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A Perturbative Approach for the Solution of Sturm-Liouville Problems

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Abstract: Spectral analysis extends the theory of eigenvectors and eigenvalues of a square matrix to a broader theory involving operators. In particular, a branch of spectral analysis is devoted to Sturm-Liouville (SL) problems, which are eigenvalue problems for differential operators. In this study, we propose a numerical method to solve SL problems. This method uses a simple perturbative approach. Starting from an SL problem having differential operator L_0 and known eigensystem, the proposed iterative algorithm considers M SL problems having differential operators L_m, $m = 1, 2, \dots, M$, such that L_m is a perturbation of L_{m-1}, and L_M is the differential operator of the SL problem that we want to solve. Each step of this algorithm is based on the well-known Jacobi orthogonal component correction method, which acts on the refinement of approximated eigensystems. Moreover, the proposed method depends on the choice of L_0 and the representation basis for the eigenfunctions, thus giving rise to different approximation schemes. We show the performance of the proposed method both in the solution of some selected SL problems and the refinement of approximated eigensystems computed by other numerical methods. In these numerical experiments, the perturbative method is compared with a classical approximation technique and the obtained results are strongly promising in terms of accuracy.

Keywords: Sturm-Liouville Problem, Eigenvalue Problem, Perturbative Approach

1. Introduction

Spectral analysis is a very important tool in applied mathematics and engineering computations. It is strictly connected to vibrations, and objects like violin strings, drums, bridges, and skyscrapers can swing. The natural frequencies of a musical instrument are related to the quality of the sound produced by the instrument itself [2, 9, 18]. In the case of buildings, the frequencies of the mechanical structure are related to their resistance with respect to external forces like wind, earthquake, traffic of heavy vehicles, and so on, [4, 5, 7, 10, 12, 16]. Another important application in applied mathematics is also the solution of time-dependent problems for partial differential equations by the separation of variables, see [1, 14, 19] for details. SL problems provide the most simple example of eigenvalue problems for differential equations. So they can be used to set up numerical methods for the spectral analysis of more complex operators like partial differential operators.

We define some basic notations. Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{R}$ be the set of real numbers. Let $\mathbb{R}^N$, $N \in \mathbb{N}$, be the N-dimensional real Euclidean space, let $\mathbb{R}^{N\times N}$, be the space of square matrices of order $N \in \mathbb{N}$. Let $\underline{x} \in \mathbb{R}^N$, $N \in \mathbb{N}$, we denote with $\underline{x}^T$ its transpose and with $||\underline{x}|| = \sqrt{\underline{x}^T\underline{x}}$ its norm. Let $\underline{x} \in \mathbb{R}^N$, $\underline{x} \neq \underline{0}$, and $A \in \mathbb{R}^{N\times N}$ we denote with $\rho_A(\underline{x}) = \frac{\underline{x}^T A\underline{x}}{\underline{x}^T\underline{x}}$ the Rayleigh quotient. We denote with $\delta_{n,m}$, $n, m \in \mathbb{N}$, the Kronecker delta. Let (a, b) be an open interval of $\mathbb{R}$ and let $Ł^2(a, b)$ be the space of square-integrable functions from (a, b) to $\mathbb{R}$. We denote with $\langle\cdot,\cdot\rangle$ the scalar product on $Ł^2(a, b)$, and with $|\cdot|$ the norm on $Ł^2(a, b)$.

We consider the SL problem

$$\begin{cases} Ly = \lambda y, & a < x < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1}$$

where $Ly = -(p(x)y')' + q(x)y$ is a symmetric differential operator determined by functions $p(x)$ and $q(x)$, such that $p(x) > 0$, $p'(x)$ and $q(x) \geq 0$ are continuous. For the SL problem (1) there exists an infinite sequence of nonnegative eigenvalues $0 \leq \lambda_1 < \lambda_2 < \lambda_3 \ldots$; the corresponding eigenfunctions $\{y_n(x)\}_{n\in\mathbb{N}}$ are twice continuously differentiable, and satisfy the orthogonality relations $\langle y_n, y_m \rangle = \delta_{n,m}$, $n, m = 1, 2, \ldots$; moreover the nth eigenfunction has $n-1$ distinct zeros in (a, b), see [3] for a proof.

We propose a numerical method to solve (1) by using a perturbative approach similar to that introduced in [6]. This is an iterative method that starts from an SL problem with operator L_0 whose eigensystem is known. At each step, $m = 1, 2, \ldots, M$, the method computes the approximated eigensystem of the SL problem with operator L_m as a perturbation of the approximated eigensystem of the SL problem with operator L_{m-1}, where L_m is a small perturbation of L_{m-1}, and $L_M = L$. This method is based on the Jacobi orthogonal component correction, see [11, 17]. For the convenience of the reader, we sketch this method. The Jacobi orthogonal component correction is a numerical method to improve an approximation of an eigenpair, $\underline{x} \in \mathbb{R}^N$ eigenvector and $\lambda \in \mathbb{R}$ eigenvalue, of a symmetric matrix $A \in \mathbb{R}^{N\times N}$. Let $\underline{u} \in \mathbb{R}^N$, with $||\underline{u}|| = 1$, be an approximation of the eigenvector $\underline{x}$, the correction $\underline{t} \in \mathbb{R}^N$ of $\underline{u}$ must satisfy

$$\underline{t} \perp \underline{u}, \qquad A(\underline{u} + \underline{t}) = \lambda(\underline{u} + \underline{t}). \tag{2}$$

