




An approach to p -adic qubits from irreducible representations of $SO(3)_p$

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We introduce the notion of p -adic quantum bit (p -qubit) in the context of the p -adic quantum mechanics initiated and developed after the seminal paper of Volovich [Theor. Math. Phys. 71, 574 (1987)]. In this approach, physics takes place in three-dimensional p -adic space rather than Euclidean space. Based on our prior work describing the p -adic special orthogonal group [Di Martino *et al.*, arXiv:2104.06228 [math.NT] (2021)], we outline a program to classify its continuous unitary projective representations, which can be interpreted as a theory of p -adic angular momentum. The p -adic quantum bit arises from the irreducible representations of minimal nontrivial dimension two, of which we construct examples for all primes p .

I. INTRODUCTION

A qubit at its most basic is the state space of a two-level quantum system, i.e., one with two-dimensional complex Hilbert space \mathcal{H} . This abstracts from all physical characteristics such a system might have, be it an electron spin, or that of a neutron, the polarization of a photon, a quantum dot, etc. Its characteristic is that the symmetry group $SU(2)$ acts transitively on all pure states

$$\mathcal{P}(\mathcal{H}) = \{\rho = |\psi\rangle\langle\psi| : \rho \geq 0, \text{Tr } \rho = 1, \rho \text{ rank-one}\},$$

which geometrically is the surface of the Bloch sphere, \mathbb{R}^3 , and the conjugation action on the density matrices is equivalent to the action of $SO(3)$ on the Bloch sphere. Furthermore, for a spin- $\frac{1}{2}$ particle, this action coincides with the one induced by the spatial rotations, and as an irreducible representation (irrep) gives rise to all other irreps by tensor powers $U^{\otimes n}$ and their Clebsch-Gordan decomposition into irreps. This has been long noticed and has even been taken as a possible foundation of quantum mechanics [1].

However, this link of the most basic quantum system to the geometry and, indeed, symmetry of physical space [2] opens up other possibilities. For example, why should we accept *a priori* that physical space is described by the Euclidean \mathbb{R}^3 ? Indeed, the real field is not the only possible metric completion of the rationals: by Ostrowski's Theorem [3], there is a countable family of other completions, those of the p -adic numbers \mathbb{Q}_p . The fractal geometry of the p -adic numbers made them interesting for attempts at “new” quantum physics with a space-time that is decidedly non-Euclidean at very short or very large distances, as seems possible in quantum gravity [4, 5]. In these approaches, the focus is on the translation symmetry of phase space, which in ordinary quantum mechanics leads to the framework of the Heisenberg-Weyl group of position

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and momentum displacements, parametrized by real numbers, but which here are displacements by p -adic amounts [6].

Here, we consider the rotation symmetry of \mathbb{Q}_p^3 , encoded in the special orthogonal group $SO(3)_p$ in dimension three. Recently, we have characterized the geometry of this group [7] based on the crucial observation that, up to equivalence, there is a unique quadratic form on \mathbb{Q}_p^3 with a compact symmetry group. Motivated by the analogy with the real case, our program is to find its unitary complex projective representations, which should constitute a p -adic theory of quantum angular momentum. In particular, the irreps of dimension two are crucial as p -adic analogs of spin- $\frac{1}{2}$ particles, which we interpret as p -adic qubits. In this spirit, we call any pair (\mathcal{H}, ψ) , where \mathcal{H} is a two-dimensional Hilbert space and ψ is a projective irrep on \mathcal{H} , a qubit. In the present paper, we do not carry out this entire program, leaving the bulk of it for future work, but contend ourselves with the construction of the simplest possible two-dimensional irreps of $SO(3)_p$ for every prime p . We emphasize that the angular momentum representations that we are looking for map the p -adic group $SO(3)_p$ to groups of linear transformations on complex vector spaces. Taking this route, moving from p -adic numbers to complex numbers, is different and alternative to the ones that can be considered more conventional, such as (i) defining p -adic Hilbert spaces and working on them [8–12] and (ii) defining p -adic Lie groups [13].

The structure of the rest of the paper is as follows. In Sec. II, we recall the basic definitions and facts about the three-dimensional special orthogonal groups $SO(3)_p$, among them the crucial insight that the coefficients of special orthogonal matrices are p -adic integers. In Sec. III, we exploit the latter property to define a hierarchy of projected groups by reducing the matrix entries modulo p^k , $k \geq 1$; each of them is a finite group, offering an easy way to constructing irreps. In Sec. IV, we turn our attention, in particular, to the smallest reduction, $SO(3)_p \bmod p$, for odd primes $p > 2$; we show that each has a homomorphism to some dihedral group that is onto. This implies, in particular, that they all have two-dimensional irreps. In the subsequent sections, we explicitly compute the groups $SO(3)_3 \bmod 3$ (Sec. V) and $SO(3)_5 \bmod 5$ (Sec. VI) and their representation theory in both cases exhibiting qubits, i.e., two-dimensional irreps. Finally, in Sec. VII, we do the same for $SO(3)_2 \bmod 2$, which turns out to be isomorphic to \mathbb{S}_3 , the permutation group of three elements. We conclude in Sec. VIII with an outlook to future work and open questions.

II. PRELIMINARIES ON THE COMPACT GROUP $SO(3)_p$

We recall here some basic notions and results drawn from Ref. [7].

Theorem II.1 *For every prime p , there is a unique quadratic form on \mathbb{Q}_p^3 , up to linear equivalence and scaling, which does not represent zero,*

$$Q(\mathbf{x}) = \begin{cases} x_1^2 - vx_2^2 + px_3^2 & \text{if } p > 2, \\ x_1^2 + x_2^2 + x_3^2 & \text{if } p = 2, \end{cases} \quad (1)$$

where

$$v := \begin{cases} -1 & \text{if } p \equiv 3 \pmod{4}, \\ -u & \text{if } p \equiv 1 \pmod{4}, \end{cases} \quad (2)$$

where $u \in \mathbb{U}$ (the group of p -adic units, i.e., the group of invertible elements of \mathbb{Z}_p) is not a square.

Recalling that every quadratic form Q induces a special orthogonal group as the set of determinant 1 matrices that preserve Q , we have the following result.

Theorem II.2 *There is a unique (up to isomorphisms) compact special orthogonal group on \mathbb{Q}_p^3 for every prime p ,*

$$SO(3)_p = \{L \in \mathcal{M}_{3 \times 3}(\mathbb{Q}_p) : A = L^T A L, \det L = 1\} \quad (3)$$

endowed with the common matrix product, where A is the matrix representation in the canonical basis of \mathbb{Q}_p^3 of the preserved quadratic form Q ,

$$A = \begin{cases} \text{diag}(1, -v, p) & \text{if } p > 2, \\ \mathbf{1} & \text{if } p = 2. \end{cases} \quad (4)$$

Since $SO(3)_p$ is a topological group, it is natural that the representations we look for are continuous [with respect to the p -adic metric on the domain and the Lie group metric of the complex $SU(d)$]. All of the representations we find will turn out to be continuous for trivial reasons.

Another crucial result proved in Ref. [7] is the following.

Theorem II.3 *All special orthogonal matrices have p -adic integer entries,*

$$SO(3)_p \subset SL(3, \mathbb{Z}_p).$$

Next we state a fundamental fact about the geometry of $SO(3)_p$.

Theorem II.4 *The elements of $SO(3)_p$ are rotations, i.e., they always have an eigenvalue 1, and the corresponding eigenspace is the rotation axis.*

We recall the parameterization of matrices of $SO(3)_p$.

Theorem II.5 *A rotation of $SO(3)_p$ around $\mathbf{n} \in \mathbb{Q}_p^3 \setminus \{\mathbf{0}\}$ takes the following matrix form with respect to an orthogonal basis $(\mathbf{g}, \mathbf{h}, \mathbf{n})$ of \mathbb{Q}_p^3 :*

$$\mathcal{R}_{\mathbf{n}}(\sigma) = \begin{pmatrix} \frac{1-\alpha\sigma^2}{1+\alpha\sigma^2} & -\frac{2\alpha\sigma}{1+\alpha\sigma^2} & 0 \\ \frac{2\sigma}{1+\alpha\sigma^2} & \frac{1-\alpha\sigma^2}{1+\alpha\sigma^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5)$$

with $\sigma \in \mathbb{Q}_p \cup \{\infty\}$ and $\alpha = Q(\mathbf{h})/Q(\mathbf{g})$. The set $SO(3)_{p,\mathbf{n}}$ of rotations around a given $\mathbf{n} \in \mathbb{Q}_p^3 \setminus \{\mathbf{0}\}$ forms an Abelian subgroup of $SO(3)_p$

Last, in parallel to the real orthogonal case, we found that only certain main angle decompositions of rotations around the reference axes hold for $SO(3)_p$.

Theorem II.6 *For an odd prime $p > 2$, every $M \in SO(3)_p$ can be written as any of the Cardano type compositions,*

$$\mathcal{R}_x \mathcal{R}_y \mathcal{R}_z, \quad \mathcal{R}_z \mathcal{R}_y \mathcal{R}_x, \quad \mathcal{R}_z \mathcal{R}_x \mathcal{R}_y, \quad \mathcal{R}_y \mathcal{R}_x \mathcal{R}_z,$$

respectively, with certain rotation parameters $\sigma, \tau, \omega \in \mathbb{Q}_p \cup \{\infty\}$.

For a prime $p \equiv 1 \pmod{4}$, every $M \in SO(3)_p$ can also be written as any of the Cardano-type compositions,

$$\mathcal{R}_x \mathcal{R}_z \mathcal{R}_y, \quad \mathcal{R}_y \mathcal{R}_z \mathcal{R}_x,$$

respectively, with certain rotation parameters $\sigma, \tau, \omega \in \mathbb{Q}_p \cup \{\infty\}$.

Note that none of the other possible forms of main angle decomposition, familiar from the real Euclidean case, hold universally for $SO(3)_p$. In particular, no Euler angle decomposition exists for primes $p > 2$, and none of the Euler and Cardano decompositions exist when $p = 2$.

III. QUOTIENTING THE GROUP $SO(3)_p$

A fundamental tool to understand the algebraic aspects of $SO(3)_p$ is quotienting modulo p^k the group [14]: this is possible because the matrix entries are p -adic integers (Theorem II.3). These quotients are finite groups and, thus, much easier to study while generating $SO(3)_p$ as a projective limit. For us, these projections are crucial as their representations induce representations of the whole $SO(3)_p$ and, thus, allow us to find irreps easily.

We can write a p -adic integer $a \in \mathbb{Z}_p$ as a sequence $a = (a_1, a_2, a_3, \dots)$ with $a_k \in \mathbb{Z}/p^k$ such that $a_{k+1} \equiv a_k \pmod{p^k}$ for all $k > 0$, with component-wise addition and multiplication in the respective rings [3]. This means that the functions

$$\begin{aligned} \pi'_k : \mathbb{Z}_p &\rightarrow \mathbb{Z}/p^k \\ a = (a_i)_{i=1}^\infty &\mapsto a_k \end{aligned} \quad (6)$$

are ring homomorphisms,

$$\pi'_k(a + b) = \pi'_k((a_i)_i + (b_j)_j) = \pi'_k((a_i + b_i)_i) = a_k + b_k = \pi'_k(a) + \pi'_k(b), \quad (7)$$

$$\pi'_k(ab) = \pi'_k((a_i)_i (b_j)_j) = \pi'_k((a_i b_i)_i) = a_k b_k = \pi'_k(a) \pi'_k(b). \quad (8)$$

This can be extended to multiplicative groups of matrices over \mathbb{Z}_p : the matrix product is defined through sums and products of elements, for which π'_k are homomorphisms; then the maps

$$\pi_k(M) = \pi_k((m_{ij})_{ij}) := (\pi'_k(m_{ij}))_{ij} \quad (9)$$

are group homomorphisms on any group contained in $\mathcal{M}_{n \times n}(\mathbb{Z}_p)$ to some other group contained in $\mathcal{M}_{n \times n}(\mathbb{Z}/p^k)$.

Remark III.1 Note that all the maps π_k are continuous with respect to the p -adic metric on $SO(3)_p$ and the discrete topology on the range, which is a finite set.

Theorem II.3 means that the projection maps π_k are well defined on $SO(3)_p$,

$$\begin{aligned} \pi_k : SO(3)_p &\rightarrow SO(3)_p \pmod{p^k} \subset SL(3, \mathbb{Z}/p^k), \\ L = (a_{ij})_{ij} &\mapsto (a_{ij} \pmod{p^k})_{ij} \end{aligned} \quad (10)$$

since all matrix entries of eligible L are p -adic integers. This allows us to build a projective sequence of quotient groups,

$$G_{p^k} := \pi_k(SO(3)_p) = SO(3)_p \pmod{p^k}. \quad (11)$$

These are finite groups as anticipated, being subgroups of $SL(3, \mathbb{Z}/p^k)$. By G_{n,p^k} we will denote the Abelian subgroup of G_{p^k} of π_k -projected rotations $\mathcal{R}_n(\sigma) \in SO(3)_p$ around a fixed $n \in \mathbb{Q}_p^3 \setminus \{\mathbf{0}\}$,

$$G_{n,p^k} := \pi_k(SO(3)_{p,n}). \quad (12)$$

The Cardano decompositions (Theorem II.6) hold in G_{p^k} too.

It is possible to rewrite the parameterization of the subgroups $SO(3)_{p,n}$ of rotations around a given $n \in \mathbb{Q}_p^3 \setminus \{\mathbf{0}\}$ (Theorem II.5) only in terms of p -adic integers [7]. To wit, if κ denotes the determinant of the quadratic form preserved by p -adic planar rotations, in the plane orthogonal to the rotation axis, we have

$$SO(2)_p^\kappa = \{\mathcal{R}_\kappa(\sigma) : \sigma \in \mathbb{Z}_p\} \cup \{-\mathcal{R}_\kappa(\sigma) : \sigma \in p\mathbb{Z}_p\}$$

for $p > 2$ and $\kappa = -v$ (rotations around the z -axis), or $p = 2$ and $\kappa = 1, \pm 5$ ($\kappa = 1$ corresponds to rotations around x, y and z axes), while

$$SO(2)_p^\kappa = \{\mathcal{R}_\kappa(\sigma) : \sigma \in \mathbb{Z}_p\} \cup \{-\mathcal{R}_\kappa(\sigma) : \sigma \in \mathbb{Z}_p\}$$

for $p > 2$ and $\kappa = p, up$ (rotations around x and y axes), or $p = 2$ and $\kappa = \pm 2, \pm 10$. We also recall that $\mathcal{R}_n(-\frac{1}{\alpha\sigma}) = \mathcal{R}_n(\infty)\mathcal{R}_n(\sigma)$, which allows us to write for every $p > 2$,

$$\begin{aligned} G_{z,p^k} &= \left\{ \mathcal{R}_z(\sigma) \pmod{p^k} : \sigma \in \mathbb{Z}_p \right\} \cup \left\{ \mathcal{R}_z(\infty)\mathcal{R}_z(\sigma) \pmod{p^k} : \sigma \in p\mathbb{Z}_p \right\} \\ &= \left\{ \mathcal{R}_z(\sigma) \pmod{p^k} : \sigma \in \mathbb{Z}/p^k \right\} \cup \left\{ \mathcal{R}_z(\infty)\mathcal{R}_z(\sigma) \pmod{p^k} : \sigma \in p(\mathbb{Z}/p^k) \right\}, \end{aligned} \quad (13)$$

and similarly,

$$G_{n,p^k} = \left\{ \mathcal{R}_n(\sigma) \pmod{p^k} : \sigma \in \mathbb{Z}/p^k \right\} \cup \left\{ \mathcal{R}_n(\infty)\mathcal{R}_n(\sigma) \pmod{p^k} : \sigma \in \mathbb{Z}/p^k \right\} \quad (14)$$

for the x and y axes.

