

ON THE GEOMETRIC ORDER OF TOTALLY NONDEGENERATE CR MANIFOLDS

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ABSTRACT. A CR manifold M , with CR distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$, is called *totally nondegenerate of depth μ* if: (a) the complex tangent space $T^{\mathbb{C}}M$ is generated by all complex vector fields that might be determined by iterated Lie brackets between at most μ fields in $\mathcal{D}^{10} + \overline{\mathcal{D}^{10}}$; (b) for each integer $2 \leq k \leq \mu - 1$, the families of all vector fields that might be determined by iterated Lie brackets between at most k fields in $\mathcal{D}^{10} + \overline{\mathcal{D}^{10}}$ generate regular complex distributions; (c) the ranks of the distributions in (b) have the *maximal values* that can be obtained amongst all CR manifolds of the same CR dimension and satisfying (a) and (b) – this maximality property is the *total nondegeneracy* condition. In this paper, we prove that, for any Tanaka symbol $\mathfrak{m} = \mathfrak{m}^{-\mu} + \dots + \mathfrak{m}^{-1}$ of a totally nondegenerate CR manifold of depth $\mu \geq 4$, the full Tanaka prolongation of \mathfrak{m} has trivial subspaces of degree $k \geq 1$, i.e. it has the form $\mathfrak{m}^{-\mu} + \dots + \mathfrak{m}^{-1} + \mathfrak{g}^0$. This result has various consequences. For instance it implies that any (local) CR automorphism of a regular totally nondegenerate CR manifold is uniquely determined by its first order jet at a fixed point of the manifold. It also gives a complete proof of a conjecture by Beloshapka on the group of automorphisms of homogeneous totally nondegenerate CR manifolds.

1. INTRODUCTION

An (*abstract*) CR manifold is a real manifold M , equipped with a \mathcal{C}^∞ complex distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ such that: (i) $\mathcal{D}^{10} \cap \overline{\mathcal{D}^{10}} = \{0\}$; (ii) for all vector fields $X, Y \in \mathcal{D}^{10}$ the Lie bracket $[X, Y]$ is also in \mathcal{D}^{10} . The rank of the distribution \mathcal{D}^{10} is called *CR dimension*. The most natural and studied examples are the *embedded CR manifolds*, which are the smooth real submanifolds $M \subset \mathbb{C}^N$, $N \geq 2$, satisfying appropriate constant rank conditions that guarantee that the family of all complex vector fields of the holomorphic distribution of \mathbb{C}^N with real and imaginary parts tangent to M , generate a complex distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ of constant rank.

In this paper, we study the geometric structures of the *totally nondegenerate CR manifolds of depth μ* , a class of CR manifolds introduced by Beloshapka in [2]. For motivation and main reasons of interest for such important class of CR structures we refer to the original paper (see also [3, 10, 11]). The properties which characterise a totally nondegenerate CR manifold (M, \mathcal{D}^{10}) of depth μ can be shortly described as follows (see §3 for the detailed definition):

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- a) the complexified tangent bundle $T^{\mathbb{C}}M$ is spanned by the vector fields in $\mathcal{D}^{10} + \overline{\mathcal{D}^{10}}$ and by all iterated Lie brackets determined by sets of at most μ vector fields in $\mathcal{D}^{10} + \overline{\mathcal{D}^{10}}$;
- b) for each $2 \leq k \leq \mu - 1$, the vector fields in $\mathcal{D}^{10} + \overline{\mathcal{D}^{10}}$ and all iterated Lie brackets determined by sets of at most k vector fields in $\mathcal{D}^{10} + \overline{\mathcal{D}^{10}}$ span a regular complex distribution, denoted by $(\mathcal{D}^k)^{\mathbb{C}}$;
- c) the ranks of the distributions $(\mathcal{D}^k)^{\mathbb{C}}$, $2 \leq k \leq \mu - 1$, are the maximal possible ones that can occur in any other CR manifold of the same CR dimension and satisfying (a) and (b) – this maximality property is called *total nondegeneracy condition*.