The solution $\underline{t} \in \mathbb{R}^N$ of problem (2) is computed as follows

$$(I - \underline{u}\underline{u}^T)(A - \theta I)(I - \underline{u}\underline{u}^T)\underline{t} = -\underline{r} \tag{3}$$

where I is the identity matrix, $\theta = \rho_A(\underline{u})$ is the approximation of λ, and $\underline{r} = (A - \theta I)\underline{u}$. Note that equation (3) is consistent if $A - \theta I$ is nonsingular, in fact $||\underline{u}|| = 1$ and $\underline{r} \perp \underline{u}$. Let $\underline{t}$ be the solution of equation (3), the new approximation of eigenvector $\underline{x}$ is given by $\underline{u} + \underline{t}$.

A numerical experiment is used to compare the performance of the proposed method and a classical approximation technique like the finite difference method.

In Section 2 we describe the finite difference method to solve SL problems. In Section 3 we present the proposed algorithm. In Section 4 we present the numerical results obtained with the two methods described in Sections 2, 3. In Section 5 we give some conclusions and future developments.

2. The Finite Difference Method

In this section, we recall the finite difference method for the solution of the SL problem. This is a very classical method where the SL problem is approximated by an eigenvalue problem for a symmetric matrix.

The finite difference method for two-point boundary value problems can be easily adapted to reduce SL problem (1) into an eigenvalue problem for a symmetric matrix. Let J be a positive integer, and

$$x_j = a + jh, j = 0, 1, \ldots, J+1, h = \frac{b-a}{J+1},$$

by using central difference operators with nodes x_j, $j = 0, 1, \ldots, J+1$, problem (1) can be approximated by the following one

$$\begin{cases} L_h u_j = h^2 \Lambda u_j, & j = 1, 2, \ldots, J, \\ u_0 = u_{J+1} = 0, \end{cases} \tag{4}$$

where

$$L_h u_j = (\alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1}),$$
$$\alpha_j = -p\left(x_j - \frac{h}{2}\right), \quad \gamma_j = -p\left(x_j + \frac{h}{2}\right),$$
$$\beta_j = -\alpha_j - \gamma_j + h^2 q(x_j),$$

u_j is the approximation of eigenfunction y at node x_j, and Λ is the approximation of the corresponding eigenvalue λ. Let $\underline{u} = (u_1, u_2, \ldots, u_J)^T \in \mathbb{R}^J$ and $A = (a_{i,j}) \in \mathbb{R}^{J\times J}$ given by

$$\begin{array}{ll} a_{j,j-1} = \alpha_j, & j = 2, 3, \ldots, J, \\ a_{j,j} = \beta_j, & j = 1, 2, \ldots, J, \\ a_{j,j+1} = \gamma_j, & j = 1, 2, \ldots, J-1, \\ a_{i,j} = 0, & \text{if } |i-j| > 1, \end{array} \tag{5}$$

problem (4) becomes

$$A\underline{u} = h^2 \Lambda \underline{u}. \tag{6}$$

This is an eigenvalue problem for a symmetric tridiagonal and positive definite matrix A. Taking advantage of the properties of matrix A, very efficient methods can be considered for the solution of (6), see [8] for details. The eigenvalues of A provide an approximation for the eigenvalues of L, in particular, we have the following theorem [13].

Theorem 1. *For each fixed eigenvalue λ of Sturm-Liouville problem with corresponding eigenfunction $y(x)$, there exists an eigenvalue, say $h^2\Lambda$, of matrix A given in (5) such that for h sufficiently small*

$$|\Lambda - \lambda| \leq \frac{||\underline{\tau}\{y\}||}{||\underline{y}||},$$

where the jth component of the truncation error $\underline{\tau}\{y\}$ is $L_h(y(x_j)) - (Ly)(x_j)$, $j = 1, 2, \ldots, J$.

Note that a similar result holds between the eigenvectors of A and the eigenfunctions of L. From standard arguments on approximation theory, we have that $\underline{\tau}$ is proportional to h^2 and to the fourth order derivative of eigenfunction y. So, from the properties of the zeros of eigenfunctions, we have the usual loss of accuracy for high-order eigenvalues and eigenfunctions.

3. The Perturbative Method

In this section, we describe the method proposed to solve the SL problem, and some results that justify this choice.

Lemma 3.1. Let L be the differential operator of an SL problem, with eigensystem $\lambda_n,\ y_n(x),\ n = 1, 2, \dots,\ x \in (a, b)$, where $Ly_n = \lambda_n y_n,\ \langle y_n, y_m\rangle = \delta_{n,m},\ m, n = 1, 2, \dots$. Let $\lambda = \lambda_m$ for some $m \in \mathbb{N}$, $f : [a, b] \to \mathbb{R}$ be a given function. The following boundary value problem:

$$\begin{cases} Lv - \lambda v = f, & x \in (a, b), \\ v(a) = v(b) = 0, \end{cases} \tag{7}$$

has a solution if and only if $\langle y_m, f\rangle = 0$, and

$$v(x) = \sum_{n\neq m} \frac{\langle y_n, f\rangle}{\lambda_n - \lambda} y_n(x), \qquad x \in (a, b). \tag{8}$$

Proof The proof follows from the orthogonality of eigenfunctions of SL problems, and (8) is the least norm solution.