Moreover, when $p = 2$, we have the following for each of the reference axes:

$$\begin{aligned} G_{n,2^k} &= \left\{ \mathcal{R}_n(\sigma) \pmod{2^k} : \sigma \in \mathbb{Z}_2 \right\} \cup \left\{ \mathcal{R}_n(\infty)\mathcal{R}_n(\sigma) \pmod{2^k} : \sigma \in 2\mathbb{Z}_2 \right\} \\ &= \left\{ \mathcal{R}_n(\sigma) \pmod{2^k} : \sigma \in 2(\mathbb{Z}/2^k) \right\} \\ &\quad \cup \left\{ \mathcal{R}_n(\infty)\mathcal{R}_n(\sigma) \pmod{2^k} : \sigma \in 2(\mathbb{Z}/2^k) \right\} \\ &\quad \cup \left\{ \mathcal{R}_n(\sigma) \pmod{2^k} : \sigma \in \mathbb{Z}/2^k, \sigma \equiv 1 \pmod{2} \right\}, \end{aligned} \quad (15)$$

with the elements in the last set of the union in the plane orthogonal to n written as

$$\left(\begin{array}{cc} \frac{-2(\sigma+\sigma^2)}{1+2(\sigma+\sigma^2)} & -\frac{1+2\sigma}{1+2(\sigma+\sigma^2)} \\ \frac{1+2\sigma}{1+2(\sigma+\sigma^2)} & \frac{-2(\sigma+\sigma^2)}{1+2(\sigma+\sigma^2)} \end{array} \right) \pmod{2^k}, \sigma \in \mathbb{Z}/2^{k-1}. \quad (16)$$

Eqs. (13), (14) and (15) provide a parameterization of G_{p^k} for every prime p and $k \geq 1$, thanks to the existing Cardano decompositions.

We conclude this section by introducing the groups of solutions modulo p^k , $k \geq 1$, of the defining equations of $SO(3)_p$,

$$\widehat{G}_{p^k} = \{L \in \mathcal{M}_{3 \times 3}(\mathbb{Z}/p^k) : A \equiv L^T A L \pmod{p^k}, \det L \equiv 1 \pmod{p^k}\}. \quad (17)$$

These groups will turn out to be fundamental in the discussion about the representations of $SO(3)_p$ at least for $k = 1$ in this paper.

Remark III.2 *By virtue of Theorem II.3*

$$G_{p^k} \subseteq \widehat{G}_{p^k}, \quad (18)$$

and a fundamental open question is whether these groups coincide or not, by varying the values of prime $p > 2$ and $k \geq 1$. This question has the character of Hensel's Lemma [15]: do the integer solutions modulo p of $A = L^T A L$ with $\det L = 1$ lift to p -adic solutions?

This boils down to checking if $|G_{p^k}| = |\widehat{G}_{p^k}|$ for some p and k . Focusing on primes $p > 2$, this can be done by comparing the order of \widehat{G}_{p^k} with the order of G_{p^k} given by one of its possible Cardano decompositions. One of the main aspects left to be discovered in this matter is, then, the multiplicity of the Cardano decompositions for G_{p^k} for every prime $p > 2$, $k \geq 1$, that is, how many distinct Cardano decompositions around the same axes exist for each matrix of G_{p^k} . Once known that, it will not be hard to find the order of the subgroups G_{n,p^k} for the x, y and z axes from Eqs. (13) and (14) and, then, deduce $|G_{p^k}|$.

IV. THE GROUP $SO(3)_p \pmod p$ FOR $p > 2$

We specialize Eqs. (13) and (14) for odd prime $p > 2$ in the case $k = 1$: for the x or y axes,

$$G_{n,p} = \{\mathcal{R}_n(\sigma) \pmod p : \sigma \in \mathbb{Z}/p\} \cup \{\mathcal{R}_n(\infty)\mathcal{R}_n(\sigma) \pmod p : \sigma \in \mathbb{Z}/p\}, \quad (19)$$

while for the z axis,

$$G_{z,p} = \{\mathcal{R}_z(\sigma) \pmod p, \sigma \in \mathbb{Z}/p\} \cup \{\mathcal{R}_z(\infty) \pmod p\}. \quad (20)$$

By using (5), Eqs. (19) and (20) can be rewritten as follows:

$$G_{x,p} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2\xi & 1 \end{pmatrix} \pmod p : \xi \in \mathbb{Z}/p \right\} \cup \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -2\xi & -1 \end{pmatrix} \pmod p : \xi \in \mathbb{Z}/p \right\}, \quad (21)$$

$$G_{y,p} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\eta & 0 & 1 \end{pmatrix} \pmod p : \eta \in \mathbb{Z}/p \right\} \cup \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\eta & 0 & -1 \end{pmatrix} \pmod p : \eta \in \mathbb{Z}/p \right\}, \quad (22)$$

$$G_{z,p} = \left\{ \begin{pmatrix} \frac{1+v\zeta^2}{1-v\zeta^2} & \frac{2v\zeta}{1-v\zeta^2} & 0 \\ \frac{2\zeta}{1-v\zeta^2} & \frac{1+v\zeta^2}{1-v\zeta^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod p : \zeta \in \mathbb{Z}/p \right\} \cup \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod p \right\}. \quad (23)$$

We will refer to the first set of each of these three unions as “first branch” and to the second one as “second branch.” They provide a parameterization of $G_p = SO(3)_p \pmod p$ thanks to the existing Cardano decompositions.

Remark IV.1 *It is trivial to check that matrices within each of the two branches of $G_{x,p}$ and $G_{y,p}$ are all distinct by varying the parameter in \mathbb{Z}/p . Moreover a matrix from the first branch cannot be equal to a matrix of the second branch, for both $G_{x,p}$ and $G_{y,p}$, because $1 \not\equiv -1 \pmod p$ when $p > 2$. Furthermore, by equating two matrices from the first branch of $G_{z,p}$, we get*

$$\frac{1+v\zeta^2}{1-v\zeta^2} \equiv \frac{1+v\zeta'^2}{1-v\zeta'^2} \pmod p, \quad (24)$$

which is equivalent to $\zeta^2 \equiv \zeta'^2 \pmod p$. Plugging this into

$$\frac{2\zeta}{1-v\zeta^2} \equiv \frac{2\zeta'}{1-v\zeta'^2} \pmod p, \quad (25)$$

we obtain $\zeta \equiv \zeta' \pmod p$. Finally, we equate a matrix from the first branch of $G_{z,p}$ to the one of the second branch,

$$\frac{1+v\zeta^2}{1-v\zeta^2} \equiv -1 \pmod p, \quad (26)$$

which is equivalent to $1 \equiv -1 \pmod p$, which is impossible. Therefore, the matrices in $G_{x,p}$, $G_{y,p}$ and $G_{z,p}$ are all distinct by varying the respective parameters. Then, it follows that

$$|G_{x,p}| = |G_{y,p}| = 2p, \quad |G_{z,p}| = p + 1. \quad (27)$$

As anticipated, a fundamental point in this matter is to discover the multiplicity of a modulo- p -reduced Cardano decomposition for G_p , for example that of the kind $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$. A systematic ambiguity of order 2 in the Cardano representations is based on the relation

$$\mathcal{R}_x(\infty)\mathcal{R}_y(\infty)\mathcal{R}_z(\infty) \equiv \mathbb{1} \pmod{p}. \quad (28)$$

Moreover, in the calculations below, we are going to use the following commutation relation

$$[\mathcal{R}_z(\infty), \mathcal{R}_y(\eta)] \equiv \mathcal{R}_z(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(\infty)^{-1}\mathcal{R}_y(\eta)^{-1} \equiv \mathcal{R}_y(-2\eta) \pmod{p} \quad (29)$$

so that

$$\mathcal{R}_z(\infty)\mathcal{R}_y(\eta) \equiv [\mathcal{R}_z(\infty), \mathcal{R}_y(\eta)]\mathcal{R}_y(\eta)\mathcal{R}_z(\infty) \equiv \mathcal{R}_y(-\eta)\mathcal{R}_z(\infty) \pmod{p}. \quad (30)$$

Now, we start analyzing all the possibilities for the branches of the three rotations involved in the Cardano decomposition of the kind $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$. A triple ijk with $i, j, k \in \{1, 2\}$ will denote a Cardano representation of the kind $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$, where the x , y and z rotation modulo p are taken from the i th, j th and k th branch, respectively. There are $8 = 2^3$ possible Cardano representations $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$ with respect to the branches of each of the three involved rotations. Thus, there are 36 possibilities of equating two triple products, i.e., of equating modulo p two Cardano representations depending on their rotation branches.

Suppose that a given $M \in G_p$ admits a Cardano representation of the kind 111, i.e., there exist $\xi, \eta, \zeta \in \mathbb{Z}/p$ such that

$$M \equiv \mathcal{R}_x(\xi)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \pmod{p}. \quad (31)$$

Then, there exists at least another distinct Cardano representation of M along the same axes: by using Eqs. (28) and (30), we get

$$\begin{aligned} M &\equiv \mathcal{R}_x(\xi) \mathcal{R}_x(\infty)\mathcal{R}_y(\infty)\mathcal{R}_z(\infty) \mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \\ &\equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(-\eta)\mathcal{R}_z(\infty)\mathcal{R}_z(\zeta) \pmod{p}. \end{aligned} \quad (32)$$

The latter is a Cardano representation of M of the kind either 221 if $\zeta \not\equiv 0 \pmod{p}$ or 222 if $\zeta \equiv 0 \pmod{p}$. The reverse is also true: if $M \in G_p$ admits a Cardano representation 221 with non-zero parameter for the z -rotation (or a Cardano representation 222), then it admits a Cardano representation 111 too.

Similarly, if for a given $M \in G_p$ there exist $\xi, \eta, \zeta \in \mathbb{Z}/p$ such that M has Cardano representation of the kind 121,

$$M \equiv \mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \pmod{p}, \quad (33)$$

then also

$$\begin{aligned} M &\equiv \mathcal{R}_x(\xi) \mathcal{R}_x(\infty)\mathcal{R}_y(\infty)\mathcal{R}_z(\infty) \mathcal{R}_y(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \\ &\equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(\infty)^2\mathcal{R}_z(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \\ &\equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(-\eta)\mathcal{R}_z(\infty)\mathcal{R}_z(\zeta) \pmod{p}, \end{aligned} \quad (34)$$

which is a Cardano representation of M of the kind either 211 if $\zeta \not\equiv 0 \pmod{p}$ or 212 if $\zeta \equiv 0 \pmod{p}$.

Moreover, if for a given $M \in G_p$ there exist $\xi, \eta \in \mathbb{Z}/p$ such that M has Cardano representation 112,

$$M \equiv \mathcal{R}_x(\xi)\mathcal{R}_y(\eta)\mathcal{R}_z(\infty) \pmod{p}, \quad (35)$$

then also

$$\begin{aligned} M &\equiv \mathcal{R}_x(\xi) \mathcal{R}_x(\infty) \mathcal{R}_y(\infty) \mathcal{R}_z(\infty) \mathcal{R}_y(\eta) \mathcal{R}_z(\infty) \\ &\equiv \mathcal{R}_x(\infty) \mathcal{R}_x(\xi) \mathcal{R}_y(\infty) \mathcal{R}_y(-\eta) \mathcal{R}_z(0) \pmod{p}, \end{aligned} \quad (36)$$

that is, a Cardano representation 221 of M .

Finally, if for a given $M \in G_p$ there exist $\xi, \eta \in \mathbb{Z}/p$ such that M has Cardano representation 122,

$$M \equiv \mathcal{R}_x(\xi) \mathcal{R}_y(\infty) \mathcal{R}_y(\eta) \mathcal{R}_z(\infty) \pmod{p}, \quad (37)$$

then also

$$\begin{aligned} M &\equiv \mathcal{R}_x(\xi) \mathcal{R}_x(\infty) \mathcal{R}_y(\infty) \mathcal{R}_z(\infty) \mathcal{R}_y(\infty) \mathcal{R}_y(\eta) \mathcal{R}_z(\infty) \\ &\equiv \mathcal{R}_x(\infty) \mathcal{R}_x(\xi) \mathcal{R}_y(\infty)^2 \mathcal{R}_z(\infty) \mathcal{R}_y(\eta) \mathcal{R}_z(\infty) \\ &\equiv \mathcal{R}_x(\infty) \mathcal{R}_x(\xi) \mathcal{R}_y(-\eta) \mathcal{R}_z(0) \pmod{p}, \end{aligned} \quad (38)$$

that is, a Cardano representation 211 of M .

We have analyzed each of the eight initial triples ijk since the reverse of each of the above reasonings is also valid. This provides six different modular congruences of Cardano representations with respect to certain triples of branches. We deduce the following.

Remark IV.2 *Given a Cardano representation of M of the kind $\mathcal{R}_x \mathcal{R}_y \mathcal{R}_z$ of parameters ξ, η, ζ , respectively, then M admits at least another distinct Cardano representation of the kind $\mathcal{R}_x \mathcal{R}_y \mathcal{R}_z$ with parameters ξ', η', ζ' , respectively: that obtained by changing the branches of the x - and y -rotation and*

- *by changing the branch of the z -rotation if its parameter ζ is $0 \in \mathbb{Z}/p$ or ∞ and*
- *by fixing the branch of the z -rotation if $\zeta \in \mathbb{Z}/p$, $\zeta \not\equiv 0 \pmod{p}$,*

and with parameters

$$\xi' \equiv \xi, \quad \eta' \equiv -\eta, \quad \zeta' \equiv -\frac{1}{\alpha\zeta} \pmod{p}. \quad (39)$$

Now, we show that a given $M \in G_p$ cannot admit two distinct Cardano representations of the kind $\mathcal{R}_x \mathcal{R}_y \mathcal{R}_z$ with respect to the same three branches. Here we need to compare each of the eight branch triples ijk with itself. The thesis is trivial to check in the four cases $ij2$ where the z -rotation is $\mathcal{R}_z(\infty)$. We now focus on the case 111 where the three rotations of the Cardano representations all come from the first branch since the calculations for the remaining three cases $ij1$ are very similar.

