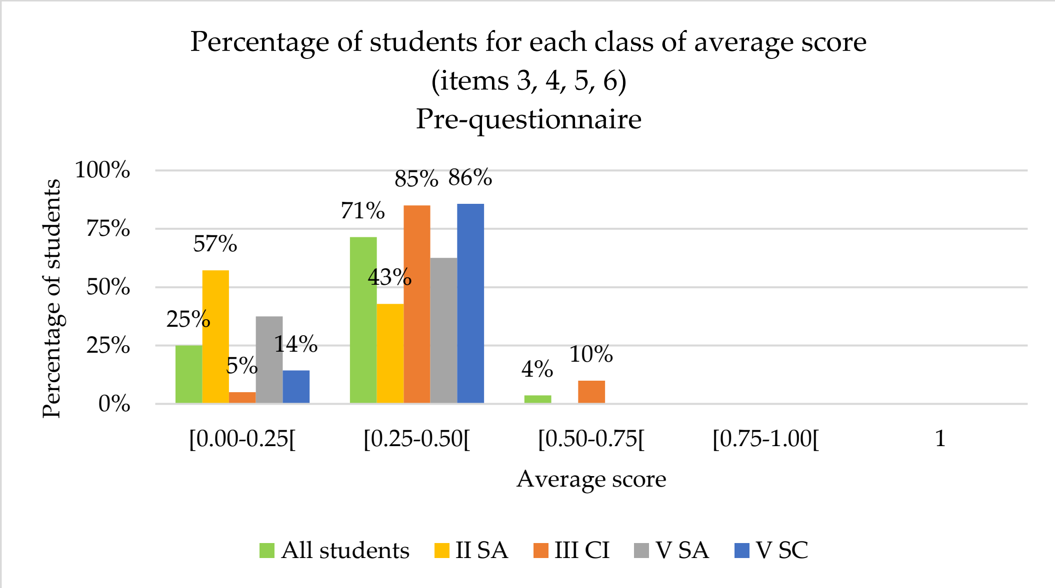


Figure 57. Percentage of students for each class of average score (items 3, 4, 5, 6) - Pre-questionnaire.

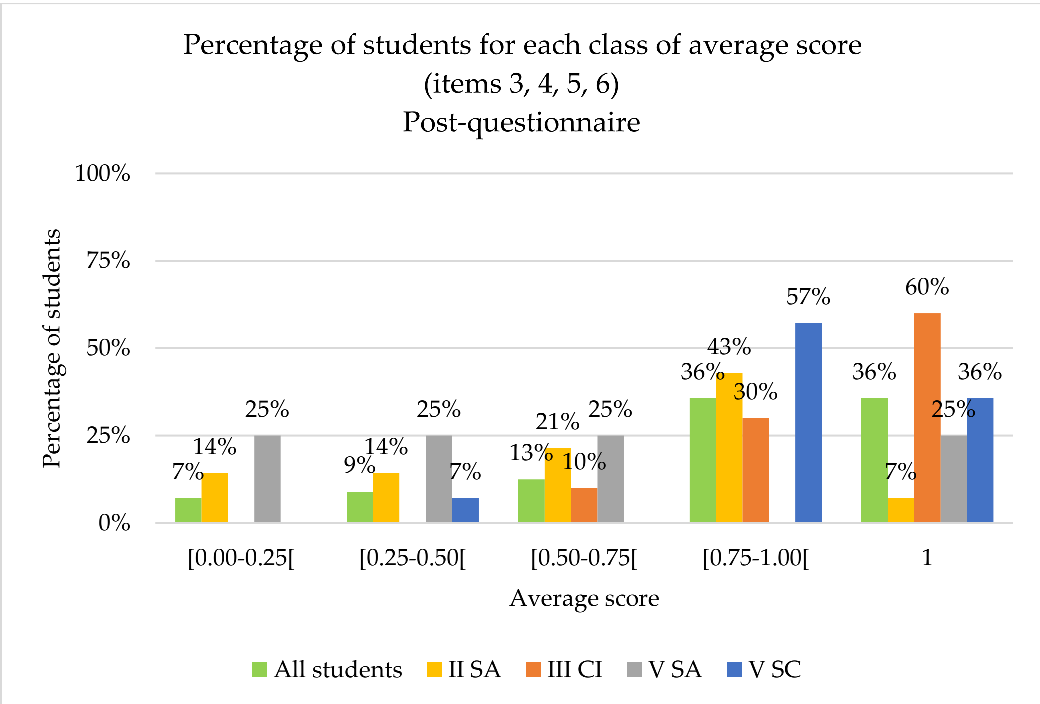


Figure 58. Percentage of students for each class of average score (items 3, 4, 5, 6) - Post-questionnaire.

I give more details on each item from 3 to 7 in Appendix 2.

Item 3, sub-items c, d, and e

Table 24 (shown in Appendix 2) shows data related to students who, in sub-item 3.c of the pre-questionnaire, selected that it is not possible to state the negation of the fifth Euclidean postulate. Specifically, this table shows how students justified their selection and their answers to sub-items 3.c-d/e of the post-questionnaire. By their justifications I understand that not all students gave question 3.c the same interpretation as it was intended by me. Some students answer that it is not possible to state the negation because it would be false. They interpret "Is it possible to state […]" (in Italian: *"E' possible enunciare […]?"*) as "Is it true […]" (in Italian: *"E' vera […]?")*. Follow some student's justifications:

- SM43IISA: *"[It is not possible to state the negation of the fifth Euclidean postulate] because in order to negate a statement, a geometric case must be found in which the opposite is proved"*.
- SF41IIICI: *"Because denying the statement it would become: given a straight line and a point P external to r there is no single line passing through P and parallel to r and this implies that there can be more parallel straight lines passing through P"*.
- SF18VSA: *"Because the statement at the end would be false"* (note that she is assuming that a statement cannot be false).

Noting this misunderstanding leads me to say that the question would not be phrased as I did in the second study. It was not useful to discern students who do not think possible to express the negation of a postulate. Moreover, it also did not allow me to see how some students would deny the given statement. Nevertheless, in Figure 73 and in Figure 74 (shown in Appendix 2) I show students' answers respectively to question 3.c. and to question 3.d.

What is possible to say for sure is that students that give a negation in the pre-questionnaire, improve their ability to correctly deny the given sentence (see Table 4 in Section 5.3.5.2).

6.2.1 Discussion

The first two items are related to the concept of axiom. The number of students who seem to gain a modern view of the concept of *axiom* rises after the NEG course.

Items 3-7(a, b) are related to notions of non-Euclidean geometries. Before the course, almost all students have a Euclidean view. This can be seen from their score

(see Figure 56), which is 0.25 for almost all students (please note that an answer given according to a Euclidean point of view was assigned a score of 0.25). After the course, the average score increases significantly, reaching 0.77 on average, with more than one third of the entire population reaching a score in the highest quartile. These students correctly understand that the propositions I give in the questionnaire are assumptions or merely consequences of a specific axiomatic system (the Euclidean one) and not necessarily true proposition. These students correctly answer to all questions, motivate their point of view, and give correct examples and counterexamples.
Set 2 (III CI class) is the one that reaches the best results both before and after the course. 60% of their students correctly answer to all questions and give correct examples and counterexamples. If we consider the scores [0.75:1], 90% of the students of Class III CI and 93% of class V SC are in this category.

In conclusion, an introductory course on non-Euclidean geometries can provide students for a deeper understanding of the nature of axiomatic system in geometry.

6.3 *PROOF questionnaire*

The *PROOF questionnaire* – combined with the *NEG questionnaire* – addresses to the research question *RQ4* (*To what extent do students' critical thinking and proof skills improve over the duration of the course?*).

In this section – divided in four subsections – we analyse the students' answers in the *PROOF questionnaire*. The first subsection is on students' answers to items 1.a and 2.a of the pre-questionnaire and of the post-questionnaire; the third one on the answers to items 1.b-c and 2.b-c of the pre-questionnaire and of the post-questionnaire.

Items 1.a and 2.a

Figure 59 shows students' answers to questions 1.a and 2.a of the pre-questionnaire. Three of the five options have been selected the same times in question 1.a and in question 2.a. Moreover, if we denote by X the 5x1 vector composed by the number of students x_i who selected option *i* (i= 1, …, 5) in item 1.a, and by Y the 5x1 vector composed by the number of students y_i who selected option *i* (i= 1, …, 5) in item 2.a, the Pearson correlation coefficients performed between *X* and *Y* is 0.96. The Pearson correlation coefficient performed is positive and very high. Nevertheless, we cannot conclude that students were for the most part coherent answering the two questions. Indeed, from Figure 60 we understand that only 23 students (41.07%) answer coherently, selecting the same option in the two questions. Only 8 students (14.29%) give the right answer to the both questions. Yet, one of these students (SM25IIICI[69]) is not coherent answering the next item 2.b. Indeed, he selects the option "*The proof is wrong because there are surfaces without parallel lines*".

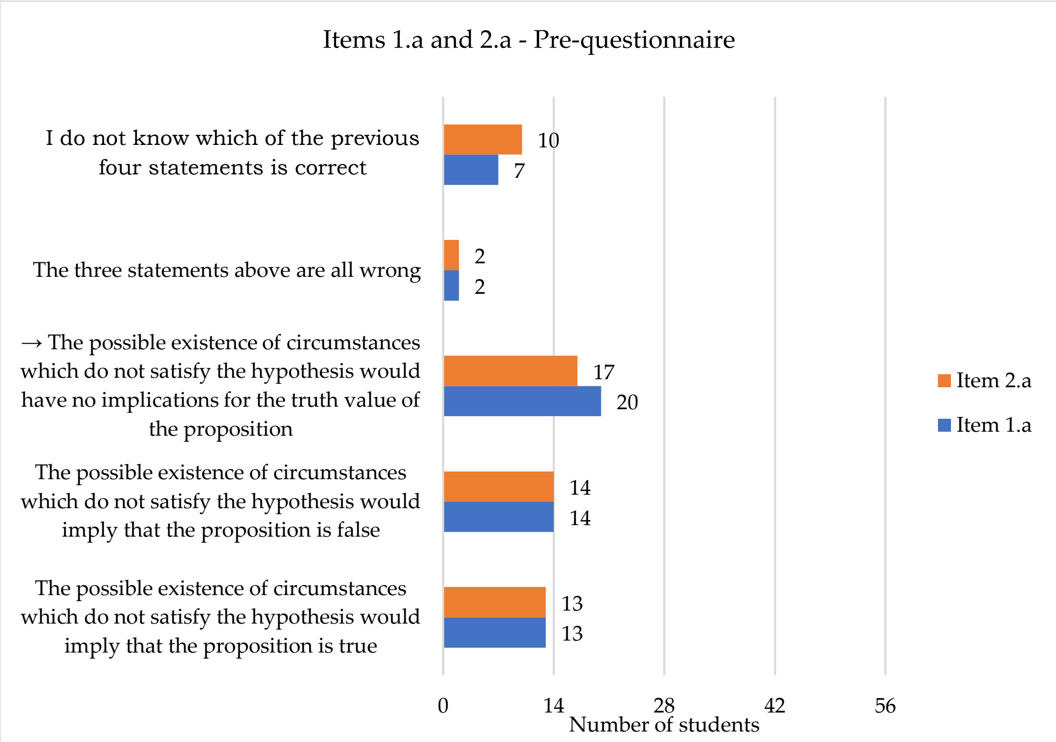


Figure 59. Student's answer to items 1.a and 2.a of the pre-questionnaire.

[69] For the sake of anonymity, the students are indicated by a alphanumeric code.

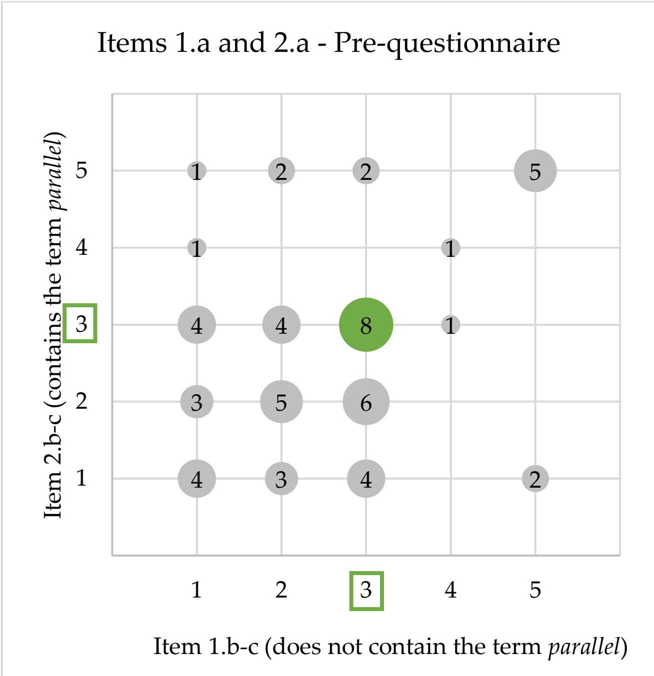


Figure 60. Student's answer to items 1.a and 2.a of the pre-questionnaire (more details). (1: The possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is true. 2: The possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is false. 3: The possible existence of circumstances which do not satisfy the hypothesis would have no implications for the truth value of the proposition. 4: The three statements above are all wrong. 5: I do not know which of the previous statements is correct.)

Figure 61 shows students' answers to questions 1.a and 2.a of the post-questionnaire. The number of students who answer well to question 1.a is 18 (32.14%), while 27 (48.21%) is the number of students who answer well to question 2.a (the one that contains the term "parallel" in the statement). The term "parallel" in item 2.a could have raised their attention. Moreover, the fact that question 2.a deal with a property already discussed in *NEG questionnaire*, could have helped them.

Figure 62 gives more details on students' answers to questions 1.a and 2.a of the post-questionnaire. It shows that only 21 students (37.50%) answers coherently, selecting the same option in the two questions. 16 students (28.57%) correctly answer question 2.a (the one that contains the term "parallel") but get wrong selecting option 2 ("The possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is false"), in item 1.a. The term "parallel" that appear in item 2.a could have alerted the students and helped them in answering.

Only 9 students (16.07%) give the right answer to the both questions. I add that none of these give the right answer to the both questions of the pre-questionnaire. It is worth observing that the number of students who coherently select the second option (the possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is false) in both the questions doubles from the pre-questionnaire to the post questionnaire (from 5 to 10).

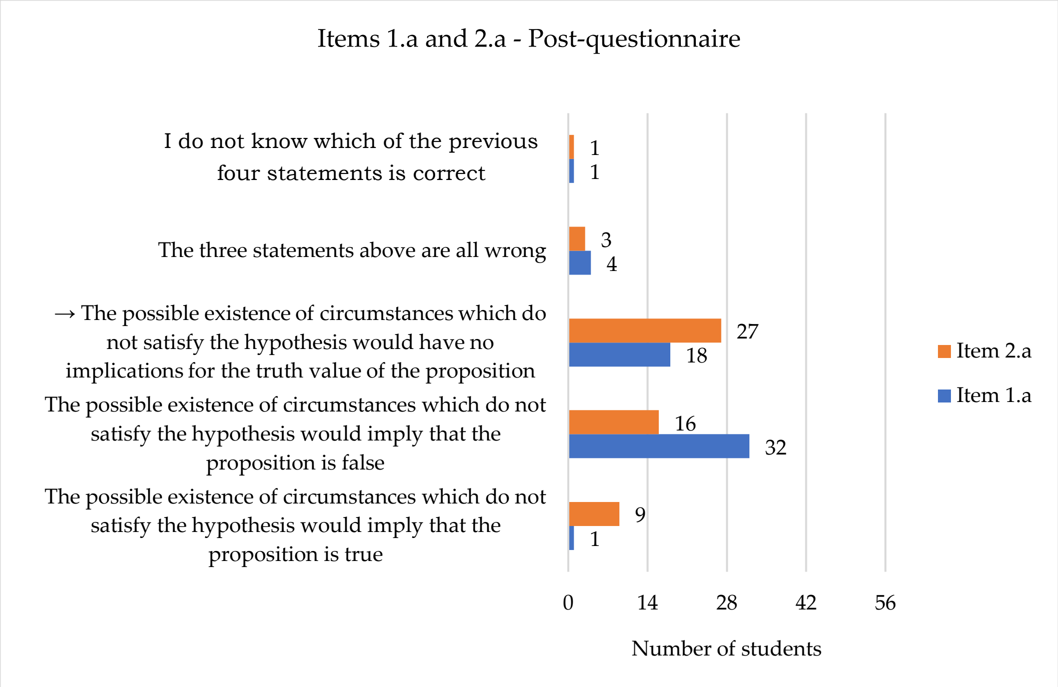


Figure 61. Student's answer to items 1.a and 2.a of the post-questionnaire.

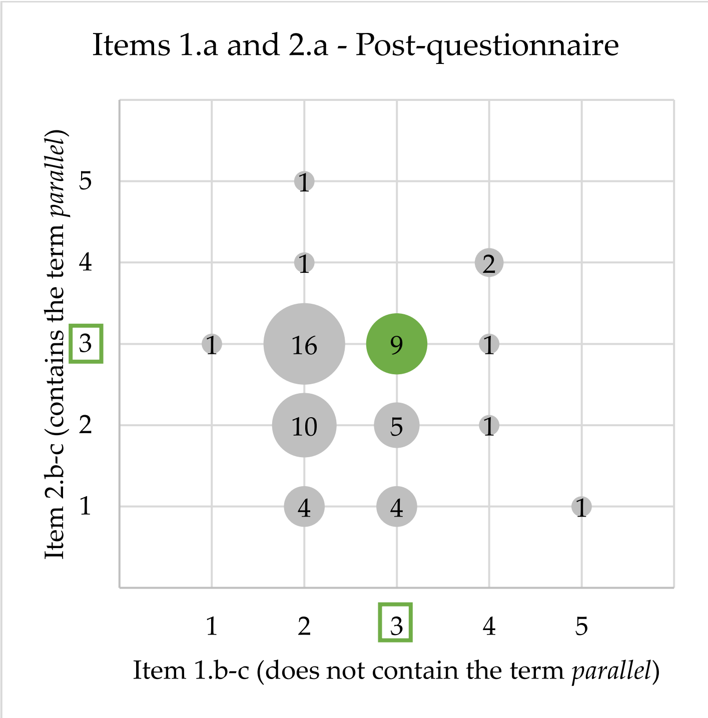


Figure 62. Student's answer to items 1.a and 2.a of the post-questionnaire (more details). (1: The possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is true;. 2: The possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is false. 3: The possible existence of circumstances which do not satisfy the hypothesis would have no implications for the truth value of the proposition. 4: The three statements above are all wrong. 5: I donot know which of the previous four statements is correct.)

Items 1.b-c and 2.b-c

Figure 63 shows students' answers to questions 1.b-c and 2.b-c of the pre-questionnaire. None of the students answer correctly, not even one of the two questions. Figure 64 gives more details on students' answers. Specifically, it shows that 25 students (44.64%) answer to both questions that the proof is correct. This was predictable because the *NEG questionnaire* has revealed that none of the students had dealt with geometry different from the Euclidean one, two students are only aware of the existence of non-Euclidean geometries. These two students are wrong in their answer to question 2.b-c since they both select that the proof is not correct because there exist surfaces without parallel straight lines (option 3).

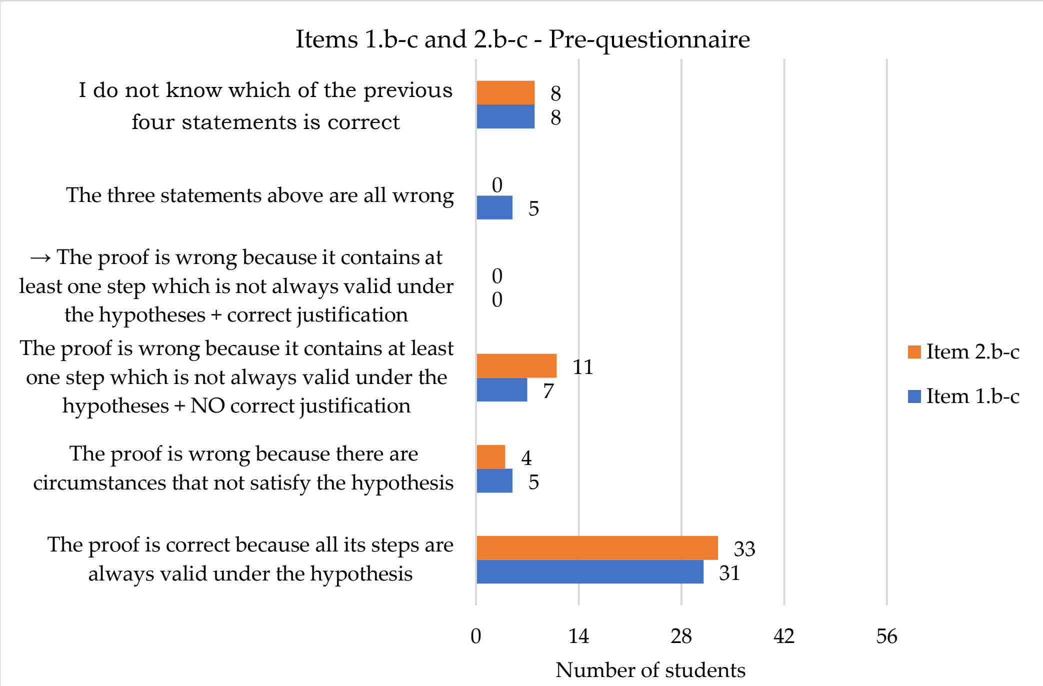


Figure 63. Student's answer to items 1.b-c and 2.b-c of the pre-questionnaire.

Figure 65 shows student's answer to items 1.b-c and 2.b-c of the post-questionnaire. The most relevant data are the following:

a) the number of students who answer as if they still have a Euclidean view of geometry: 21 (37.5%) in item 1; 14 (25%) in item 2.

b) The number of students who think that the proof is wrong because there exist circumstances that do not satisfy the hypothesis: 11 (19.6%) in item 1; 16 (28.6%) in item 2.

c) The number of students who give a completely correct answer (right selection in sub-items *b* and right justification in sub-items *c*): 4 (7.1%) in item 1; 9 (16.1%) in item 2. I suppose that the best performances on item 2 depend on the fact that the wrong step in the "proof" was already discussed in the previous *NEG questionnaire*. Moreover, in item 2, the term "parallel" appear both in the statement and in the potential proof. It could have alerted the students and helped them in answering.

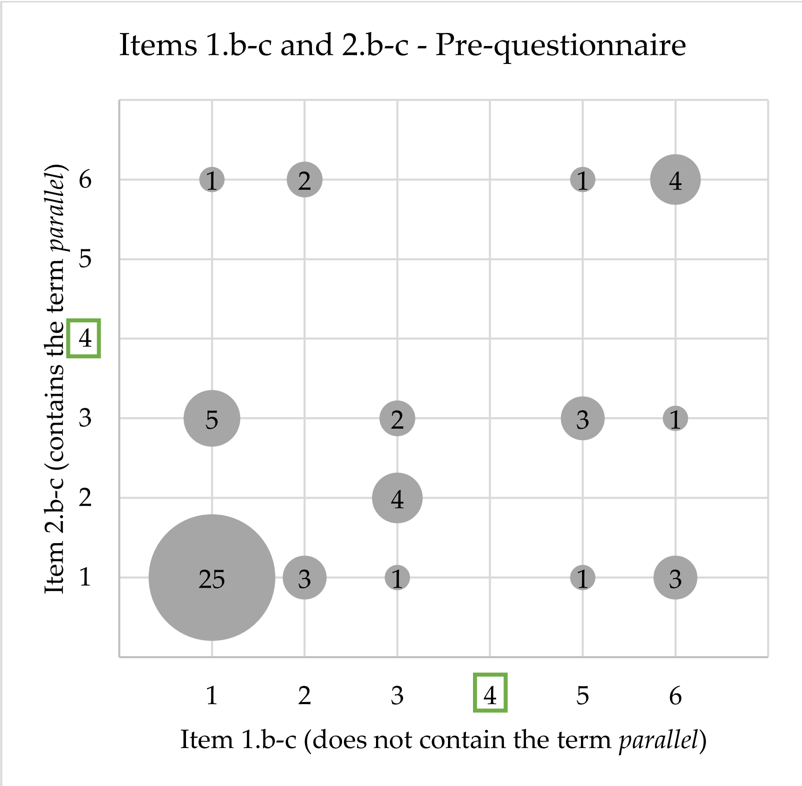


Figure 64. Student's answer to items 1.b-c and 2.b-c of the pre-questionnaire (more details). (1: The proof is correct because all its steps are always valid under the hypotesis. 2: The proof is wrong because there are circumstances that not satisfy the hypothesis. 3: The proof is wrong because it contains at least one step which is not always valid under the hypotheses + NO correct justification. 4: The proof is wrong because it contains at least one step which is not always valid under the hypotheses + correct justification. 5: The three statements above are all wrong. 6: I do not know which of the previous four statements is correct.)