We propose a perturbation method, for the computation of the eigensystem of an SL problem. This is an iterative algorithm based on the Jacobi orthogonal component correction. In particular, we have the following.

Theorem 2. *Let $L(\epsilon) = L_0 + \epsilon S$, where $0 < \epsilon \ll 1$, and $L_0,\ S$ be symmetric differential operators. Let $\lambda_{n,\epsilon} \in \mathbb{R}$, $y_{n,\epsilon}$, $n = 1, 2, \dots$ be the eigensystem of the SL problem associated to $L(\epsilon)$ and $\lambda_n^{(0)} \in \mathbb{R}$, $y_n^{(0)}$, $n = 1, 2, \dots$ be the eigensystem of the SL problem associated with L_0. If there exist $\lambda_n^{(k)} \in \mathbb{R}$, $\alpha_n > 0$ and functions $y_n^{(k)}$, $n, k \in \mathbb{N}$, such that*

$$y_n^{(k)}(a) = y_n^{(k)}(b) = 0, \qquad n, k \in \mathbb{N}, \tag{9}$$

$$\lambda_{n,\epsilon} = \sum_{k=0}^{\infty} \lambda_n^{(k)} \epsilon^k, \quad y_{n,\epsilon} = \alpha_n \sum_{k=0}^{\infty} y_n^{(k)} \epsilon^k, \qquad n \in \mathbb{N}. \tag{10}$$

Then, for $k = 1, 2, \dots$,

$$\lambda_n^{(k)} = \langle y_n^{(0)}, S y_n^{(k-1)}\rangle, \qquad n \in \mathbb{N}, \tag{11}$$

$$y_n^{(k)} = \sum_{m\neq n} \frac{\langle y_m^{(0)}, \sum_{h=1}^{k-1} \lambda_n^{(h)} y_n^{(k-h)} - S y_n^{(k-1)}\rangle}{\lambda_m^{(0)} - \lambda_n^{(0)}} y_m^{(0)} \qquad n \in \mathbb{N}. \tag{12}$$

Moreover, we can write

$$y_{n,\epsilon}(x) = \alpha_n \left(y_n^{(0)}(x) + v_{n,\epsilon}(x) \right), \tag{13}$$

where $\alpha_n > 0$ is the normalization factor and $v_{n,\epsilon}$ is a function orthogonal to $y_n^{(0)}$, such that also $Lv_{n,\epsilon}$ is orthogonal to $y_n^{(0)}$ and $|v_{n,\epsilon}| \to 0$ as $\epsilon \to 0$. In particular, for $n \in \mathbb{N}$,

$$v_{n,\epsilon} = \sum_{m\neq n} \frac{\langle y_m^{(0)} + v_{m,\epsilon}, r_{n,\epsilon}\rangle + \epsilon\langle y_n^{(0)}, S v_{n,\epsilon}\rangle\langle v_{m,\epsilon}, y_n^{(0)}\rangle}{\rho_{L(\epsilon)}(y_m^{(0)}) - \rho_{L(\epsilon)}(y_n^{(0)}) + \epsilon\langle y_m^{(0)}, S v_{m,\epsilon}\rangle - \epsilon\langle y_n^{(0)}, S v_{n,\epsilon}\rangle} y_{m,\epsilon}, \tag{14}$$

where $r_{n,\epsilon} = -L(\epsilon) y_n^{(0)} + \rho_{L(\epsilon)}(y_n^{(0)}) y_n^{(0)}$, and $\rho_{L(\epsilon)}(f) = \frac{\langle f, L(\epsilon) f\rangle}{|f|^2}$ is the Rayleigh quotient.

Proof From (10) we have

$$L(\epsilon) y_{n,\epsilon} = \alpha_n (L_0 + \epsilon S) \left(\sum_{k=0}^{\infty} y_n^{(k)} \epsilon^k \right) = \alpha_n \left(\lambda_n^{(0)} y_n^{(0)} + \sum_{k=1}^{\infty} \left(L_0 y_n^{(k)} + S y_n^{(k-1)} \right) \epsilon^k \right), \tag{15}$$

$$\lambda_{n,\epsilon} y_{n,\epsilon} = \alpha_n \left(\sum_{k=0}^{\infty} \lambda_n^{(k)} \epsilon^k \right) \left(\sum_{k=0}^{\infty} y_n^{(k)} \epsilon^k \right) = \alpha_n \left(\lambda_n^{(0)} y_n^{(0)} + \sum_{k=1}^{\infty} \left(\sum_{h=0}^{k} \lambda_n^{(h)} y_n^{(k-h)} \right) \epsilon^k \right). \tag{16}$$