Suppose that a given $M \in G_p$ can be represented as

$$M \equiv \mathcal{R}_x(\xi) \mathcal{R}_y(\eta) \mathcal{R}_z(\zeta) \equiv \mathcal{R}_x(\xi') \mathcal{R}_y(\eta') \mathcal{R}_z(\zeta') \pmod{p} \quad (40)$$

for certain $\xi, \xi', \eta, \eta', \zeta, \zeta' \in \mathbb{Z}/p$. It means that

$$\begin{aligned} M &\equiv \begin{pmatrix} \frac{1+v\zeta^2}{1-v\zeta^2} & \frac{2v\zeta}{1-v\zeta^2} & 0 \\ \frac{2\zeta}{1-v\zeta^2} & \frac{1+v\zeta^2}{1-v\zeta^2} & 0 \\ \frac{2}{1-v\zeta^2}(\eta(1+v\zeta^2) + 2\xi\zeta) & \frac{2}{1-v\zeta^2}(\xi(1+v\zeta^2) + 2v\eta\zeta) & 1 \end{pmatrix} \\ &\equiv \begin{pmatrix} \frac{1+v\zeta'^2}{1-v\zeta'^2} & \frac{2v\zeta'}{1-v\zeta'^2} & 0 \\ \frac{2\zeta'}{1-v\zeta'^2} & \frac{1+v\zeta'^2}{1-v\zeta'^2} & 0 \\ \frac{2}{1-v\zeta'^2}(\eta'(1+v\zeta'^2) + 2\xi'\zeta') & \frac{2}{1-v\zeta'^2}(\xi'(1+v\zeta'^2) + 2v\eta'\zeta') & 1 \end{pmatrix} \pmod{p}. \end{aligned}$$

By equating the entries of the 2×2 upper left minors, we get Eqs. (24) and (25), giving $\zeta \equiv \zeta' \pmod p$, i.e., $\mathcal{R}_z(\zeta) \equiv \mathcal{R}_z(\zeta') \pmod p$. Then the following are equivalent:

$$\mathcal{R}_x(\xi)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \equiv \mathcal{R}_x(\xi')\mathcal{R}_y(\eta')\mathcal{R}_z(\zeta') \pmod p, \quad (41)$$

$$\mathcal{R}_x(\xi)\mathcal{R}_y(\eta) \equiv \mathcal{R}_x(\xi')\mathcal{R}_y(\eta') \pmod p, \quad (42)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\eta & 2\xi & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\eta' & 2\xi' & 1 \end{pmatrix} \pmod p, \quad (43)$$

$$\xi' \equiv \xi \pmod p \quad \text{and} \quad \eta' \equiv \eta \pmod p, \quad (44)$$

$$\mathcal{R}_x(\xi') \equiv \mathcal{R}_x(\xi) \pmod p \quad \text{and} \quad \mathcal{R}_y(\eta') \equiv \mathcal{R}_y(\eta) \pmod p. \quad (45)$$

Thus each $M \in G_p$ has a unique Cardano representation of the kind $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$ with respect to a certain triple of rotation branches.

There are $36 - 6 - 8 = 22$ possibilities left of equating two triples ijk , representing Cardano decompositions $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$ with respect to those triples of branches. For all of them, equating modulo p the first or last matrix entry of the two Cardano products leads to $1 \equiv -1 \pmod p$, which is impossible. It means that each $M \in G_p$ does not have Cardano representations of the kind $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$ with respect to different triples of branches, except from the cases of Remark IV.2.

We have, thus, proved the following.

Theorem IV.3 *Every $M \in G_p$, for every odd prime p , has exactly two distinct Cardano representations of the kind $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$. They are one of the following six (depending on M) according to Remark IV.2, for certain $\xi, \eta, \zeta \in \mathbb{Z}/p$:*

$$\begin{aligned} 111 \leftrightarrow 221 & \quad \mathcal{R}_x(\xi)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(-\eta)\mathcal{R}_z\left(\frac{1}{v\zeta}\right) \pmod p \text{ if } \zeta \not\equiv 0 \pmod p, \\ 111 \leftrightarrow 222 & \quad \mathcal{R}_x(\xi)\mathcal{R}_y(\eta)\mathcal{R}_z(0) \equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(-\eta)\mathcal{R}_z(\infty) \pmod p, \\ 121 \leftrightarrow 211 & \quad \mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(\zeta) \equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(-\eta)\mathcal{R}_z\left(\frac{1}{v\zeta}\right) \pmod p \text{ if } \zeta \not\equiv 0 \pmod p, \\ 121 \leftrightarrow 212 & \quad \mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(0) \equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(-\eta)\mathcal{R}_z(\infty) \pmod p, \\ 112 \leftrightarrow 221 & \quad \mathcal{R}_x(\xi)\mathcal{R}_y(\eta)\mathcal{R}_z(\infty) \equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(-\eta)\mathcal{R}_z(0) \pmod p, \\ 122 \leftrightarrow 211 & \quad \mathcal{R}_x(\xi)\mathcal{R}_y(\infty)\mathcal{R}_y(\eta)\mathcal{R}_z(\infty) \equiv \mathcal{R}_x(\infty)\mathcal{R}_x(\xi)\mathcal{R}_y(-\eta)\mathcal{R}_z(0) \pmod p. \end{aligned}$$

Corollary IV.4 *Given a prime $p > 2$, the order of the group $SO(3)_p \pmod p = G_p$ is given by*

$$|G_p| = \frac{1}{2}|G_{x,p}||G_{y,p}||G_{z,p}| = 2p^2(p+1). \quad (46)$$

Proof The rotation groups $G_{x,p}$, $G_{y,p}$, and $G_{z,p}$ are subgroups of G_p ; hence, each product $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$ such that $\mathcal{R}_x \in G_{x,p}$, $\mathcal{R}_y \in G_{y,p}$, and $\mathcal{R}_z \in G_{z,p}$ is an element of G_p . By the the Cardano decomposition $\mathcal{R}_x\mathcal{R}_y\mathcal{R}_z$, each element of G_p has a decomposition into a product of these three, and Theorem IV.3, more precisely, states that each element has exactly two such decompositions. Therefore, $|G_p| = \frac{1}{2}|G_{x,p}||G_{y,p}||G_{z,p}|$. \square

A. Equivalence with \widehat{G}_p

Now we consider the groups of Eq. (17) for $k = 1$: we want to find a parameterization of these groups \widehat{G}_p too. Being $\pi_1(A) = \text{diag}(1, -v, 0)$ and setting $L = (a_{ij})_{i,j=1}^3 \in SO(3)_p$, the defining

conditions of $SO(3)_p$ reduced modulo p translate in the following system of equations for \widehat{G}_p :

$$\begin{cases} a_{11}^2 - va_{21}^2 \equiv 1 \pmod{p}, \\ a_{12}^2 - va_{22}^2 \equiv -v \pmod{p}, \\ a_{13}^2 - va_{23}^2 \equiv 0 \pmod{p}, \\ a_{11}a_{12} - va_{21}a_{22} \equiv 0 \pmod{p}, \\ a_{11}a_{13} - va_{21}a_{23} \equiv 0 \pmod{p}, \\ a_{12}a_{13} - va_{22}a_{23} \equiv 0 \pmod{p}, \\ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \equiv 1 \pmod{p}. \end{cases} \quad (47)$$

The elements a_{31}, a_{32}, a_{33} appear only in the last equation of (47), the one deriving from the determinant: this means that we could at first try to find the possible solutions for the other two rows, looking at the other equations.

If $a_{23} \not\equiv 0 \pmod{p}$, the third equation of (47) would give $v \equiv a_{13}^2 a_{23}^{-2} \pmod{p}$, which is in contradiction to the fact that v is not a square modulo p . Hence it must be $a_{13}, a_{23} \equiv 0 \pmod{p}$, and from (47), we have

$$\begin{cases} a_{11}^2 - va_{21}^2 \equiv 1 \pmod{p}, \\ a_{12}^2 - va_{22}^2 \equiv -v \pmod{p}, \\ a_{13}, a_{23} \equiv 0 \pmod{p}, \\ a_{11}a_{12} - va_{21}a_{22} \equiv 0 \pmod{p}, \\ a_{33}(a_{11}a_{22} - a_{21}a_{12}) \equiv 1 \pmod{p}. \end{cases} \quad (48)$$

With $a_{33}(a_{11}a_{22} - a_{21}a_{12})$ being non-zero and, thus, with its two factors, we can write $a_{33} \equiv (a_{11}a_{22} - a_{21}a_{12})^{-1}$. Actually, we can say more about these values: let us multiply the first two equations of (48) member by member, obtaining

$$\begin{aligned} -v &\equiv a_{11}^2 a_{12}^2 - va_{11}^2 a_{22}^2 - va_{12}^2 a_{21}^2 + v^2 a_{21}^2 a_{22}^2 \\ &\equiv a_{11}^2 a_{12}^2 - va_{11}^2 a_{22}^2 - va_{12}^2 a_{21}^2 + v^2 a_{21}^2 a_{22}^2 + 2va_{11}a_{12}a_{21}a_{22} - 2va_{11}a_{12}a_{21}a_{22} \\ &\equiv (a_{11}a_{12} - va_{21}a_{22})^2 - v(a_{11}a_{22} - a_{12}a_{21})^2 \\ &\equiv -v(a_{11}a_{22} - a_{12}a_{21})^2 \pmod{p}. \end{aligned}$$

We can, thus, deduce

$$a_{11}a_{22} - a_{21}a_{12} \equiv a_{33} \equiv \pm 1. \quad (49)$$

Now, the whole remaining study concerns the submatrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (50)$$

with

$$a_{11}^2 - va_{21}^2 \equiv 1 \pmod{p}, \quad (51)$$

$$a_{12}^2 - va_{22}^2 \equiv -v \pmod{p}, \quad (52)$$

$$a_{11}a_{12} - va_{21}a_{22} \equiv 0 \pmod{p}, \quad (53)$$

$$a_{11}a_{22} - a_{12}a_{21} \equiv \pm 1 \pmod{p}. \quad (54)$$

Here, a_{11} and a_{21} cannot be both divisible by p ; otherwise, Eq. (51) would be impossible: suppose that $a_{11} \not\equiv 0 \pmod{p}$. Equation (53) provides $a_{12} \equiv va_{11}^{-1}a_{21}a_{22} \pmod{p}$. We plug this into Eq. (52) and use Eq. (51),

$$\begin{aligned} -v &\equiv v^2a_{11}^{-2}a_{21}^2a_{22}^2 - va_{22}^2 \equiv -va_{22}^2(1 - va_{11}^{-2}a_{21}^2) \\ &\equiv -va_{22}^2(1 + a_{11}^{-2}(1 - a_{11}^2)) \equiv -va_{11}^{-2}a_{22}^2 \pmod{p}, \end{aligned}$$

which implies that

$$a_{22} \equiv \pm_2 a_{11}. \quad (55)$$

Now, from Eqs. (51) and (52), we get $a_{12}^2 \equiv v(a_{22}^2 - 1) \equiv v(a_{11}^2 - 1) \equiv v^2a_{21}^2 \pmod{p}$, and so

$$a_{12} \equiv \pm_3 va_{21} \pmod{p}. \quad (56)$$

The choice is now only about a_{11}, a_{21} and three signs. However, we show that these three signs \pm_1, \pm_2, \pm_3 are actually the same. Equations (55) and (56) make Eq. (53) become $va_{11}a_{21}(\pm_3 1 \mp_2 1) \equiv 0 \pmod{p} \Leftrightarrow (\pm_3 1 \mp_2 1)a_{21} \equiv 0 \pmod{p}$. If $a_{21} \not\equiv 0 \pmod{p}$, then $\pm_2 = \pm_3$; if $a_{21} \equiv 0 \pmod{p}$, then the choice of \pm_3 is meaningless. Hence, we can write $\pm_2 1 = \pm_3 1$, in general. Now, Eqs. (51) and (54) give $\pm_1 1 \equiv \pm_2 (a_{11}^2 - va_{21}^2) \equiv \pm_2 1$; therefore $\pm_1 = \pm_2 = \pm_3$.

Equation (53) becomes an identity, while Eqs. (52) and (54) become equivalent to Eq. (51). Very similar steps in the hypothesis that $a_{21} \not\equiv 0 \pmod{p}$ provide the same results, which we collect in the following equations:

$$\begin{cases} a_{11}^2 - va_{21}^2 \equiv 1 \pmod{p}, \\ a_{22} \equiv \pm_1 a_{11} \pmod{p}, \\ a_{12} \equiv \pm_1 va_{21} \pmod{p}, \\ a_{13} \equiv a_{23} \equiv 0 \pmod{p}, \\ a_{33} \equiv \pm_1 \pmod{p}. \end{cases} \quad (57)$$

In addition, $\pm = \pm_1$: $a_{33}^{-1} \equiv a_{11}a_{22} - a_{12}a_{21} \Leftrightarrow \pm 1 \equiv \pm_1(a_{11}^2 - va_{21}^2) \equiv \pm_1 1 \pmod{p}$. We conclude that matrix solutions of Eq. (47) are of the form

$$\begin{pmatrix} a_{11} & \pm va_{21} & 0 \\ a_{21} & \pm a_{11} & 0 \\ a_{31} & a_{32} & \pm 1 \end{pmatrix} \in \mathcal{M}_{3 \times 3}(\mathbb{Z}/p) \quad (58)$$

with $a_{11}^2 - va_{21}^2 \equiv 1 \pmod{p}$.

All of the above can be summarized in the following result.

Theorem IV.5 *The matrices of \widehat{G}_p for any prime $p > 2$ are uniquely defined by $a, b, c, d \in \mathbb{Z}/p$ (the first two solutions of $a^2 - vb^2 \equiv 1 \pmod{p}$ and the others chosen freely) and by a sign $s = \pm 1$,*

$$M(a, b, c, d, s) = \begin{pmatrix} a & svb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix}. \quad (59)$$

Corollary IV.6 *Given a prime $p > 2$, the order of the matrix group \widehat{G}_p is*

$$|\widehat{G}_p| = 2p^2(p + 1). \quad (60)$$

Proof For a fixed odd prime p , the number of solutions of the equation $a^2 - vb^2 \equiv 1 \pmod{p}$ is $p+1$ (Appendix A). Then, the total number of combinations of a, b, c, d, s is $(p+1) \cdot p \cdot p \cdot 2$ such that every choice represents a distinct matrix. Then the number of distinct matrices is $2p^2(p+1)$. \square

Remark IV.7 Proving that G_p coincides with \widehat{G}_p is equivalent to proving that $|G_p| = |\widehat{G}_p|$ (Remark III.2). This is true for every prime $p > 2$, by comparing Eqs. (46) and (60). This implies that Eq. (59) is an equivalent parameterization for G_p .

The study of $G_p := SO(3)_p \pmod{p}$ regarding its order and parameterization turns out to be fundamental to build a first list of representations of $SO(3)_p$. Examining the commutators and the abelianization of a group is enough to exhaust the list of one-dimensional representations [16]: this will be helpful to understand how many other representations are present and their form. It is for this reason that here we anticipate the structure of the commutators between matrices of G_p for every prime $p > 2$ by exploiting Remark IV.7. We calculate $[M, N] = MNM^{-1}N^{-1}$ with $M, N \in G_p$ written through the parameterization (59) as

$$M = \begin{pmatrix} a_{11} & tva_{21} & 0 \\ a_{21} & ta_{11} & 0 \\ a_{31} & a_{32} & t \end{pmatrix}, \quad N = \begin{pmatrix} b_{11} & rvb_{21} & 0 \\ b_{21} & rb_{11} & 0 \\ b_{31} & b_{32} & r \end{pmatrix}.$$

We get

$$[M, N] = \begin{pmatrix} c_{11} & vc_{21} & 0 \\ c_{21} & c_{11} & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}, \quad (61)$$

where

$$\begin{aligned} c_{11} &= (b_{11}^2 - tvb_{21}^2)(a_{11}^2 - rva_{21}^2) + (t + r - tr - 1)va_{11}a_{21}b_{21}b_{11}, \\ c_{21} &= (t - 1)(a_{11}^2 - rva_{21}^2)b_{21}b_{11} - (r - 1)(b_{11}^2 - tvb_{21}^2)a_{11}a_{21}. \end{aligned} \quad (62)$$

B. From p -adic rotations to dihedral groups

We introduce a homomorphism, which will turn out to be useful to our aim of finding two-dimensional irreducible representations of $SO(3)_p$.

Proposition IV.8 *The map*

$$\begin{aligned} K'_p : G_p &\rightarrow GL((\mathbb{Z}/p)^2), \\ M &\mapsto M_{33}, \end{aligned} \quad (63)$$

where M_{33} is the the 2×2 upper-left minor of M , is a group homomorphism.