By the more detailed Figure 66, we understand that:

a) 6 students (10.71%) answer both the questions as if they still have a Euclidean view of geometry.
b) 10 students (17,86%) select in both the questions that the proof is wrong because there exist circumstances that do not satisfy the hypothesis. It is worth observing that this case corresponds to the largest bubble of the chart.
c) 4 students (7.14 %) give a completely correct answer: a student of Set 4 (V SC), and three students of Set 2 (III CI). Only 2 of these students answer well to all the questions of the post-questionnaire: a student of Set 4 and a student of Set 2.

The previous point b) has an aspect in common with what we have observed analazyng questions 1.a and 2.a: we observed, after the course, a doubling of the number of students who answer to both questions that the possible existence of circumstances which do not satisfy the hypothesis would imply that the proposition is false (from 5 to 10).

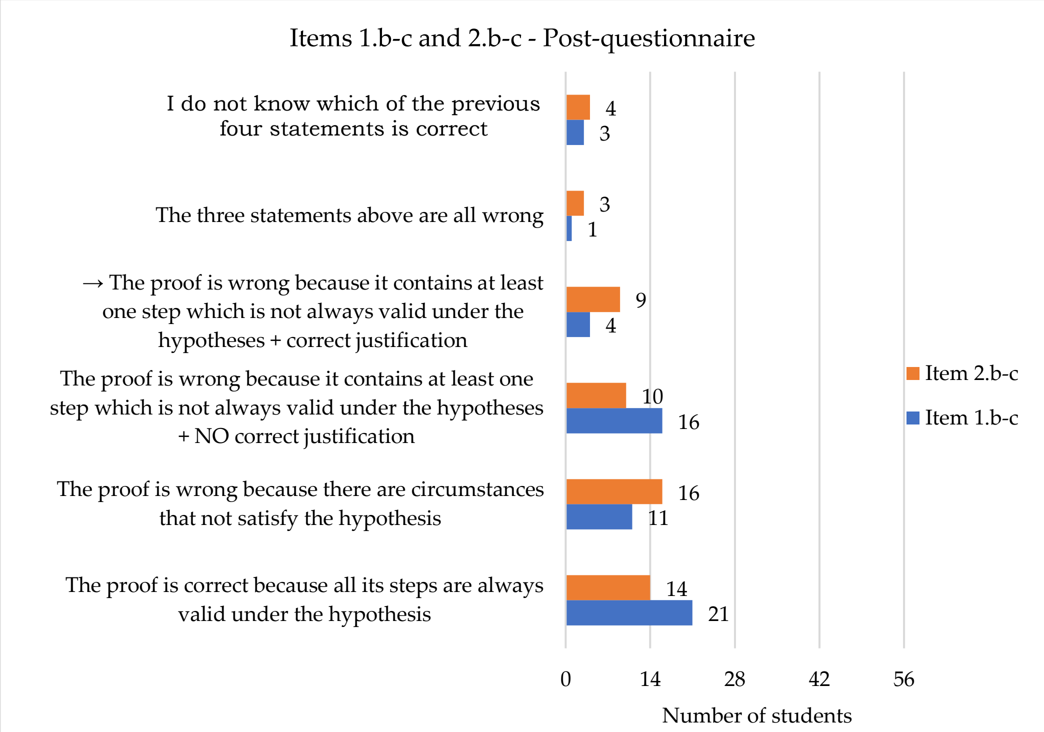


Figure 65. Student's answer to items 1.b-c and 2.b-c of the post-questionnaire.

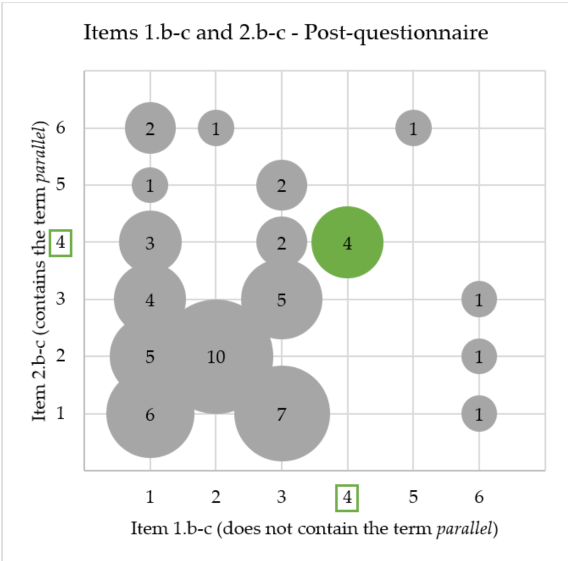


Figure 66. Student's answer to items 1.b-c and 2.b-c of the post-questionnaire (more details). (1: The proof is correct because all its steps are always valid under the hypotesis. 2: The proof is wrong because there are circumstances that not satisfy the hypothesis. 3: The proof is wrong because it contains at least one step which is not always valid under the hypotheses + NO correct justification. 4: The proof is wrong because it contains at least one step which is not always valid under the hypotheses + correct justification. 5: The three statements above are all wrong. 6: I do not know which of the previous four statements is correct.)

6.3.1 Discussion

The pilot study let us suppose that a large part of the students is unaware of the following concept: the existence of circumstances that does not satisfy the hypothesis of a proposition has no implications for the truth value of the proposition. This concept is the object of this questionnaire. Students' answers to the *PROOF questionnaires* confirm my supposition. Most of the students do not have the logical skills necessary to build counterexamples when they are needed. They seem to miss a fundamental concept, i.e. that a counterexample to a statement is an example that meets the statement's hypothesis but not the statement's thesis. I compared the number of students who, before and after the course, seem to be aware of the aforementioned concept: it is almost the same, 8 in the pre-questionnaire and 9 in the post-questionnaire. Moreover, none of these students gave the right answers in both questions (pre-questionnaire and post-questionnaire). Therefore, some students seem to improve their knowledge, but a similar number of students seem to worsen. Specifically, after the course, there was a doubling of the number of students who *think* that the possible existence of circumstances which do not satisfy the hypothesis of a proposition would imply that the proposition is false (from 5 to 10).

We remember that 20 students (36%), after the NEG course, correctly understand that the propositions I give in the *NEG questionnaire* are assumptions or merely consequences of a specific axiomatic system (the Euclidean one) and not necessarily true proposition. Despite this, after the course, most of the students still take propositions for granted when they cannot. Only 4 students (7.14 %) find out all the mistakes contained in the potential proofs I give. Only these students seem able to find out, in the potential proofs, properties that are valid only in some circumstances among those admitted by the hypotheses. Only 2 students give all correct answers in the post-questionnaire.

The term "parallel" contained in a proposition or in a proof seems to alert students' attention and help them in answering. Indeed, students seem to answer better to question related to such propositions.

After my introductory course on non-Euclidean geometries, I did not observe a general improvement of students' critical thinking and proof skills. One obstacle I encountered is the students' difficulty in finding counterexamples. This difficulty is rooted in the fact that most students do not know that a counterexample to a statement, to be such, must not only fail to satisfy the thesis but must also satisfy the statement's hypotheses. The ability to construct counterexamples is essential for proving mathematical results.

6.4 *BELIEFS questionnaire*

The *BELIEFS questionnaire* addresses to the research question *RQ5* (*Do students' beliefs about mathematics change over the duration of the course?*). Moreover, item 2 of the *BELIEFS questionnaire* – combined with items 1 and 2 of the *NEG questionnaire* and item 19 of the *VHL test* – addresses the research question *RQ2* (*To what extent do students gain a new perspective on the concept of axiomatic system?).*

This section examines the results related to the *BELIEFS questionnaire* obtained from the definitive experimental phase of the project. The results are detailed and commented in the following subsections. For the sake of clarity, the answers are denoted with the following acronyms: "CNA" stands for "I cannot answer"; "P" for "prescriptive account of mathematics"; "D" for "descriptive account of mathematics". I have noted that all the students who answered "Other", provided a justification that can be classified intermediate between "P" and "D", therefore I will refer to these answers as "P/D" for "nor exclusively prescriptive nor exclusively descriptive account of mathematics".

Item 1

Question 1 aims to test whether, in the student's opinion, mathematics is discovered or invented. Figure 67 reports the results.

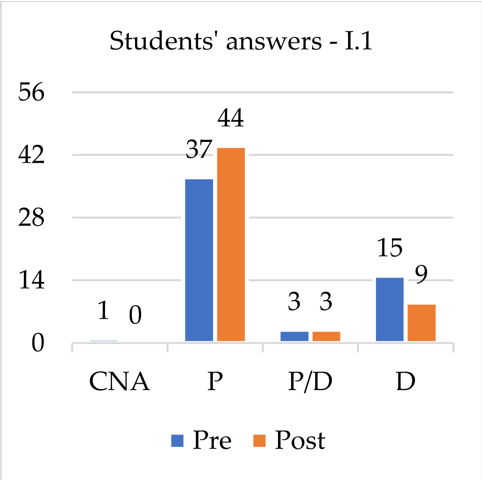
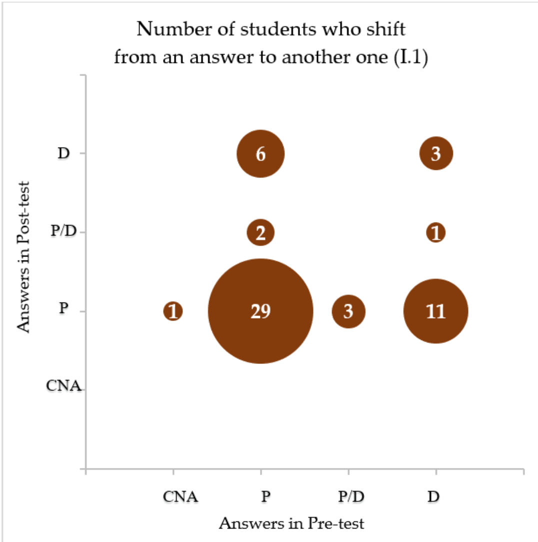


Figure 67. Students' answers to question 1 (Mathematics is discovered or invented). Above, the column chart. One the right, the bubble chart.



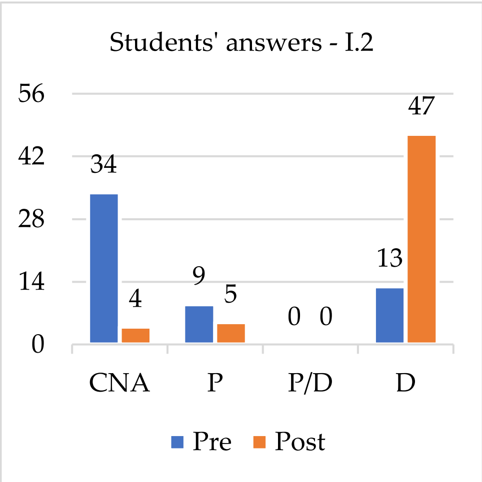
Students' view of Mathematics is predominantly Platonic both before and after the course on non-Euclidean geometries. The Platonic view is even more common after the course. Some students, regardless of whether they change their view, justify their answer in a deeper way (please note that I ask each answer to be justified). We see examples in the Table 19.

Student	Pre	Post
SF4VLS	- I agree. - In every action we perform during the day, mathematical calculations are intrinsic *("In ogni azione che compiamo durante la giornata i calcoli matematici sono intrinsechi")*	- I do not agree. - as we have understood in this course, there is no single "right" mathematics but different mathematics can be invented based on different axioms. *("come abbiamo avuto modo di capire in questo corso, non esiste una sola matematica "giusta" ma possono essere inventate diverse matematiche basate su assiomi differenti.")*
SM10VLS	- I agree. - Mathematics is present in nature. *("La matemática è presente nella natura.")*	Other: I can't answer because I don't think anyone can answer. The presence or absence of the idea in the entity is an eternal ontological problem and I think there is evidence for both positions: mathematics is in Nature, proof of this is the golden ratio that is found innumerable times in the cosmos or the properties of number of Nepero; yet the fact that there is no unique geometry and therefore that there is no absolute mathematical truth would be clear proof that the mathematician creates properties and objects *("Non so rispondere perché penso che nessuno sappia rispondere. È un problema ontologico eterno quello della presenza o assenza dell' idea nell' ente e penso ci siano prove per entrambe le posizioni: la matematica è nella Natura, ne è prova il rapporto aureo che si ritrova innumerevoli volte nel cosmo o le proprietà del numero di nepero; eppure il fatto che non esista una geometria unica e quindi che non esista una verità matematica assoluta sarebbe la prova evidente che il matematico crea proprietà e oggetti")*
SF47IISA	- I do not agree. - in my opinion, mathematics can be found in anything, in fact it applies to human life on any occasion. *("secondo me la matematica si piò trovare in qualsiasi cosa, infatti si applica alla vita dell'uomo in qualsiasi occasione.")*	- I do not agree. - in my opinion the mathematician asks himself questions by observing what surrounds him and from these questions he begins to obtain a lot of information that will then be discussed, expanded and affirmed by people like him who have the same objectives. But from the opinions of the whole mathematical community it is possible to study or enunciate objects that are not seen in real life. An example is the hyperbolic geometry that we can compare to a black hole that had never been observed at the time of the discovery. *("secondo me il matematico si pone delle domande osservando quello che lo circonda e da queste domande inizia a ricavare tantissime informazioni che poi saranno discusse, ampliate e affermate da persone che come lui che hanno gli stessi obbiettivi. Ma dai pareri di tutta la comunità matematica si può arrivare a studiare o ad enunciare oggetti che non si vedono nella vita reale. Un esempio è la geometria iperbolica che possiamo paragonare ad un buco nero che al tempo della scoperta non si era mai osservato.")*

Table 19. Some students' answers to question 1.

Item 2

Question 2 aims to test whether in the student's opinion in mathematics it is interesting to establish whether an axiomatic system is true/false versus consistent/non-consistent. This aspect is explicitly stated during the course, therefore it may not be surprising that in the pre-questionnaire most of the students select that in mathematics it is interesting to establish whether an axiomatic system is true/false, while in the post-questionnaire most of them select that it is interesting to establish whether it is consistent/inconsistent.
From Figure 68 we can observe a large shift from CNA in the pre-questionnaire to D in the post-questionnaire, which tells us that the students were attentive during the course.



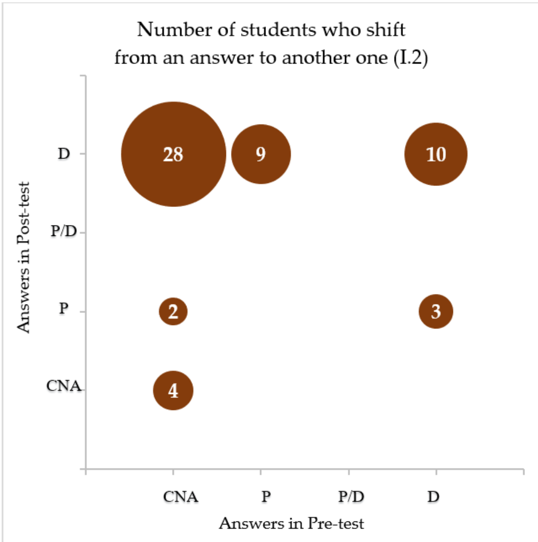


Figure 68. Students' answers to question 2 (Mathematical concepts are true/false versus consistent/non consistent). Above, the column chart. One the right, the bubble chart.

Moreover, three students who select the option consistent/inconsistent in the pre-questionnaire, do not justify their answer while one other gives it wrong. This is the case of student SM30IIICI, who states: "An axiomatic system is in any case TRUE. For this reason, we can only discuss it, but it must still be assumed truth. Having said that, it can be defined as consistent/inconsistent, but this is based on our personal idea"[70].

[70] Translated from the Italian "*Un sistema assiomatico è in ogni caso VERO. Per questo motivo noi possiamo solamente discuterlo, ma deve essere comunque preso per vero. Detto ciò, lo si può definire coerente/incoerente, questo però in base alla nostra personale idea*".

We observe that 5 students (8.93%) select, in the pre-questionnaire, the option "I cannot answer" and justify their answers stating that they do not know what "axiomatic system" means. One of them belongs to Set 4 (class V SC); one other to Set 2 (class III CI), three of them to Set 1 (class II SA). Moreover, it is interesting observing how a student of Set 2 justifies its selection "true/false. He state: "Honestly, I do not know what axiomatic system means, but when we do geometry or algebra it sometimes happens to say if it is true or false and not consistent or inconsistent"[71].

In the Table 20 I list answers given by some students in the post-questionnaire and which show that they have not grasped the concept of consistency. I also added their answer in the pre-questionnaire.

Student	**Pre**	**Post**
SF9VLS	I cannot answer.	- Consistent/inconsistent - it is useful to better understand what we are dealing with *("è utile per capire al meglio di cosa stiamo tratando")*
SM12VSA	- Consistent/inconsistent - (She does not justify)	- True/false - An axiomatic system to be defined as such must be consistent so it is not possible to establish if it is [consistent] or not *("Un sistema assiomatico per essere definito tale deve essere coerente quindi non è possibile stabilire se lo sia o meno")*
SM15VSA	- Consistent/inconsistent - (He does not justify)	- Consistent/inconsistent - the true and false are relative, since one could proceed with proofs by reductio ad absurdum *("il vero e falso è relativo, poichè si potrebbe procedere con dimostrazioni per assurdo")*

Table 20. In the third column I list students' answers given in the post-questionnaire which show that they have not grasped the concept of consistency. In the second column I add their answers in the pre-questionnaire.

Item 3

Question 3 aims to test whether in the student's opinion mathematical concepts are subject to historical revisions.

[71] Translated from the Italian *"Sinceramente non so cosa vuol dire sistema assiomatico, ma quando facciamo geometria o algebra avvolte capita di dire se è vero o falso e non coerente o incoerente"*

Figure 69 shows that, before the course, 43 students (76.78 %) do not agree with the statement *"Mathematical knowledge is definitive and is not subject to any revision"*. The number of students with a descriptive view decreases after the non-Euclidean geometries course: 38 (68.86 %) are the students who do not agree with the statement *"Mathematical knowledge is an immutable set of truths"*. Specifically, 5 students pass from a prescriptive to a descriptive vision. While 7 students pass from a descriptive to a prescriptive vision. 32 students (57.14 %) do not change their vision.

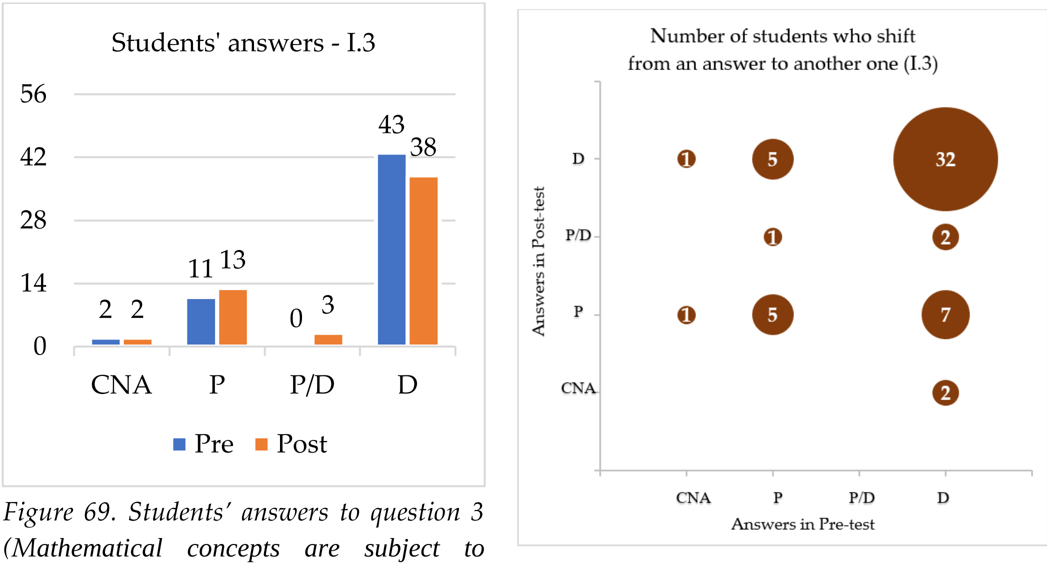


Figure 69. Students' answers to question 3 (Mathematical concepts are subject to historical revisions). Above, the column chart. One the right, the bubble chart.

We observe that while in the pre-questionnaire 4 students talk of the possibility for a theorem to be disproven, it happens only one time in the post-questionnaire. I list what students wrote to justify why they selected "I do not agree":

SF5VSC – pre-questionnaire: "I disagree because it may be that in a few years those laws and theorems are no longer valid or need modification".[72]

SF5VSC – pre-questionnaire: "Mathematical theorems are often revisited even after a long time to correct or complete them"[73].

[72] Translated from the Italian *"Non condivido perché può essere che tra qualche anno quelle leggi e teoremi non siano più validi o abbiano bisogno di modifiche"*.

[73] Translated from the Italian *"I teoremi matematici vengono spesso rivisitati anche a distanza di molto tempo per correggerli o completarli"*.

SF18VSA – pre-questionnaire: "On the basis of different means and knowledge, a theorem can be revised and modified"[74].
SF44IISA – pre-questionnaire: "I do not agree because for me mathematical knowledge is subject to revision, as over the years some theorems can be resumed and reformulated"[75].
SM204VSA – post-questionnaire: "I think in the future there will be new discoveries that will determine the falsity of other old theorems"[76].

Student SM204VSA could have understood, along the course, that some theorems were disproved (I underline that he writes "other old theorems"). This is a possible misunderstanding to be taken into account talking about the history of the fifth postulate and of the birth of non-Euclidean geometries. Of course, proof may be revised over time and judged to be incorrect. However, this is not the teaching what I referred to during the course, and it is not what the history of the fifth postulate and the birth of non-Euclidean geometries teaches.

In the pre-questionnaire, only one student refers to non-Euclidean geometries. In the post questionnaire five students refer to non-Euclidean geometries. In Table 21, I transcribe the answers of students who refer to non-Euclidean geometries.

Student	Pre	Post
SF5VSC	I do not agree because it may be that in a few years those laws and theorems are no longer valid or need to be modified *("Non condivido perché può essere che tra qualche anno quelle leggi e teoremi non siano più validi o abbiano bisogno di modifiche")*	I do not agree because some aspects of Euclidean geometry are not true in spherical geometry. Therefore, even in mathematics there is no an absolute truth, an immutable truth *("non condivido perché esempio alcuni aspetti della geometria euclidea non sono veri nella geometria sferica. Quindi neanche in matematica esiste un vero assoluto, una verità immutabile")*
SM13VSC	Mathematical knowledge is definitive, what has been previously proved in fact cannot be subject to revision	I do not agree because as proved by the historical events concerning the birth of non-Euclidean geometries, mathematics is a mutable knowledge. Some properties referring to Euclid's postulates were denied in other geometries: for example, as

[74] Translated from the Italian *"In base a mezzi e conoscenze diverse un teorema può essere revisionato e modificato"*.
[75] Translated from the Italian *"Non condivido perché per me la conoscenza matematica viene soggetta a revisione, poichè negli anni possono essere ripresi e riformulati alcuni teoremi"*.
[76] Translated from the Italian *"Secondo me in futuro ci saranno nuove scoperte che determineranno la falsità di altri vecchi teoremi"*.