By equalizing (15) and (16), we obtain

$$L_0 y_n^{(k)} - \lambda_n^{(0)} y_n^{(k)} = -S y_n^{(k-1)} + \sum_{h=1}^{k} \lambda_n^{(h)} y_n^{(k-h)}, \quad k = 1, 2, \dots. \tag{17}$$

From Lemma 3.1, problem (9), (17) has solution if and only if

$$\left\langle y_n^{(0)}, -S y_n^{(k-1)} + \sum_{h=1}^{k} \lambda_n^{(h)} y_n^{(k-h)} \right\rangle = 0, \qquad k = 1, 2, \dots, \tag{18}$$

and so formulas (11), (12) follow from (17), (18) and Lemma 3.1. Let n be a generic positive integer. Let $v_{n,\epsilon} = \sum_{k=1}^{\infty} y_n^{(k)} \epsilon^k$, let $y_{n,\epsilon}$ be the function defined in (13), we have that $v_{n,\epsilon}$ is orthogonal to $y_n^{(0)}$, and also $Ly_{n,\epsilon}$ is orthogonal to $y_n^{(0)}$, moreover $v_{n,\epsilon}(a) = v_{n,\epsilon}(b) = 0$. Let $P_n^{\perp}$ be the projection operator on the space of functions that are orthogonal to $y_n^{(0)}$, that is $P_n^{\perp}(f) = f - \langle f, y_n^{(0)} \rangle y_n^{(0)}$. Then, from $L(\epsilon)y_{n,\epsilon} = \lambda_{n,\epsilon} y_{n,\epsilon}$ and formula (13), we have

$$
\begin{aligned}
&P_n^{\perp}\left(L(\epsilon)\left(y_n^{(0)} + v_{n,\epsilon}\right) - \lambda_{n,\epsilon}\left(y_n^{(0)} + v_{n,\epsilon}\right)\right) = 0,\\
&P_n^{\perp}\left(L(\epsilon)v_{n,\epsilon} - \lambda_{n,\epsilon}v_{n,\epsilon}\right) = -P_n^{\perp}\left(L(\epsilon)y_n^{(0)} - \lambda_{n,\epsilon}y_n^{(0)}\right),\\
&L(\epsilon)v_{n,\epsilon} - \lambda_{n,\epsilon}v_{n,\epsilon} - \langle L(\epsilon)v_{n,\epsilon}, y_n^{(0)}\rangle y_n^{(0)} = -P_n^{\perp}\left(L(\epsilon)y_n^{(0)}\right),\\
&L(\epsilon)v_{n,\epsilon} - \lambda_{n,\epsilon}v_{n,\epsilon} = \epsilon\langle Sv_{n,\epsilon}, y_n^{(0)}\rangle y_n^{(0)} - L(\epsilon)y_n^{(0)} + \rho_{L(\epsilon)}\left(y_n^{(0)}\right) y_n^{(0)}.
\end{aligned} \tag{19}
$$

Note that, for the problem defined by (19) and $v_{n,\epsilon}(a) = v_{n,\epsilon}(b) = 0$, hypoteses of Lemma 3.1 are satisfied, in fact

$$
\begin{aligned}
&\langle y_n^{(0)} + v_{n,\epsilon}, \epsilon\langle Sv_{n,\epsilon}, y_n^{(0)}\rangle y_n^{(0)} - L(\epsilon)y_n^{(0)} + \rho_{L(\epsilon)}\left(y_n^{(0)}\right) y_n^{(0)}\rangle =\\
&= \epsilon\langle Sv_{n,\epsilon}, y_n^{(0)}\rangle - \rho_{L(\epsilon)}\left(y_n^{(0)}\right) + \rho_{L(\epsilon)}\left(y_n^{(0)}\right) - \epsilon\langle v_{n,\epsilon}, Sy_n^{(0)}\rangle = 0,
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
&L(\epsilon)\left(y_n^{(0)} + v_{n,\epsilon}\right) = \lambda_{n,\epsilon}\left(y_n^{(0)} + v_{n,\epsilon}\right),\\
&\langle L(\epsilon)\left(y_n^{(0)} + v_{n,\epsilon}\right), y_n^{(0)}\rangle = \lambda_{n,\epsilon}\left\langle\left(y_n^{(0)} + v_{n,\epsilon}\right), y_n^{(0)}\right\rangle,\\
&\lambda_{n,\epsilon} = \rho_{L(\epsilon)}\left(y_{n,0}\right) + \langle L(\epsilon)v_{n,\epsilon}, y_{n,0}\rangle = \rho_{L(\epsilon)}\left(y_{n,0}\right) + \epsilon\langle Sv_{n,\epsilon}, y_{n,0}\rangle.
\end{aligned} \tag{21}
$$

Hence, formula (14) follows from (19), (20), (21) and Lemma 3.1.