Proof According to the parameterization (59) and Remark IV.7, the map K'_p acts as follows:

$$\begin{pmatrix} a & svb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} \mapsto \begin{pmatrix} a & svb \\ b & sa \end{pmatrix}.$$

The image $\mathfrak{J}(K'_p)$ is parameterized by $(a, b) \in (\mathbb{Z}/p)^2$, which are solutions of $a^2 - vb^2 \equiv 1 \pmod{p}$ and $s \in \{\pm 1\}$.

- $\mathfrak{J}(K'_p) \subset GL(\mathbb{Z}/p)$,

$$\det \begin{pmatrix} a & svb \\ b & sa \end{pmatrix} = s(a^2 - vb^2) \equiv \pm 1 \neq 0.$$

- K'_p is a group homomorphism,

$$K'_p \begin{pmatrix} a & svb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} K'_p \begin{pmatrix} a' & s'vb' & 0 \\ b' & s'a' & 0 \\ c' & d' & s' \end{pmatrix} = \begin{pmatrix} a & svb \\ b & sa \end{pmatrix} \begin{pmatrix} a' & s'vb' \\ b' & s'a' \end{pmatrix} = \begin{pmatrix} aa' + vsbb' & vs'(ab' + sa'b) \\ a'b + sab' & s'(saa' + vbb') \end{pmatrix},$$

which equals

$$K'_p \left(\begin{pmatrix} a & svb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} \begin{pmatrix} a' & s'vb' & 0 \\ b' & s'a' & 0 \\ c' & d' & s' \end{pmatrix} \right) = K'_p \begin{pmatrix} aa' + vsbb' & vs'(ab' + sa'b) & 0 \\ a'b + sab' & s'(saa' + vbb') & 0 \\ \dots & \dots & ss' \end{pmatrix}.$$

□

A point of interest lies in the behaviour of a and b as solutions of $a^2 - vb^2 \equiv 1 \pmod{p}$ while composing matrices in $\mathfrak{J}(K'_p)$, [14]. In $\mathfrak{J}(K'_3)$ there are only the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (64)$$

up to sign changes on the components, and it is trivial that products of matrices of this form give back matrices of this form. However, which mathematical property guarantees that a product

$$\begin{pmatrix} a & svb \\ b & sa \end{pmatrix} \begin{pmatrix} a' & s'vb' \\ b' & s'a' \end{pmatrix} = \begin{pmatrix} aa' + vsbb' & vs'(ab' + sa'b) \\ a'b + sab' & s'(saa' + vbb') \end{pmatrix} \quad (65)$$

still gives a matrix coherent with the definition in $\mathfrak{J}(K'_p)$? Can we find some deeper meaning about it?

As proved in Appendix A, solutions of $a^2 - vb^2 \equiv 1$ in \mathbb{Z}/p form a cyclic group: the key to prove it lies in looking at the pair (a, b) as a single “complex” number $a + ib$, $i^2 = v$, and introducing products between pairs as products between complex numbers. As a consequence, the “product” of two solutions gives another solution according to the product of matrices. As a matter of fact, another way to represent the complex product is by writing the components in a 2×2 matrix of the form

$$\begin{pmatrix} a & vb \\ b & a \end{pmatrix} \quad (66)$$

and applying the common matrix product.

This is of great interest for us, as these are exactly half of the matrices in $\mathfrak{J}(K'_p)$, the ones with $s = 1$. The other half, the matrices with $s = -1$, are obtained simply by multiplying these by the (Pauli) matrix Z ,

$$\begin{pmatrix} a & -vb \\ b & -a \end{pmatrix} = \begin{pmatrix} a & vb \\ b & a \end{pmatrix} Z, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (67)$$

Note that $Z \in \mathfrak{J}(K'_p)$ because $\mathcal{R}_x(\infty) = \text{diag}(1, -1, -1)_{\mathbb{Z}_p} \in SO(3)_p$ for every prime p .

This means that the group $\mathfrak{J}(K'_p)$ is generated by two matrices,

$$C = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (68)$$

where C is the matrix obtained by (a_0, b_0) , a generator of the cyclic group of the solutions (a, b) .

Finally we observe that

- $C^{p+1} = \mathbb{1}$ because of the isomorphism with the solutions subgroup of $\mathbb{Z}/p[i]^*$,
- $Z^2 = \mathbb{1}$, and
- $ZCZ = C^{-1}$, as can be checked readily.

This gives us a complete presentation for our group,

$$\mathfrak{J}(K'_p) = \langle C, Z \mid C^{p+1} = Z^2 = \mathbb{1}, ZCZ = C^{-1} \rangle, \quad (69)$$

which is exactly the presentation for the dihedral group D_{p+1} : in other words, the group $\mathfrak{J}(K'_p)$ is isomorphic to the group of symmetries of a $(p+1)$ -gon. Denoting this isomorphism with φ , we have

$$\varphi : \mathfrak{J}(K'_p) \xrightarrow{\sim} D_{p+1}, \quad C \mapsto a, \quad Z \mapsto x, \quad (70)$$

where a and x are the generators (rotation and reflection, respectively) of D_{p+1} .

This will allow us to exploit the known representation theory of dihedral groups for the p -adic group $SO(3)_p$, some basic facts about which are reported in Appendix B. While we will not discuss this in detail, Appendix B1 shows that $\mathfrak{J}(K'_p)$ has always four one-dimensional representations. Furthermore, Appendix B2 shows that there always exists a two-dimensional irrep of $\mathfrak{J}(K'_p)$, and for $p > 3$, there are more than one 2-irreps, leading to different p -adic qubits.

Remark IV.9 *We summarize the preceding discussion by recording that*

$$J_p^{(i)} := \rho_i \circ \varphi \circ K'_p \circ \pi_1, \quad (71)$$

where ρ_i 's are the 2-irreps of D_{p+1} , are 2-irreps of $SO(3)_p$, providing examples of p -adic qubits for every $p > 2$.

V. REPRESENTATIONS FOR $SO(3)_3 \pmod{3}$

In the present section, we exploit the previous results for the case $p = 3$ to study some irreducible representations of $SO(3)_3$. From these will emerge one irrep of degree 2, which we will propose as the foundation for the 3-adic qubit.

A. Abelianization

As we anticipated, a starting point in the study of the irreducible representations of $SO(3)_3$ is through those of its projection G_3 , which is a finite group. First, we find its abelianization to understand how many one-dimensional representations G_3 has, which benefits the analysis of the degrees of its remaining irreps.

We specify Eq. (59) for $p = 3$: by Remark IV.7, the matrices of G_3 are parametrized by

$$M(a, b, c, d, s) = \begin{pmatrix} a & -sb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} \quad (72)$$

with $(a, b) \in \{(\pm 1, 0), (0, \pm 1)\}$. Now we consider Eqs. (61) and (62) for $p = 3$: $c_{21} \equiv 0 \pmod{3}$ because $a_{11}a_{21} \equiv b_{11}b_{21} \equiv 0 \pmod{3}$. Moreover, $c_{11} \equiv \pm 1 \pmod{3}$, so a commutator of G_3 is of the form

$$[M, N] = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}.$$

The set of commutators of this form is closed under multiplication, meaning that

$$[G_3, G_3] = \left\{ \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{11} & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}, c_{11} \in \{\pm 1\}, c_{31}, c_{32} \in \mathbb{Z}/3 \right\}. \quad (73)$$

By comparing this with Eq. (72), we can state the following.

Proposition V.1 $M(a, b, c, d, s) \in G_3$ belongs to the commutator subgroup $[G_3, G_3]$ if and only if $s = 1$ and $(a, b) = (\pm 1, 0)$.

The order of $[G_3, G_3]$ is 18, so the abelianization $\text{Ab}(G_3) := G_3/[G_3, G_3]$ is of order $72/18 = 4$.

Proposition V.2 $\text{Ab}(G_3)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$: a class of matrices is characterized by s and $t := a^2 - b^2$ using the form in (72). The following equations are possible representatives for the classes $\text{Ab}(s, t)$:

$$\begin{aligned} \text{Ab}(1, 1) &\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{Ab}(-1, 1) &\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \text{Ab}(1, -1) &\rightsquigarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{Ab}(-1, -1) &\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Proof According to Eqs. (72) and (73) we have

$$M(a, b, c, d, s)[G_3, G_3] = \begin{pmatrix} a & -sb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{11} & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} ac_{11} & -sc_{11}b & 0 \\ c_{11}b & sac_{11} & 0 \\ c_{11}c + sc_{31} & c_{11}d + sc_{32} & s \end{pmatrix}$$

with $c_{11}, s \in \{\pm 1\}$, $(a, b) \in \{(\pm 1, 0), (0, \pm 1)\}$.

With fixed a, b and s , we have $M(a, b, c, d, s)[G_3, G_3] = M(a, b, c', d', s)[G_3, G_3]$ for every $c, c', d, d' \in \mathbb{Z}/3$. Moreover, for every $c, c', d, d' \in \mathbb{Z}/3$,

$$M(1, 0, c, d, 1)[G_3, G_3] = M(-1, 0, c', d', 1)[G_3, G_3] = \left\{ \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{11} & 0 \\ c'_{31} & c'_{32} & 1 \end{pmatrix}, c_{11} \in \{\pm 1\}, c'_{31}, c'_{32} \in \mathbb{Z}/3 \right\}.$$

A representative of this class is $\mathbf{1}$, and the class is located by $s = 1$ and $t = a^2 - b^2 = 1$.

Similarly, for every $c, c', d, d' \in \mathbb{Z}/3$,

$$M(1, 0, c, d, -1)[G_3, G_3] = M(-1, 0, c', d', -1)[G_3, G_3] = \left\{ \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & -c_{11} & 0 \\ c'_{31} & c'_{32} & -1 \end{pmatrix}, c_{11} \in \{\pm 1\}, c'_{31}, c'_{32} \in \mathbb{Z}/3 \right\},$$

$$M(0, 1, c, d, 1)[G_3, G_3] = M(0, -1, c', d', 1)[G_3, G_3] = \left\{ \begin{pmatrix} 0 & -c_{11} & 0 \\ c_{11} & 0 & 0 \\ c'_{31} & c'_{32} & 1 \end{pmatrix}, c_{11} \in \{\pm 1\}, c'_{31}, c'_{32} \in \mathbb{Z}/3 \right\},$$

$$M(0, 1, c, d, -1)[G_3, G_3] = M(0, -1, c', d', -1)[G_3, G_3] = \left\{ \begin{pmatrix} 0 & c_{11} & 0 \\ c_{11} & 0 & 0 \\ c'_{31} & c'_{32} & 1 \end{pmatrix}, c_{11} \in \{\pm 1\}, c'_{31}, c'_{32} \in \mathbb{Z}/3 \right\}.$$

□

As a consequence of Proposition V.2, we have the following further result.

Proposition V.3 G_3 cannot have more than four orthogonal one-dimensional irreducible representations $U : G_3 \rightarrow \mathbb{C}^*$: a possible choice for the four 1-irreps is as follows:

- $\det : M(a, b, c, d, s) \mapsto 1$,
- $s : M(a, b, c, d, s) \mapsto s$,
- $t : M(a, b, c, d, s) \mapsto a^2 - b^2$, and
- $st : M(a, b, c, d, s) \mapsto s(a^2 - b^2)$.

Proof A 1-irrep for a group is always the trivial representation mapping any element of the group to 1. This can be seen as the determinant on matrices of G_3 .

The irreducible character χ_U of a 1-irrep U is U itself: in fact, the trace of a 1×1 matrix is equal to the only value it contains. Since the product in \mathbb{C} is commutative, the 1-irreps of G_3 have to be invariant on abelianized classes. As consequences, we can actually build the character table for $\text{Ab}(G_3)$ in order to find the 1-irreps of G_3 , and the two characterizing quantities s and t in Proposition V.2 can be good candidates as characters. Indeed, we show that \det , s , t , st , as defined in the Proposition, are the four 1-irreps of this group.

Now, \det , s , t , st are homomorphisms from G_3 to \mathbb{C}^* because

$$M(a, b, c, d, s)M(a', b', c', d', s') = \begin{pmatrix} aa' - sbb' & \dots & 0 \\ sab' + a'b & \dots & 0 \\ \dots & \dots & ss' \end{pmatrix},$$

so $s(MM') = ss' = s(M)s(M')$, $t(MM') = (aa' - sbb')^2 - (sab' + a'b)^2 = (a^2 - b^2)(a'^2 - b'^2) = t(M)t(M')$, and similarly for st .

Furthermore, in accordance with general Peter-Weyl theory [16], \det , s , t , st are orthogonal representations. This can be seen by looking at the character table of $\text{Ab}(G_3)$.

$\text{Ab}(G_3) \simeq \{(s, t)\}$	(1, 1)	(1, -1)	(-1, 1)	(-1, -1)
\det	1	1	1	1
s	1	1	-1	-1
t	1	-1	1	-1
st	1	-1	-1	1

The rows are all different, and they correspond to the different values the 1-irreps associate with each element of G_3 .

We have shown four 1-irreps of G_3 : any other 1-irrep would be equivalent to these because of Proposition V.2. □

B. Degree of representations

The Peter-Weyl theorem [16] states that the sum of the squares of the degrees d_α of the irreducible representations of G located by α is equal to the order of G .

The order of G_3 is 72 by Corollary IV.4, and there are 4 irreducible representations of G_3 of degree 1 by Proposition V.3. Then,

$$\sum_{\alpha|d_\alpha>1} d_\alpha^2 = 68. \quad (74)$$

Unfortunately, this is not sufficient to deduce the nature of higher-degree irreps: we need another proposition (proved in Appendix C).

Proposition V.4 *G_3 is partitioned in nine conjugacy classes. Their orders and lists are in Appendix C.*

As a consequence of the fact that the number of conjugacy classes of a group equals the number of its irreducible representations [16], there are nine non equivalent irreps of G_3 , among which five have degree greater than 1. Suppose, then, to sort the degrees of these irreps from the smaller to the greater as d_1, \dots, d_9 : we can state that

$$d_1 = d_2 = d_3 = d_4 = 1, \quad (75)$$

$$d_5^2 + d_6^2 + d_7^2 + d_8^2 + d_9^2 = 68. \quad (76)$$

It is known that d_α divides $|G|$ when α locates an irreducible representation of G . This is enough to deduce all of the d_α :

Proposition V.5 *The only solution for (76) for integers d_α divisors of 72 greater than 1 is the following up to permutations:*

$$d_5 = 2, \quad d_6 = d_7 = d_8 = d_9 = 4. \quad (77)$$

The proof of this proposition is in Appendix D. Looking at d_5 , we can now state the following.

Proposition V.6 *There exists a two-dimensional irrep V of G_3 , that is,*

$$K_3 : G_3 \rightarrow GL(V), \quad V \simeq \mathbb{C}^2,$$

such that for any subspace $Y \subset V$,

$$(K_3(G_3))Y = Y \text{ implies } Y = \{0_V\} \text{ or } Y = V.$$

C. A 3-adic qubit

Now it is not hard to find an explicit form of a 2-irrep for $SO(3)_3$, [14]. An important consequence of the way $G_3 = SO(3)_3 \pmod{3} = \pi_1(SO(3)_3)$ was constructed according to Sec. III is as follows.