The CR manifolds M that are appropriately osculated at each point $x \in M$ by a fixed homogeneous CR manifold satisfying (a) – (c) are called *regular* (or *of uniform type*).

To be rigorous, we have to mention that in [2] Beloshapka did not really give the above definition, but actually introduced the notion of *germ of a totally nondegenerate embedded CR submanifold* of \mathbb{C}^N and presented it in terms of some appropriate normal forms for its defining equations. Nonetheless, one can directly check that, if one translates everything into the language of germs of embedded submanifolds, our definition becomes completely equivalent to Beloshapka’s one.

According to the results by Tanaka on the equivalence problem for non-integrable distributions and for the CR structures ([14]; see also [1]), any totally nondegenerate CR manifold (M, \mathcal{D}^{10}) of depth μ is canonically associated with a family of pairs (\mathfrak{m}_x, J_x) , one per each $x \in M$, formed by:

- a negatively graded Lie algebra $\mathfrak{m}_x = \mathfrak{m}_x^{-\mu} + \mathfrak{m}_x^{-\mu+1} + \dots + \mathfrak{m}_x^{-1}$, called *Tanaka’s symbol at x* , with $\dim \mathfrak{m}_x = T_x M$ and with $(\mathfrak{m}_x^{-1})^{\mathbb{C}} \simeq \mathcal{D}_x^{10} \oplus \overline{\mathcal{D}_x^{10}}$;
- a complex structure $J_x : \mathfrak{m}_x^{-1} \rightarrow \mathfrak{m}_x^{-1}$, whose $(\pm i)$ -eigenspaces \mathfrak{m}_x^{10} and \mathfrak{m}_x^{01} in the complexified space $(\mathfrak{m}_x^{-1})^{\mathbb{C}} \simeq \mathcal{D}_x^{10} \oplus \overline{\mathcal{D}_x^{10}}$ coincide with the spaces \mathcal{D}_x^{10} and $\overline{\mathcal{D}_x^{10}}$, respectively (see §2 for details).

A CR manifold is regular if and only if all pairs (\mathfrak{m}_x, J_x) are isomorphic to the same one, say (\mathfrak{m}, J) . Hence, if the CR manifold is regular, we may say that at each point x it is “osculated” by the CR manifold, given by the simply connected Lie group $G^{\mathfrak{m}}$ with $\text{Lie}(G^{\mathfrak{m}}) = \mathfrak{m}$ equipped with the unique $G^{\mathfrak{m}}$ -invariant distribution $\mathcal{D}^{\mathfrak{m}10} \subset T^{\mathbb{C}}G^{\mathfrak{m}}$ with $\mathcal{D}^{\mathfrak{m}10}|_e = \mathfrak{m}^{10}$.

We remark that, in [2], Beloshapka considered a particular class of embedded totally nondegenerate CR manifolds, called *model surfaces*, and proved that all of them are acted on transitively by a group of CR transformations whose Lie algebra is isomorphic to one of the above described graded Lie algebras $\text{Lie}(G^{\mathfrak{m}}) = \mathfrak{m}$. This immediately implies that, up to coverings, Beloshapka’s model surfaces are nothing but embedded realisations in \mathbb{C}^N of the above described osculating manifolds $G^{\mathfrak{m}}$.

This paper is devoted to the proof of the following (Theorem 3.5): *if $\mu \geq 4$, the Lie algebra of the group $\text{Aut}(G^{\mathfrak{m}})$ of CR automorphisms of the homogenous CR manifold $G^{\mathfrak{m}}$ of depth μ is the graded Lie algebra*

$$\text{Lie}(\text{Aut}(G^{\mathfrak{m}})) = \mathfrak{m} + \mathfrak{g}^0 = \mathfrak{m}^{-\mu} + \dots + \mathfrak{m}^{-1} + \mathfrak{g}^0, \quad (*)$$

where \mathfrak{g}^0 is the Lie algebra of the group of automorphisms of the graded Lie algebra \mathfrak{m} that leave invariant the complex structure $J : \mathfrak{m}^{-1} \rightarrow \mathfrak{m}^{-1}$. Note that if n is the CR dimension, \mathfrak{g}^0 is naturally identifiable with a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

Our main result naturally includes and completes all previous results on the automorphisms of totally nondegenerate CR manifolds of depth $\mu \geq 4$, obtained in diverse papers by Kossovskiĭ and by the first author ([7, 10, 11]; see also [12, 13]). Indeed, our proof has been built on several ideas of the first author and it can be considered as a realisation of a project, which has been outlined in his previous papers.