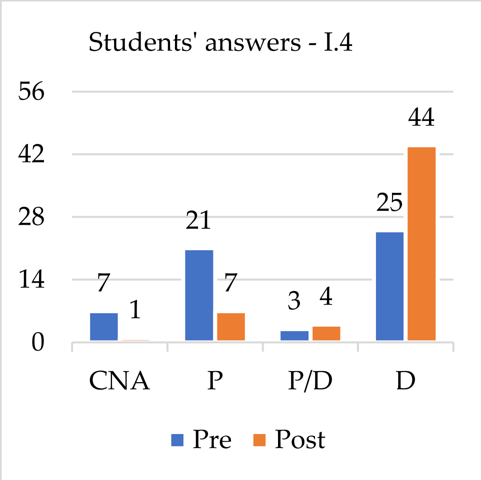
	as mathematics is an exact science. *("La conoscenza matematica è di tipo definitivo, ciò che è stato dimostrato in precedenza infatti non può essere soggetto a revisione essendo la matematica una scienza esatta.")*	the transitive property of parallelism which in hyperbolic geometry is not always valid. *("Non condivido perché come dimostrato dagli eventi storici che riguardano la nascita delle geometrie Non Euclidee la matematica è una conoscenza mutabile. Alcune proprietà riferite ai postulati d'Euclide vennero negate in altre geometrie: ad esempio come la proprietà transitiva del parallelismo che nella geometria iperbolica non è sempre valida.")*
SM14VSC	Any knowledge in science can be challenged by new knowledge. *("Ogni conoscenza in campo scientifico può essere messa in discussione da nuove conoscenze.")*	in mathematics there is no an absolute truth but only a relative one. For example, in spherical geometry the sum of the internal angles of a triangle is greater than 180 degrees, while in hyperbolic geometry it is less than 180 degrees. *("in matematica non esiste una verità assoluta ma solo relativa. Ad esempio in geometria sferica la somma degli angoli interni di un triangolo è maggiore di 180 gradi, mentre nella geometria iperbolica è minore di 180 gradi.")*
SM23IIICI	In my opinion, there is always new room for revision in anything, as our knowledge progresses, maybe our mathematical knowledge will change *("sono dell'opinione che ci sia sempre nuovo margine di revisione in qualsiasi cosa, con il progredire delle nostre conoscenze, magari la nostra conoscenza matematica varierà")*	we have seen how mathematics itself is very changeable, ranging from Euclidean geometry to spherical and hyperbolic geometry, consequently I believe that it is not immutable, but it is a multiform set of hypotheses given that to be an absolute truth a postulate should be true in all the geometries *("abbiamo visto come la matematica stessa sia molto mutevole, spaziando dalla geometria euclidea a quella sferica ed iperbolica, di conseguenza credo che essa non sia immutabile, ma sia un insieme multiforme di ipotesi dato che per essere una verità assoluta un postulato dovrebbe essere vero in tutte le geometrie.")*
SM42IIICI	Mathematics is constantly evolving and, if we had based ourselves on this statement, we would not have had the non-Euclidean geometries that are based on the transgression of the initial Euclidean axioms *("La matematica è in continua evoluzione e, se ci si fosse basati su questa affermazione, non avremmo avuto le geometrie non euclidee che si basano sulla trasgressione degli iniziali assiomi euclidei")*	Mathematics is constantly evolving and, often, great mathematical discoveries have arisen from the denial of established orthodoxy. It is not immutable and, after all, we do not even know if it is true. *("La matematica è in costante evoluzione e, spesso, grandi scoperte matematiche sono nate dalla negazione dell'ortodossia costituita. Non è immutabile e, in fondo, non sappiamo neanche se sia vera.")*
SF47IISA	mathematical knowledge is definitive, in fact everyone will study it, but over the	I do not agree with this statement because I think that mathematics evolves with time, in fact before

	time and the increase of new technologies it could be modified *("la conoscenza matematica è definitiva, infatti tutti la studieranno, ma con il passare del tempo e l'incremento delle nuove tecnologie potrebbe essere modificata.")*	this course I did not even know about the existence of the pseudosphere or its characteristics and I certainly knew the sphere but I would never have imagined its properties. Personally, I think that mathematics is always based on something that we can define as foundations, but which, based on society, uses and curiosities, expand to the construction of a building. This statement also varies because in my opinion with the acquisition of new knowledge of the mathematician everything can be seen with a different eye. Therefore everything can be disproved or discussed. For example, today a geometry completely independent from the Euclidean one could be born overnight, and we cannot consider it false. *("personalmente non condivido questa affermazione perché penso che la matematica si evolva con il tempo, infatti prima di iniziare questo corso non sapevo neppure dell'esistenza delle pseudosfera o delle sue caratteristiche e certo conoscevo la sfera ma non mi sarei mai immaginata le sue proprietà. Personalmente penso che la matematica si basi sempre su un qualcosa che possiamo definire fondamenta, ma che in base alla società, agli usi e le curiosità si vanno ad ampliare fino alla costruzione di un palazzo. Questa affermazione varia anche perché secondo la mia opinione con l'acquisire di nuove conoscenze del matematico tutto può essere vista con un occhio diverso. Quindi tutto può essere smentito o discusso. Per esempio oggi potrebbe nascere da un giorno all'altro una geometria completamente indipendente da quella euclidea, e noi non possiamo considerarla falsa.")*

Table 21. Answers of students who refer to non-Euclidean geometries.

Item 4

Questions 4 aims to test whether in the student's opinion socio-cultural factors influence mathematical knowledge. Figure 70 reports the results.



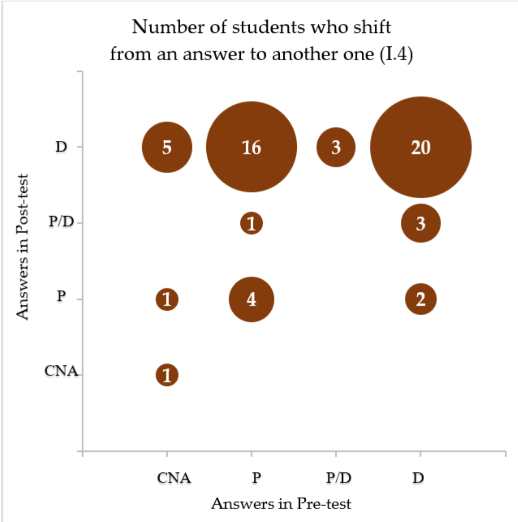


Figure 70. Students' answers to question 4 (Socio-cultural factors influence mathematical knowledge). Above, the column chart. One the right, the bubble chart.

Item 4 is the one in which most of the students view change from a prescriptive one to a descriptive one. Indeed, in the pre-questionnaire, 25 students (44.64 %) do not agree with the following statement: "The development of mathematical knowledge is not influenced by socio-cultural factors". Figure 70 shows that 44 of them (78.57%) do not agree, in the post questionnaire, with the following statement: "Mathematical knowledge is not linked to the social and cultural context".

Table 22 lists answers of students who, in the post-questionnaire, explicitly mention non-Euclidean geometries or considerations relating to their birth.

Student	Pre	Post
SF4VSC	There are a thousand factors that will influence as I wrote in the previous motivation *("Ci sono mille fattori che andranno ad influenzare come ho scritto nella motivazione precedente")*	Based on the means and the amount of knowledge at our disposal, we can make more or less considerations. For example, Thales, for the dogmas of the time, would not have been able to admit another geometry other than the Euclidean one. *("In base ai mezzi e alla quantità di nozioni a nostra disposizione possiamo fare più o meno considerazioni. Per esempio, Talete, non sarebbe riuscito, per i dogmi dell'epoca, ad ammettere un'altra geometria diversa da quella euclidea.")*
SF7VSC	I cannot answer	It is not possible to speak of mathematicians without referring to the historical context, for example non-Euclidean geometries were initially criticized as were the mathematicians who supported them *("per parlare di matematici non si può fare a meno del contesto storico, per esempio le geometrie non euclidee inizialmente erano criticate così come i matematici che le sostenevano")*
SM22VSA	Mathematics is used on a daily basis according to the context in which one finds oneself *("La matematica viene adoperata quotidianamente in base al contesto in cui ci si trova")*	Ancient populations would not have had the same means and the same freedom of thought to be able to express new geometries that would affect the dogmatized ones *("Popolazioni antiche non avrebbero avuto gli stessi mezzi e la stessa libertà di pensiero per poter esprimere nuove geometrie che andassero ad intaccare quelle dogmatizzate")*
SF35IIICI	We find mathematics in everyday life and I think this is precisely what we have led to the study of mathematics *("La matematica la troviamo nella vita di tutti i giorni e penso che proprio questo abbiamo portato allo studio della matematica.")*	Mathematical knowledge is also influenced by everyday life and by the social and cultural situation, such as, for example, when the knowledge of hyperbolic geometry took place *("Il sapere matematico è influenzato anche dalla vita di tutti i giorni e dalla situazione sociale e culturale, come, per esempio, quando è avvenuta la conoscenza della geometria iperbolica.")*
SF47IISA	in my opinion, society and culture have been the characteristics for which mathematics has developed since the beginning of time, in fact everything arises from some questions, curiosities or needs on the part of society. *("la società e la cultura secondo me sono state le caratteristiche per cui la matematica si è sviluppata dall'inizio dei tempi,*	The mathematical knowledge as I have already mentioned in the previous question in my opinion is very much connected to the cultural life of the time. In fact, everything arises from a need or a curiosity that varies according to our thinking and our lifestyle, therefore goes hand in hand with our cultural life. Furthermore, we have testimonies of many mathematicians who, despite having concluded extraordinary conclusions and theories, persisted in [not] making them public for fear that society would

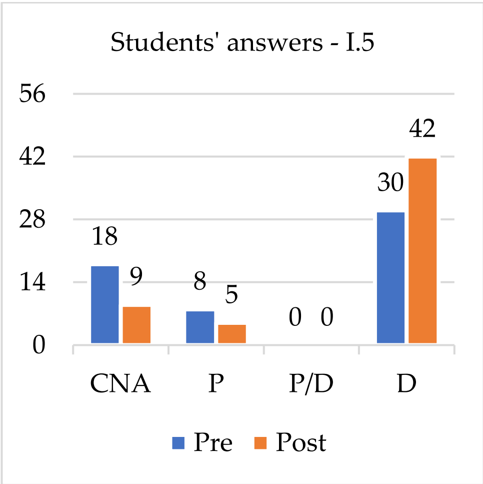
	infatti tutto nasce da alcune domande, curiosità o esigenze da parte della società.")	not accept them, because they questioned all the bases on which everyone had always referred. *("il sapere matematico come ho già accennato nella domanda precedente secondo me si collega molto alla vita culturale dell'epoca. Infatti tutto nasce da un'esigenza o da una curiosità che varia in base al nostro pensiero e al nostro stile di vita, quindi va di pari passo con la nostra vita culturale. Inoltre noi abbiamo delle testimonianze di molti matematici che nonostante avessero concluso delle straordinarie conclusioni e teorie, si ostinavano a non renderle pubbliche per la paura che la società non le accettasse, perché mettevano in discussione tutte le basi su cui tutti avevano sempre fatto riferimento.")*
SF51IISA	In my opinion the development of mathematical knowledge is influenced by socio-cultural factors, in fact a type of population or, if anything, a single individual can be hindered by society or by the historical period in which it finds itself. *("Secondo me lo sviluppo del sapere matematico è influenzato da fattori socio-culturali, infatti un tipo di popolazione o semmai un unico individuo può essere ostacolato dalla società o dal periodo storico in cui si trova.")*	Mathematical knowledge is connected to the social and cultural context, because many mathematicians, scientists have not been able to publish their reports regarding discoveries or insights precisely because of the will of society, which insisted on denying the evidence, persisted in deny a new "truth". *("Il sapere matematico si collega al contesto sociale e culturale, perché molti matematici, scienziati non hanno potuto pubblicare le loro relazioni riguardo a scoperte o approfondimenti proprio per il volere della società, la quale si ostinava a negare l'evidenza, si ostinava a negare una nuova "verità".")*

Table 22. Answers of students who, in the post-questionnaire, explicitly mention non-Euclidean geometries or considerations relating to their birth

Item 5

Question 5 aims to test whether in the student's opinion the existence of revolutionary changes within the development of mathematical knowledge.

Figure 71 shows how students' answer before and after the course. Before the course, 30 students (53.57 %) believe that in mathematics there have been no theories or studies so revolutionary as to constitute an obstacle to their acceptance by the scientific community. After the course 42 students (75.00 %) believe that, also for the development of mathematical knowledge, some particularly revolutionary scientific concepts have marked the scientific community through a radical change, a discontinuous leap, a break with the past. Among these 42 students, 16 mention non-Euclidean geometries.



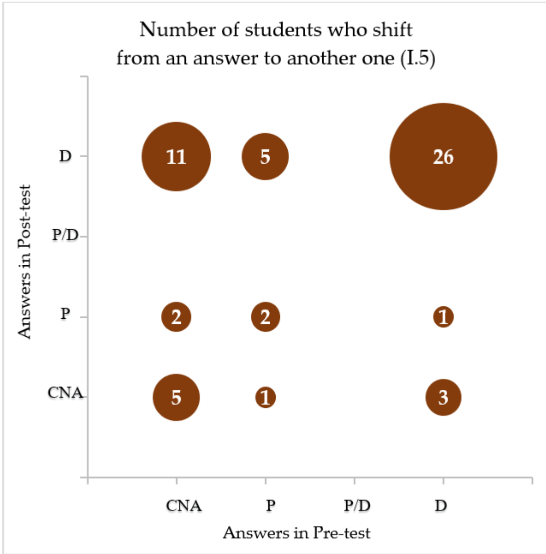


Figure 71. Students' answers to question 5 (The existence of revolutionary changes within the development of mathematical knowledge). Above, the column chart. One the right, the bubble chart.

4 of the 5 students who, in the post-questionnaire, agree with the statement written in question 5 (no breaks with the past exist in the development of mathematical knowledge), do not justify their answers or seem not having understand the question. Only one student (SM33IIICI) gives an interesting justification. Indeed, he writes: "I agree because I believe that all the mathematical knowledge we have available today is derived from laws already existing in antiquity, so in my opinion there has never been a critical break with the past"[77].

6.4.2 Discussion

The data that emerges from the *BELIEFS questionnaire* is quite varied and very interesting, especially when looking at the individual answers of the students which provide an impressive quantity of material for future discussion.

In some items we see a clear shift towards a descriptive view of mathematics, while in some other this does not occur. Therefore, each individual question must be treated separately. The statistics show that for some of the topics covered by the *BELIEFS questionnaire*, the course provoked a remarkable shift from CNA or P to

[77] Translated from the Italian *"Condivido perché credo che tutte le conoscenze matematiche che abbiamo a disposizione oggi siano derivate da leggi già esistenti nell'antichità, quindi secondo me non c'è mai stata una rottura critica con il passato".*

D. This is the case for items 2, 4 and 5. For items 1 is less important to see a shift in the students’ view because these is related to a topic that is still debated in the community (discovery vs. invention).

In this section I want to highlight what I believe to be the most salient points.

One aspect worth noting is the fact that the students’ answers were richer and elaborate in the post-questionnaire. Their ability to discuss has improved and some ideas have sparked from the course. On the other hand, I observe that some foundations are sometimes missing and, in general, my short course was not sufficient to instruct them properly, but rather, it made them more aware of the issues and progression of mathematics as a science.

As an example, we notice that some students admit they do not know what an axiomatic system is. This clashes with the Italian national guidelines which require the students to learn about axiomatic systems and their basic components (axioms, definitions, theorems).

Through the course, the students may have been hit by the role of non-Euclidean geometries in the evolution of mathematics. This is reflected in several answers where explicit references to non-Euclidean geometries is made. Specifically, this is more frequent in items 3 and 4, where the evolution of truths in mathematics and the role of culture and society is discussed.

6.5 Discussion of the results and answers to the research questions

In this last section I will try to resume all the results seen so far, to provide an interpretation of the data. Having different questionnaires that cover different topics allows to conclude on the numerous aspects and impacts of the teaching of non-Euclidean geometries. Naturally, these results are related to this specific formulation of the NEG course and are surely affected by the way it was conducted through a remote teaching platform. For this reason, more weight is given to the evaluation of the students’ answers, and to the individual interviews.

Firstly, I conducted the *VHL test* to assess how individuals think over geometrical concepts. The starting level of the classes are quite different as seen in Figure 45 (p. 162). The highest average level is obtained by Class III CI, shortly followed by

Class V SC. Class II SA has a lower average level. This was expected, since its students are younger. However, the lowest average level is achieved by Class V SA. This shows that the level of a class is not directly related to the age of its students. On the opposite, the history of the class and the capacities of the individuals can lead to very different outcomes in the *VHL test*. Several reasons can lead to the fact that the van Hiele level is not directly proportional to the grade: (1) since classes III have recently concluded its studies on the axiomatic formulation of Euclidean geometry (this is taught in I and II class), they are more prepared on the subject; (2) the classes in these tests have different level and preparations due to random statistical factors (different teachers and overall level of the individual students).

For what concerns the overall improvement of the students' performance, the statistical analysis shows that it improved by a small but statistically significant amount after the course. It is interesting to observe this change, even though the course duration was short and the course did not concentrate on Euclidean geometry. Looking into the data in more detail we observe from Figure 45 (p. 162) that a high level of preparation of the class is also correlated to a larger increase in the *VHL test* performance. After the course, Class III CI, which had the higher starting level was also the one with the larger increase in the level after the course. Similarly, Class V SC, which scored second, had a noticeable improvement. Class II SA had no change, while Class V SA had a slightly negative change. This data seems to confirm a trend and certainly constitutes an interesting finding, but it must be verified in future studies with a larger number of students and classes. If the above is true, we can say that not all the students were prepared for a NEG course, therefore, it may be useful to find a threshold under which the intervention has no effect or even a negative effect. Unfortunately, the tragic occurrence of the Covid-19 pandemic forced many teachers, who were engaged in the study, to give up on participating, reducing dramatically the number of subjects involved.

In any case, the large difference in the students' level is a good precondition for the rest of the test, since it tells us that the sample is sufficiently heterogeneous to give rigor to the whole experiment. Indeed, many seminars and workshops on non-Euclidean geometries that are taught in high schools are directed to a selected pool of students, those with the highest marks, impairing the validity of any statistical analysis conducted on this sample (if ever this is done). Furthermore, those seminars are generally taught to fifth degree, since older students are considered more apt to understanding the abstract concepts implied by axiomatic geometry. A more neutral and varied sample is necessary for quantitative experiments like the one that has been described here.

If the *VHL test* allows to assess the initial level of the students, the *NEG questionnaire* allows to answer to *RQ3* (*How well do students learn the taught concepts of non-Euclidean geometries?*) by testing what they learned about basic notions of non-Euclidean geometries. In the post questionnaire they can answer questions (e.g., what is the sum of the internal angles of a triangle) by following their Euclidean knowledge, the newly learned non-Euclidean knowledge and show they can discern among all the seen geometries. The result of this test is very positive. The pre-questionnaire clearly shows that they are all consistently answering in the light of the only geometry they know. After the course, a large part of the students (72%) can answer correctly (score [0.75-1]), showing that they can tell that some statements are true only in some geometries and they can tell which one correctly. By combining the *pre-VHL test* and the *post-NEG questionnaire*, we can say that most students had an acceptable starting level and that they were able to learn the different features of the geometries they were taught. Considering that the classes were taught remotely and during the students' spare time, this is a result that could not be given for granted and is a precondition for the validity of the analysis.

A common mistake made by students, in the *NEG questionnaire*, is to select the wrong example when they are asked, e.g. whether the transitivity of parallel straight lines is false. Many students in this case give the spherical geometry as an example, bringing as an argument the fact that parallel straight lines do not exist on a spherical surface. This is clearly an error: if parallel lines cannot exist it is senseless to talk about the transitivity of parallelism. This and many other logical errors are found in the questionnaires and raise a doubt regarding their proficiency in logics and the time that is allocated to the teaching of logical thinking in schools.

This takes us to the next point, which is the evaluation of the questionnaires related to *RQ4* (*To what extent do students' critical thinking and proof skills improve over the duration of the course?*). One of the aims of this project, is to stimulate logical thinking, an ability that lays the foundations for the development of axiomatic geometry. The *PROOF questionnaire* was conceived for testing one specific ability, i.e. that of understanding the validity of a logical step in a proof in relation to the validity of a hypothesis. More specifically, in this questionnaire, the students are shown a proof of a statement which is applicable only in a geometry and they have to tell if the proof is always valid and if not, tell when. Differently from the previous two questionnaire, this one did not yield good results. The outcome of this questionnaire is that after the course the students still prevalently think in Euclidean terms. Many of them got both the pre and post questionnaire wrong, and many of those who were right in the pre-questionnaire gave a wrong answer in the post questionnaire and vice versa. Therefore, after the introductory course on non-Euclidean geometries, I did not observe a general improvement of students' critical thinking and proof skills. One obstacle I encountered is the students' difficulty in finding counterexamples. This difficulty is rooted in the fact that most students do not know that a counterexample to a statement, to be such, must not only fail to satisfy the thesis but must also satisfy the statement's hypotheses. The ability to construct counterexamples is essential for proving mathematical results. These results are not unexpected; indeed proof skills and logical skills require much more time than a short course to build up. It is worth reflecting on the fact that many students were attending the last class of high-school and, nevertheless, they have limited proof logical and logical skills. One reason that may have impaired the students' results is their attitude to reply quickly, without concentrating on the logical implications of their answer. Another aspect that may have affected the results of the *PROOF questionnaire* is the insufficient ability of the students to comprehend a long text and focus on it for a sufficient amount of time. Some of them informally stated that the text is excessively long. However, it is not possible to express questions of such complexity in a shorter or simpler form.

The *BELIEFS questionnaire* shows a certain degree of development in the students in relation to their beliefs on mathematics. For a detailed discussion for each of the items – each related to an aspect (social, historical, axiomatic, etc.) – please refer to the discussion in Section 6.4.2, and the analysis of the results of the single items (Section 6.4). I can, thus, say that the answer to *RQ5* (*Do students' beliefs about mathematics change over the duration of the course?*) is generally positive.
Additionally, I observed that their ability to discuss has improved and some ideas have sparked from the course. The students are now more prone to motivating their answers and cite non-Euclidean geometries as examples of a descriptive mathematics view.

Up to this point I have not addressed results related to *RQ2* and *RQ1* because these required a discussion of all the questionnaires. As for *RQ2* (*To what extent do students gain a new perspective on the concept of axiomatic system*?) I am particularly interested in two aspects (requiring a complete understanding of an axiomatic system would be an excessive goal for high school students): *RQ2.1)* the fact that some terms must be left undefined and it is necessary to assume that some statements are valid; *RQ2.2)* the fact that postulates are not required to be self-evident. To answer to *RQ2.1* I refer to items 19 of *VHL*; to answer to *RQ2.2* I refer to items 1 and 2 of *NEG*, item 2 of *BELIEFS*.
Item 19 of *VHL test* created difficulties for most of the students, however it was the item with the largest improvement between the pre-test and the post-test (from 5% to 30% of correct answers). Therefore, the short course provided a remarkable improvement. For *RQ2.2* considering together the various items, they provide a positive picture.