We note that this theorem defines the eigensystem of the perturbed operator $L(\epsilon)$ in terms of the eigensystem of the unperturbed operator L, where the correction term $v_{n,\epsilon}$ is computed by the Jacobi orthogonal component correction method. In the proposed method, we consider the approximation $Y_{n,\epsilon}$ of $y_{n,\epsilon}$ obtained by neglecting the $O(\epsilon^2)$ terms in (14), that is

$$
V_{n,\epsilon} = \sum_{k \neq n} \frac{\langle y_k^{(0)}, r_{n,\epsilon}\rangle}{\rho_{L(\epsilon)}\left(y_k^{(0)}\right) - \rho_{L(\epsilon)}\left(y_n^{(0)}\right)} y_k^{(0)}.
$$

So the corresponding approximated eigenfunctions of $L(\epsilon)$ are

$$
Y_{n,\epsilon} = \frac{y_n^{(0)} + V_{n,\epsilon}}{||y_n^{(0)} + V_{n,\epsilon}||}, \qquad n = 1, 2, \ldots.
$$

These are the fundamental formulas in the following recursive method.

Given an SL problem with operator L we choose an operator L_0 close, in some sense, to operator L, and having a known eigensystem, we define

$$
S = L - L_0.
$$

Starting from L_0, an iterative algorithm considers M, SL problems with different operators L_m, where

$$
L_m = L_0 + m\epsilon S = L_{m-1} + \epsilon S, \qquad m = 1, 2, \ldots, M, \tag{22}
$$

and $\epsilon = 1/M$. At each step, $m = 1, 2, \ldots, M$, the proposed algorithm computes the approximated eigensystem of L_m from the approximated eigensystem of L_{m-1}.

Algorithm 1. *Given the SL problem with operator L, determined by functions $p(x)$, $q(x)$; given $M \in \mathbb{N}$, and the SL problem with operator L_0, determined by functions $p_0(x)$, $q_0(x)$, and with orthonormal eigenfunctions $y_{n,0}$, $n \in \mathbb{N}$. Compute the approximated eigenvalues $\lambda_{n,M}$ and the approximated eigenfunctions $y_{n,M}$, $n = 1, 2, \ldots$, of the SL problem associated to L in the following way:*

$\epsilon = 1/M$, $S = L - L_0$,
for $m = 1, 2, \ldots, M$

$$\begin{aligned}
&L_m = L_0 + m\epsilon S,\\
&r_{n,m} = -L_m y_{n,m-1} + \rho_{L_m}(y_{n,m-1})\, y_{n,m-1}, \quad n \in \mathbb{N},\\
&w_{n,m} = y_{n,m-1} + \sum_{k\in\mathbb{N}, k\neq n} \frac{\langle y_{k,m-1}, r_{n,m}\rangle}{\rho_{L_m}(y_{k,m-1}) - \rho_{L_m}(y_{n,m-1})} y_{k,m-1}, \quad n \in \mathbb{N},\\
&y_{n,m} = w_{n,m}/|w_{n,m}|, \quad n \in \mathbb{N},
\end{aligned}$$

end for
$\lambda_{n,M} = \rho_L(y_{n,M}), \quad n \in \mathbb{N}.$

Note that in a practical implementation of this algorithm, the computation is truncated to a finite upper bound N of n. Moreover the eigenfunctions $y_{n,m}$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M$, are expressed in terms of a finite basis having N_B elements. So, this scheme depends on the choice of the basis and of the initial SL problem.

Some variants of the above algorithm can be considered. The simplest one is obtained by introducing an internal iteration within the main iteration of Algorithm 1. More precisely at each step of this internal iteration, multiple Jacobi orthogonal component correction steps can be performed for the same differential operator L_m. This simple modification of Algorithm 1 allows a substantial improvement in the accuracy of the final eigensystem by performing only one or two steps in the internal iteration. In the following algorithm, we describe a variant of Algorithm 1 where the discretization step ϵ is chosen automatically by a simple adaptive strategy.

Algorithm 2. *Given the SL problem with operator L, determined by functions $p(x)$, $q(x)$; given $\epsilon_0 < 1$, and the SL problem with operator L_0, determined by functions $p_0(x)$, $q_0(x)$, and having orthonormal eigenfunctions $y_{n,0}$, $n = 1, 2, \ldots$. Compute the first N approximated eigenvalues Λ_n and the approximated eigenfunctions Y_n, $n = 1, 2, \ldots, N$, of the SL problem associated to L in the following way.*
$t = \epsilon_0$, $S = L - L_0$, $m = 0$
while $t < 1$ *do*

$$\begin{aligned}
&m = m + 1\\
&C = \max_{1\le n\le N} \left\{ \sum_{k=1,k\neq n}^{N} \left(\frac{\langle y_{k,m-1}, S y_{n,m-1}\rangle}{\rho_{L_{m-1}}(y_{k,m-1}) - \rho_{L_{m-1}}(y_{n,m-1})} \right)^2 \right\}\\
&\epsilon = \min\{1/C, \epsilon_0, 1 - t\}\\
&t = t + \epsilon\\
&L_m = L_{m-1} + \epsilon S,\\
&\textit{for } n = 1, 2, \ldots, N\\
&\quad r_{n,m} = -L_m y_{n,m-1} + \rho_{L_m}(y_{n,m-1})\, y_{n,m-1},\\
&\quad w_{n,m} = y_{n,m-1} + \sum_{k=1,k\neq n}^{N} \frac{\langle y_{k,m-1}, r_{n,m}\rangle}{\rho_{L_m}(y_{k,m-1}) - \rho_{L_m}(y_{n,m-1})} y_{k,m-1},\\
&\quad y_{n,m} = w_{n,m}/||w_{n,m}||,\\
&\textit{end for}
\end{aligned}$$

end while
$Y_n = y_{n,m}, \quad \Lambda_n = \rho_L(Y_n), \quad n = 1, 2, \ldots N.$