Proposition V.7 *There exists a 2-irrep J_3 of $SO(3)_3$,*

$$J_3 := K_3 \circ \pi_1.$$

Proof J_3 is a homomorphism because it is composition of homomorphisms. J_3 is irreducible since K_3 is so: if $Y \subset V$, then the following are equivalent:

$$(J_3(SO(3)_3))Y = Y, \quad (78)$$

$$(K_3(\pi_1(SO(3)_3)))Y = Y, \quad (79)$$

$$(K_3(G_3))Y = Y. \quad (80)$$

The latter is equivalent to $Y = \{0_V\}$ or $Y = V$. \square

The first ingredient to find an explicit form for a 2-irrep K_3 is the homomorphism K'_3 of Eq. (63). Furthermore, one considers the embedding of the integers modulo 3 into the complex numbers

$$\iota : \mathbb{Z}/3 \hookrightarrow \mathbb{C}, \quad 0_{\mathbb{Z}/3} \mapsto 0_{\mathbb{C}}, \quad \pm 1_{\mathbb{Z}/3} \mapsto \pm 1_{\mathbb{C}}. \quad (81)$$

This can be extended to a map between groups of matrices on these fields, respectively, which we will always call ι with abuse of notation. Note that the solutions $(a, b) \in \{(\pm 1, 0), (0, \pm 1)\} \subset (\mathbb{Z}/3)^2$ of $a^2 + b^2 \equiv 1 \pmod{3}$ are such that their embedding $\{(\pm 1, 0), (0, \pm 1)\} \subset \mathbb{C}^2$ is solution of $a^2 + b^2 = 1$ in \mathbb{C} .

Proposition V.8 $\iota : \mathcal{J}(K'_3) \rightarrow GL(\mathbb{C}^2)$ is a group homomorphism.

Proof

- $\mathcal{J}(\iota) \subset GL(\mathbb{C}^2)$,

$$\det \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix}_{\mathbb{C}} = s(a^2 + b^2) = s \neq 0,$$

where $(a, b) \in \{(\pm 1, 0), (0, \pm 1)\} \subset \mathbb{C}^2$.

- $\{0, \pm 1\} \subset \mathbb{C}$ is not a subfield since it is not closed under the addition operation. However, if we look at the entries of $\begin{pmatrix} aa' - sbb' & -s'(ab' + sa'b) \\ a'b + sab' & s'(saa' - bb') \end{pmatrix}$, when these operations are to be performed both in $\mathbb{Z}/3$ and in \mathbb{C} , they never present sums between ± 1 and ± 1 because $(a, b), (a', b') \in \{(\pm 1, 0), (0, \pm 1)\}$. It follows that

$$\begin{aligned} \iota \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix}_{\mathbb{Z}/3} \iota \begin{pmatrix} a' & -s'b' \\ b' & s'a' \end{pmatrix}_{\mathbb{Z}/3} &= \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix}_{\mathbb{C}} \begin{pmatrix} a' & -s'b' \\ b' & s'a' \end{pmatrix}_{\mathbb{C}} = \begin{pmatrix} aa' - sbb' & -s'(ab' + sa'b) \\ a'b + sab' & s'(saa' - bb') \end{pmatrix}_{\mathbb{C}} \\ &= \iota \begin{pmatrix} aa' - sbb' & -s'(ab' + sa'b) \\ a'b + sab' & s'(saa' - bb') \end{pmatrix}_{\mathbb{Z}/3} = \iota \left(\begin{pmatrix} a & -sb \\ b & sa \end{pmatrix}_{\mathbb{Z}/3} \begin{pmatrix} a' & -s'b' \\ b' & s'a' \end{pmatrix}_{\mathbb{Z}/3} \right). \end{aligned}$$

\square

Now we are able to state the following.

Proposition V.9 The function

$$K_3 : G_3 \rightarrow GL(\mathbb{C}^2), \quad K_3 := \iota \circ K'_3 \quad (82)$$

is the unitary 2-irrep of G_3 up to isomorphisms.

Proof As said, parameterization (72) is fundamental to understand the action of G_3 , that is,

$$\begin{pmatrix} a & -sb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix}_{\mathbb{Z}/3} \xrightarrow{K'_3} \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix}_{\mathbb{Z}/3} \xrightarrow{\iota} \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix}_{\mathbb{C}}.$$

The image $\mathcal{J}(K_3)$ of K_3 is parameterized by $(a, b) \in \{(\pm 1, 0), (0, \pm 1)\} \subset \mathbb{C}^2, s \in \{\pm 1\} \subset \mathbb{C}$.

- $\mathfrak{J}(K_3) \subset GL(\mathbb{C}^2)$ as noticed in Proposition V.8.
- K_3 is a group homomorphism, being the composition of two group homomorphisms K'_3 and ι .
- K_3 is irreducible:
if Y is a subrepresentation such that $Y \neq \{0\}, \mathbb{C}^2$, then $\dim(Y) = 1$, i.e., $Y = \mathbb{C}y$ for any $y \in Y$. By definition of subrepresentation, $K_3(G_3)Y \subset Y$ means that $K_3(M)y \in Y$ for every $M \in G_3$ and $y \in Y$: it must be $K_3(M)y = \lambda y$ for some $\lambda \in \mathbb{C}$ depending on y and M . In other words, y is a common eigenvector for every action of the group. This must be true, in particular, for

$$K_3(M(1, 0, c, d, 1)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\mathbb{C}} \text{ of eigenvectors } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and for

$$K_3(M(0, 1, c, d, 1)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{C}} \text{ of eigenvectors } \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

These two group actions have no common eigenvectors. This is a contradiction: the representation K_3 is irreducible.

- K_3 is unitary,
namely, $\mathfrak{J}(K_3)$ is composed by unitary linear transformations on \mathbb{C}^2 ,

$$\begin{aligned} \left\langle \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & -sb \\ b & sa \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} ax_1 - sbx_2 \\ bx_1 + sax_2 \end{pmatrix}, \begin{pmatrix} ay_1 - sb y_2 \\ by_1 + say_2 \end{pmatrix} \right\rangle \\ &= (ax_1 - sbx_2)(ay_1 - sb y_2) + (bx_1 + sax_2)(by_1 + say_2) \\ &= (a^2 + b^2)(x_1 y_1 + x_2 y_2) = x_1 y_1 + x_2 y_2 = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle \end{aligned}$$

for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{C}^2$ according to its Euclidean scalar product. □

We have, then, an explicit form for J_3 too: it is a function that makes a 2×2 complex matrix out of the first components of the p -adic sequences in the upper-left 2×2 part of a matrix in $SO(3)_3$, that is,

$$J_3 := K_3 \circ \pi_1 : SO(3)_3 \rightarrow GL(\mathbb{C}^2)$$

$$\begin{pmatrix} (a_{111}, a_{112}, \dots) & (a_{121}, a_{122}, \dots) & (a_{131}, a_{132}, \dots) \\ (a_{211}, a_{212}, \dots) & (a_{221}, a_{222}, \dots) & (a_{231}, a_{232}, \dots) \\ (a_{311}, a_{312}, \dots) & (a_{321}, a_{322}, \dots) & (a_{331}, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_3} \xrightarrow{J_3} \begin{pmatrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{pmatrix}_{\mathbb{C}}. \quad (83)$$

We, thus, have the following final result.

Proposition V.10 *The pair (\mathbb{C}^2, J_3) with J_3 defined as in (83) is a 3-adic qubit, that is, a continuous unitary irreducible linear (and then projective) representation of degree 2 on \mathbb{C} of the 3-adic special orthogonal group $SO(3)_3$ defined in (3).*

The isomorphism $\varphi : \mathfrak{J}(K'_3) \rightarrow D_4$ of Eq. (70) gives us an equivalent way to derive the 2-irrep J_3 of $SO(3)_3$, applying the representation theory of dihedral groups of even degree to $\mathfrak{J}(K'_3)$. However, first, Appendix B1 confirms the four one-dimensional representations we found in Proposition V.3 for the case $p = 3$. Then, Appendix B2 shows that for $p = 3$ there is a unique 2-irrep of $\mathfrak{J}(K'_3)$,

$$C_{\mathbb{Z}/3} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{C}}, \quad Z_{\mathbb{Z}/3} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathbb{C}}. \quad (84)$$

This coincides with ι of Proposition V.8 if we choose $(a_0, b_0) = (0, 1)$, leading to the same qubit representation J_3 of Eq. (83), which, then, can be written as $J_3 = \rho \circ \varphi \circ K'_3 \circ \pi_1$ too, where ρ denotes the 2-irrep of D_4 .

VI. REPRESENTATIONS FOR $SO(3)_5 \pmod{5}$

We have $\widehat{G}_5 = G_5$ according to Remark IV.7. We choose $u = 2$ as non-square invertible 5-adic integer. In this way, the parameterization (59) for the matrices in G_5 becomes

$$M(a, b, c, d, s) = \begin{pmatrix} a & -2sb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix}, \quad (85)$$

where $(a, b) \in \{(\pm 1, 0), (\pm 2, 1), (\pm 2, -1)\}$ are the solutions of $a^2 + 2b^2 \equiv 1 \pmod{5}$.

A. Abelianization

We specify Eqs. (61) and (62) for $p = 5$: it turns out that $(c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}$, so a commutator of G_5 is of one of the following forms:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}. \quad (86)$$

The set of commutators of these forms is closed under multiplication, meaning that

$$[G_5, G_5] = \left\{ \begin{pmatrix} c_{11} & -2c_{21} & 0 \\ c_{21} & c_{11} & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix}, (c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}, c_{31}, c_{32} \in \mathbb{Z}/5 \right\}. \quad (87)$$

By comparing this with Eq. (85), we can state the following.

Proposition VI.1 *$M(a, b, c, d, s) \in G_5$ belongs to the commutator subgroup $[G_5, G_5]$ if and only if $s = 1$ and $(a, b) \in \{(1, 0), (2, \pm 1)\}$.*

The order of $[G_5, G_5]$ is 75, so the abelianization $\text{Ab}(G_5) := G_5/[G_5, G_5]$ is of order $300/75 = 4$.

Proposition VI.2 $Ab(G_5)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$: a class of matrices is characterized by s and $t := \text{sign}(a)$ (taking the representatives of the elements in $\mathbb{Z}/5$ as $-2, -1, 0, 1, 2$) using the form in (85). The following are possible representatives for the classes $Ab(s, t)$:

$$\begin{aligned} Ab(1, 1) &\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & Ab(-1, 1) &\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ Ab(1, -1) &\rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & Ab(-1, -1) &\rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Proof According to Eqs. (85) and (87), we have

$$M(a, b, c, d, s)[G_5, G_5] = \begin{pmatrix} a & -2sb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} \begin{pmatrix} c_{11} & -2c_{21} & 0 \\ c_{21} & c_{11} & 0 \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} ac_{11} - 2sbc_{21} & -2ac_{21} - 2sc_{11}b & 0 \\ c_{11}b + sac_{21} & -2bc_{21} + sac_{11} & 0 \\ c_{11}c + c_{21}d + sc_{31} & c_{11}d - 2c_{21}c + sc_{32} & s \end{pmatrix}$$

with $(c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}$, $s \in \{\pm 1\}$, $(a, b) \in \{(\pm 1, 0), (\pm 2, 1), (\pm 2, -1)\}$.

With fixed a, b and s , we have $M(a, b, c, d, s)[G_5, G_5] = M(a, b, c', d', s)[G_5, G_5]$ for every $c, c', d, d' \in \mathbb{Z}/5$. Moreover, for every $c, c', c'', d, d', d'' \in \mathbb{Z}/5$,

$$\begin{aligned} M(1, 0, c, d, 1)[G_5, G_5] &= M(2, 1, c', d', 1)[G_5, G_5] = M(2, -1, c'', d'', 1)[G_5, G_5] \\ &= \left\{ \begin{pmatrix} c_{11} & -2c_{21} & 0 \\ c_{21} & c_{11} & 0 \\ c'_{31} & c'_{32} & 1 \end{pmatrix}, (c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}, c'_{31}, c'_{32} \in \mathbb{Z}/5 \right\}. \end{aligned}$$

A representative of this class is $\mathbb{1}$, and the class is located by $s = 1$, $t = \text{sign}(a) = 1$.

Similarly, for every $c, c', c'', d, d', d'' \in \mathbb{Z}/5$,

$$\begin{aligned} M(-1, 0, c, d, 1)[G_5, G_5] &= M(-2, 1, c', d', 1)[G_5, G_5] = M(-2, -1, c'', d'', 1)[G_5, G_5] \\ &= \left\{ \begin{pmatrix} -c_{11} & 2c_{21} & 0 \\ -c_{21} & -c_{11} & 0 \\ c'_{31} & c'_{32} & 1 \end{pmatrix}, (c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}, c'_{31}, c'_{32} \in \mathbb{Z}/5 \right\}, \\ M(1, 0, c, d, -1)[G_5, G_5] &= M(2, 1, c', d', -1)[G_5, G_5] = M(2, -1, c'', d'', -1)[G_5, G_5] \\ &= \left\{ \begin{pmatrix} c_{11} & -2c_{21} & 0 \\ -c_{21} & -c_{11} & 0 \\ c'_{31} & c'_{32} & -1 \end{pmatrix}, (c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}, c'_{31}, c'_{32} \in \mathbb{Z}/5 \right\}, \\ M(-1, 0, c, d, -1)[G_5, G_5] &= M(-2, 1, c', d', -1)[G_5, G_5] = M(-2, -1, c'', d'', -1)[G_5, G_5] \\ &= \left\{ \begin{pmatrix} -c_{11} & 2c_{21} & 0 \\ c_{21} & c_{11} & 0 \\ c'_{31} & c'_{32} & -1 \end{pmatrix}, (c_{11}, c_{21}) \in \{(1, 0), (2, \pm 1)\}, c'_{31}, c'_{32} \in \mathbb{Z}/5 \right\}. \end{aligned}$$

□

As a consequence of Proposition VI.2, we have the following further result.

Proposition VI.3 G_5 has four orthogonal one-dimensional irreducible representations $U : G_5 \rightarrow \mathbb{C}^*$: a possible choice for the four 1-irreps is as follows:

- $\det : M(a, b, c, d, s) \mapsto 1$,
- $s : M(a, b, c, d, s) \mapsto s$,
- $t : M(a, b, c, d, s) \mapsto \text{sign}(a)$ (taking the representatives of the elements in $\mathbb{Z}/5$ as $-2, -1, 0, 1, 2$), and
- $st : M(a, b, c, d, s) \mapsto s \cdot \text{sign}(a)$.

Proof As for the case $p = 3$, the 1-irreps of G_5 correspond to the irreducible characters of $\text{Ab}(G_5)$. The character table of $\text{Ab}(G_5) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ is as follows.

$\text{Ab}(G_5) \simeq \{Ab(s, t)\}$	$Ab(1, 1)$	$Ab(1, -1)$	$Ab(-1, 1)$	$Ab(-1, -1)$
\det	1	1	1	1
s	1	1	-1	-1
t	1	-1	1	-1
st	1	-1	-1	1

The bijective correspondence between the 1-irreps of G_5 and the irreps of $\text{Ab}(G_5)$ is as follows: if $\tilde{\rho} : \text{Ab}(G_5) \rightarrow \mathbb{C}^*$ is a 1-irrep of $\text{Ab}(G_5)$, then $\rho(M) := \tilde{\rho}(M[G_5, G_5])$ is a 1-irrep of G_5 (and vice versa). This concludes the proof. \square

B. Degree of representations

The Peter-Weyl theorem [16] for G_5 states that $\sum_{\alpha} d_{\alpha}^2 = |G_5| = 300$, where d_{α} are the degrees of the irreducible representations of G_5 located by α . Then,

$$\sum_{\alpha | d_{\alpha} > 1} d_{\alpha}^2 = 296 \quad (88)$$

according to Proposition VI.3.

This is not enough to deduce the nature of higher-degree irreps, but another proposition (proved in Appendix E) comes to our help.