Finally, to reply to *RQ1* (*What features of a short introductory course in non-Euclidean geometries are effective in engaging high-school students?*), the features of the short course that were found to be most engaging and fruitful for the students are the following:

- The first session of the course, that is the interactive session during which students understood what a circle and a straight line look like on a spherical surface;
- The workshop on the sphere.
- The argument on the consistency of postulates.
- The historical references.

Working on the sphere, especially in a practical setting, captures the attention of the students. The spherical surface allows to redefine concepts that students are used to work with on the plane surface. This has the potential of opening their mind to non-Euclidean geometries and this can be seen by the many references that they make in the post-tests while answering to the open questions. Some of the students also declared that the spherical surface is the non-Euclidean geometry that they could understand better.
The argument on the consistency of postulates generated a lot of questions and interactions with the students during the classes. It stayed in the students minds and this can be seen by the answers they gave in the post-tests.

Regarding the historical references, the students mentioned them in some of their answers, and after the course, during the interviews some of the students stated that they appreciated this part of the course.

Before concluding the chapter, it is worth to report a last observation not related to the original research questions. An important part of the course involved practical activities. When these were conducted in presence, the students were asked to solve problems practically, manipulating objects mimicking a positive and negative curvature surface. The task seemed not to be trivial for the students. These difficulties worsened in the distance learning courses. In the distance learning courses the students had some of the objects available, one per student (those that could be inexpensively bought for all). Not being able to work in the same place reduced the ability of the students to communicate visually and non-verbally. Each one seemed to work more on her/his own with respect to the in-presence courses. The students were forced to communicate orally. This, however, requires the ability to formalize the thoughts and adopt the correct lexicon, which in some cases was not sufficiently developed to allow the solution of the practical problems.

7 Conclusions

This thesis deals with the teaching of non-Euclidean geometries at Italian high school with the intent of enhancing the proficiency of the students in logics, proof skills and providing them with a more accurate view of axiomatic systems and their role in the development of modern Science.

The idea stems from the observation that students find the study of geometry more troublesome than, e.g., the study of algebra. In the introduction I have indicated some clues why this is the case, i.e.: (i) non algorithmic approach; (ii) difficulty of conjugating linguistic and logical skills; (iii) self-evidence of Euclidean geometry. A proposal that I put forward is that introducing non-Euclidean geometries and let students confront with them can contribute to solve these issues. Non-Euclidean geometries show the students the necessity for an axiomatic system, whenever some properties are not self-evident. It also exercises their ability on a non-algorithmic approach to the solution of problems that stimulates creativity and visualization on one side, and logically correct propositions on the other side. Non-Euclidean geometries can also be stimulating in understanding how different conclusions can be drawn from different assumptions, without any of the conclusions being "wrong" in the originating context. This is also a quality of mathematics that can teach a lot to citizens (Gallo, 2012). Unfortunately, in the Italian school national guidelines at the moment there is no mention to non-Euclidean geometries, even if sometimes they are proposed by individual teachers in the form of seminars. What are the objectives of these seminars? Do they fulfill them? What would be the best way to reach the desired goals? Are these short courses sufficiently deep? What remains to the students afterwards? Are they appropriate for the age and competences of the students?

Based on these motivations, the thesis proposed and implemented a new format, also drawing from previous research, for teaching non-Euclidean geometries in short courses for secondary school according to a set of research questions that will be later resumed. This is one of the main contributions of the work. Furthermore, with the intent to foster open research, all materials are described in the text or provided in the appendices. In order to address the research questions, a thorough evaluation was done, mainly following quantitative methods. To the best of my knowledge, this is one of the first attempts in this specific field. Should these be repeated on a larger scale they could better orient policy makers, in case non-Euclidean geometries will be reconsidered in future reforms of the school system.

An outline of the thesis will help review the framework and concepts that have been introduced and will help the reader outline the achievements of the conducted work.

Chapter 1 motivates in great detail what I briefly described in the above paragraphs, introduces the role of geometry in developing a rational mind, and quotes the opinions of several intellectuals along the history of mathematics.

A historical overview of the evolution of geometries has been provided in Chapter 2, for the sake of a concise but complete introduction to the topic, showing different points of view and interpretations of axiomatic systems. This chapter collects an abundancy of quotes and statements and provides a solid historical reference for the history of Euclidean and non-Euclidean geometries. It also questions the definition of revolution in mathematics and provides the opinions of several important authors about the birth of non-Euclidean geometries as such.

Chapter 3 dives further in history to reconstruct a historical overview of the teaching of geometry in the Italian school context. This chapter provides a timely and concise history up to the most recent days, especially useful to foreign researchers. One of the key aspects of this historical overview is the long-debated teaching of non-Euclidean geometries in school. The outcome of the overview is that in the current days no explicit reference to non-Euclidean geometries is made in the national guidelines.

In Chapter 4 I outlined some of the questions arising from the theoretical background shed so far. After conducting a survey of experimental works related to this topic in Section 4.3, I formulated the following research questions:

- *RQ1: What features of a short introductory course in non-Euclidean geometries are effective in engaging high-school students?*
- *RQ2: To what extent do students gain a new perspective on the concept of axiomatic system?*
- *RQ3: How well do students learn the taught concepts of non-Euclidean geometries?*
- *RQ4: To what extent do students' critical thinking and proof skills improve over the duration of the course?*

- *RQ5: Do students' beliefs about mathematics change over the duration of the course?*

Alongside the choice of a quantitative method, which is motivated in Chapter 5, I have designed the experimental phase with several objectives in mind that have been gradually updated through a series of pilot experiments that allowed to iteratively revise the material. The fundamental idea behind this short course was to lead the students to discover several of the surprising properties of non-Euclidean geometries, to let them formulate their doubts and finally to provide them with answers. Answering to their questions means showing that the apparent contradictions are inherent in the axioms and that each axiomatic theory is consistent starting from these. The work with the classes was structured as a 10-hour workshop with both frontal lessons and practical laboratory lessons. The whole experience revolved around the concept of curvature, dealing with different geometries (spherical, hyperbolic and Euclidean) which were wrapped up in a last lesson, where the historical evolution (and revolutions) of geometries were analysed under the same light. The practical activities involved objects that allowed to visually and manually experiment the concepts of positive and negative curved surface and running straight on such surfaces.

The protocol and questionnaires were finalized in the end of 2019, and a large-scale study was proposed to many schools and classes.

Unfortunately, at this point, the global Covid-19 pandemic started, making impossible to conduct the study as desired for more than 1 year to come. For this reason, I modified part of the study to adapt to a distance learning paradigm. The activities were now taught during the afternoon, to accommodate the changed needs of the schools during the pandemic. The practical activities were reduced in favor of more frontal lessons. The number of students dropped significantly due to the difficulties of the teachers in re-organizing the didactical activities under the pandemic.

This second round of experimental courses was divided in five meetings of two hours each, beginning with an interactive session whose main objective was to understand what a circle and a straight line look like on a spherical surface, during this session we also deal with the definitions of some basic geometrical objects on a spherical surface. During the second session, the students, divided into heterogeneous online groups, were engaged in tasks to be carried out on polystyrene spheres. These tasks allowed each student to explore the spherical surface. The third session revolved around the question of why, some geometric figures' properties hold on a plane surface do not hold on a spherical surface, and vice versa. We refreshed the basic elements of the Euclidean geometry and discussed on the possible validity of the five postulates of Euclid on a spherical surface. Following an easy flow of reasoning, we could mention Hilbert's formalization of Euclidean geometry and the meaning of consistency, completeness, and independence of an axiomatic system. Connecting to the concept of independence of an axiomatic system, the fourth session focused on the controversy surrounding Euclid's fifth postulate, on the birth of the hyperbolic geometry, and on the importance of having models for an axiomatic system. 3D-printing models of pseudospheres were used to show geometric figures' properties that hold on a plane surface but that do not hold on a pseudosphere and vice versa. Finally, the last meeting consisted of a workshop on the Poincaré disk model and a final discussion to resume the whole course. The aim of the workshop on the Poincaré disk model was to let the students become more familiar with hyperbolic geometry, understand that there can be more than a model for a geometry, and avoid the misconception of identifying a geometry with one of its models.

The results of this last incarnation of the study are reported in Chapter 6, where they are discussed in detail. The research work was impaired by the pandemic in several regards. First of all, a large part of the teachers had to withdraw their participation. The urgent need to revise their schedules and activities allowed no spare time to dedicate to this project. The laboratories could not be done within their normal class hours due to the fact that the pandemic had reduced the schooling time. Due to the distance learning it was not possible to provide all objects to all students. Those objects that were bought in quantity (one per student, e.g. the spheres and pins) were provided to all students through their teachers, while the 3D-printed one, which were available in a few number could not. The work groups were conducted with virtual rooms, limiting the capacity of the students to interact together. This method also reduced the visibility of their activities, since the software tool only allows to be present in one room at once. Anyway, these tools had an impact: they enabled to conduct the activities even if the pandemics forced all the students to stay at home. I also observed that the impossibility to interact manually and visually forces the students to interact orally, thus forcing them to improve their communication and language skills. This made them less capable of working together, since they are not always used to express concepts in a formal way. In the distance learning setting, the students showed some difficulty in working manually, and their interactions were limited, due to the online conduction of the laboratory. However, I observed a lack of practical abilities also in the pilot studies, that were all conducted in presence. This was confirmed by their teachers which, similarly, observed their difficulties in manipulating the object and solve the problems practically. Finally, the number of interviews with the students were reduced not to increase their burden and their online time, since these could not be done in presence.

Notwithstanding this, the research work has brought results. It is worth mentioning some details of these results, especially those that will foster a discussion for future works. In Section 6.5 a complete summary of the results is provided, linking these to the aforementioned research questions. The answers to *RQ2*, *RQ3* and *RQ5* are positive. The outcome of *RQ4* is negative: in finding an answer to *RQ4* I discovered that some logical skills are insufficient, or – at least – students tend to answer too quickly or are fatigued by reading carefully the given texts. Addressing logical and proofing skills requires more than a short course, thus, must be done in the appropriate ways inside the regular school programs. Finally, the features that engaged most the students and were retained after some time (*RQ1*) are the works on the spherical surface, the historical discussion and the argument on the coherency of postulates.
More details follow:

i. The implementation of several tests with the same students allowed to evaluate both their knowledge and their skills. The practical experiences and the frontal activities increased their knowledge, enriching their cultural background, which they showed in the questionnaires, giving more elaborate answers and showing they learned what was discussed during the course.
ii. On the other hand, in the context of a proof, the students are not yet capable of turning their non-Euclidean knowledge into an ability, and thus they still prevalently reason in Euclidean terms. Developing skills requires time and exercise.
iii. Another observation related to proving a theorem: it seems that students did not understand the need for formal rigor that is required. For example,

they have problems in formulating counterexamples. A common issue is the belief that when a hypothesis is false the whole theorem is false.

iv. After the course, their ability to discuss has improved and some ideas have sparked from the course. The students are now more prone to motivating their answers and cite non-Euclidean geometries as examples of a descriptive mathematics view.

The first of the above highlights is deduced from the analysis of the *VHL* and *NEG questionnaires*. The *VHL* shows that each of the classes involved in the study have a different average level and the levels are not directly proportional to the age of the students. The results from the *NEG questionnaire* show an evident change in the students' view of the geometries, from a mostly Euclidean framework to a new framework open to non-Euclidean geometries. It must be noted that the classes that start from a higher level in the *VHL test* are able to improve better in the *NEG questionnaire*, therefore, in a perspective study, students should be selected based on their initial preparation assessed by the *VHL test*.

As to point ii, data from the *PROOF questionnaire* shows that only 7% of the students find out all the mistakes in the proofs proposed by the test. This test does not show a significant change after the course. This means that logical and proof skills must be cultivated with more time and dedicated efforts. One common mistake is believing that a theorem is false if the hypothesis is false, therefore according to some students, a single example that invalidates the hypothesis, invalidates the whole theorem. In the questions of the test, the number of the students making this mistake oscillates from 17% to 28%. In the experiments I conducted, the questionnaires were the last activity that was done with the students, except from the personal interviews done with some of them later. In future experimentation I suggest to conduct a debriefing with the students, discussing their answers to find together the right answers and highlighting their logics pitfalls.

To conclude, the goal of the work was to provide answers to questions in the teaching of non-Euclidean geometries. The methods aimed at providing a quantitative output, particularly useful when the institutions need an input for policy making. Unfortunately, the Covid-19 pandemic made a large scale experiment impossible, however in the future, the same method could be re-enacted on a large scale to have feedback useful for policy-making and, in particular, for a re-introduction of non-Euclidean geometries in high school as a mean to reinforce logic skills, reasoning and the ability of proving a theorem in a mathematical context. The outcome of this whole project seems to validate the original ideas, i.e. that introducing non-Euclidean geometries to high school students can challenge their conceptions of mathematics and improve some of their capabilities as listed above. With respect to the state of the art, this work provides a first reproducible method for teaching non-Euclidean geometries in school and evaluating them accordingly.

7.1 Future works

The research work conducted for this thesis motivates for future works in this field. The experimental phase was hard to accomplish and I would like to provide some suggestions for the realization of further studies. In particular, it is of paramount importance that further studies will have a larger statistical basis, and this requires a thorough planning of the school involvement, some funding to support the purchase of the material and for the preparation of the teachers. Indeed, at first the aim of the project was to instruct educators and teachers in the realization of the seminars themselves. However, it soon turned out that such a work – especially in the conditions of a pandemic – would be overwhelming for the teachers. Planning a series of lessons for the teachers, supporting this extra work and launching a campaign of investigations with their classes can be done only within the context of a project, either national or at the European level.

Having a large statistical basis allows a high likelihood of the data, allows to dissect them and evaluate them separately for different school types and curricula. Another suggestion I would like to give is to prepare some personnel other than the class teachers to interview the students after the course. Interviewing the students requires a large effort, both in the interview itself, and in the transcription and analysis, since this part of the work cannot be automated using spreadsheets or statistical analysis tools.

An even more ambitious project would be to introduce in the whole secondary school a series of lessons with the aim of introducing different perspectives and subjects according to the age of the students. By bringing into the classroom some evidences of spherical geometry from the first three years of the secondary school

(Italian “scuola secondaria di primo grado”, ages 11-14) and then expanding in the high school years (ages 14-19) adding more complexity, the students would gain a more elaborate and mature view of geometry a little at a time, letting concept build one onto another.

More specifically, with students in the age 11-14, I would recommend laboratory exercises on the spherical surface with the intent of playfully discovering its properties, enhancing manual abilities and experiencing another geometry. Later, in the first two years of the high school (“biennio”), the program of these lessons would advance together with the Euclidean geometry lessons, trying to pose questions related to these two different worlds, and analysing some interesting conclusions related to the validity of their postulates. Finally, in the last years of high school, the students would be able to abstract, formalize and grasp the evolution of mathematics and its rules, bridging it with their studies in modern philosophy, especially in the last year. In this regard I want to mention Vinicio Villani, who affirmed that the formalization of a theory comes at last, after experiences of intuition, conjectures and construction: “Anyone who has ever tried to independently prove some mathematical result, however simple it may be, knows well that the search for a proving path is made up of intuitions, conjectures, checks on particular cases, constructions of examples and counterexamples, and only at the end of formalization”[78] (Villani, 1993).

[78] Traslated from the Italian “Chiunque abbia mai cercato di dimostrare autonomamente qualche risultato matematico, per semplice che fosse, sa bene che la ricerca di un percorso dimostrativo si compone di intuizioni, congetture, verifiche su casi particolari, costruzioni di esempi e controesempi, e solo alla fine di formalizzazione”.

Appendix 1

This appendix reports definitions, common notions (or axioms) and postulates found in the version of the *Elements* translated by Thomas Heath.

Definitions that appear in Book I numbering according to Thomas Heat's translation of Euclid's *Elements* (Heath, 1956):

1. *A point is that which has no part.*
2. *A line is breadthless length.*
3. *The extremities of a line are points.*
4. *A straight line is a line which lies evenly with the points on itself.*
5. *A surface is that which has length and breadth only.*
6. *The extremities of a surface are lines.*
7. *A plane surface is a surface which lies evenly with the straight lines on itself.*
8. *A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*
9. *And when the lines containing the angle are straight, the angle is called rectilinear.*
10. *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.*
11. *An obtuse angle is an angle greater than a right angle.*
12. *An acute angle is an angle less than a right angle.*
13. *A boundary is that which is an extremity of anything.*
14. *A figure is that which is contained by any boundary or boundaries.*

The text of the five postulates, from Thomas Heath's translation of Euclid's *Elements*:

Let the following be postulated:

1. *To draw a straight line from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a circle with any centre and distance.*
4. *That all right angles are equal to one another.*
5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

The text of the five axioms, from Thomas Heath's translation of Euclid's *Elements*:

1. *Things which are equal to the same thing are also equal one another.*
2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*
4. *Things which coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

Appendix 2

This appendix reports details on students' answers to items 3 to 7 of the *NEG questionnaires* (second study).

Item 3 (a, b)

Figure 72 shows the scoring to students' answers to the pair of items 3a-3b. The average score in the pre-questionnaire is 0.25 and the standard deviation is 0.13. The average score rises to 0.83 in the post questionnaire, where the standard deviation is 0.31. In the pre-questionnaire, only one of the 56 students (about the 0.02%) correctly writes a case in which the given proposition is true and a case in which it is false. In the post-questionnaire the number of students which correctly answer rises to 44 (about the 78.56%).

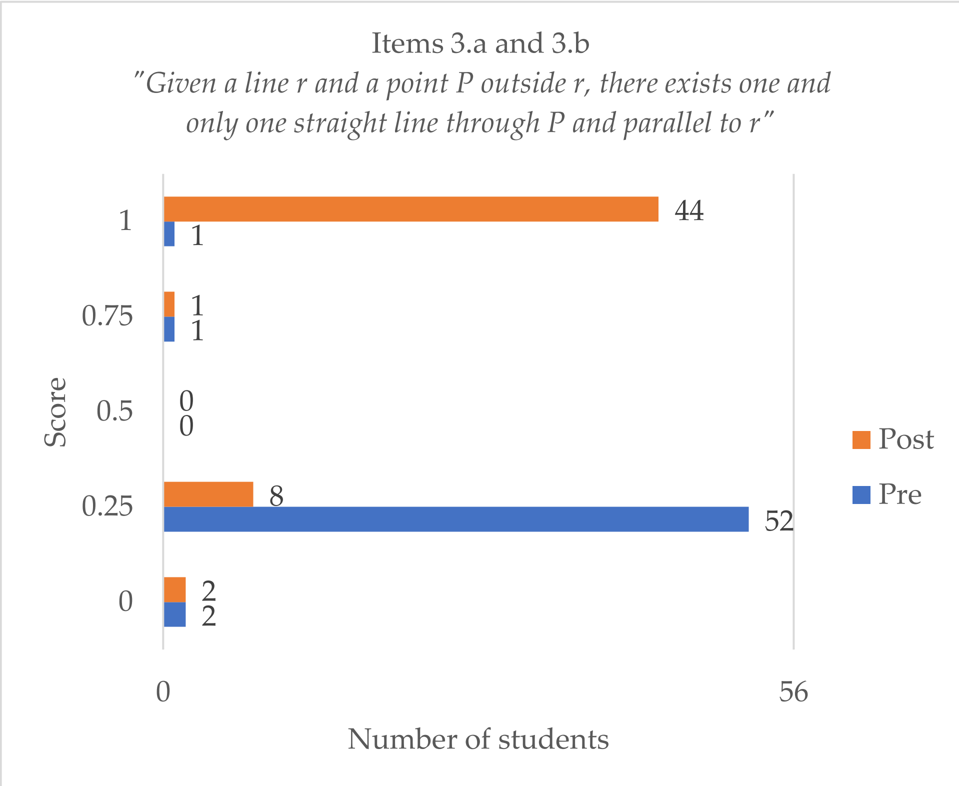


Figure 72. Scoring to students' answers to the pair of items 3a-3b.

Table 23 shows some relevant answers given by the students in the pre-questionnaire and the relative score we assign. We can see that, filling out the pre-questionnaire, students SF9VSC and SM19VSA show not having read or not having understood the definition of parallel straight line I gave. Probably, assuming that two straight line are parallel if and only if they do not intersect (than skew lines are parallel lines), think the case of skew lines as an example of circumstance in which the given proposition (the modern formulation of the fifth Euclidean postulate) is false. Moreover, SF9VSC take for granted that a surface is a Euclidean plane. Instead, student SM39IIICI show to have awareness of the existence of geometry different from the Euclidean one but his answer is not completely correct. Note that two of the three students cited in Table 23 refer to the "Cartesian plane".

Item 3.a-b	Score	Student	True	False
Pre-questionnaire	0.25	SF9VSC	*If the point and the line belong to the same surface*	*If the point and the line do not belong to the same surface*
		SM19VSA	*In case a Cartesian plane with x and y axis is taken into consideration*	*In case the point and the straight line are in the Cartesian plane with x, y and z axes*
	0.75	SM39IIICI	*If I assume that the Cartesian plane is indeed a plane then the statement is true*	*If I assume that the Cartesian plane is a sphere or any three-dimensional figure, the statement is false*

Table 23. Relevant wrong or not completely correct answers given by the students in the pre-questionnaire to sub-item 3.b and the relative score I assign.

Item 3 (c, d, and e)

Table 25 reports data related to students who, in sub-item 3.c of the pre-questionnaire, selected that it is possible to state the negation of the fifth Euclidean postulate, Table 26 reports data related to students who, in sub-item 3.c of the pre-questionnaire, selected that they do not know if it is not possible to state the negation of the fifth Euclidean postulate, Table 27 reports data related to students who, in sub-item 3.c select "Other" as answer. Figure 73 and Figure 74 report, respectively, students' answers to question 3.c students' answers to question 3.d.