Note that in this algorithm the adaptive procedure to choose ϵ is based on the following observation. At each step m, when $k \neq n$ we have

$$\langle y_{k,m-1}, r_{n,m}\rangle = \langle y_{k,m-1}, -L_m y_{n,m-1} + \rho_{L_m}(y_{n,m-1})\, y_{n,m-1}\rangle \approx -\epsilon \langle y_{k,m-1}, S y_{n,m-1}\rangle,$$

so the norm of the correction of the eigenfunction $y_{n,m-1}$ satisfies

$$\left| \sum_{k=1,k\neq n}^{N} \frac{\langle y_{k,m-1}, r_{n,m}\rangle}{\rho_{L_m}(y_{k,m-1}) - \rho_{L_m}(y_{n,m-1})} y_{k,m-1} \right|^2 \le C\epsilon,$$

where C is the constant defined in Algorithm 2. Note that the eigenfunctions have a unit norm, so a reasonable requirement is that $C\epsilon \leq 1$, from which arises the adaptive procedure.

We conclude this section with another algorithm, where the same perturbative technique is used to correct a given approximation of an SL problem.

Algorithm 3. *Given the SL problem with operator L, determined by functions $p(x)$, $q(x)$; given $M \geq 1$, and eigensystem $\lambda_{n,0}$, $y_{n,0}$, $n = 1, 2, \dots, N$. Compute the approximated eigenvalues Λ_n and the approximated eigenfunctions Y_n, $n = 1, 2, \dots N$, of the SL problem associated to L in the following way:*

$\epsilon = \frac{1}{M}$,

for $m = 1, 2, \dots, M$

$$
\begin{aligned}
&\textit{for } n = 1, 2, \dots, N \\
&\quad \rho_{n,m} = (1-\epsilon)\lambda_{n,m-1} + \epsilon\langle y_{n,m-1}, Ly_{n,m-1}\rangle, \\
&\quad w_{n,m} = y_{n,m-1} - \epsilon \sum_{k=1, k\neq n}^{N} \frac{\langle y_{k,m-1}, Ly_{n,m-1}\rangle}{\rho_{k,m} - \rho_{n,m}} y_{k,m-1}, \\
&\quad \lambda_{n,m} = \rho_{n,m} + \epsilon\langle y_{n,m-1}, Lw_{n,m}\rangle, \quad n \in \mathbb{N}, \\
&\quad y_{n,m} = w_{n,m}/||w_{n,m}||, \\
&\textit{end for}
\end{aligned}
$$

end for

$Y_n = y_{n,M}, \quad \Lambda_n = \lambda_{n,M}, \quad n = 1, 2, \dots N.$

Note that when $\lambda_{n,0}$, $y_{n,0}$, $n = 1, 2, \dots, N$ is a good approximation of the eigensystem of L then we can usually choose $M = 1$.

4. Numerical Results

We present two numerical experiments with the methods described in the previous sections. These experiments aim to compare the accuracy of the numerical solutions obtained by these methods.

In the first experiment, we consider the numerical results obtained by these methods for two particular SL problems with known analytical solutions. The first problem is the following one

$$
\begin{cases} -((1+x)^2 y')' = \lambda y, & 0 < x < 1, \\ y(0) = y(1) = 0, \end{cases} \tag{23}
$$

whose eigensystem, for $n \in \mathbb{N}$, is

$$
\lambda_n = (n\pi/\ln 2)^2 + \frac{1}{4},\ y_n(x) = \sqrt{\frac{2}{\ln 2}} \frac{\sin\left(\frac{n\pi \ln(1+x)}{\ln 2}\right)}{\sqrt{1+x}}. \tag{24}
$$

The second problem is the following one

$$
\begin{cases} -((1+x^2)^2 y')' - (1+2\,x^2)y = \lambda y, & 0 < x < 1, \\ y(0) = y(1) = 0, \end{cases} \tag{25}
$$

whose eigensystem, for $n \in \mathbb{N}$, is

$$
\lambda_n = 16n^2,\ y_n(x) = \frac{\sin(4n \arctan x)}{\sqrt{1+x^2}}. \tag{26}
$$

In the finite difference method, the eigenvalue problem (6) is solved by the QR method implemented in routine F08JEF of NAG library [15].

In the perturbative method, we start from the differential operator L_0 having $p_0(x) = 1$, $q_0(x) = 0$, $x \in (0, 1)$, the corresponding SL problem has eigensystem

$$
\lambda_{n,0} = (n\pi)^2, \qquad y_{n,0}(x) = \sqrt{2}\sin(n\pi x), \qquad n \in \mathbb{N}. \tag{27}
$$

For the sake of brevity, we report only the results obtained by Algorithm 2, which is implemented by using two different representation bases. Note that the iterative procedure in Algorithm 2 can be easily rewritten as an iterative procedure for the coefficients, with respect to the basis taken into account, of the eigenfunctions appearing in the algorithm. The numerical results are reported in Figures 1-4.