By Tanaka's theory ([14, 1]), our result has immediate interesting consequences on the geometry of *any* regular totally nondegenerate CR manifold. For instance, it implies that for any n CR dimensional manifold M of this kind, the dimension of the automorphism group is less than or equal to $\dim M + 2n^2$ and any infinitesimal CR transformation is completely determined by its first order jet at some point. An expanded discussion of such geometric outcomes will be given in a future paper.

We also remark that, by the existence of CR equivalences (up to coverings) between the homogeneous CR manifolds G^m and Beloshapka's model surfaces, our result gives also a complete proof for the cases $\mu \geq 4$ of a conjectured property, called 'maximum conjecture' in [3]. There, Beloshapka wrote that the Lie algebra of the automorphism groups of *any* model surface of depth $\mu \geq 3$ was expected to be precisely of the form (*). Since this property has been proved for $\mu = 3$ by Gammel' and Kossovskiĭ in [4], by the results of this paper we may claim that Beloshapka's conjecture is now confirmed in all cases.

The paper is structured as follows. In §2 and §3, we give a short review of all elements of Tanaka's theory of fundamental Lie algebras, prolongations, etc. , which we need for our proof, and we introduce the notion of *universal fundamental CR algebra*. This can be considered as a CR analogue of the notion of free Lie algebra (also called *universal fundamental Lie algebra* in [14]) and allows a very convenient characterisation of Tanaka's symbols of totally nondegenerate CR structures. Section §3 ends with a detailed definition of regular totally nondegenerate CR manifolds and the statement of our main result. In §4 we give a crucial result on linear independent sets in universal fundamental CR algebras and in totally nondegenerate fundamental Lie algebras. The proof of the main theorem is given in §5.

Acknowledgement. After posting our paper on arXiv, we noticed that, independently and practically simultaneously to us, our main result has been proven also by Jan Gregorovič in [5]. The authors are grateful to Ilya Kossovskiĭ for pointing this preprint to us. We are also very grateful to Joël Merker and the referee for careful readings and truly helpful and constructive remarks.

2. PRELIMINARIES

2.1. Fundamental algebras, CR Tanaka algebras and CR manifolds.

2.1.1. *Fundamental algebras and CR Tanaka algebras.* We recall that a *fundamental algebra* of depth μ is a negatively graded Lie algebra over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} of the form

$$\mathfrak{m} := \mathfrak{m}^{-\mu} + \mathfrak{m}^{-\mu+1} + \dots + \mathfrak{m}^{-1}$$

with $[\mathfrak{m}^{-i}, \mathfrak{m}^{-1}] = \mathfrak{m}^{-i-1}$ for each i . Further, denoting by $\text{aut}(\mathfrak{m}) = \text{Lie}(\text{Aut}(\mathfrak{m}))$ the Lie algebra of the group $\text{Aut}(\mathfrak{m})$ of all automorphisms of the graded Lie algebra \mathfrak{m} and given a subalgebra $\mathfrak{g}^0 \subset \text{aut}(\mathfrak{m})$, the *Tanaka algebra* with isotropy \mathfrak{g}^0 is the non-positively graded Lie algebra

$$\mathfrak{m} + \mathfrak{g}^0 = \mathfrak{m}^{-\mu} + \dots + \mathfrak{m}^{-1} + \mathfrak{g}^0.$$