Proposition VI.4 G_5 is partitioned in 14 conjugacy classes. Their orders and lists are in Appendix E.

It means that there are 14 non-equivalent irreps of G_5 , among which four have degree 1. Sorting the degrees of these irreps from the smaller to the greater as d_1, \dots, d_{14} , we have

$$d_1 = d_2 = d_3 = d_4 = 1, \quad (89)$$

$$\sum_{\alpha=5}^{14} d_{\alpha}^2 = 296. \quad (90)$$

Solving Eq. (90) for d_{α} divisors of $|G_5|$ greater than 1 provides 18 possible solutions, among which four present no degree equal to 2, which we are interested relatively to 5-adic qubits. However, there exists a stronger result due to Itô [17], namely, d_{α} divides the index in G_5 of any of its (maximal) Abelian normal subgroups. In order to exploit it, we prove the following.

Proposition VI.5

$$A := \{M(1, 0, c, d, 1) : c, d \in \mathbb{Z}/5\} \quad (91)$$

is the maximal normal Abelian subgroup of G_5 .

Proof Normal subgroups are unions of conjugacy classes. The elements in each $C_i, i = 6, \dots, 14$, C_i conjugacy classes of G_5 in Appendix E, do not commute, in general. These classes, then, are not of interest for our purposes here. On the other hand, the matrices in $\bigcup_{i=1}^5 C_i = \{M(1, 0, c, d, 1) : c, d \in \mathbb{Z}/5\}$ commute. This set is closed under matrix multiplication and contains the identity matrix. \square

The index of A in G_5 is $|G_5 : A| = |G_5|/|A| = 300/25 = 12$. It follows by Itô's theorem that $d_\alpha \in \{2, 3, 4, 6, 12\}$ for every $\alpha = 5, \dots, 14$.

Proposition VI.6 Equation (90) has three solutions for integers $d_\alpha \in \{2, 3, 4, 6, 12\}$ up to permutations.

	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}
First	2	2	3	3	3	3	6	6	6	12
Second	2	2	6	6	6	6	6	6	6	6
Third	2	4	4	4	4	4	4	4	6	12

The proof of this proposition is in Appendix F.

In order to further restrict this set of solutions, we recall Eq. (70): here $\mathfrak{J}(K'_5)$ is isomorphic to the dihedral group D_6 . Appendix (B 1) confirms the four one-dimensional representations we found in (VI.3). Moreover, Appendix (B 2) shows that $\mathfrak{J}(K'_5)$ has $\frac{p-1}{2} = 2$ irreps of degree 2; hence, G_5 has at least these two 2-irreps. This reduces the possibilities of Proposition VI.6 to the following solutions.

	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}
First	2	2	3	3	3	3	6	6	6	12
Second	2	2	6	6	6	6	6	6	6	6

In any case, we conclude that G_5 leads exactly to two qubit representations: those coming from the two unitary 2-irreps of the dihedral group D_6 .

C. 5-Adic qubits

Similarly to Proposition V.7, we have the following.

Proposition VI.7 There exist two unitary 2-irreps $J_5^{(1)}, J_5^{(2)}$ of $SO(3)_5$,

$$J_5^{(i)} := \rho_i \circ \varphi \circ K'_5 \circ \pi_1, \quad (92)$$

where φ represents isomorphism (70) and $\rho_i, i = 1, 2$ are the two 2-irreps of D_6 (Appendix B 2),

$$\begin{aligned} \rho_1 : \quad a &\mapsto \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \rho_2 : \quad a &\mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Proof $J_5^{(i)}$ for $i = 1, 2$ are group homomorphisms from $SO(3)_5$ to $GL(\mathbb{C}^2)$ because they are compositions of group homomorphisms. They are irreducible since the ρ_i are so: if $Y \subset \mathbb{C}^2$, then

the following are equivalent:

$$\left(J_5^{(i)}(SO(3)_5) \right) Y = Y, \quad (93)$$

$$\rho_i((\varphi \circ K'_5 \circ \pi_1)(SO(3)_5)) Y = Y, \quad (94)$$

$$\rho_i(D_6) Y = Y. \quad (95)$$

The latter is equivalent to $Y = \{0\}$ or $Y = \mathbb{C}^2$. \square

To write them explicitly, it is enough to tell the action of the generators C, Z of $\mathfrak{J}(K'_5)$. Choosing $(a_0, b_0) = (-2, 1)$ as the generator of the solutions (a, b) to $a^2 + 2b^2 \equiv 1 \pmod{5}$ and writing 5-adic numbers as sequences of integer numbers modulo 5^k , we get

$$J_5^{(1)} : \begin{pmatrix} (-2, a_{112}, \dots) & (-2, a_{122}, \dots) & (0, a_{132}, \dots) \\ (1, a_{212}, \dots) & (-2, a_{222}, \dots) & (0, a_{232}, \dots) \\ (a_{311}, a_{312}, \dots) & (a_{321}, a_{322}, \dots) & (1, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_5} \xrightarrow{K'_5 \circ \pi_1} C = \begin{pmatrix} -2 & -2 \\ 1 & -2 \end{pmatrix}_{\mathbb{Z}/5} \xrightarrow{\rho_1 \circ \varphi} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}_{\mathbb{C}}$$

$$\begin{pmatrix} (1, a_{112}, \dots) & (0, a_{122}, \dots) & (0, a_{132}, \dots) \\ (0, a_{212}, \dots) & (-1, a_{222}, \dots) & (0, a_{232}, \dots) \\ (a_{311}, a_{312}, \dots) & (a_{321}, a_{322}, \dots) & (-1, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_5} \xrightarrow{K'_5 \circ \pi_1} Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathbb{Z}/5} \xrightarrow{\rho_1 \circ \varphi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathbb{C}} \quad (96)$$

and

$$J_5^{(2)} : \begin{pmatrix} (-2, a_{112}, \dots) & (-2, a_{122}, \dots) & (0, a_{132}, \dots) \\ (1, a_{212}, \dots) & (-2, a_{222}, \dots) & (0, a_{232}, \dots) \\ (a_{311}, a_{312}, \dots) & (a_{321}, a_{322}, \dots) & (1, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_5} \xrightarrow{K'_5 \circ \pi_1} C = \begin{pmatrix} -2 & -2 \\ 1 & -2 \end{pmatrix}_{\mathbb{Z}/5} \xrightarrow{\rho_2 \circ \varphi} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_{\mathbb{C}}$$

$$\begin{pmatrix} (1, a_{112}, \dots) & (0, a_{122}, \dots) & (0, a_{132}, \dots) \\ (0, a_{212}, \dots) & (-1, a_{222}, \dots) & (0, a_{232}, \dots) \\ (a_{311}, a_{312}, \dots) & (a_{321}, a_{322}, \dots) & (-1, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_5} \xrightarrow{K'_5 \circ \pi_1} Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathbb{Z}/5} \xrightarrow{\rho_2 \circ \varphi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathbb{C}}. \quad (97)$$

The matrix $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ is of order 6 as C , and ρ_1 is a faithful representation of $D_6 \simeq \mathfrak{J}(K'_5)$. On

the other hand, $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ is of order 3, so ρ_2 is not injective.

We, thus, arrive to the following final result.

Proposition VI.8 *The pairs $(\mathbb{C}^2, J_5^{(1)})$, $(\mathbb{C}^2, J_5^{(2)})$, with $J_5^{(i)}$ defined as in (96) and (97), are 5-adic qubits, i.e., continuous unitary irreducible linear (and then projective) representations of degree 2 on \mathbb{C} of the 5-adic special orthogonal group $SO(3)_5$ defined in (3).*

Note that, in contrast to the case $p = 3$, $\iota' : \mathbb{Z}/5 \hookrightarrow \mathbb{C}$, $0_{\mathbb{Z}/5} \mapsto 0_{\mathbb{C}}$, $\pm 1_{\mathbb{Z}/5} \mapsto \pm 1_{\mathbb{C}}$, $\pm 2_{\mathbb{Z}/5} \mapsto \pm 2_{\mathbb{C}}$ does not lead to a group homomorphism $\iota' \circ K'_5$; the four solutions $(a, b) \in \{(\pm 2, 1), (\pm 2, -1)\} \subset (\mathbb{Z}/5)^2$ of $a^2 + 2b^2 \equiv 1 \pmod{5}$ are not embedded by ι' to solutions of $a^2 + 2b^2 = 1$ in \mathbb{C} .

VII. REPRESENTATIONS OF $SO(3)_2 \pmod{2}$

We exploit Eqs. (15) and (16) for $k = 1$. Since $\mathcal{R}_n(\infty) \equiv \mathbf{1} \pmod{2}$, we just have

$$G_{n,2} = \{ \mathcal{R}_n(\sigma) \pmod{2} : \sigma \in \mathbb{Z}/2 \}. \quad (98)$$

The elements of $G_{n,2}$ written on the plane orthogonal to n are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (99)$$

which are the identity and the exchange of the two reference axes orthogonal to n . This holds for every n of the canonical basis of \mathbb{Q}_2^3 , for which the groups $G_{n,2}$ provide the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (100)$$

These exchanges generate the group of permutations of the three reference axes of \mathbb{Q}_2^3 . We conclude that there exists an isomorphism ϕ between $SO(3)_2 \pmod 2$ and the group \mathbb{S}_3 of permutations of 3 elements,

$$\begin{aligned} \phi : SO(3)_2 \pmod 2 &\xrightarrow{\sim} \mathbb{S}_3, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &\mapsto (12), \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} &\mapsto (23), \end{aligned} \quad (101)$$

where (12), (23) are the exchanges that generate \mathbb{S}_3 .

This isomorphism allows us to apply the representation theory of \mathbb{S}_3 to $SO(3)_2 \pmod 2$. In particular, we know that \mathbb{S}_3 has a unique 2-irrep, which we propose as the foundation for the 2-adic qubit.

In detail [18], $\sum_{\alpha} d_{\alpha}^2 = |\mathbb{S}_3| = 6$, where d_{α} are the degrees of all the irreducible representations of \mathbb{S}_3 located by α . This means that no irreducible representation of degree higher or equal than 3 is allowed for \mathbb{S}_3 . The commutator subgroup $[\mathbb{S}_3, \mathbb{S}_3]$ is the group of even permutations, so $|\mathbb{S}_3/[\mathbb{S}_3, \mathbb{S}_3]| = 2$ is the number of 1-irreps of \mathbb{S}_3 . Hence, \mathbb{S}_3 has two 1-irreps and one 2-irrep.

The complex 1-irreps for \mathbb{S}_3 are as follows:

- the trivial representation

$$\text{TRIV} : \mathbb{S}_3 \rightarrow \mathbb{C}^*, \quad \pi \mapsto 1,$$

- the sign representation

$$\text{SIGN} : \mathbb{S}_3 \rightarrow \mathbb{C}^*, \quad \pi \mapsto \text{sign}(\pi) = \begin{cases} +1, & \text{if } \pi \text{ even,} \\ -1, & \text{if } \pi \text{ odd.} \end{cases}$$

A complex representation of degree 3 of \mathbb{S}_3 is

$$\tau : \mathbb{S}_3 \rightarrow GL(\mathbb{C}^3), \quad \tau(\pi)(e_i) = e_{\pi(i)},$$

which permutes the basis elements $\{e_i\}_{i=1}^3$ of \mathbb{C}^3 . This representation is reducible, in fact there exists the subrepresentation τ' on

$$\mathbb{C}^2 \simeq \text{span}\{e_1 - e_2, e_2 - e_3\} = \{x^1 e_1 + x^2 e_2 + x^3 e_3 \in \mathbb{C}^3 : x^1 + x^2 + x^3 = 0\}. \quad (102)$$

This has the following matrix form for the generators (12) and (23) of \mathbb{S}_3 with respect to the basis $\{e_1 - e_2, e_2 - e_3\}$:

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

τ' can be easily shown to be the unitary irreducible representation of degree 2 of \mathbb{S}_3 . In fact, as in the proof of irreducibility of Proposition V.9, the actions of (12) and (23) have no common eigenvectors.

Proposition VII.1 *There exists a unitary 2-irrep J_2 of $SO(3)_2$,*

$$J_2 := \tau' \circ \phi \circ \pi_1. \quad (103)$$

Proof J_2 is a homomorphism because it is composition of homomorphisms. J_2 is irreducible since τ' is so: if $Y \subset \mathbb{C}^2$, then the following are equivalent:

$$(J_2(SO(3)_2)Y = Y, \quad (104)$$

$$\tau'((\phi \circ \pi_1)(SO(3)_2))Y = Y, \quad (105)$$

$$\tau'(\mathbb{S}_3)Y = Y. \quad (106)$$

The latter is equivalent to $Y = \{0\}$ or $Y = \mathbb{C}^2$. □

To write J_2 explicitly, it is enough to tell the action of the generators $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ of $SO(3)_2 \pmod{2}$,

$$J_2 : \begin{pmatrix} (0, a_{112}, \dots) & (1, a_{122}, \dots) & (0, a_{132}, \dots) \\ (1, a_{212}, \dots) & (0, a_{222}, \dots) & (0, a_{232}, \dots) \\ (0, a_{312}, \dots) & (0, a_{322}, \dots) & (1, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_2} \xrightarrow{\phi \circ \pi_1} (12) \in \mathbb{S}_3 \xrightarrow{\tau'} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}_{\mathbb{C}} \quad (107)$$

$$\begin{pmatrix} (1, a_{112}, \dots) & (0, a_{122}, \dots) & (0, a_{132}, \dots) \\ (0, a_{212}, \dots) & (0, a_{222}, \dots) & (1, a_{232}, \dots) \\ (0, a_{312}, \dots) & (1, a_{322}, \dots) & (0, a_{332}, \dots) \end{pmatrix}_{\mathbb{Z}_2} \xrightarrow{\phi \circ \pi_1} (23) \in \mathbb{S}_3 \xrightarrow{\tau'} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}_{\mathbb{C}}.$$

We, thus, arrive to the following final result.

Proposition VII.2 *The pair (\mathbb{C}^2, J_2) with J_2 defined as in (107) is a 2-adic qubit, i.e., a continuous unitary irreducible linear (and then projective) representation of degree 2 on \mathbb{C} of the 2-adic special orthogonal group $SO(3)_2$ defined in (3).*

VIII. DISCUSSION

We have examined the p -adic special orthogonal group $SO(3)_p$ in three dimensions, outlining some of its geometric and algebraic properties with the help of modular arithmetics and representation theory. Our aim was to show, for each p , at least one two-dimensional continuous unitary projective irrep, thus showing the existence of p -adic qubits, in the sense of the construction of a spin- $\frac{1}{2}$ particle through an angular momentum representation.

As a matter of fact, for $p \geq 5$, we found that there is more than one such p -adic qubit, and it remains an important open question to classify all two-dimensional projective irreps for every p , not to mention the higher-dimensional ones.

A special role is played by the one-dimensional irreps, which act by multiplication on the other irreps and which themselves form an Abelian group, isomorphic to the abelianization of $SO(3)_p$. It remains largely unknown, except that we know for odd $p > 2$ that it contains the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$.

In any case, the present work has the potential to open new lines of investigations, as follows:

- *p-adic spin*: while there are works treating representations of $SO(3)_p$ as a characterizing property of particles, such as Ref. [19], they do not work to distinguish the behaviour of the group for different values of p .
- *p-adic quantum computation*: the qubit is the simplest object in this subject, and computational gates can be formulated in this framework with the potential of leading to new eventually more efficient algorithms.
- *p-adic quantum information*: stochastic processes over the field of p -adic numbers [20] can be formalized for qubit, thus enlarging the perspectives of quantum communication.

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Author Declarations

Conflict of Interest

The authors have no conflicts to disclose.

Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A: Cyclic structure of solutions of the equation $a^2 - vb^2 \equiv 1$ in \mathbb{Z}/p , $p > 2$ prime

Recall that v is not a square modulo p for every prime $p > 2$. Thus, we can construct the field extension of the Galois field $\mathbb{F}_p = \mathbb{Z}/p$ [21], with i being a fixed root of $x^2 \equiv v$. This is denoted as $\mathbb{F}_p[i]$ and can be seen as $\mathbb{F}_p + i\mathbb{F}_p$. The Galois group $\text{Aut}(\mathbb{F}_p[i]/\mathbb{F}_p) = \{\text{automorphisms } \psi : \mathbb{F}_p[i] \rightarrow \mathbb{F}_p[i] : \psi(x) = x \forall x \in \mathbb{F}_p\}$ contains only two elements: the identity and $i \mapsto -i$.

Next, $\mathbb{F}_p[i]^*$ with the multiplicative operation is a cyclic group (since $\mathbb{F}_p[i]$ is a finite field) of order $p^2 - 1$. This means there exists an element $\alpha + i\beta$, $\alpha, \beta \in \mathbb{F}_p^*$ of order $p^2 - 1$ which generates the whole group:

$$\mathbb{F}_p[i]^* = \langle \alpha + i\beta \rangle = \left\{ (\alpha + i\beta)^k : k = 0, \dots, p^2 - 2 \right\}. \quad (\text{A1})$$

Our aim is to find all the solutions of $1 \equiv a^2 - vb^2 \equiv (a + ib)(a - ib) \pmod{p}$. We have

$$a + ib \equiv (\alpha + i\beta)^m, \quad a - ib \equiv (\alpha - i\beta)^m \quad (\text{A2})$$

for some $m = 0, \dots, p^2 - 2$ because $i \mapsto -i$ is the only nontrivial automorphism of the field $\mathbb{F}_p[i] \supset \mathbb{F}_p$. Hence, our equation becomes $(\alpha^2 - v\beta^2)^m \equiv 1 \pmod{p}$, from which we see that m is a multiple of the order of $\alpha^2 - v\beta^2$.

Proposition A.1 *If $\alpha + i\beta$ is generator of $\mathbb{F}_p[i]^*$, then $\alpha^2 - v\beta^2$ is generator of \mathbb{F}_p^* .*

Proof First, we want to show that the map

$$f : \mathbb{F}_p[i] \rightarrow \mathbb{F}_p[i], \quad x \mapsto x^p \quad (\text{A3})$$

corresponds to the non trivial automorphism $i \mapsto -i$.

$f \in \text{Aut}(\mathbb{F}_p[i]/\mathbb{F}_p)$ because of the following conditions:

- $f(x + y) \equiv (x + y)^p \equiv \sum_{j=0}^p \binom{p}{j} x^j y^{p-j} \equiv x^p + y^p \equiv f(x) + f(y)$, $\forall x, y \in \mathbb{F}_p[i]$, since the binomial coefficients $\binom{p}{j}$ for $j = 1, \dots, p - 1$ are multiples of p .
- $f(xy) \equiv \underbrace{xy \dots xy}_{p \text{ times}} \equiv x^p y^p \equiv f(x)f(y)$, $\forall x, y \in \mathbb{F}_p[i]$.
- $f(x) \equiv 0 \Leftrightarrow x \equiv 0$.
- For every $x \in \mathbb{F}_p^*$, we have $f(x) \equiv x^p \equiv x^{p-1}x \equiv x$ since \mathbb{F}_p^* is a multiplicative cyclic group of order $p - 1$.

Furthermore, $f \neq id$; otherwise $x^p \equiv x$ for every $x \in \mathbb{F}_p[i]$ would imply $x^{p-1} \equiv 1$ for every $x \in \mathbb{F}_p[i]^*$, which leads to a contradiction with the fact that the order of $\mathbb{F}_p[i]^*$ is $p^2 - 1$.

It follows that $a - ib \equiv f(a + ib) \equiv (a + ib)^p$, $\forall a, b \in \mathbb{F}_p$.

If $\alpha + i\beta$ is generator of $\mathbb{F}_p[i]^*$, it is an element of order $p^2 - 1$ and $1 \equiv (\alpha + i\beta)^{p^2-1} \equiv [(\alpha + i\beta)^{(p+1)]^{(p-1)}}$. Therefore $\alpha^2 - v\beta^2 \equiv (\alpha + i\beta)(\alpha - i\beta) \equiv (\alpha + i\beta)^{p+1}$ is a generator of \mathbb{F}_p^* . \square

Since $\alpha^2 - v\beta^2 \in \mathbb{F}_p^*$ is of order $p - 1$, together with $(\alpha^2 - v\beta^2)^m \equiv 1$, we can write $m = (p - 1)n$ where $n = 0, \dots, p$, and

$$a + ib \equiv ((\alpha + i\beta)^{p-1})^n := (a_0 + ib_0)^n. \quad (\text{A4})$$

We have $a_0^2 - vb_0^2 \equiv (\alpha + i\beta)^{p-1}(\alpha - i\beta)^{p-1} \equiv (\alpha^2 - v\beta^2)^{p-1} \equiv 1$. The pairs (a, b) such that $a + ib \equiv (a_0 + ib_0)^n$ are all distinct by varying n because $(a_0 + ib_0)^n \equiv (a_0 + ib_0)^{n'} \Leftrightarrow 1 \equiv (a_0 + ib_0)^{n-n'} \equiv$

$(\alpha + i\beta)^{(p-1)(n-n')} \Leftrightarrow (n-n')(p-1) = M(p^2-1) \Leftrightarrow n-n' = M(p+1)$, which is impossible for every $M \in \mathbb{Z}_{>0}$. Furthermore, $(a_0 + ib_0)^n \equiv (\alpha + i\beta)^{(p-1)n} \equiv 1 \Leftrightarrow (p-1)n = q(p^2-1) \Leftrightarrow n = q(p+1)$ for $q \in \mathbb{Z}$.

This means that the solutions $(a, b) \in \mathbb{F}_p^2$ of $a^2 - vb^2 \equiv 1 \pmod{p}$ form a cyclic group of order $p+1$, endowed with the product

$$(a, b) \cdot (c, d) := (ac + vbd, ad + bc). \quad (\text{A5})$$

This is because the pairs (a, b) are regarded as a single number $a + ib$ and $(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac + vbd) + i(ad + bd)$. Another equivalence is

$$a + ib \leftrightarrow \begin{pmatrix} a & vb \\ b & a \end{pmatrix} \quad (\text{A6})$$

that gives again

$$(a, b) \cdot (c, d) = \begin{pmatrix} a & vb \\ b & a \end{pmatrix} \begin{pmatrix} c & vd \\ d & c \end{pmatrix} = \begin{pmatrix} ac + vbd & v(ad + bc) \\ ad + bc & ac + vbd \end{pmatrix}. \quad (\text{A7})$$

Appendix B: Representations of dihedral groups of even degree

Consider the dihedral group D_n , where n is even [18],

$$D_n := \langle a, x \mid a^n = x^2 = e, xax = a^{-1} \rangle. \quad (\text{B1})$$

This group has $(n+6)/2$ conjugacy classes: the identity element, the element $a^{n/2}$, $(n-2)/2$ other conjugacy classes in $\langle a \rangle$, and two conjugacy classes outside $\langle a \rangle$, with representatives x and ax . The representations we are going to show are all unitary.

1. The four one-dimensional representations

The commutator subgroup is $\langle a^2 \rangle$, which has index four, and the quotient group (the abelianization) is a Klein four-group. There are, thus, four one-dimensional representations:

- The trivial representation, sending all elements to the 1×1 matrix (1).
- The representation sending all elements in $\langle a \rangle$ to (1) and all the others to (-1) .
- The representation sending all elements in $\langle a^2, x \rangle$ to (1) and all the others to (-1) .
- The representation sending all elements in $\langle a^2, ax \rangle$ to (1) and all the others to (-1) .

2. Two-dimensional representations

There are $(n-2)/2$ irreducible two-dimensional representations. The k th representation into real orthogonal matrices is as follows:

$$a \mapsto \begin{pmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B2})$$

k ranges between 1 and $(n-2)/2$, because k and $n-k$ form equivalent complex irreducible representations. Furthermore, for $k=0$ and $k=n/2$, the representation is not irreducible and can be broken down into one-dimensional representations.

Appendix C: Conjugacy classes of G_3

In Proposition V.4, we need to know the number of conjugacy classes of

$$G_3 = \pi_1(SO(3)_3) = SO(3)_3 \pmod{3}.$$

G_3 is partitioned in the following nine conjugacy classes [using the form in (72)]:

- one class of order 1, composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Two classes of order 4,

$$\{M(1, 0, c, d, 1) : (c, d) \in \{(0, \pm 1), (\pm 1, 0)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\{M(1, 0, c, d, 1) : (c, d) \in \{(1, \pm 1), (-1, \pm 1)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

- Two classes of order 6,

$$\{M(a, b, c, d, -1) : (a, b) \in \{(0, \pm 1)\}, (c, d) \in \{(0, 0), (\pm 1, \pm b)\}\}$$

composed of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix},$$

$$\{M(1, 0, c, d, -1) : (c, d) \in \{(0, 0), (\pm 1, 0)\}\} \cup \{M(-1, 0, c, d, -1) : (c, d) \in \{(0, 0), (0, \pm 1)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

- One class of order 9,

$$\{M(-1, 0, c, d, 1) : c, d \in \mathbb{Z}/3\}$$

composed of

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

- Two classes of order 12,

$$\{M(0, \pm 1, c, d, -1) : (c, d) \in \{(0, \pm 1), (\pm 1, 0), (\pm 1, \mp b)\}\}$$

composed of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix},$$

$$\{M(1, 0, c, \pm 1, -1) : c \in \mathbb{Z}/3\} \cup \{M(-1, 0, \pm 1, d, -1) : d \in \mathbb{Z}/3\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

- One class of order 18,

$$\{M(0, \pm 1, c, d, 1) : c, d \in \mathbb{Z}/3\}$$

composed of

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Appendix D: Proof of Proposition V.5

We need to solve $\sum_{\alpha|d_\alpha>1} d_\alpha^2 = 68$ for integers d_α divisors of 72 greater than 1 up to permutation, $|\{\alpha \mid d_\alpha > 1\}| = 5$. Without loss of generality, we can suppose the d_α in non-decreasing order. We will study case by case, starting from the largest value, d_9 , with decreasing values.

d_9 cannot be larger than 8, because its square would exceed 68, which is the sum of positive numbers.

Case $d_9 = 8$: this would mean that $d_5^2 + d_6^2 + d_7^2 + d_8^2 = 68 - 64 = 4$, but that is not possible as $d_\alpha > 1$ for any $\alpha = 5, 6, 7, 8$.

Case $d_9 = 6$: assuming this, we have $d_5^2 + d_6^2 + d_7^2 + d_8^2 = 32$. What can we say about d_8 ?

- $d_8 = 4$: then, $d_5^2 + d_6^2 + d_7^2 = 16$.

- $d_7 = 3$: $d_5^2 + d_6^2 = 7$, with only the contradictory chance $d_5 = d_6 = 2$.
- $d_7 = 2$: then, $d_5 = d_6 = 2$, but $d_5^2 + d_6^2 + d_7^2$ results as 12 and not 16.
- $d_8 = 3$: $d_5^2 + d_6^2 + d_7^2 = 23$.
 - $d_7 = 3$: $d_5^2 + d_6^2 = 14$.
 - * $d_6 = 3$: $d_5^2 = 5$, impossible.
 - * $d_6 = 2$: it would be $d_5^2 + d_6^2 = 8 = 14$, absurd.
 - $d_7 = 2$: then, $d_5^2 + d_6^2 + d_7^2 = 12$, while it should be 23.
- $d_8 = 2$: in this case, $d_5^2 + d_6^2 + d_7^2 + d_8^2 = 16$ instead of 32.

Case $d_9 = 4$: consequently, $d_5^2 + d_6^2 + d_7^2 + d_8^2 = 52$.

- $d_8 = 4$: $d_5^2 + d_6^2 + d_7^2 = 36$.
 - $d_7 = 4$: $d_5^2 + d_6^2 = 20$.
 - * $d_6 = 4$: then, $d_5^2 = 4 \Rightarrow d_5 = 2$, solution.
 - * $d_6 = 3$: then, $d_5^2 = 11$, impossible.
 - * $d_6 = 2$: then, $d_5^2 + d_6^2 = 8 < 16$.
 - $d_7 = 3$: then, $d_5^2 + d_6^2 = 27$. The odd sum forces $d_6 = 3 \Rightarrow d_5^2 = 18$, impossible.
 - $d_7 = 2$: then, $d_5^2 + d_6^2 + d_7^2 = 12 < 36$.
- $d_8 = 3$: $d_5^2 + d_6^2 + d_7^2$ should be 43, but $d_5^2 + d_6^2 + d_7^2 \leq 27$.
- $d_8 = 2$: then, $d_5^2 + d_6^2 + d_7^2 + d_8^2 = 16 < 52$.

Case $d_9 = 3$: if the larger value is 3, $d_5^2 + d_6^2 + d_7^2 + d_8^2 + d_9^2$ can be at most $5 \cdot 9 = 45 < 68$.

Case $d_9 = 2$: it is ruled out for the same reason as $d_9 = 3$.

The only solution is $d_5 = 2$, $d_6 = d_7 = d_8 = d_9 = 4$.

Appendix E: Conjugacy classes of G_5

In Proposition VI.4, we need to know the number of conjugacy classes of

$$G_5 = \pi_1(SO(3)_5) = SO(3)_5 \pmod{5}.$$

G_5 is partitioned in the following 14 conjugacy classes [referring to the form in (85)]:

- one class of order 1, say C_1 , composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Four classes of order 6,

$$C_2 = \{M(1, 0, c, d, 1) : (c, d) \in \{(1, \pm 1), (-1, \pm 1), (\pm 2, 0)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix},$$

$$C_3 = \{M(1, 0, c, d, 1) : (c, d) \in \{(\pm 1, 0), (2, \pm 2), (-2, \pm 2)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix},$$

$$C_4 = \{M(1, 0, c, d, 1) : (c, d) \in \{(0, \pm 2), (2, \pm 1), (-2, \pm 1)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix},$$

$$C_5 = \{M(1, 0, c, d, 1) : (c, d) \in \{(0, \pm 1), (1, \pm 2), (-1, \pm 2)\}\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$$

- Two classes of order 15,

$$C_6 = \{M(a, b, c, sbc, -1) : (a, b) \in \{(1, 0), (2, \pm 1)\}, c \in \mathbb{Z}/5\}$$

composed of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \\ \begin{pmatrix} 2 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & -2 & 0 \\ -2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & -2 & 0 \\ -1 & 1 & -1 \end{pmatrix}, \\ \begin{pmatrix} 2 & -2 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -1 & -2 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -1 & -2 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -1 & -2 & 0 \\ -2 & -2 & -1 \end{pmatrix},$$

$$C_7 = \{M(-1, 0, 0, d, -1) : d \in \mathbb{Z}/5\} \cup \{M(a, b, c, abc, -1) : (a, b) \in \{(-2, \pm 1), c \in \mathbb{Z}/5\}\}$$

composed of

$$\begin{aligned} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix}, \\ & \begin{pmatrix} -2 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 2 & 0 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 2 & 0 \\ 1 & 2 & 0 \\ -2 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & -2 & -1 \end{pmatrix} \\ & \begin{pmatrix} -2 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -2 & 0 \\ -1 & 2 & 0 \\ 1 & 2 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -2 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -2 & 0 \\ -1 & 2 & 0 \\ -2 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -2 & 0 \\ -1 & 2 & 0 \\ -1 & -2 & -1 \end{pmatrix}. \end{aligned}$$

- One class of order 25,

$$C_8 = \{M(-1, 0, c, d, 1) : c, d \in \mathbb{Z}/5\}$$

composed of

$$\begin{aligned} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \\ & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix}, \\ & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \\ & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \\ & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}. \end{aligned}$$

- Four classes of order 30,

$$\begin{aligned} C_9 = & \{M(-1, 0, \pm 2, d, -1) : d \in \mathbb{Z}/5\} \\ & \cup \{M(-2, 1, c, d, -1) : (c, d) \in \{(0, \pm 2), (\pm 1, 0), (\pm 1, \pm 1), (\pm 2, \mp 1), (\pm 2, \mp 2)\}\} \\ & \cup \{M(-2, -1, c, d, -1) : (c, d) \in \{(0, \pm 2), (\pm 1, 0), (\pm 1, \mp 1), (\pm 2, \pm 1), (\pm 2, \pm 2)\}\} \end{aligned}$$

composed of

$$\begin{aligned} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -2 & -2 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -2 & -2 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -2 & 2 & 1 \end{pmatrix}.
\end{aligned}$$

Appendix F: Proof of Proposition VI.6

To solve $\sum_{\alpha=5}^{14} d_{\alpha}^2 = 296$ for $d_{\alpha} \in \{2, 3, 4, 6, 12\}$ up to permutations, we proceed analogously to the Proof of Proposition V.5 (Appendix D).