Pre-questionnaire		Post-questionnaire
Students who select "No" in sub-item 3.c	**Students' justification (answer to sub-item 3.e):**	**Students' answers to sub-items 3.c and 3.d or 3.e**
SM49IISA	*Because this statement is true ("Perché questo enunciato è vero")*	3.c: *Yes* 3.d: *The statement A*
SM43IISA	*Because in order to deny a statement, a geometric case must be found in which the opposite is proved ("Perché per negare un enunciato bisogna trovare il caso geometrico nel quale venga dimostrato il contrario. Questo enunciato è impossibile da negare")*	
SM1VSC	*Because it is true and evident ("Perché è vero ed evidente")*	3.c: *Yes* 3.d: *The statement C*
SF46IISA	*Because through a point no more than one straight line parallel to r can pass ("Perché in un punto non può passare più di una retta parallela a r")*	3.c: *Yes* 3.d: *The statement E*
SF5VSC	*Because it is not true that several straight lines parallel to r pass through p ("Perché non è vero che per p passano più rette parallele a r")*	3.c: *Yes* 3.d: *The union of the statement A with the statement C*
SM10VSC	*Because to state the negation would be to negate an axiom ("Perché enunciare la negazione significherebbe negare un assioma")*	
SF12VSC	*Because it is the consequence of an axiom ("Perché è la conseguenza di un assioma")*	
SM13VSC	*Because I have considered the previous statement to be true and I cannot consider both statements to be true at the same time ("Perché ho considerato l'enunciato precedente vero e non posso considerare entrambi gli enunciati veri allo stesso tempo")*	
SM14VSC	Does not justify	
SM23IIICI	*Because this statement is always true when applied in a Euclidean plane ("Perché questo enunciato è sempre vero se applicato in un piano euclideo")*	
SF24IIICI	*Because infinite straight lines pass through a point, but only one, passing through that point, can be parallel to another ("Perché per un punto passano infinite rette, ma solo una, passante per quel punto, può essere parallela a un'altra")*	
SM26IIICI	*Because there is only one straight line that is parallel to another at a certain distance ("Perché vi è una sola retta che è parallela ad un'altra ad una determinata distanza")*	
SM29IIICI	*Because infinite straight lines pass through a point, and only a straight line passing through P can be parallel to r because*	

	they have the same inclination ("Perché per un punto passano infinite rette, e solo una retta passante per P può essere parallela a r perché possiedono stessa inclinazione")	
SF31IIICI	*Because it exists ("Perché esiste")*	
SF32IIICI	*Because there will always be a straight line passing through P and parallel to r ("Perché ci sarà sempre una retta passante per P e parallela ad r")*	
SF34IIICI	*Because there is no point P outside a straight line r for which you pass a more than line parallel to r) ("Perché non esiste un punto P esterno a una retta r per cui passi una più di retta parallela a r")*	
SF35IIICI	*Because only a straight line that is parallel to r can pass through the point P ("Perché per il punto P può passare solo una retta che sia parallela ad r")*	
SM37IIICI	*Since it is Euclid's fifth postulate, and as such it cannot be proved, but must always be considered true ("Poiché è il quinto postulato di Euclide, e come tale non può essere dimostrato, ma deve sempre essere considerato vero")*	
SF41IIICI	*Because denying the statement it would become: given a straight line and a point P external to r there is no single line passing through P and parallel to r and this implies that there can be more parallel straight lines passing through P ("Perché negando l'enunciato diventerebbe: data una retta e un punto P esterno a r non esiste una sola retta passante per P e parallela a r e questo implica che ci possono essere più rette parallele e passanti per P"*	
SM5VSC	*Because it can only be true ("Perché non può che essere vero")*	3.c: *Yes* 3.d: *The union of the statement A with the statement E*
SM16VSA	*Because only a straight line parallel to r can pass at the point p ("Perché può passare solo una retta parallela a r nel punto p")*	
SF18VSA	*Because the statement at the end would be false ("Perché l'enunciato alla fine sarebbe falso")*	
SM21VSA	*Because only one straight line passes through a point ("Perché per un punto passa una sola retta")*	3.c: *Yes* 3.d: *The union of the statement B with the statement D*
SF53IISA	*Because in a point they cannot pass more than one parallel to r ("Perché in un punto non possono passare più di una retta parallela a r")*	3.c: *Yes* 3.d: *The union of the statement B with the statement E*
SM15VSA	*We have a graphic answer, we can see that drawing on P n. straight lines, only one will be parallel to r ("Abbiamo una*	3.c: *Yes*

	risposta grafica, possiamo notare che disegnando su P n. rette, una sola sarà parallela ad r")	3.d: *The union of the statement C with the statement D*
SF27IIICI	*Because given a straight line r a point P external to r, there is only one straight line passing through P and parallel to r. In theory, one could add a "not" somewhere in the sentence, but then the statement would not be true ("Perché data una retta r un punto P esterno a r, esiste una sola retta passante per P e parallela a r. In linea teorica, si potrebbe aggiungere un "non" da qualche parte nella frase, ma poi l'enunciato non sarebbe vero.")*	3.c: *Yes* 3.d: *None of the previous option*
SM11VSC	*Because using the graphical verification test, drawing a straight line r and a point P external to this line, only one straight line parallel to the given line will pass through it ("Perché utilizzando la prova di verifica grafica, tracciando una retta r e un punto P esterno a questa retta, per esso passerà una sola retta parallela ala retta data")*	3.c: *I cannot answer*
SM44IISA	*because the statement is true therefore it cannot be denied ("Perché l'enunciato è vero quindi non si può negare")*	
SM48IISA	*Because the straight lines cannot be accidents, since P is not inside r. Then there can be only one straight line passing through P and parallel to r ("Perché le rette non posso essere incidenti, non essendo P all'interno r. Poi ci può essere una sola rette passante per P e parallela a r")*	
SF45IISA	*Because it is impossible for a straight line r and a point P external to it not to pass any straight line with those characteristics ("Perché è impossibile che per una retta r e un punto P esterno a essa non passi nessuna retta con quelle caratteristiche"*	
SM55IISA	Does not justify	
SM20IIICI	*Because the previous statement is true as there is only one straight line passing through P parallel to r ("Perché l'enunciato precedente è vero in quanto esiste una sola retta passante per P parallela a r")*	3.c: *No* 3.e: *because if we are in a plane the statement is always true ("perché se siamo in un piano l'enunciato è sempre vero")*
SF8VSC	*Because this happens in Euclidean geometry. the sentence is a postulate of Euclid ("Perché è in geometria euclidea avviene ciò. la frase è un postulato di Euclide")*	3.c: *No* 3.e: Does not justify
SM9VSC	*I have to add 'belonging to the same surface' ("devo aggiungere 'appartenenti alla stessa superficie'")*	

SM17VSA	*Because if it is a postulate, the statement is always true and verified "Perché se è un postulato l'enunciato è sempre vero e verificato")*	3.c: Other: *It is possible to state the negation only in some cases, as when we consider for example the spherical surface ("é possibile enunciare la negazione solo in alcuni casi, come quando consideriamo per esempio la superficie sferica")*
SM30IIICI	*Since two parallel lines are equidistant, then only one line, to be parallel to the line r, can satisfy this property (Euclid's fifth postulate). Indeed it must have a certain type of angle, and only in this way will the two lines be parallel ("Poichè due rette parallele sono equidistanti, allora solo una retta, per essere parallela alla retta r, può soddisfare questa proprietà (quinto postulato di Euclide). Infatti deve avere un certo tipo di angolazione, e solo in questo modo le due rette saranno parallele")*	3.c: Other: *it depends on which geometry we are referring to ("dipende a quale geometria facciamo riferiamo")*
SF47IISA	*Because a statement is always true, in fact there are infinite lines that can pass through the point P but only one of these is parallel to the line r ("Perché un enunciato è sempre vero, infatti ci sono infinite rette che possono passare per il punto P ma solo una di queste è parallela alla retta r")*	

Table 24. Data related to students who – in sub-item 3.c – selected that it is not possible to state the negation of the fifth Euclidean postulate (NEG pre-questionnaire, second study).

<table>
<tr><th colspan="2">Pre-questionnaire</th><th>Post-questionnaire</th></tr>
<tr><th>Students who select "Yes" in sub-item 3.c</th><th>Students' answer in sub-item 3.d</th><th>Students' answer in sub-items 3.c and 3.d</th></tr>
<tr><td>SF2VSC</td><td>The union of the statement A with the statement C</td><td rowspan="7">3.c: Yes
3.d: The union of the statement A with the statement C</td></tr>
<tr><td>SF6VSC</td><td>The union of the statement C with the statement E</td></tr>
<tr><td>SM25IIICI</td><td rowspan="2">The union of the statement D with the statement E</td></tr>
<tr><td>SM42IISA</td></tr>
<tr><td>SM33IIICI</td><td rowspan="2">The statement B</td></tr>
<tr><td>SM39IIICI</td></tr>
<tr><td>SM28IIICI</td><td>The statement E</td></tr>
<tr><td>SM52IISA</td><td>The statement B</td><td>3.c: No
3.d: Because it is impossible ("Perché è impossibile")</td></tr>
<tr><td>SM56IISA</td><td>The statement D</td><td>3.c: I cannot answer</td></tr>
<tr><td>SM54IISA</td><td>The statement E</td><td>3.c: Other: Yes, but only in certain circumstances ("si ma in certe circostanze")</td></tr>
</table>

Table 25. Data related to students who, in sub-item 3.c of the pre-questionnaire, selected that it is possible to state the negation of the fifth Euclidean postulate.

<table>
<tr><th>Students who select "I cannot answer" in item 3.c of the pre-questionnaire</th><th>Students' answers to sub-items 3.c and 3.d or 3.e of the post-questionnaire</th></tr>
<tr><td>SF7VSC</td><td rowspan="4">3.c: Yes
3.d: The union of the statement A with the statement C</td></tr>
<tr><td>SM36IIICI</td></tr>
<tr><td>SM38IIICI</td></tr>
<tr><td>SM40IIICI</td></tr>
<tr><td>SM19VSA</td><td>3.c: I cannot answer</td></tr>
</table>

Table 26. Data related to students who, in sub-item 3.c of the pre-questionnaire, selected that they do not know if it is not possible to state the negation of the fifth Euclidean postulate.

Students who select "Other" in item 3.c of the pre-questionnaire	Students' justification in the pre-questionnaire	Students' answers to sub-items 3.c and 3.d or 3.e of the post-questionnaire
SF6VSC	*"Non capisco il senso della domanda" (I don't understand the meaning of the question)*	*Sì → L'unione dell'affermazione A con l'affermazione C.*
SM22VSA		
SM50IISA	*"Dipende se può essere considerata la retta stessa" (It depends on whether the line itself can be considered)*	*3.c: Other: "si ma in certe circostanze" (Yes but only in certain circumstances)*
SF51IISA	*"Allora direi si nel caso si può disegnare una seconda retta coincidente con quella parallela a r, questa retta viene chiamata differente." (Then I would say yes in case you can draw a second line coinciding with the one parallel to r, this line is called different.)*	*3.c: Other: "Sì, a seconda della geometria che prendiamo in considerazione, nella geometria euclidea ciò sarebbe impossibile mentre possibile in quella sferica e iperbolica." (Yes, depending on the geometry we take into consideration, in Euclidean geometry this would be impossible while possible in the spherical and hyperbolic one.)*

Table 27. Data related to students who, in sub-item 3.c select "Other" as answer.

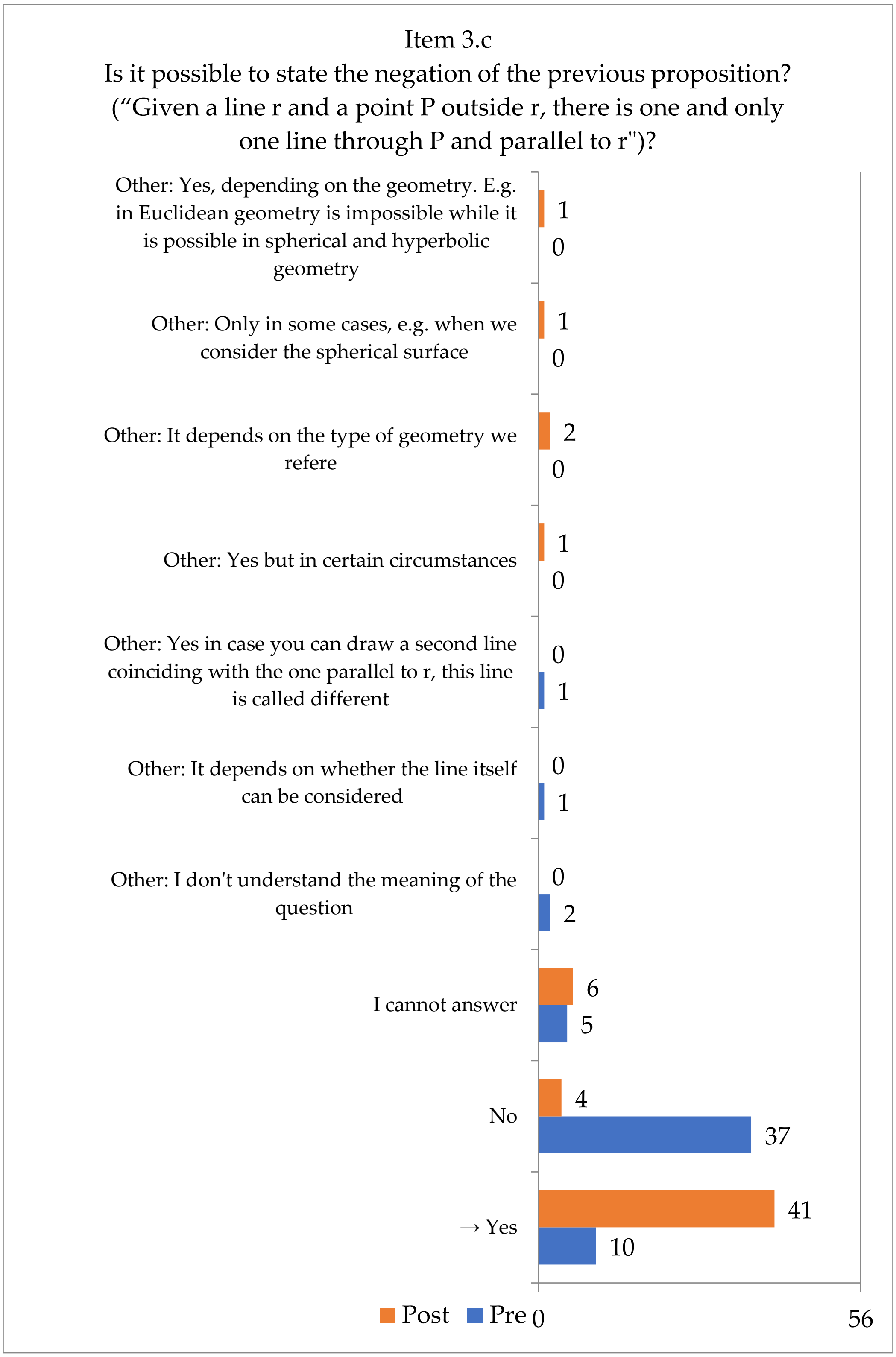


Figure 73. Students' answers to question 3.c.

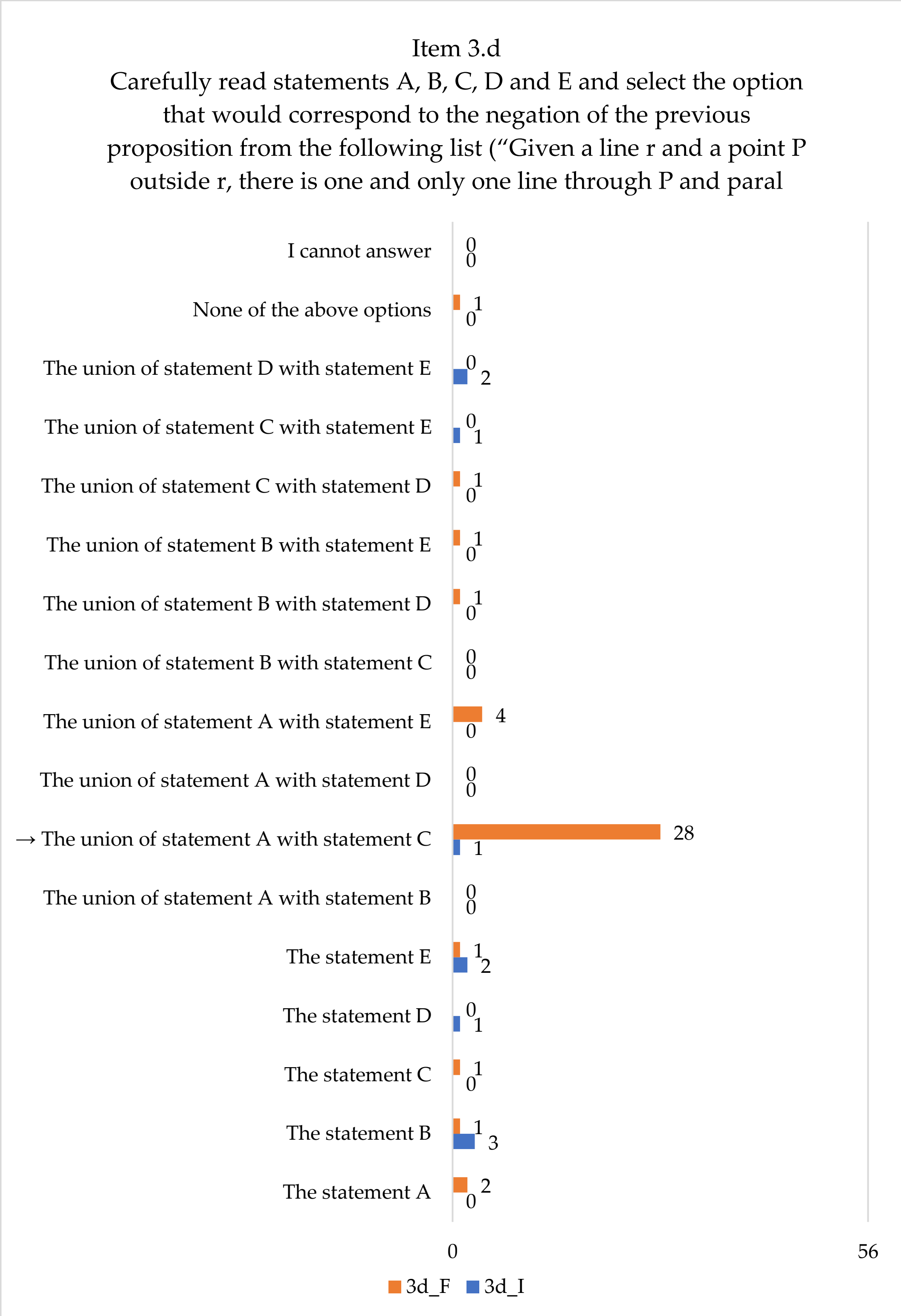


Figure 74. Students' answers to question 3.d. This question is activated only if "Yes" was selected in sub-item 3.c.

Item 4

Figure 75 shows the scoring to students' answers to the pair of items 4a-4b. The average score in the pre-questionnaire is 0.24 and the standard deviation is 0.11. The average score rises to 0.71 in the post questionnaire, where the standard deviation is 0.33. In the pre-questionnaire no students correctly write a case in which the given proposition is true and a case in which it is false. In the post-questionnaire the number of students which correctly answer rises to 44 (about the 82.76%).

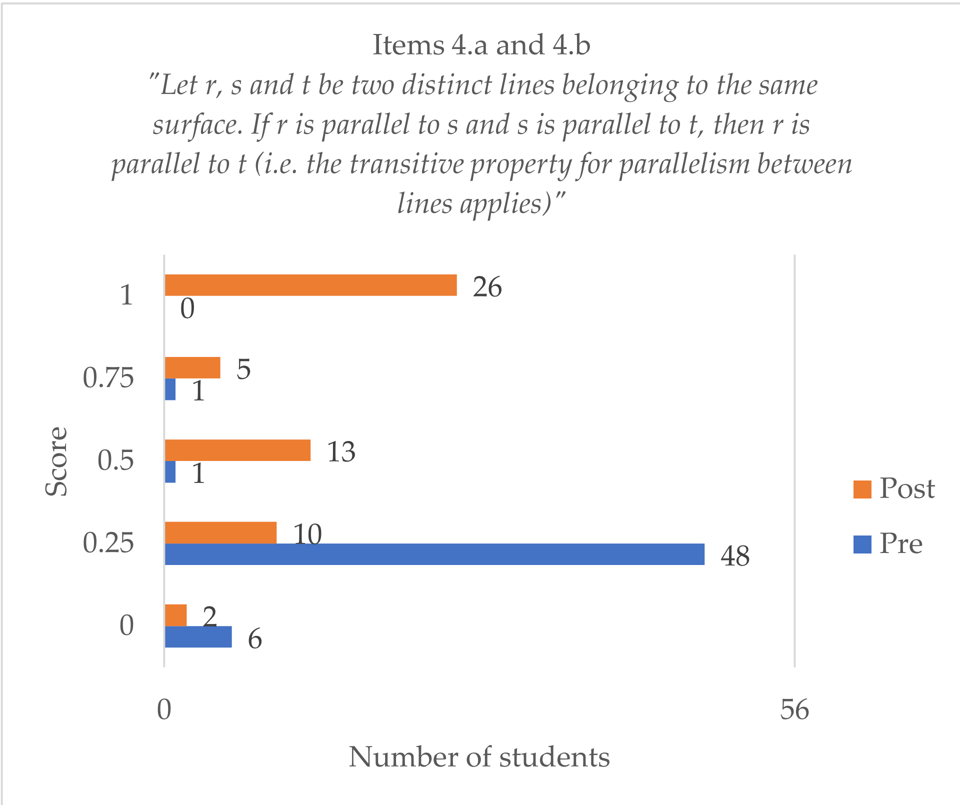


Figure 75. Scoring to students' answers to the paoir of items 4a-4b.

Item 5

Figure 76 shows the scoring to students' answers to the pair of items 5a-5b. The average score in the pre-questionnaire is 0.22. This rises to 0.73 in the post questionnaire. In the pre-questionnaire no students correctly write a case in which the given proposition is true and a case in which it is false. In the post-questionnaire the number of students which correctly answer rises to 30 (about the 53.57%).

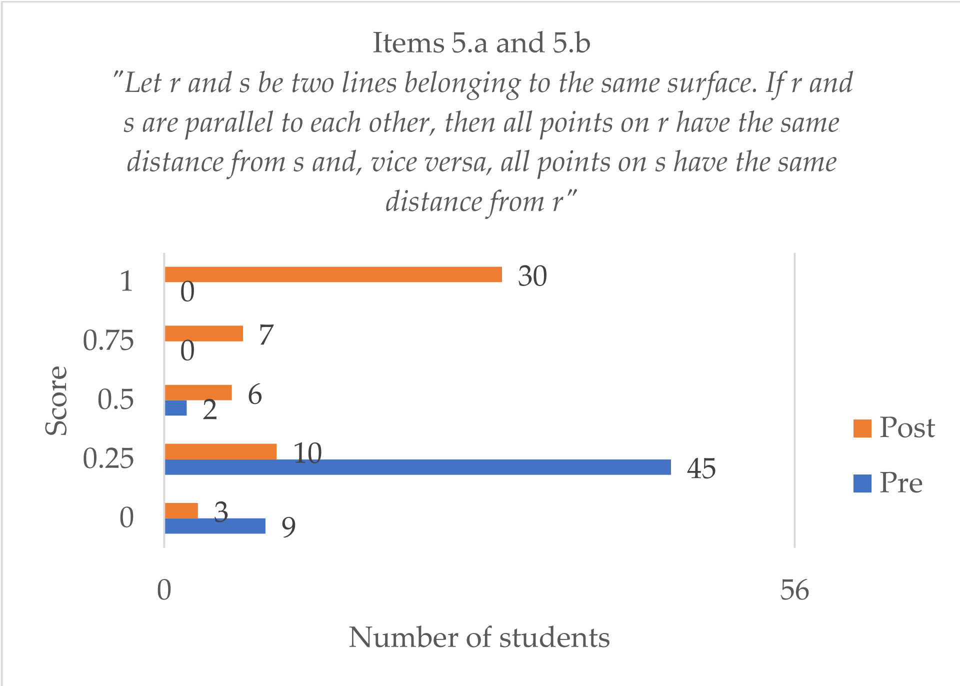


Figure 76. Scoring to students' answers to the paoir of items 3a-3b.

Item 6

Figure 77 shows the scoring to students' answers to the pair of items 6a-6b. The average score in the pre-questionnaire is 0.24 and the standard deviation is 0.11. The average score rises to 0.82 in the post questionnaire, where the standard deviation is 0.34. In the pre-questionnaire no students correctly write a case in which the given proposition is true and a case in which it is false. In the post-questionnaire the number of students which correctly answer rises to 44 (about the 78.57%).