When \mathfrak{m} is real and is equipped with a complex structure $J : \mathfrak{m}^{-1} \rightarrow \mathfrak{m}^{-1}$ on the subspace \mathfrak{m}^{-1} that satisfies the so-called *integrability condition*

$$[JX, JY] = [X, Y] \quad \text{for all } X, Y \in \mathfrak{m}^{-1}, \quad (2.1)$$

the pair (\mathfrak{m}, J) is called *fundamental CR algebra*. The Tanaka algebra $\mathfrak{m} + \mathfrak{g}^0$, with \mathfrak{g}^0 equal to the Lie algebra $\mathfrak{g}^0 = \text{aut}(\mathfrak{m}, J) = \text{Lie}(\text{Aut}(\mathfrak{m}, J))$ of the group $\text{Aut}(\mathfrak{m}, J)$ of J -preserving automorphisms is called *CR Tanaka algebra* determined by (\mathfrak{m}, J) .

2.1.2. Regular CR manifolds of given type and associated Tanaka algebras. Let M be a real n -dimensional manifold. A (integrable) *CR structure* on M is a pair (\mathcal{D}, J) , formed by a distribution $\mathcal{D} \subset TM$ of rank $2k \leq n$ and a smooth family of complex structures $J_x : \mathcal{D}_x \rightarrow \mathcal{D}_x$, $x \in M$, with the property that the complex distribution $\mathcal{D}^{1,0} \subset T^{\mathbb{C}}M$ formed by the $+i$ -eigenspaces of the complex structures $J_x : \mathcal{D}_x \rightarrow \mathcal{D}_x$, is involutive. A diffeomorphism $f : M \rightarrow M$ of a CR manifold is called *CR automorphism* if $f_*(\mathcal{D}) \subset \mathcal{D}$ and $f_*(J) = J$. A vector field $X \in \mathfrak{X}(M)$, whose (local) flow consists of 1-parameter families of (local) CR automorphisms is called an *infinitesimal (local) CR automorphism* of (M, \mathcal{D}, J) . The space of all infinitesimal CR transformations is known to be a Lie algebra and we denote it by $\text{aut}(M, \mathcal{D}, J)$.

We recall that the most relevant examples of CR manifolds are given by real submanifolds of \mathbb{C}^N . More precisely, if we denote by J_{st} the standard complex structure of \mathbb{C}^n , any \mathcal{C}^∞ real submanifold $M \subset \mathbb{C}^N$, for which the family \mathcal{D}^o of tangent subspaces $\mathcal{D}_x^o = \{ v \in T_x M : J_{\text{st}}v = \sqrt{-1}v \in T_x M \}$ is a regular distribution of constant rank $2k$, then the pair $(\mathcal{D}^o, J := J_{\text{st}}|_{\mathcal{D}^o})$ is a CR structure on M . It is called the *induced CR structure* of the regularly embedded submanifold $M \subset \mathbb{C}^N$.

We also remind that if (\mathcal{D}, J) is a CR structure on M , the integer $k = \dim M - \text{rank } \mathcal{D}$ is called *CR codimension* and, for induced CR structures of regularly embedded CR generic submanifolds, this is also equal to the real codimension of the submanifold.

There exists a canonical relation between CR Tanaka algebras and CR manifolds satisfying appropriate regularity conditions. Consider an n -dimensional CR manifold (M, \mathcal{D}, J) . Further, for each (real or complex) distribution \mathcal{K} on M , let us adopt the notational convention of indicating by $\underline{\mathcal{K}}$ the class of all local smooth vector fields taking values of \mathcal{K} . We may now consider the sequence of spaces \mathfrak{D}_{-j} , $1 \leq j$ of vector fields, defined inductively by

$$\mathfrak{D}_{-1} := \underline{\mathcal{D}}, \quad \mathfrak{D}_{-(j+1)} := \mathfrak{D}_{-j} + [\mathfrak{D}_{-1}, \mathfrak{D}_{-j}]. \quad (2.2)$$