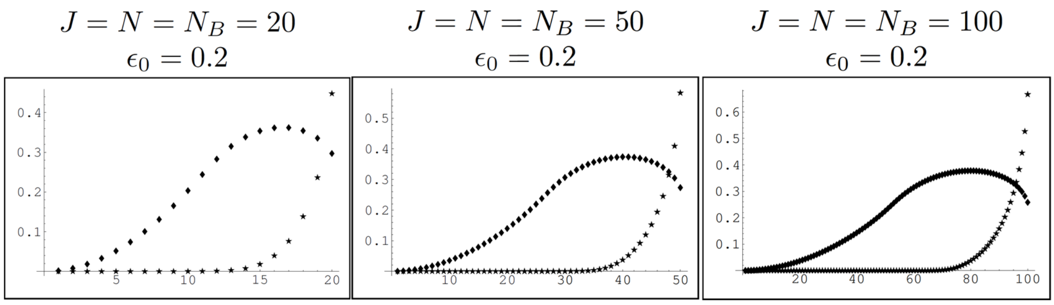

Figure 1. *The numerical results obtained by the finite difference method (⧫) and by Algorithm 2 (⋆) for problem (23). The diagrams show the relative errors in the eigenvalues for different choices of the discretization parameters. In Algorithm 2 the eigenfunctions reported in (27) are used also as the representation basis.*

In particular, Figures 1, 3 show the results obtained for problems (23) and (25), respectively, with basis $\{y_{n,0}(x)\}_{n=1,2,...,N_B}$, where $y_{n,0}$ is given by (27). We always choose $N_B = N$. Figures 2, 4 show the results obtained for problems (23) and (25), respectively, with the piecewise linear basis functions, constructed on a uniform subdivision of interval $(0, 1)$ with step $1/(N_B + 1)$.

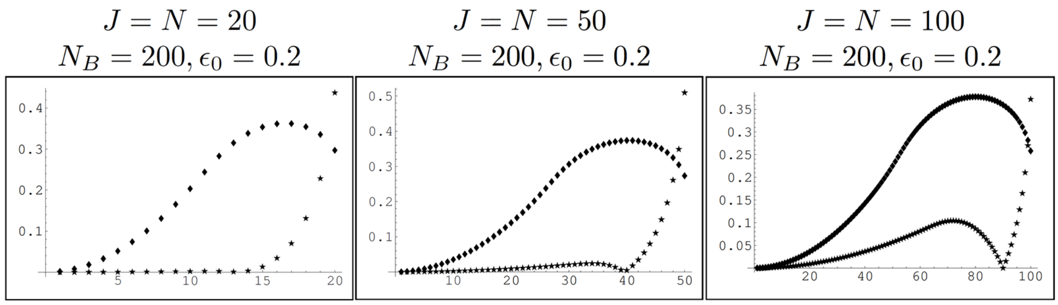

Figure 2. *The numerical results obtained by the finite difference method (⧫) and by Algorithm 2 (⋆) for problem (23). The diagrams show the relative errors in the eigenvalues for different choices of the discretization parameters. In Algorithm 2 the piecewise linear basis functions are used as the representation basis.*

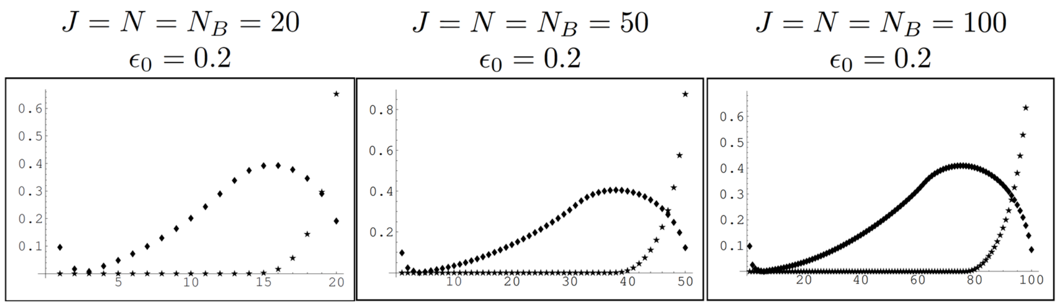

Figure 3. *The numerical results obtained by the finite difference method (⧫) and by Algorithm 2 (⋆) for problem (25). The diagrams show the relative errors in the eigenvalues for different choices of the discretization parameters. In Algorithm 2 the eigenfunctions reported in (27) are used also as the representation basis.*