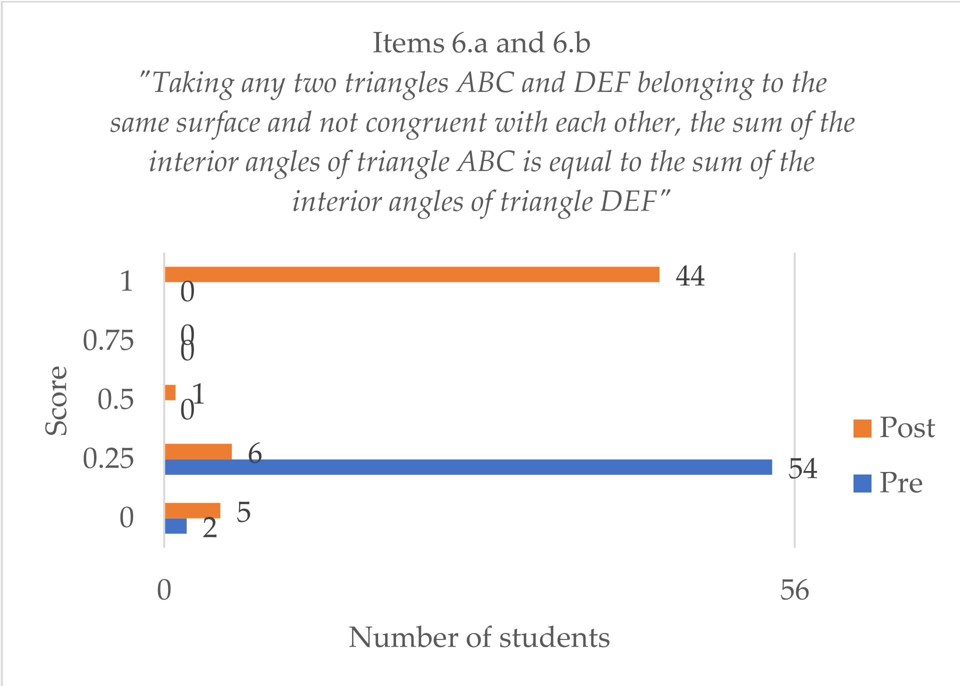


Figure 77. Scoring to students' answers to the paoir of items 3a-3b.

Item 7

Figure 78 shows the scoring to students' answers to the pair of items 7a-7b of the post-questionnaire.

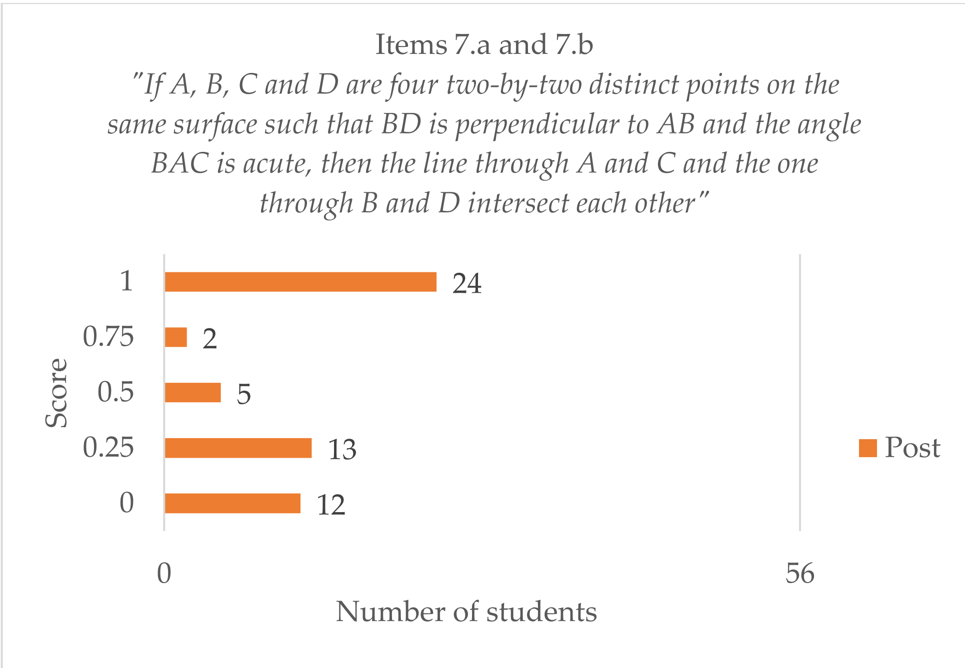


Figure 78. Scoring to students' answers to the paoir of items 7a-7b.

We note that this is the question with the lower number of completely correct answers. To monitor whether the scores I assign to each student for her/his answer to question 7 is linearly related to the ones assigned to questions 3, 4, 5, and 6, I performed the Pearson correlation coefficient r. Given the pair of variables (X, Y), the formula for r is:

$$r = \frac{\rho_{X,Y}}{\rho_X \rho_Y}$$

where:

ρ_X is the standard deviation of X: $\rho_X = \sqrt{\sum_i^n (x_i - \bar{x})^2}$ ($\bar{x}$ is the arithmetic average of X);

$\rho_{X,Y}$ is the covariance between X and Y: $\rho_{X,Y} = \sum_i^n [(x_i - \bar{x})(y_i - \bar{y})]$.

In this case, n is the number of students ($n = 56$), and the pair of variables we consider are (X, Y_q), where:

X is the 56x1 vector composed by the score assigned to each student for her/his answer to question 7;

Y is the 56x1 vector composed by the average of the score assigned to each student for her/his answer to question q, $q = 3,4,5,6$.

The Pearson correlation coefficients performed between X and Y is $r = 0.51$. The Pearson correlation coefficient performed is positive. This despite, unlike what concern proposition in item from 3 to 6, no laboratory activities were carried out on the proposition object of question 7. Moreover, answering question 7 requires greater attention. Indeed, students also must draw an appropriate graph before answering.

An observation on students' answers to question 7 of the final questionnaire deserves attention. More details follow. Of the 24 students who answer correctly showing the Euclidean surface as a circumstance in which the given proposition given is true, and showing the hyperbolic surface as a circumstance in which the given proposition is false. 7 of these students also add the spherical surface as a circumstance in which the proposition is true. None of these students attend the V class, 5 of them attend class III and 2 the class II.

Appendix 3

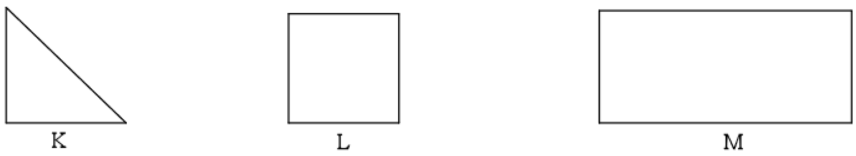
This appendix reports the *VHL test*.

VHL test – Original version[79]

TEST (25 quesiti – 45 minuti)

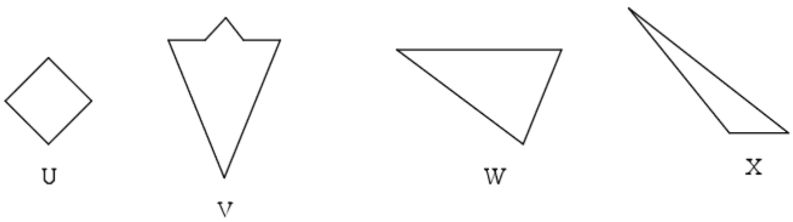
1) Nel piano euclideo, quali di questi sono quadrati?

a) Solo K.
b) Solo L.
c) Solo M.
d) Solo L e M.
e) Sono tutti quadrati.



2) Nel piano euclideo, quali di questi sono triangoli?

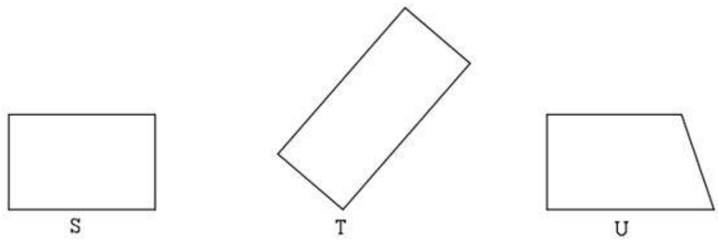
a) Nessuno.
b) Solo V.
c) Solo W.
d) Solo W e X.
e) Solo V e W.



[79] Administrated via Google Form.

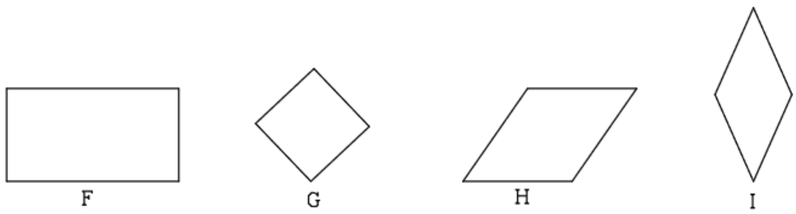
3) Nel piano euclideo, quali di questi sono rettangoli?

a) Solo S.
b) Solo T.
c) Solo S e T.
d) Solo S e U.
e) Tutti.



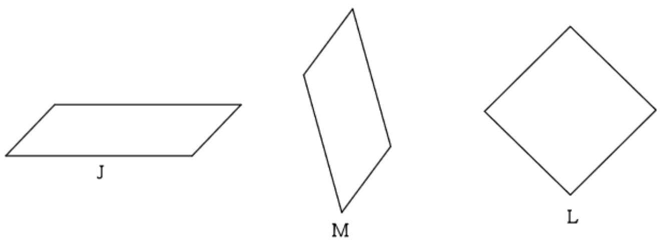
4) Nel piano euclideo, quali di questi sono quadrati?

a) Nessuno.
b) Solo G.
c) Solo F e G.
d) Solo G e I.
e) Tutti.



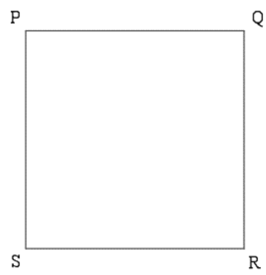
5) Nel piano euclideo, quali di questi sono parallelogrammi?

a) Solo J.
b) Solo L.
c) Solo J e M.
d) Nessuno.
e) Tutti.

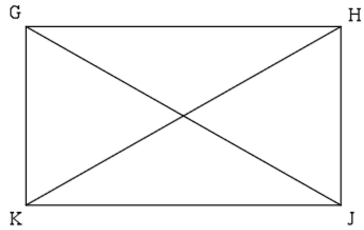


6) PQRS è un generico quadrato nel piano euclideo.

Tra le seguenti relazione, quale è vera?
a) PR e RS hanno la stessa lunghezza.
b) QS e PR sono perpendicolari.
c) PS e QR sono perpendicolari.
d) PS e QS hanno la stessa lunghezza.
e) L’angolo in Q è più grande dell’angolo in R.

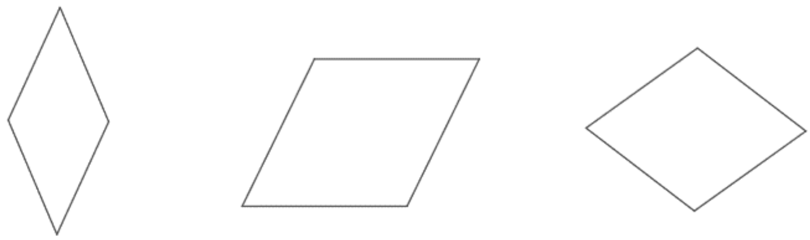


7) GHJK è un generico rettangolo nel piano euclideo, GJ e HK sono le sue diagonali. Tra le prime quattro seguenti affermazioni, quale NON è vera? [Indica la quinta opzione se credi che le precedenti siano tutte vere.]



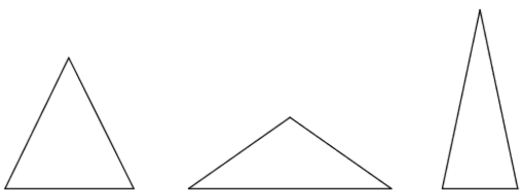
a) In ogni rettangolo del piano euclideo ci sono quattro angoli retti.
b) In ogni rettangolo del piano euclideo ci sono quattro lati.
c) In ogni rettangolo del piano euclideo le diagonali hanno la stessa lunghezza.
d) In ogni rettangolo del piano euclideo i lati opposti hanno la stessa lunghezza.
e) Le precedenti affermazioni sono tutte vere.

8) Nel piano euclideo, un rombo è una figura di quattro lati con tutti i lati della stessa lunghezza. Nella figura qui sotto ci sono tre esempi. Tra le prime quattro seguenti affermazioni, quale NON è vera? [Indica la quinta opzione se credi che le precedenti siano tutte vere.]



a) Nel piano euclideo, in ogni rombo le due diagonali hanno la stessa lunghezza.
b) Nel piano euclideo, in ogni rombo ogni diagonale biseca due angoli del rombo.
c) Nel piano euclideo, in ogni rombo le due diagonali sono perpendicolari.
d) Nel piano euclideo, in ogni rombo gli angoli opposti hanno la stessa misura.
e) Le precedenti affermazioni sono tutte vere.

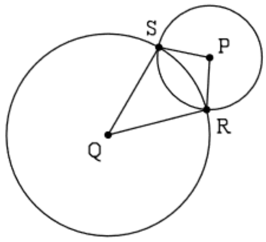
9) Nel piano euclideo, un triangolo isoscele è un triangolo con due lati della stessa lunghezza. Nella figura qui sotto ci sono tre esempi. Tra le prime quattro seguenti affermazioni, quale è vera? [Indica la quinta opzione se credi che nessuna delle precedenti sia vera.]

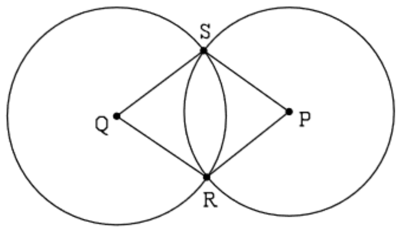


a) Nel piano euclideo, in ogni triangolo isoscele i tre lati devono avere la stessa lunghezza.

b) Nel piano euclideo, in ogni triangolo isoscele un lato deve essere lungo il doppio di un altro lato.
c) Nel piano euclideo, in ogni triangolo isoscele ci devono essere almeno due angoli della stessa Misura.
d) Nel piano euclideo, in ogni triangolo isoscele i tre angoli devono avere la stessa misura.
e) Nessuna delle precedenti affermazioni è vera.

10) Consideriamo il caso in cui due cerchi del piano euclideo di centri P e Q si intersecano in due punti (R e S) formando una figura di quattro lati PRQS. In figura ci sono due esempi. Tra le prime quattro seguenti affermazioni, quale NON è sempre vera? [Indica la quinta opzione se credi che le precedenti siano tutte sempre vere.]





a) PRQS avrà due coppie di lati di uguale lunghezza.
b) PRQS avrà almeno due angoli della stessa misura.
c) I segmenti PQ e RS saranno perpendicolari.
d) L'angolo in P e l'angolo in Q avranno la stessa misura.
e) Le precedenti affermazioni sono tutte sempre vere.

11) Leggi i due enunciati scritti qui sotto (Enunciato 1 ed Enunciato 2) e indica quale delle affermazioni di seguito riportate è corretta.

Enunciato 1: "La Figura F è un rettangolo del piano euclideo".

Enunciato 2: "La Figura F è un triangolo del piano euclideo".

a) Se 1 è vero allora 2 è vero.
b) Se 1 è falso allora 2 è vero.
c) 1 e 2 non possono essere entrambi veri.
d) 1 e 2 non possono essere entrambi falsi.
e) Nessuna delle precedenti affermazioni è corretta.

12) Leggi i due enunciati scritti qui sotto (Enunciato S ed Enunciato T) e indica quale delle affermazioni di seguito riportate è corretta.

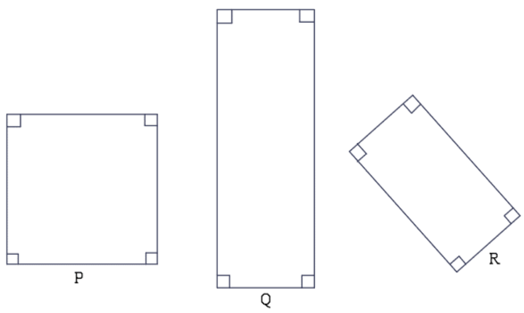
Enunciato S: “Il triangolo ABC del piano euclideo ha tre lati della stessa lunghezza”.

Enunciato T: “Nel triangolo ABC del piano euclideo gli angoli $\widehat{B}$ e $\widehat{C}$ hanno la stessa misura”.

a) Gli enunciati S e T non possono essere entrambi veri.
b) Se S è vero allora T è vero.
c) Se T è vero allora S è vero.
d) Se S è falso allora T è falso.
e) Nessuna delle precedenti affermazioni è corretta.

13) Quali di questi possono essere chiamati rettangoli nel piano euclideo?

a) Tutti.
b) Solo Q.
c) Solo R.
d) Solo P e Q.
e) Solo Q e R.



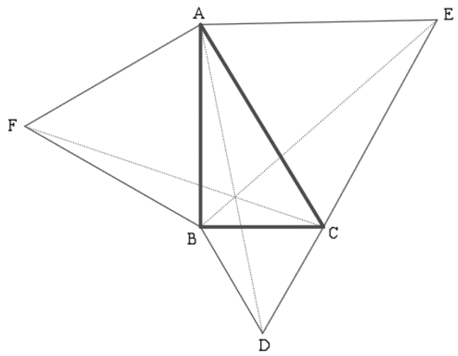
14) Tra le prime quattro seguenti affermazioni, quale è vera? [Indica la quinta opzione se credi che nessuna delle precedenti sia vera.]

a) Nel piano euclideo, le proprietà dei rettangoli sono proprietà anche dei quadrati.
b) Nel piano euclideo, le proprietà dei quadrati sono proprietà anche dei rettangoli.
c) Nel piano euclideo, le proprietà dei rettangoli sono proprietà anche dei parallelogrammi.
d) Nel piano euclideo, le proprietà dei quadrati sono proprietà anche dei parallelogrammi.
e) Nessuna delle precedenti affermazioni è vera.

15) Tra le prime quattro seguenti proprietà, quale è valida per ogni rettangolo ma NON è valida per ogni parallelogramma nel piano euclideo? [Indica la quinta opzione se credi che nessuna delle precedenti soddisfi la richiesta indicata.]

a) Lati opposti congruenti.
b) Diagonali congruenti.
c) Lati opposti paralleli.
d) Angoli opposti congruenti.
e) Nessuna delle precedenti proprietà soddisfa la richiesta indicata.

16) Sia ABC un generico triangolo rettangolo del piano euclideo sui cui lati sono stati costruiti tre triangoli equilateri ACE, ABF e BCD, ognuno dei quali ha in comune con ABC solo un lato. È stato dimostrato che i segmenti AD, BE e CF hanno un punto in comune. Quale delle seguenti affermazioni è vera?



a) Solo per il triangolo rettangolo (del piano euclideo) qui rappresentato possiamo essere sicuri che AD, BE e CF hanno un punto in comune.
b) In alcuni ma non in tutti i triangoli rettangoli del piano euclideo, AD, BE e CF hanno un punto in comune.
c) In ogni triangolo rettangolo del piano euclideo, AD, BE e CF hanno un punto in comune.
d) In ogni triangolo del piano euclideo, AD, BE e CF hanno un punto in comune.
e) In ogni triangolo equilatero del piano euclideo, AD, BE e CF hanno un punto in comune.

17) Leggi i tre enunciati, riferiti a una figura, scritti qui sotto (Enunciato D, Enunciato Q ed Enunciato R):

Enunciato D: “Ha diagonali della stessa lunghezza”.
Enunciato Q: “È un quadrato”.
Enunciato R: “È un rettangolo”.

Quale tra le seguenti affermazioni è vera?

a) D implica Q che implica R.
b) D implica R che implica Q.
c) Q implica R che implica D.
d) R implica D che implica Q.
e) R implica Q che implica D.

18) Leggi i due enunciati scritti qui sotto (I e II) e indica quale delle affermazioni di seguito riportate è corretta.

I. "Se una figura è un rettangolo nel piano euclideo allora le sue diagonali si bisecano vicendevolmente".

II. "Se le diagonali di una figura nel piano euclideo si bisecano vicendevolmente allora la figura è un rettangolo".

a) Per dimostrare che I è vero sarebbe sufficiente dimostrare che II è vero.
b) Per dimostrare che II è vero sarebbe sufficiente dimostrare che I è vero.
c) Per dimostrare che II è vero sarebbe sufficiente trovare un rettangolo nel piano euclideo le cui diagonali si bisecano vicendevolmente.
d) Per dimostrare che II è falso sarebbe sufficiente trovare una figura nel piano euclideo che non sia un rettangolo le cui diagonali si bisecano vicendevolmente.
e) Nessuna delle precedenti affermazioni è corretta.

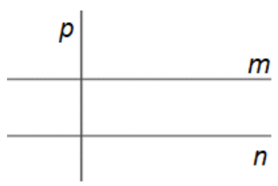
19) In geometria, quale delle seguenti affermazioni è corretta?

a) Ogni termine può essere definito ed è possibile dimostrare la verità di ogni enunciato vero.
b) Ogni termine può essere definito ma è necessario assumere che alcuni enunciati siano veri.
c) Qualche termine deve essere lasciato non definito ma si deve poter dimostrare la verità di ogni enunciato vero.
d) Qualche termine deve essere lasciato non definito ed è necessario assumere che alcuni enunciati siano veri.
e) Nessuna delle affermazioni (a), (b), (c), (d), è corretta.

20) Esamina le tre seguenti proposizioni. Esamina le tre seguenti proposizioni (1, 2 e 3) poi osserva la figura del piano euclideo rappresentata nell'immagine. Sia dato che le rette m e p sono perpendicolari tra loro e che le rette n e p sono perpendicolari tra loro. Quale delle seguenti affermazioni può essere la ragione per cui la retta m è parallela alla retta n?

(1) "Nel piano euclideo, due rette perpendicolari a una stessa retta sono parallele".
(2) "Nel piano euclideo, una retta che è perpendicolare a una di due rette tra loro parallele è perpendicolare anche all'altra".
(3) "Nel piano euclideo, se due rette sono equidistanti allora sono parallele".

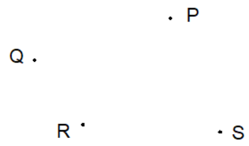
a) Solo (1).
b) Solo (2).
c) Solo (3).
d) Sia (1) sia (2).
e) Sia (2) sia (3).



21) In una F-geometria, una geometria diversa da quella euclidea, ci sono esattamente quattro punti e sei rette. Ogni retta contiene esattamente due punti. Se i punti sono P, Q, R e S, le rette sono {P,Q}, {P,R}, {P,S}, {Q,R}, {Q,S}, e {R,S}. Qui di seguito è scritto come i termini "intersecarsi" e "parallelo" sono usati nella F-geometria:

"Le rette {P,Q} e {P,R} si intersecano in P perché {P,Q} e {P,R} hanno in comune il punto P";
"Le rette {P,Q} e {R,S} sono parallele perché non hanno punti in comune".

In base alle precedenti informazioni, quale delle seguenti affermazioni è corretta?



a) {P,R} e {Q,S} si intersecano.
b) {P,R} e {Q,S} sono parallele.
c) {Q,R} e {R,S} sono parallele.
d) {P,S} e {Q,R} si intersecano.
e) Nessuna delle affermazioni (a), (b), (c), (d),è corretta.

22) TRISECARE un angolo significa dividerlo in tre parti di uguale misura. Nel 1847, P. L. Wantzel dimostrò che, in generale, è impossibile trisecare angoli utilizzando soltanto un compasso e un righello NON graduato. Quale delle seguenti affermazioni si può dedurre dalla sua dimostrazione? [Leggi con attenzione.]

a) In generale, è impossibile BISECARE angoli usando solamente un compasso e un righello non graduato.
b) In generale, è impossibile TRISECARE angoli usando solamente un compasso e un righello graduato.
c) In generale, è impossibile TRISECARE angoli usando qualsiasi strumento da disegno.
d) È ancora possibile che in futuro qualcuno possa trovare un modo generale per TRISECARE angoli usando solamente un compasso e un righello non graduato.
e) Nessuno potrà mai trovare un metodo generale per TRISECARE angoli usando solamente un compasso e un righello non graduato.

23) Esiste una geometria inventata da un matematico J in cui il seguente enunciato è vero: "La somma delle misure degli angoli di un triangolo è minore di 180°". Quale delle prime quattro seguenti affermazioni se ne può dedurre? [Indica la quinta opzione se credi che non è possibile dedurre nessuna delle affermazioni precedenti.]

a) J ha commesso un errore nel misurare gli angoli del triangolo.
b) J ha commesso un errore di logica nel suo ragionamento.
c) J ha una idea errata di che cosa si debba intendere con il termine "vero".
d) J è partito da assunzioni differenti da quelle della geometria euclidea.
e) Non è possibile dedurre nessuna delle precedenti affermazioni.