Case $d_{14} = 12$: this gives $\sum_{\alpha=5}^{13} d_{\alpha}^2 = 296 - 12^2 = 152$.

- $d_{13} = 12$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 152 - 12^2 = 8 < 8 \cdot 2^2 = \min\left(\sum_{\alpha=5}^{12} d_{\alpha}^2\right)$, impossible.
- $d_{13} = 6$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 152 - 6^2 = 116$.
 - $d_{12} = 6$: $\sum_{\alpha=5}^{11} d_{\alpha}^2 = 116 - 6^2 = 80$.
 - * $d_{11} = 6$: $\sum_{\alpha=5}^{10} d_{\alpha}^2 = 80 - 6^2 = 44$.
 - $d_{10} = 6$: $\sum_{\alpha=5}^9 d_{\alpha}^2 = 44 - 6^2 = 8 < 5 \cdot 2^2$ impossible.
 - $d_{10} = 4$: $\sum_{\alpha=5}^9 d_{\alpha}^2 = 44 - 4^2 = 28$.
 1. $d_9 = 4$: $\sum_{\alpha=5}^8 d_{\alpha}^2 = 28 - 4^2 = 12 < 4 \cdot 2^2$ impossible.
 2. $d_9 = 3$: $\sum_{\alpha=5}^8 d_{\alpha}^2 = 28 - 3^2 = 19$.
 - (a) $d_8 = 3$: $\sum_{\alpha=5}^7 d_{\alpha}^2 = 19 - 3^2 = 10 < 3 \cdot 2^2$, impossible.
 - (b) $d_8 = 2$: this forces $\sum_{\alpha=5}^8 d_{\alpha}^2 = 4 \cdot 2^2 = 16 = 19$, absurd.
 3. $d_9 = 2$: this forces $\sum_{\alpha=5}^9 d_{\alpha}^2 = 5 \cdot 2^2 = 20 = 28$, absurd.

- $d_{10} = 3: \sum_{\alpha=5}^9 d_\alpha^2 = 44 - 3^2 = 35.$
- 1. $d_9 = 3: \sum_{\alpha=5}^8 d_\alpha^2 = 35 - 3^2 = 26.$
 - (a) $d_8 = 3: \sum_{\alpha=5}^7 d_\alpha^2 = 26 - 3^2 = 17.$
 $d_7 = 3$ gives $d_5^2 + d_6^2 = 17 - 3^2 = 8 \Rightarrow d_5 = d_6 = 2, \text{ solution.}$
 $d_7 = 2$ forces $\sum_{\alpha=5}^7 d_\alpha^2 = 3 \cdot 2^2 = 12 = 17, \text{ absurd.}$
 - (b) $d_8 = 2: \text{ this forces } \sum_{\alpha=5}^8 d_\alpha^2 = 4 \cdot 2^2 = 16 = 26, \text{ absurd.}$
- 2. $d_9 = 2: \text{ this forces } \sum_{\alpha=5}^9 d_\alpha^2 = 5 \cdot 2^2 = 20 = 35.$
- $d_{10} = 2: \text{ this forces } \sum_{\alpha=5}^{10} d_\alpha^2 = 6 \cdot 2^2 = 24 = 44, \text{ absurd.}$
- * $d_{11} = 4: \sum_{\alpha=5}^{10} d_\alpha^2 = 80 - 4^2 = 64.$
- $d_{10} = 4: \sum_{\alpha=5}^9 d_\alpha^2 = 64 - 4^2 = 48.$
 - 1. $d_9 = 4: \sum_{\alpha=5}^8 d_\alpha^2 = 48 - 4^2 = 32.$
 - (a) $d_8 = 4: \sum_{\alpha=5}^7 d_\alpha^2 = 32 - 4^2 = 16.$
 $\text{If } d_7 = 3, 4, \text{ then } d_5^2 + d_6^2 < 8 \text{ is impossible.}$
 $d_7 = 2 \text{ forces } \sum_{\alpha=5}^7 d_\alpha^2 = 3 \cdot 2^2 = 12 = 16, \text{ absurd.}$
 - (b) $d_8 = 3: \sum_{\alpha=5}^7 d_\alpha^2 = 32 - 3^2 = 23.$
 $\text{If } d_7 = 3, \text{ then } d_5^2 + d_6^2 = 23 - 3^2 = 14: \text{ this is impossible both for } d_6 = 3$
 $(\text{for which } d_5^2 = 14 - 3^2 = 5 \Rightarrow d_5 \notin \mathbb{Z}) \text{ and for } d_6 = 2 \text{ (which forces}$
 $d_5^2 + d_6^2 = 2 \cdot 2^2 = 8 = 14).$
 $d_7 = 2 \text{ forces } \sum_{\alpha=5}^7 d_\alpha^2 = 3 \cdot 2^2 = 12 = 23, \text{ absurd.}$
 - (c) $d_8 = 2: \text{ this forces } \sum_{\alpha=5}^8 d_\alpha^2 = 4 \cdot 2^2 = 16 = 32, \text{ absurd.}$
 - 2. $d_9 = 3: \sum_{\alpha=5}^8 d_\alpha^2 = 48 - 3^2 = 39.$
 - (a) $d_8 = 3: \sum_{\alpha=5}^7 d_\alpha^2 = 39 - 3^2 = 30 > 3 \cdot 3^2 = \max\left(\sum_{\alpha=5}^7 d_\alpha^2\right), \text{ impossible.}$
 - (b) $d_8 = 3: \text{ this forces } \sum_{\alpha=5}^8 d_\alpha^2 = 4 \cdot 2^2 = 16 = 39, \text{ absurd.}$
 - 3. $d_9 = 2: \text{ this forces } \sum_{\alpha=5}^9 d_\alpha^2 = 5 \cdot 2^2 = 20 = 48, \text{ absurd.}$
- $d_{10} = 3: \sum_{\alpha=5}^9 d_\alpha^2 = 64 - 9 = 55 > 5 \cdot 3^2 = \max\left(\sum_{\alpha=5}^9 d_\alpha^2\right), \text{ impossible.}$
- $d_{10} = 2: \text{ this forces } \sum_{\alpha=5}^{10} d_\alpha^2 = 6 \cdot 2^2 = 24 = 64, \text{ absurd.}$
- * $d_{11} = 3: \sum_{\alpha=5}^{10} d_\alpha^2 = 80 - 3^2 = 71 > 6 \cdot 3^2 = \max\left(\sum_{\alpha=5}^{10} d_\alpha^2\right), \text{ impossible.}$
- * $d_{11} = 2: \text{ this forces } \sum_{\alpha=5}^{11} d_\alpha^2 = 7 \cdot 2^2 = 28 = 80, \text{ absurd.}$
- $d_{12} = 4: \sum_{\alpha=5}^{11} d_\alpha^2 = 116 - 4^2 = 100.$
- * $d_{11} = 4: \sum_{\alpha=5}^{10} d_\alpha^2 = 100 - 4^2 = 84.$
- $d_{10} = 4: \sum_{\alpha=5}^9 d_\alpha^2 = 84 - 4^2 = 68.$
 - 1. $d_9 = 4: \sum_{\alpha=5}^8 d_\alpha^2 = 68 - 4^2 = 52.$
 - (a) $d_8 = 4: \sum_{\alpha=5}^7 d_\alpha^2 = 52 - 4^2 = 36.$ $\text{It must be } d_7 = 4 \text{ otherwise at most}$
 $\sum_{\alpha=5}^7 d_\alpha^2 = 3 \cdot 3^2 = 27 < 36.$ $\text{Then, } d_5^2 + d_6^2 = 36 - 4^2 = 20, \text{ which implies}$
 $d_6 = 4, d_5 = 2, \text{ solution.}$
 - (b) $d_8 = 3: \sum_{\alpha=5}^7 d_\alpha^2 = 52 - 3^2 = 43 > 3 \cdot 3^2 = \max\left(\sum_{\alpha=5}^7 d_\alpha^2\right), \text{ impossible.}$
 - (c) $d_8 = 2: \text{ this forces } \sum_{\alpha=5}^8 d_\alpha^2 = 4 \cdot 2^2 = 16 = 52, \text{ absurd.}$
 - 2. $d_9 = 3: \sum_{\alpha=5}^8 d_\alpha^2 = 68 - 3^2 = 59 > 4 \cdot 3^2 = \max\left(\sum_{\alpha=5}^8 d_\alpha^2\right), \text{ impossible.}$
 - 3. $d_9 = 2: \text{ this forces } \sum_{\alpha=5}^9 d_\alpha^2 = 5 \cdot 2^2 = 20 = 52, \text{ absurd.}$
- $d_{10} = 3: \sum_{\alpha=5}^9 d_\alpha^2 = 84 - 3^2 = 75 > 5 \cdot 3^2 = \max\left(\sum_{\alpha=5}^9 d_\alpha^2\right), \text{ impossible.}$

- $d_{10} = 2$: this forces $\sum_{\alpha=5}^{10} d_{\alpha}^2 = 24 = 84$, absurd.
- * $d_{11} = 3$: $\sum_{\alpha=5}^{10} d_{\alpha}^2 = 100 - 3^2 = 91 > 6 \cdot 3^2$, impossible.
- * $d_{11} = 2$: this forces $\sum_{\alpha=5}^{11} d_{\alpha}^2 = 28 = 100$, absurd.
- $d_{12} = 3$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 116 - 3^2 = 107 > 8 \cdot 3^2$, impossible.
- $d_{12} = 2$: this forces $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 32 = 112$, absurd.
- $d_{13} = 4$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 152 - 4^2 = 136 > 8 \cdot 4^2$, impossible.
- $d_{13} = 3$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 152 - 3^2 = 143 > 8 \cdot 3^2$, impossible.
- $d_{13} = 2$: this forces $\sum_{\alpha=5}^{13} d_{\alpha}^2 = 36 = 152$, absurd.

Case $d_{14} = 6$: this gives $\sum_{\alpha=5}^{13} d_{\alpha}^2 = 296 - 6^2 = 260$.

- $d_{13} = 6$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 260 - 6^2 = 224$.
 - $d_{12} = 6$: $\sum_{\alpha=5}^{11} d_{\alpha}^2 = 224 - 6^2 = 188$.
 - * $d_{11} = 6$: $\sum_{\alpha=5}^{10} d_{\alpha}^2 = 188 - 6^2 = 152$.
 - $d_{10} = 6$: $\sum_{\alpha=5}^9 d_{\alpha}^2 = 152 - 6^2 = 116$.
 1. $d_9 = 6$: $\sum_{\alpha=5}^8 d_{\alpha}^2 = 116 - 6^2 = 80$.
 - (a) $d_8 = 6$: $\sum_{\alpha=5}^7 d_{\alpha}^2 = 80 - 6^2 = 44$.
 - If $d_7 = 6$ then, $d_5^2 + d_6^2 = 44 - 6^2 = 8$, which implies $d_5 = d_6 = 2$, solution.
 - If $d_7 = 4$ then, $d_5^2 + d_6^2 = 44 - 4^2 = 28$ is impossible: $28 < 2 \cdot 4^2$ and $28 > 3^2 + 4^2$.
 - If $d_7 = 3$ then, $d_5^2 + d_6^2 = 44 - 3^2 = 35 > 2 \cdot 3^2$, impossible.
 - $d_7 = 2$ forces $\sum_{\alpha=5}^7 d_{\alpha}^2 = 12 = 44$, absurd.
 - (b) $d_8 = 3, 4$: $\sum_{\alpha=5}^7 d_{\alpha}^2 > 3 \cdot 4^2$, impossible.
 - (c) $d_8 = 2$: this forces $\sum_{\alpha=5}^8 d_{\alpha}^2 = 16 = 80$, absurd.
 2. $d_9 = 3, 4$: $\sum_{\alpha=5}^8 d_{\alpha}^2 > 4 \cdot 4^2$, impossible.
 3. $d_9 = 2$: this forces $\sum_{\alpha=5}^9 d_{\alpha}^2 = 20 = 116$, absurd.
 - $d_{10} = 3, 4$: $\sum_{\alpha=5}^9 d_{\alpha}^2 > 5 \cdot 4^2$, impossible.
 - $d_{10} = 2$: this forces $\sum_{\alpha=5}^{10} d_{\alpha}^2 = 24 = 152$, absurd.
 - * $d_{11} = 3, 4$: $\sum_{\alpha=5}^{10} d_{\alpha}^2 > 6 \cdot 4^2$, impossible.
 - * $d_{11} = 2$: this forces $\sum_{\alpha=5}^{11} d_{\alpha}^2 = 28 = 188$, absurd.
 - $d_{12} = 3, 4$: $\sum_{\alpha=5}^{11} d_{\alpha}^2 > 7 \cdot 4^2$, impossible.
 - $d_{12} = 2$: this forces $\sum_{\alpha=5}^{12} d_{\alpha}^2 = 32 = 224$, absurd.
- $d_{13} = 3, 4$: $\sum_{\alpha=5}^{12} d_{\alpha}^2 > 8 \cdot 4^2$, impossible.
- $d_{13} = 2$: this forces $\sum_{\alpha=5}^{13} d_{\alpha}^2 = 36 = 260$, absurd.

Cases $d_{14} = 3, 4$: $\sum_{\alpha=5}^{13} d_{\alpha}^2 > 9 \cdot 4^2$, impossible.

Case $d_{12} = 2$: this forces $\sum_{\alpha=5}^{14} d_{\alpha}^2 = 40 = 296$, absurd.

We found three distinct solutions:

$$d_5 = d_6 = 2, d_7 = d_8 = d_9 = d_{10} = 3, d_{11} = d_{12} = d_{13} = 6, d_{14} = 12,$$

$d_5 = d_6 = 2$, $d_7 = d_8 = d_9 = d_{10} = d_{11} = d_{12} = d_{13} = d_{14} = 6$, and
 $d_5 = 2$, $d_6 = d_7 = d_8 = d_9 = d_{10} = d_{11} = d_{12} = 4$, $d_{13} = 6$, $d_{14} = 12$.

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