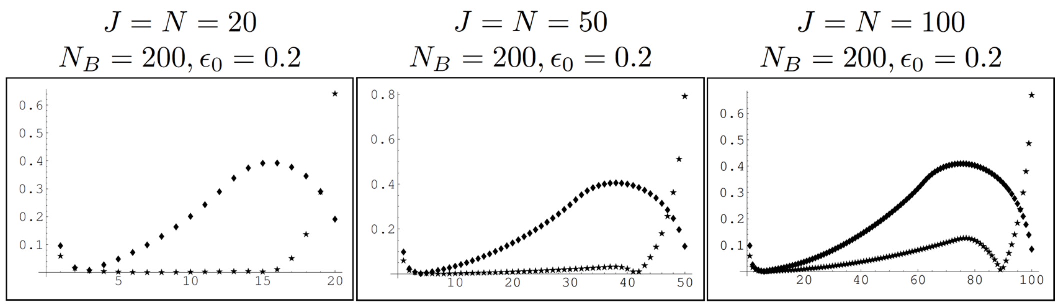

Figure 4. *The numerical results obtained by the finite difference method (⧫) and by Algorithm 2 (⋆) for problem (25). The diagrams show the relative errors in the eigenvalues for different choices of the discretization parameters. In Algorithm 2 the piecewise linear basis functions are used as the representation basis.*

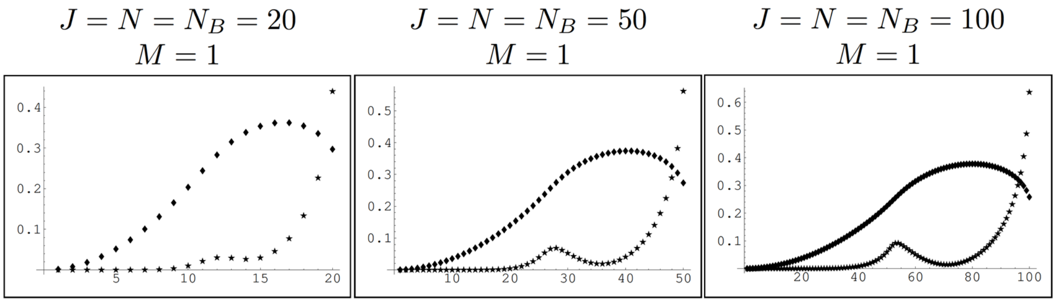

Figure 5. *The numerical results obtained by the finite difference method (♦) and by Algorithm 3 (⋆) for problem (23). The diagrams show the relative errors in the eigenvalues for different choices of the discretization parameters. In Algorithm 3 only $M = 1$ step is considered and the eigenfunctions reported in (27) are used also as the representation basis.*

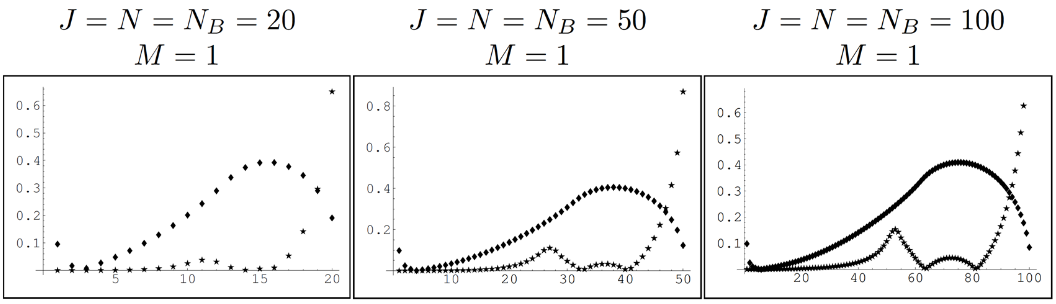

Figure 6. *The numerical results obtained by the finite difference method (♦) and by Algorithm 3 (⋆) for problem (25). The diagrams show the relative errors in the eigenvalues for different choices of the discretization parameters. In Algorithm 3 only $M = 1$ step is considered and the eigenfunctions reported in (27) are used also as the representation basis.*

In the second experiment, we start from the approximated eigensystems of problems (23), (25), computed by finite difference scheme, and we correct these approximations by Algorithm 3. The numerical results are reported in Figures 5, 6; they are relative to problems (23) and (25), respectively, and the representation basis $\{y_{n,0}(x)\}_{n=1,2,\ldots,N_B}$, where $N_B = N$ and $y_{n,0}$ is given by (27).

We note that Figures 5, 6 are obtained by applying Algorithm 3 with only $M = 1$ step; of course with more correction steps we obtain a further reduction of these relative errors. Figures 1-6 shows interesting results; in fact, the approximated eigenvalues computed by the proposed method are usually more accurate than the ones computed by the finite difference method. This is always true except for the last eigenvalues, where the proposed method has abnormal behavior. From numerical experiments not reported for brevity, the computational cost of the proposed method is higher than the cost of the finite difference scheme. So, the next study should be devoted to improving the efficiency of this method.

5. Conclusions

A perturbative method for the solution of SL problems has been proposed. Starting from an initial SL problem for the differential operator L_0 and known eigensystem, at each step, the proposed method uses the Jacobi orthogonal component correction to compute the eigensystem of a differential operator L_m from the eigensystem of L_{m-1}. This method can be used to obtain different approximation schemes depending on the choice of L_0 and the basis used in the numerical implementation. The numerical results obtained with this method are usually more accurate than the results obtained by the finite difference method.

Further studies of this method should consider the improvement of its efficiency and the application of similar techniques to the solution of different eigenvalue problems, such as for examples: matrix eigenvalue problems, and eigenvalue problems for partial differential operators.

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