24) Due testi di geometria definiscono uno stesso termine in maniera differente. Quale delle seguenti affermazioni è corretta?

a) Una delle due definizioni è certamente errata.

b) È certo che almeno una proprietà della figura geometrica indicata con il termine in questione in uno dei due testi non vale per la figura geometrica indicata dallo stesso termine nell'altro testo.

c) È certo che tutte le proprietà della figura geometrica indicata con il termine in questione in uno dei due testi devono valere anche per la figura geometrica indicata con lo stesso termine nell'altro testo.

d) Le proprietà della figura geometrica indicata con il termine in questione in uno dei due testi potrebbero essere differenti dalle proprietà della figura geometrica indicata con lo stesso termine nell'altro testo.

e) Nessuna delle precedenti affermazioni è corretta.

25) Supponi di avere dimostrato gli enunciati I e II scritti qui sotto. Quale degli enunciati riportati tra le cinque opzioni segue dagli enunciati I e II?

I. Se *p*, allora *q*. II. Se *s*, allora *non q*.

a) Se *p*, allora *s*.

b) Se *non p*, allora *non q*.

c) Se *p o q*, allora *s*.

d) Se *s*, allora *non p*.

e) Se *non s*, allora *p*.

Appendix 4

This appendix reports the *NEG questionnaire*(p. 244) and the *NEG questionnaire* (p. 250).

NEG pre-questionnaire – Original version[80]

QUESTIONARIO I2 (7 quesiti articolati – 50 minuti)

NB. Per evitare fraintendimenti, leggi bene la definizione di "RETTE TRA LORO PARALLELE" e quella di "RETTE TRA LORO INCIDENTI" che assumiamo valide (le troverai ripetute prima di ogni quesito, anche se i termini non faranno parte del testo). Diciamo che “due rette sono tra loro parallele" se appartengono a una stessa superficie e non hanno punti in comune. Diciamo che "due rette sono tra loro incidenti” se hanno uno e un solo punto in comune.

1. Un assioma (detto anche postulato) è un enunciato che, senza essere preventivamente dimostrato, si assume come fondamento di una teoria assiomatica (detta anche sistema assiomatico). Tale enunciato, per essere detto “assioma” (o “postulato”), deve anche essere evidente?

- Sì.
- No.
- Non so rispondere.
- Altro: ………………………………………………………………………………………………………

Motiva la precedente risposta: ……………………………………………………………………………………………
……….

2. Qual è la proprietà fondamentale che una teoria assiomatica (detta anche sistema assiomatico) dovrebbe soddisfare?

- Coerenza.
- Completezza.
- Evidenza.
- Indipendenza.

[80] Administrated via Google Form.

- Verità.
- Non so rispondere.
- Altro: …………………………………………

3.

a) Leggi il seguente enunciato precedente: "Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r".

Tra le seguenti affermazioni, seleziona quella corretta:

- L'enunciato precedente è vero.
- L'enunciato precedente è falso.
- L'enunciato precedente è vero in alcune circostanze e falso in altre.
- Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r") è vero: …………………….

- Indica almeno un caso in cui l'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r") è falso: …………………..

c) È possibile enunciare la negazione dell'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r")?

- Sì.
- No.
- Non so rispondere.
- Altro: ……………………………………………………………………………………………………

d) *[Tale quesito si attiva solo se al punto c è stato selezionato "Sì"]*

Leggi attentamente le affermazioni A, B, C, D ed E e seleziona l'opzione che, tra quelle di seguito riportate, corrisponderebbe alla negazione dell'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r").

A: "Data una retta r e un punto P esterno a r, <u>non esistono rette</u> passanti per P e parallele a r ".
B: "Data una retta r e un punto P esterno a r, <u>esiste almeno una retta</u> passante per P e parallele a r ".
C: "Data una retta r e un punto P esterno a r, <u>esiste più di una retta</u> passante per P parallele a r ".

D: “Data una retta r e un punto P esterno a r, <u>esistono molte rette</u> passanti per P e parallele a r ”.
E: “Data una retta r e un punto P esterno a r, <u>esistono infinite rette</u> passanti per P parallele a r ”.

- L’affermazione A.
- L’affermazione B.
- L’affermazione C.
- L’affermazione D.
- L’affermazione E.
- L’unione dell'affermazione A con l'affermazione B.
- L’unione dell'affermazione A con l'affermazione C.
- L’unione dell'affermazione A con l'affermazione D.
- L’unione dell'affermazione A con l'affermazione E.
- L’unione dell'affermazione B con l'affermazione C.
- L’unione dell'affermazione B con l'affermazione D.
- L’unione dell'affermazione B con l'affermazione E.
- L’unione dell'affermazione C con l'affermazione D.
- L’unione dell'affermazione C con l'affermazione E.
- L’unione dell'affermazione D con l'affermazione E.
- Nessuna delle precedenti opzioni.
- Non so rispondere.

e) *[Tale quesito si attiva solo se al punto c è stato selezionato “No”]*
Spiega perché non sarebbe possibile enunciare la negazione dell'enunciato precedente (“Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r”): ………
………

4.

a) Leggi il seguente enunciato: “Siano r, s e t rette a due a due distinte appartenenti a una stessa superficie. Se r è parallela a s e s è parallela a t allora r è parallela a t (ossia, vale la proprietà transitiva per il parallelismo tra rette)”.
Tra le seguenti affermazioni, seleziona quella corretta:

- L'enunciato precedente è vero.
- L'enunciato precedente è falso.
- L'enunciato precedente è vero in alcune circostanze e falso in altre.
- Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Siano r, s e t rette a due a due distinte appartenenti a una stessa superficie. Se r è parallela a s e s è parallela a t allora r è parallela a t (ossia, vale la proprietà transitiva per il parallelismo tre rette)") sarebbe vero:

……

……

- Indica almeno un caso in cui l'enunciato precedente ("Siano r, s e t rette a due a due distinte appartenenti a una stessa superficie. Se r è parallela a s e s è parallela a t allora r è parallela a t (ossia, vale la proprietà transitiva per il parallelismo tre rette)") sarebbe falso:

……

……

5.

a) Leggi il seguente enunciato: "Siano r e s due rette appartenenti a una stessa superficie. Se r e s sono parallele tra loro allora tutti i punti di r hanno la stessa distanza da s e, viceversa, tutti i punti di s hanno la stessa distanza da r".

Tra le seguenti affermazioni, seleziona quella corretta:

- L'enunciato precedente è vero.
- L'enunciato precedente è falso.
- L'enunciato precedente è vero in alcune circostanze e falso in altre.
- Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Siano r e s due rette appartenenti a una stessa superficie. Se r e s sono parallele tra loro allora tutti i punti di r hanno la stessa distanza da s e, viceversa, tutti i punti di s hanno la stessa distanza da r") sarebbe vero:

……

……

- Indica almeno un caso in cui l'enunciato ("Siano r e s due rette appartenenti a una stessa superficie. Se r e s sono parallele tra loro allora tutti i punti di r hanno la stessa distanza da s e, viceversa, tutti i punti di s hanno la stessa distanza da r") sarebbe falso:

……

……….

6.

a) Leggi il seguente enunciato: "Presi due qualsiasi triangoli ABC e DEF appartenenti a una stessa superficie e non congruenti tra loro, la somma degli angoli interni del triangolo ABC è uguale alla somma degli angoli interni del triangolo DEF".

Tra le seguenti affermazioni, seleziona quella corretta:

- o L'enunciato precedente è vero.
- o L'enunciato precedente è falso.
- o L'enunciato precedente è vero in alcune circostanze e falso in altre.
- o Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Presi due qualsiasi triangoli ABC e DEF appartenenti a una stessa superficie e non congruenti tra loro, la somma degli angoli interni del triangolo ABC è uguale alla somma degli angoli interni del triangolo DEF") sarebbe vero: …….
……

- Indica almeno un caso in cui l'enunciato precedente ("Presi due qualsiasi triangoli ABC e DEF appartenenti a una stessa superficie e non congruenti tra loro, la somma degli angoli interni del triangolo ABC è uguale alla somma degli angoli interni del triangolo DEF") sarebbe falso: …….
……

7.

a) Leggi il seguente enunciato: "Dato un triangolo qualsiasi, ogni suo angolo esterno è congruente alla somma degli angoli interni non adiacenti a esso". Tra le seguenti affermazioni, seleziona quella corretta:

- o L'enunciato precedente è vero.
- o L'enunciato precedente è falso.
- o L'enunciato precedente è vero in alcune circostanze e falso in altre.
- o Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l’enunciato precedente (“Dato un triangolo qualsiasi, ogni suo angolo esterno è congruente alla somma degli angoli interni non adiacenti a esso”) sarebbe vero: ……..
………

- Indica almeno un caso in cui l’enunciato precedente (“Dato un triangolo qualsiasi, ogni suo angolo esterno è congruente alla somma degli angoli interni non adiacenti a esso”) sarebbe falso: ……..
………

NEG post-questionnaire – Original version[81]

QUESTIONARIO F1 (7 quesiti articolati **–** 50 minuti)

NB. Per evitare fraintendimenti, leggi bene la definizione di "RETTE TRA LORO PARALLELE" e quella di "RETTE TRA LORO INCIDENTI" che assumiamo valide (le troverai ripetute prima di ogni quesito, anche se i termini non faranno parte del testo). Diciamo che “due rette sono tra loro parallele" se appartengono a una stessa superficie e non hanno punti in comune. Diciamo che "due rette sono tra loro incidenti” se hanno uno e un solo punto in comune.

1. Un assioma (detto anche postulato) è un enunciato che, senza essere preventivamente dimostrato, si assume come fondamento di una teoria assiomatica (detta anche sistema assiomatico). Tale enunciato, per essere detto “assioma” (o “postulato”), deve anche essere evidente?

- Sì.
- No.
- Non so rispondere.
- Altro: ………

Motiva la precedente risposta: ……………………………………………………………………………………………
……

2. Qual è la proprietà fondamentale che una teoria assiomatica (detta anche sistema assiomatico) dovrebbe soddisfare?

- Coerenza.
- Completezza.
- Evidenza.
- Indipendenza.
- Verità.
- Non so rispondere.
- Altro: ……………………………………………

[81] Administrated via Google Form.

3.

a) Leggi il seguente enunciato precedente: "Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r".

Tra le seguenti affermazioni, seleziona quella corretta:

- o L'enunciato precedente è vero.
- o L'enunciato precedente è falso.
- o L'enunciato precedente è vero in alcune circostanze e falso in altre.
- o Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r") è vero: ……………………

- Indica almeno un caso in cui l'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r") è falso: ……………….…

c) È possibile enunciare la negazione dell'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r")?

- o Sì.
- o No.
- o Non so rispondere.
- o Altro: ……………………………………………………………………………………………………

d) *[Tale quesito si attiva solo se al punto c è stato selezionato "Sì"]*

Leggi attentamente le affermazioni A, B, C, D ed E e seleziona l'opzione che, tra quelle di sotto riportate, corrisponderebbe alla negazione dell'enunciato precedente ("Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r").

A: "Data una retta r e un punto P esterno a r, <u>non esistono rette</u> passanti per P e parallele a r ".
B: "Data una retta r e un punto P esterno a r, <u>esiste almeno una retta</u> passante per P e parallele a r ".
C: "Data una retta r e un punto P esterno a r, <u>esiste più di una retta</u> passante per P parallele a r ".
D: "Data una retta r e un punto P esterno a r, <u>esistono molte rette</u> passanti per P e parallele a r ".
E: "Data una retta r e un punto P esterno a r, <u>esistono infinite rette</u> passanti per P parallele a r ".

- L’affermazione A.
- L’affermazione B.
- L’affermazione C.
- L’affermazione D.
- L’affermazione E.
- L’unione dell'affermazione A con l'affermazione B.
- L’unione dell'affermazione A con l'affermazione C.
- L’unione dell'affermazione A con l'affermazione D.
- L’unione dell'affermazione A con l'affermazione E.
- L’unione dell'affermazione B con l'affermazione C.
- L’unione dell'affermazione B con l'affermazione D.
- L’unione dell'affermazione B con l'affermazione E.
- L’unione dell'affermazione C con l'affermazione D.
- L’unione dell'affermazione C con l'affermazione E.
- L’unione dell'affermazione D con l'affermazione E.
- Nessuna delle precedenti opzioni.
- Non so rispondere.

e) *[Tale quesito si attiva solo se al punto c è stato selezionato “No”]*
Spiega perché non sarebbe possibile enunciare la negazione dell'enunciato precedente (“Data una retta r e un punto P esterno a r, esiste una e una sola retta passante per P e parallela a r”): …….
……….

4.

a) Leggi il seguente enunciato: “Siano r, s e t rette a due a due distinte appartenenti a una stessa superficie. Se r è parallela a s e s è parallela a t allora r è parallela a t (ossia, vale la proprietà transitiva per il parallelismo tra rette)”.
Tra le seguenti affermazioni, seleziona quella corretta:

- L'enunciato precedente è vero.
- L'enunciato precedente è falso.
- L'enunciato precedente è vero in alcune circostanze e falso in altre.
- Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Siano r, s e t rette a due a due distinte appartenenti a una stessa superficie. Se r è parallela a s e s è parallela a t allora r è parallela a t (ossia, vale la proprietà transitiva per il parallelismo tre rette)") sarebbe vero:

……

……

- Indica almeno un caso in cui l'enunciato precedente ("Siano r, s e t rette a due a due distinte appartenenti a una stessa superficie. Se r è parallela a s e s è parallela a t allora r è parallela a t (ossia, vale la proprietà transitiva per il parallelismo tre rette)") sarebbe falso:

……

……

5.

a) Leggi il seguente enunciato: "Siano r e s due rette appartenenti a una stessa superficie. Se r e s sono parallele tra loro allora tutti i punti di r hanno la stessa distanza da s e, viceversa, tutti i punti di s hanno la stessa distanza da r".

Tra le seguenti affermazioni, seleziona quella corretta:

- L'enunciato precedente è vero.
- L'enunciato precedente è falso.
- L'enunciato precedente è vero in alcune circostanze e falso in altre.
- Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Siano r e s due rette appartenenti a una stessa superficie. Se r e s sono parallele tra loro allora tutti i punti di r hanno la stessa distanza da s e, viceversa, tutti i punti di s hanno la stessa distanza da r") sarebbe vero:

……

……

- Indica almeno un caso in cui l'enunciato ("Siano r e s due rette appartenenti a una stessa superficie. Se r e s sono parallele tra loro allora tutti i punti di r hanno la stessa distanza da s e, viceversa, tutti i punti di s hanno la stessa distanza da r") sarebbe falso:

……

……

6.

a) Leggi il seguente enunciato: "Presi due qualsiasi triangoli ABC e DEF appartenenti a una stessa superficie e non congruenti tra loro, la somma degli angoli interni del triangolo ABC è uguale alla somma degli angoli interni del triangolo DEF".

Tra le seguenti affermazioni, seleziona quella corretta:

- o L'enunciato precedente è vero.
- o L'enunciato precedente è falso.
- o L'enunciato precedente è vero in alcune circostanze e falso in altre.
- o Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Presi due qualsiasi triangoli ABC e DEF appartenenti a una stessa superficie e non congruenti tra loro, la somma degli angoli interni del triangolo ABC è uguale alla somma degli angoli interni del triangolo DEF") sarebbe vero: ....
....

- Indica almeno un caso in cui l'enunciato precedente ("Presi due qualsiasi triangoli ABC e DEF appartenenti a una stessa superficie e non congruenti tra loro, la somma degli angoli interni del triangolo ABC è uguale alla somma degli angoli interni del triangolo DEF") sarebbe falso: ....
....

7.

a) Leggi il seguente enunciato: "*Se A, B, C e D sono quattro punti a due a due distinti appartenenti a una stessa superficie tali che BD è perpendicolare ad AB e l'angolo BAC è acuto allora la retta passante per A e per C e quella passante per B e per D si intersecano*". Tra le seguenti affermazioni, seleziona quella corretta:

- o L'enunciato precedente è vero.
- o L'enunciato precedente è falso.
- o L'enunciato precedente è vero in alcune circostanze e falso in altre.
- o Non so rispondere.

b) *[Tale quesito si attiva solo se al punto a è stata selezionata la terza affermazione]*

- Indica almeno un caso in cui l'enunciato precedente ("Dato un triangolo qualsiasi, ogni suo angolo esterno è congruente alla somma degli angoli interni non adiacenti a esso") sarebbe vero: ……..
……

- Indica almeno un caso in cui l'enunciato precedente ("Dato un triangolo qualsiasi, ogni suo angolo esterno è congruente alla somma degli angoli interni non adiacenti a esso") sarebbe falso: ……..
……

Appendix 5

This appendix reports the *PROOF questionnaire* (p. 256) and the *PROOF questionnaire* (p. 262).

PROOF pre-questionnaire – Original version[82]

QUESTIONARIO I3 (2 quesiti articolati **–** 60 minuti)

NB. Per evitare fraintendimenti, leggi bene la definizione di "RETTE TRA LORO PARALLELE" e quella di "RETTE TRA LORO INCIDENTI" che assumiamo valide (le troverai ripetute prima di ogni quesito, anche se i termini non faranno parte del testo). Diciamo che “due rette sono tra loro parallele" se appartengono a una stessa superficie e non hanno punti in comune. Diciamo che "due rette sono tra loro incidenti” se hanno uno e un solo punto in comune.

1.

Quesito 1a. Dopo avere letto attentamente l’enunciato P1 scritto qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

<u>Enunciato</u> <u>P1.</u> *Sia* ABC *un triangolo di una superficie* α. *Se* D *appartiene ad* AB *ed è tale che* $AD \cong BD \cong CD$ *allora* $\widehat{ACB} = 90°$.

- o L’eventuale esistenza di triangoli ABC per i quali nessun punto D di AB è tale che AD≅BD≅ CD sarebbe una circostanza esclusa dalle ipotesi dell’enunciato P1 quindi implicherebbe che l’enunciato P1 è vero.
- o L’eventuale esistenza di triangoli ABC per i quali nessun punto D di AB è tale che AD≅BD≅ CD sarebbe una circostanza esclusa dalle ipotesi dell’enunciato P1 quindi implicherebbe che l’enunciato P1 è falso.
- o L’eventuale esistenza di triangoli ABC per i quali nessun punto D di AB è tale che AD≅BD≅ CD sarebbe una circostanza esclusa dalle ipotesi dell’enunciato P1 quindi non avrebbe implicazioni sul valore di verità dell'enunciato P1.

[82] Administrated via Google Form.

- Le tre affermazioni precedenti sono tutte errate.
- Non so quale delle quattro affermazioni precedenti sia corretta.

Quesito 1b. Dopo avere letto attentamente di nuovo il precedente enunciato P1 e la dimostrazione scritti qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

> <u>Enunciato</u> <u>P1.</u> *Sia* ABC *un triangolo di una superficie* α. *Se* D *appartiene ad* AB *ed è tale che* $AD \cong BD \cong CD$ *allora* $\widehat{ACB} = 90°$.
>
> <u>*Dimostrazione*</u>. *Si indichi: con le lettere* α *e* β *rispettivamente gli angoli interni* $\widehat{CAB}$ *e* $\widehat{ABC}$ *del triangolo* ABC; *con le lettere* γ_1 *e* δ_1 *rispettivamente gli angoli interni* $\widehat{ACD}$ *e* $\widehat{ADC}$ *del triangolo* ACD; *con le lettere* γ_2 *e* δ_2 *rispettivamente gli angoli interni* $\widehat{BCD}$ *e* $\widehat{BDC}$ *del triangolo* BCD.
>
> *a) Poiché ogni angolo esterno è congruente alla somma degli angoli interni non adiacenti ad esso, si ricava:*
>
> $\boldsymbol{\delta_2} \cong \boldsymbol{\alpha} + \boldsymbol{\gamma_1}$ *(considerando il triangolo* ADC*);*
>
> $\boldsymbol{\delta_1} \cong \boldsymbol{\beta} + \boldsymbol{\gamma_2}$ *(considerando il triangolo* BDC***).***
>
> *b) Si consideri il triangolo* ADC:
>
> $AD \cong CD$ *(per ipotesi) quindi* $\boldsymbol{\alpha} \cong \boldsymbol{\gamma_1}$ *(conseguenza dei primi tre postulati di Euclide).*
>
> *c) Si consideri il triangolo* BDC:
>
> $BD \cong CD$ *(per ipotesi) quindi* $\boldsymbol{\beta} \cong \boldsymbol{\gamma_2}$ *(conseguenza dei primi tre postulati di Euclide).*
>
> *d) Da quanto dedotto ai punti a), b) e c) segue:*
>
> $$\boldsymbol{\delta_2} \cong \gamma_1 + \gamma_1 = \boldsymbol{2\gamma_1} \quad e \quad \boldsymbol{\delta_2} \cong \gamma_2 + \gamma_2 = \boldsymbol{2\gamma_2}.$$
>
> *e) Da quanto dedotto al punto d) si ricava:*
>
> $$\boldsymbol{\delta_2} + \boldsymbol{\delta_1} \cong 2\gamma_1 + 2\gamma_2 = \boldsymbol{2(\gamma_1 + \gamma_2)}.$$
>
> *f) Da* $\boldsymbol{\delta_2} + \boldsymbol{\delta_1} \cong \boldsymbol{2(\gamma_1 + \gamma_2)}$ *e poiché* $\delta_2 + \delta_1 = 180°$ *(conseguenza dei primi tre postulati di Euclide), otteniamo:*
>
> $$\boldsymbol{\gamma_1 + \gamma_2 = 90°}.$$
>
> *g) Da* $\boldsymbol{\gamma_1 + \gamma_2 = 90°}$ *e poiché* $\gamma_1 + \gamma_2 = \widehat{ACB}$, *otteniamo:* $\boldsymbol{\widehat{ACB} = 90°}$. ■

- La dimostrazione è corretta perché tutti i suoi passaggi sono sempre validi sotto le ipotesi indicate nell'enunciato P1.
- La dimostrazione è errata perché esistono triangoli ABC per i quali nessun punto D di AB è tale che AD≅BD≅ CD.
- La dimostrazione è errata perché contiene almeno un passaggio (a, b, c, d, e, f o g) che non è sempre valido sotto le ipotesi indicate nell'enunciato P1.
- Le tre affermazioni precedenti sono tutte errate.

- Non so quale delle quattro affermazioni precedenti sia corretta.

Quesito 1c – *[Tale quesito si attiva solo se al punto a è stata selezionata la terza opzione]*

I) Seleziona un passaggio della precedente dimostrazione (copiata qui sotto insieme all'enunciato P1) che, secondo te, non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P1.

> Enunciato P1. *Sia ABC un triangolo di una superficie α. Se D appartiene ad AB ed è tale che $AD \cong BD \cong CD$ allora $\widehat{ACB} = 90°$.*
>
> Dimostrazione. *Si indichi: con le lettere α e β rispettivamente gli angoli interni $\widehat{CAB}$ e $\widehat{ABC}$ del triangolo ABC; con le lettere γ_1 e δ_1 rispettivamente gli angoli interni $\widehat{ACD}$ e $\widehat{ADC}$ del triangolo ACD; con le lettere γ_2 e δ_2 rispettivamente gli angoli interni $\widehat{BCD}$ e $\widehat{BDC}$ del triangolo BCD.*
>
> *a) Poiché ogni angolo esterno è congruente alla somma degli angoli interni non adiacenti ad esso, si ricava:*
>
> $\boldsymbol{\delta_2} \cong \boldsymbol{\alpha + \gamma_1}$ *(considerando il triangolo ADC);*
>
> $\boldsymbol{\delta_1} \cong \boldsymbol{\beta + \gamma_2}$ *(considerando il triangolo BDC).*
>
> *b) Si consideri il triangolo ADC:*
>
> $AD \cong CD$ *(per ipotesi) quindi* $\boldsymbol{\alpha} \cong \boldsymbol{\gamma_1}$ *(conseguenza dei primi tre postulati di Euclide).*
>
> *c) Si consideri il triangolo BDC:*
>
> $BD \cong CD$ *(per ipotesi) quindi* $\boldsymbol{\beta} \cong \boldsymbol{\gamma_2}$ *(conseguenza dei primi tre postulati di Euclide).*
>
> *d) Da quanto dedotto ai punti a), b) e c) segue:*
>
> $$\boldsymbol{\delta_2} \cong \gamma_1 + \gamma_1 = \boldsymbol{2\gamma_1} \quad e \quad \boldsymbol{\delta_2} \cong \gamma_2 + \gamma_2 = \boldsymbol{2\gamma_2}.$$
>
> *e) Da quanto dedotto al punto d) si ricava:*
>
> $$\boldsymbol{\delta_2 + \delta_1} \cong 2\gamma_1 + 2\gamma_2 = \boldsymbol{2(\gamma_1 + \gamma_2)}.$$
>
> *f) Da $\boldsymbol{\delta_2 + \delta_1} \cong \boldsymbol{2(\gamma_1 + \gamma_2)}$ e poiché $\delta_2 + \delta_1 = 180°$ (conseguenza dei primi tre postulati di Euclide), otteniamo:*
>
> $$\boldsymbol{\gamma_1 + \gamma_2 = 90°}.$$
>
> *g) Da $\boldsymbol{\gamma_1 + \gamma_2 = 90°}$ e poiché $\gamma_1 + \gamma_2 = \widehat{ACB}$, otteniamo: $\boldsymbol{\widehat{ACB} = 90°}$.* ■

- a.
- b.
- c.
- d.
- e.
- f.
- g.

II) Spiega perché il passaggio da te selezionato non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P1: ……………………………………………………………………………………
……
……

2.

Quesito 2a. Dopo avere letto attentamente l'enunciato P2 scritto qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

> <u>Enunciato</u> P2. *Siano A e B due punti distinti tra loro appartenenti a una superficie α; sia r_{AB} la retta passante per A e per B; siano π_1 e π_2 le due porzioni della superficie α delimitate da r_{AB}. Sia C un punto appartenente a π_1 ed esterno alla retta r_{AB}; sia r_{AC} la retta passante per A e per C. Sia D un punto appartenente a π_1, esterno alla retta r_{AB} e tale che la retta r_{BD} sia parallela a r_{AC}. Sia E un punto distinto da B, appartenente a π_2 e su r_{BD}. Sotto le ipotesi precedenti, gli angoli $\widehat{BAC}$ e $\widehat{ABE}$ sono congruenti tra loro.*

- o L'eventuale esistenza di superfici prive di rette parallele sarebbe una circostanza esclusa dalle ipotesi dell'enunciato P2 quindi implicherebbe che l'enunciato P2 è vero.
- o L'eventuale esistenza di superfici prive di rette parallele sarebbe una circostanza esclusa dalle ipotesi dell'enunciato P2 quindi implicherebbe che l'enunciato P2 è falso.
- o L'eventuale esistenza di superfici prive di rette parallele sarebbe una circostanza esclusa dalle ipotesi dell'enunciato P2 quindi non avrebbe implicazioni sul valore di verità dell'enunciato P2.
- o Le tre affermazioni precedenti sono tutte errate.
- o Non so quale delle quattro affermazioni precedenti sia corretta.

Quesito 2b. Dopo avere letto attentamente di nuovo il precedente enunciato P2 e la dimostrazione scritti qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

> <u>Enunciato</u> P2. *Siano A e B due punti distinti tra loro appartenenti a una superficie α; sia r_{AB} la retta passante per A e per B; siano π_1 e π_2 le due porzioni della superficie α delimitate da r_{AB}. Sia C un punto appartenente a π_1 ed esterno alla retta r_{AB}; sia r_{AC} la retta passante per A e per C. Sia D un punto appartenente a π_1, esterno alla retta r_{AB} e tale che la retta r_{BD} sia parallela a r_{AC}. Sia E un punto distinto da B, appartenente a π_2 e su r_{BD}. Sotto le ipotesi precedenti, gli angoli $\widehat{BAC}$ e $\widehat{ABE}$ sono congruenti tra loro.*
>
> *<u>Dimostrazione</u>.*
> *a) **Assumiamo per assurdo che** gli angoli $\widehat{BAC}$ e $\widehat{ABE}$ non sono congruenti tra loro:*

$\widehat{\boldsymbol{BAC}} \neq \widehat{\boldsymbol{ABE}}$.

b) Da $\widehat{BAC} \neq \widehat{ABE}$ segue che uno dei due angoli deve essere meno ampio dell'altro, ***senza perdere di generalità*** *assumiamo* $\widehat{\boldsymbol{BAC}} < \widehat{\boldsymbol{ABE}}$ ***.***

c) Aggiungendo $\widehat{\boldsymbol{ABD}}$ a entrambi i membri della disuguaglianza in a), otteniamo:

$\widehat{\boldsymbol{BAC}} + \widehat{\boldsymbol{ABD}} < \widehat{\boldsymbol{ABE}} + \widehat{\boldsymbol{ABD}}$.

d) Da quanto dedotto al punto c) e poiché $\widehat{ABE} + \widehat{ABD} = 180°$ (conseguenza dei primi tre postulati di Euclide), si ricava:

$\widehat{\boldsymbol{BAC}} + \widehat{\boldsymbol{ABD}} < \mathbf{180}°$.

e) Da $\widehat{BAC} + \widehat{ABD} < 180°$ segue che $\boldsymbol{r_{AC}}$ ***e*** $\boldsymbol{r_{BD}}$ ***incidono****.*

f) ***Quanto dedotto al punto e)*** *(r_{AC} e r_{BD} incidono)* ***è assurdo*** *poiché, per ipotesi, r_{AC} e r_{BD} sono parallele tra loro.*

g) Poiché assumere che $\widehat{BAC} \neq \widehat{ABE}$ porta a un assurdo, ***gli angoli $\widehat{BAC}$ e $\widehat{ABE}$ sono congruenti tra loro.*** ■

- La dimostrazione è corretta perché tutti i suoi passaggi sono sempre validi sotto le ipotesi indicate nell'enunciato P2.
- La dimostrazione è errata perché esistono superfici prive di rette parallele.
- La dimostrazione è errata perché contiene almeno un passaggio (a, b, c, d, e, f o g) che non è sempre valido sotto le ipotesi indicate nell'enunciato P2.
- Le tre affermazioni precedenti sono tutte errate.
- Non so delle quattro affermazioni precedenti sia corretta.

Quesito 2c – *[Tale quesito si attiva solo se al punto a è stata selezionata la terza opzione]*

I) Seleziona un passaggio della precedente dimostrazione (copiata qui sotto insieme all'enunciato P2) che, secondo te, non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P2.

Enunciato P2. *Siano A e B due punti distinti tra loro appartenenti a una superficie α; sia r_{AB} la retta passante per A e per B; siano π_1 e π_2 le due porzioni della superficie α delimitate da r_{AB}. Sia C un punto appartenente a π_1 ed esterno alla retta r_{AB}; sia r_{AC} la retta passante per A e per C. Sia D un punto appartenente a π_1, esterno alla retta r_{AB} e tale che la retta r_{BD} sia parallela a r_{AC}. Sia E un punto distinto da B, appartenente a π_2 e su r_{BD}. Sotto le ipotesi precedenti, gli angoli $\widehat{BAC}$ e $\widehat{ABE}$ sono congruenti tra loro.*

Dimostrazione.

a) ***Assumiamo per assurdo che*** *gli angoli $\widehat{BAC}$ e $\widehat{ABE}$ non sono congruenti tra loro:*

$\widehat{BAC} \neq \widehat{ABE}$.

b) Da $\widehat{BAC} \neq \widehat{ABE}$ *segue che uno dei due angoli deve essere meno ampio dell'altro,* ***senza perdere di generalità*** *assumiamo* $\widehat{BAC} < \widehat{ABE}$ ***.***

c) Aggiungendo $\widehat{ABD}$ *a entrambi i membri della disuguaglianza in a), otteniamo:*

$\widehat{BAC} + \widehat{ABD} < \widehat{ABE} + \widehat{ABD}$.

d) Da quanto dedotto al punto c) e poiché $\widehat{ABE} + \widehat{ABD} = 180°$ *(conseguenza dei primi tre postulati di Euclide), si ricava:*

$\widehat{BAC} + \widehat{ABD} < 180°$.

e) Da $\widehat{BAC} + \widehat{ABD} < 180°$ *segue che* r_{AC} ***e*** r_{BD} ***incidono****.*

f) ***Quanto dedotto al punto e)*** *(*r_{AC} *e* r_{BD} *incidono)* ***è assurdo*** *poiché, per ipotesi,* r_{AC} *e* r_{BD} *sono parallele tra loro.*

g) Poiché assumere che $\widehat{BAC} \neq \widehat{ABE}$ *porta a un assurdo,* ***gli angoli*** $\widehat{BAC}$ ***e*** $\widehat{ABE}$ ***sono congruenti tra loro.*** ■

- a.
- b.
- c.
- d.
- e.
- f.
- g.

II) Spiega perché il passaggio da te selezionato non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P2: ………………………………………………………………………………………
………
………

PROOF post-questionnaire – Original version[83]

QUESTIONARIO F2 (2 quesiti articolati - 60 minuti)

NB. Per evitare fraintendimenti, leggi bene la definizione di "RETTE TRA LORO PARALLELE" e quella di "RETTE TRA LORO INCIDENTI" che assumiamo valide (le troverai ripetute prima di ogni quesito, anche se i termini non faranno parte del testo). Diciamo che "due rette sono tra loro parallele" se appartengono a una stessa superficie e non hanno punti in comune. Diciamo che "due rette sono tra loro incidenti" se hanno uno e un solo punto in comune.

1.

Quesito 1a. Dopo avere letto attentamente l'enunciato P1 scritto qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

> Enunciato P1. *Sia ABC un triangolo appartenente a una superficie α tale che il lato AC sia maggiore del lato AB. Sotto le ipotesi precedenti, l'angolo $\widehat{ABC}$ è maggiore dell'angolo $\widehat{BCA}$.*

- L'eventuale esistenza di triangoli che non soddisfino le ipotesi dell'enunciato P1 implicherebbe che l'enunciato P1 è vero.
- L'eventuale esistenza di triangoli che non soddisfino le ipotesi dell'enunciato P1 implicherebbe che l'enunciato P1 è falso.
- L'eventuale esistenza di triangoli che non soddisfino le ipotesi dell'enunciato P1 non avrebbe implicazioni sul valore di verità dell'enunciato P1.
- Le tre affermazioni precedenti sono tutte errate.
- Non so quale delle quattro affermazioni precedenti sia corretta.

[83] Administrated via Google Form.

Quesito 1b. Dopo avere letto attentamente di nuovo il precedente enunciato P1 e la dimostrazione scritti qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

> <u>Enunciato</u> <u>P1.</u> *Sia ABC un triangolo appartenente a una superficie α tale che il lato AC sia maggiore del lato AB. Sotto le ipotesi precedenti, l'angolo $\widehat{ABC}$ è maggiore dell'angolo $\widehat{BCA}$.*
>
> <u>*Dimostrazione*</u>*.*
>
> *a) Per ipotesi, il lato AC è maggiore del lato AB quindi* ***esiste un punto D sul lato AC tale che AD sia congruente al lato AB****.*
>
> *b) Grazie al primo postulato di Euclide,* ***possiamo congiungere B e D con un segmento****.*
>
> *c) Poiché l'angolo $\widehat{ADB}$ è un angolo esterno del triangolo BCD, esso è maggiore dell'angolo interno e non adiacente $\widehat{BCD}$:* $\boldsymbol{\widehat{ADB} > \widehat{BCD}}$**.**
>
> *d) Poiché, per costruzione, il lato AD è congruente al lato AB allora gli angoli $\widehat{ABD}$ e $\widehat{ADB}$ sono congruenti:* $\boldsymbol{\widehat{ABD} \cong \widehat{ADB}}$**.**
>
> *e) Da quanto dedotto in c) e in d) segue che* $\boldsymbol{\widehat{ABD} > \widehat{BCD}}$**.**
>
> *f) Poiché il tutto è maggiore della parte, l'angolo $\widehat{ABC}$ è maggiore dell'angolo $\widehat{ABD}$:* $\boldsymbol{\widehat{ABC} > \widehat{ABD}}$**.**
>
> *g) Da quanto dedotto in e) e in f) segue che* $\boldsymbol{\widehat{ABC} > \widehat{BCD}}$**.**
>
> *h) Poiché, per costruzione, gli angoli $\widehat{BCD}$ e $\widehat{BCA}$ coincidono, vale che* $\boldsymbol{\widehat{ABC} > \widehat{BCA}}$**.** ■

- La dimostrazione è corretta perché tutti i suoi passaggi sono sempre validi sotto le ipotesi indicate nell'enunciato P1.
- La dimostrazione è errata perché esistono triangoli che non soddisfano le ipotesi dell'enunciato P1.
- La dimostrazione è errata perché contiene almeno un passaggio (a, b, c, d, e, f, g o h) che non è sempre valido sotto le ipotesi indicate nell'enunciato P1.
- Le tre affermazioni precedenti sono tutte errate.
- Non so quale delle quattro affermazioni precedenti sia corretta.

Quesito 1c – *[Tale quesito si attiva solo se al punto b è stata selezionata la terza opzione]*

I) Seleziona un passaggio della precedente dimostrazione (copiata qui sotto insieme all'enunciato P1) che, secondo te, non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P1.

> <u>Enunciato</u> <u>P1.</u> *Sia ABC un triangolo appartenente a una superficie α tale che il lato AC sia maggiore del lato AB. Sotto le ipotesi precedenti, l'angolo $\widehat{ABC}$ è maggiore dell'angolo $\widehat{BCA}$.*

<u>Dimostrazione</u>.

a) Per ipotesi, il lato AC è maggiore del lato AB quindi ***esiste un punto D sul lato AC tale che AD sia congruente al lato AB****.*

b) Grazie al primo postulato di Euclide, ***possiamo congiungere B e D con un segmento****.*

c) Poiché l'angolo $\widehat{ADB}$ è un angolo esterno del triangolo BCD, esso è maggiore dell'angolo interno e non adiacente $\widehat{BCD}$: $\boldsymbol{\widehat{ADB} > \widehat{BCD}}$**.**

d) Poiché, per costruzione, il lato AD è congruente al lato AB allora gli angoli $\widehat{ABD}$ e $\widehat{ADB}$ sono congruenti: $\boldsymbol{\widehat{ABD} \cong \widehat{ADB}}$**.**

e) Da quanto dedotto in c) e in d) segue che $\boldsymbol{\widehat{ABD} > \widehat{BCD}}$**.**

f) Poiché il tutto è maggiore della parte, l'angolo $\widehat{ABC}$ è maggiore dell'angolo $\widehat{ABD}$: $\boldsymbol{\widehat{ABC} > \widehat{ABD}}$**.**

g) Da quanto dedotto in e) e in f) segue che $\boldsymbol{\widehat{ABC} > \widehat{BCD}}$**.**

h) Poiché, per costruzione, gli angoli $\widehat{BCD}$ e $\widehat{BCA}$ coincidono, vale che $\boldsymbol{\widehat{ABC} > \widehat{BCA}}$**.** ■

- a.
- b.
- c.
- d.
- e.
- f.
- g.
- h.

II) Spiega perché il passaggio da te selezionato non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P1: ……………………………………………………………………………………….
……….

2.

Quesito 2a. Dopo avere letto attentamente l'enunciato P2 scritto qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

<u>Enunciato</u> P2. *Siano l, m e k tre rette a due a due distinte appartenenti a una superficie α e tali che: l e m siano tra loro parallele; k sia incidente su m. Sotto le ipotesi precedenti, k incide anche su l.*

- L'eventuale esistenza di superfici prive di rette parallele sarebbe una circostanza esclusa dalle ipotesi dell'enunciato P2 quindi implicherebbe che l'enunciato P2 è vero.

- L’eventuale esistenza di superfici prive di rette parallele sarebbe una circostanza esclusa dalle ipotesi dell’enunciato P2 quindi implicherebbe che l’enunciato P2 è falso.
- L’eventuale esistenza di superfici prive di rette parallele sarebbe una circostanza esclusa dalle ipotesi dell’enunciato P2 quindi non avrebbe implicazioni sul valore di verità dell'enunciato P2.
- Le tre affermazioni precedenti sono tutte errate.
- Non so quale delle quattro affermazioni precedenti sia corretta.

Quesito 2b. Dopo avere letto attentamente di nuovo il precedente enunciato P2 e la dimostrazione scritti qui sotto, seleziona l'affermazione corretta tra quelle di seguito riportate.

> <u>Enunciato</u> <u>P2</u>. *Siano k, l e m tre rette a due a due distinte appartenenti a una superficie α e tali che: l e m siano tra loro parallele; k sia incidente su m. Sotto le ipotesi precedenti, k incide anche su l.*
>
> *<u>Dimostrazione</u>.*
> *a) Precisiamo che, per ipotesi,* ***k, l e m*** *sono* ***rette a due a due distinte appartenenti a una stessa superficie****.*
> *b)* ***Assumiamo per assurdo che k non incida su l.***
> *c) Da a) e da b) segue che* ***k e l sono tra loro parallele****.*
> *d) Da a), dal fatto che al punto c) si è dedotto che k e l sono tra loro parallele e dal fatto che, per ipotesi, le rette l e m sono tra loro parallele, segue che* ***le rette k e m sono tra loro parallele****.*
> *e)* ***Quanto dedotto al punto d)*** *(k e m sono tra loro parallele)* ***è assurdo*** *poiché, per ipotesi, k e m sono incidenti.*
> *f) Poiché assumere, al punto b), che k non incida su l ha condotto a un assurdo, concludiamo che* ***k incide su l****.* ■

- La dimostrazione è corretta perché tutti i suoi passaggi sono sempre validi sotto le ipotesi indicate nell’enunciato P2.
- La dimostrazione è errata perché esistono superfici prive di rette parallele.
- La dimostrazione è errata perché contiene almeno un passaggio (a, b, c, d, e o f) che non è sempre valido sotto le ipotesi indicate nell’enunciato P2.
- Le tre affermazioni precedenti sono tutte errate.
- Non so quale delle quattro affermazioni precedenti sia corretta.

Quesito 2c – *[Tale quesito si attiva solo se al punto b è stata selezionata la terza opzione]*

I) Seleziona un passaggio della precedente dimostrazione (copiata qui sotto insieme all'enunciato P2) che, secondo te, non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P2.

> <u>Enunciato</u> <u>P2</u>. *Siano k, l e m tre rette a due a due distinte appartenenti a una superficie α e tali che: l e m siano tra loro parallele; k sia incidente su m. Sotto le ipotesi precedenti, k incide anche su l.*
>
> *<u>Dimostrazione</u>.*
>
> *a) Precisiamo che, per ipotesi,* ***k, l e m*** *sono* ***rette a due a due distinte appartenenti a una stessa superficie****.*
>
> *b)* ***Assumiamo per assurdo che k non incida su l.***
>
> *c) Da a) e da b) segue che* ***k e l sono tra loro parallele****.*
>
> *d) Da a), dal fatto che al punto c) si è dedotto che k e l sono tra loro parallele e dal fatto che, per ipotesi, le rette l e m sono tra loro parallele, segue che* ***le rette k e m sono tra loro parallele****.*
>
> *e)* ***Quanto dedotto al punto d)*** *(k e m sono tra loro parallele)* ***è assurdo*** *poiché, per ipotesi, k e m sono incidenti.*
>
> *f) Poiché assumere, al punto b), che k non incida su l ha condotto a un assurdo, concludiamo che* ***k incide su l****.* $\blacksquare$

- a.
- b.
- c.
- d.
- e.
- f.

II) Spiega perché il passaggio da te selezionato non sarebbe sempre valido sotto le ipotesi indicate nell'enunciato P2: ………………………………………………………………………………………….
……….

Appendix 6

This appendix reports the *BELIEFS questionnaire* (p. 267) and the *BELIEFS questionnaire* (p. 269).

BELIEFS pre-questionnaire – Original version[84]

QUESTIONARIO I1 (5 quesiti **–** 35 minuti – LA SELEZIONE È OBBLIGATORIA)

1. Tra le seguenti affermazioni, seleziona quella che ritieni più giusta:

i) «La matematica esiste indipendentemente dall'uomo».

ii) «La matematica è una creazione umana».

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: ……………………………………………………………………………………………

2. Tra le seguenti affermazioni, seleziona quella che ritieni più giusta:

i) «In matematica, un sistema assiomatico è del tipo vero/falso».

ii) «In matematica, un sistema assiomatico è del tipo coerente/incoerente».

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: ……………………………………………………………………………………………

3. Leggi la seguente affermazione e indica se la condividi o meno: «La conoscenza matematica è di tipo definitivo e non è soggetta ad alcuna revisione».

i) Condivido.

[84] Administrated via Google Form.

ii) Non condivido.

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: …………………………………………………………………………………………….

4. Leggi la seguente affermazione e indica se la condividi o meno: «Lo sviluppo del sapere matematico non è influenzato da fattori socio-culturali».

i) Condivido.

ii) Non condivido.

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: …………………………………………………………………………………………….

5. Leggi la seguente affermazione: «Alcuni oggetti dello scibile umano (tutto ciò che può essere appreso e conosciuto dalla mente umana, come ad esempio nuove teorie, nuovi studi) sono stati talmente rivoluzionari da costituire ostacolo alla loro accettazione da parte della comunità scientifica (si pensi, ad esempio, alla rivoluzione copernicana). TALE IMPEDIMENTO NON SI È MAI PRESENTATO PER LO SVILUPPO DELLE CONOSCENZE MATEMATICHE». Condividi l'ultima frase (quella scritta in maiuscolo)?

i) Condivido.

ii) Non condivido.

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: …………………………………………………………………………………………….

BELIEFS post-questionnaire – Original version[85]

QUESTIONARIO F3 (5 quesiti – 35 minuti – LA SELEZIONE È OBBLIGATORIA)

1. Tra le seguenti affermazioni, seleziona quella che ritieni più giusta:

i) «Il matematico è uno scopritore, individua e studia proprietà e oggetti già dotati di una propria esistenza»

ii) «Il matematico introduce e crea autonomamente proprietà e oggetti»

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: ……………………………………………………………………………………………….

2. Tra le seguenti affermazioni, seleziona quella che ritieni più giusta:

i) «*In matematica è interessante stabilire se un* sistema assiomatico è del tipo vero/falso»

ii) «*In matematica è interessante stabilire se un* sistema assiomatico è del tipo coerente/incoerente»

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: ……………………………………………………………………………………………….

3. Leggi la seguente affermazione e indica se la condividi o meno: «La conoscenza matematica è un insieme immutabile di verità».

i) Condivido.

ii) Non condivido.

iii) Non so rispondere.

iv) Altro: ………………

[85] Administrated via Google Form.

Motiva la precedente risposta: …………………………………………………………………………………………………….

4. Leggi la seguente affermazione e indica se la condividi o meno: «Il sapere matematico non si collega al contesto sociale e culturale».

i) Condivido.

ii) Non condivido.

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: …………………………………………………………………………………………………….

5. Leggi la seguente affermazione: «Alcuni concetti scientifici particolarmente rivoluzionari (ad esempio, la teoria della relatività) hanno segnato la comunità scientifica attraverso un mutamento radicale, un salto discontinuo, una rottura con il passato. TALE SITUAZIONE **NON** SI È MAI PRESENTATO PER LO SVILUPPO DELLE CONOSCENZE MATEMATICHE». Condividi l'ultima frase (quella scritta in maiuscolo)?

i) Condivido.

ii) Non condivido.

iii) Non so rispondere.

iv) Altro: ………………

Motiva la precedente risposta: …………………………………………………………………………………………………….

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