If there exists a sequence of constant rank distributions $\mathcal{D}_{-j} \subset TM$ such that $\mathfrak{D}_{-j} = \underline{\mathcal{D}}_{-j}$ for each $j \geq 2$, we say that the distribution \mathcal{D} is *regular* ([14]). In this case, for

each point $x \in M$, we may define

$$\mathfrak{m}^{-1}(x) := \mathcal{D}|_x, \quad \mathfrak{m}^{-(j+1)}(x) := \mathcal{D}_{-(j+1)}|_x / \mathcal{D}_{-j}|_x$$

and, denoting by μ the first integer for which $\mathfrak{m}^{-(\mu+k)} = 0$ for all $k \geq 1$, we set

$$\mathfrak{m}(x) = \mathfrak{m}^{-\mu}(x) + \cdots + \mathfrak{m}^{-1}(x).$$

The usual Lie brackets between vector fields induce natural Lie brackets on the sum of quotient spaces on $\mathfrak{m}(x)$ and makes it a fundamental algebra of depth μ . Further, as a consequence of the integrability of J , one can check that the complex structure $J_x : \mathcal{D}_x = \mathfrak{m}^{-1}(x) \rightarrow \mathcal{D}_x = \mathfrak{m}^{-1}(x)$ makes $(\mathfrak{m}(x), J_x)$ a fundamental CR algebra.

All this allows to consider the following

Definition 2.1. Given a fundamental CR algebra (\mathfrak{m}, J) , a CR manifold (M, \mathcal{D}, J) is called *regular of type (\mathfrak{m}, J)* if

- a) the distribution \mathcal{D} is regular,
- b) all fundamental CR algebras $(\mathfrak{m}(x), J(x))$, $x \in M$, are isomorphic to \mathfrak{m} ,
- c) $\dim \mathfrak{m} = \dim M$.

The CR Tanaka algebra $\mathfrak{m} + \mathfrak{g}^0$, given by (\mathfrak{m}, J) , is called *associated to (M, \mathcal{D}, J)* .

Given a fundamental CR algebra (\mathfrak{m}, J) , let $G^{\mathfrak{m}}$ to be the unique (up to an isomorphism) simply connected Lie group with $\mathfrak{m} = \text{Lie}(G^{\mathfrak{m}})$ and consider the left invariant distribution $\mathcal{D}^{\mathfrak{m}}$, with $\mathcal{D}^{\mathfrak{m}}|_e = \mathfrak{m}^{-1} \subset T_e G^{\mathfrak{m}}$, and the left invariant family of complex structures $J^{\mathfrak{m}}|_g : \mathcal{D}_g^{\mathfrak{m}} \rightarrow \mathcal{D}_g^{\mathfrak{m}}$ for $g \in G^{\mathfrak{m}}$, with $J^{\mathfrak{m}}|_e = J$. The pair $(\mathcal{D}^{\mathfrak{m}}, J^{\mathfrak{m}})$ is directly seen to be an integrable CR structure on $G^{\mathfrak{m}}$ ([14, §10.4]). The homogeneous CR manifold $(G^{\mathfrak{m}}, \mathcal{D}^{\mathfrak{m}}, J^{\mathfrak{m}})$ constructed in this way is called the *standard CR model associated with (\mathfrak{m}, J)* .

We stress the fact that, being $(G^{\mathfrak{m}}, \mathcal{D}^{\mathfrak{m}}, J^{\mathfrak{m}})$ homogeneous, it has a natural structure of real-analytic manifold with a real-analytic CR structure. So, by classical embedding results for analytic CR manifolds, *any such standard CR model admits a local regular CR embedding in \mathbb{C}^N , with $N = \dim_{\mathbb{R}} G^{\mathfrak{m}} - \frac{1}{2} \dim_{\mathbb{R}} \mathcal{D}_o^{\mathfrak{m}}$. The germ of such embedding is unique up to local biholomorphisms* ([14, Props. 10.1 - 10.2]).

If (M, \mathcal{D}, J) is regular and of type (\mathfrak{m}, J) , then the class of all J -equivariant graded isomorphisms $\xi_x : \mathfrak{m}(x) \rightarrow \mathfrak{m}$, $x \in M$, determines a principal bundle $\pi : P^0 \rightarrow M$ over M , with structure group $G^0 := \text{Aut}(\mathfrak{m}, J)$. This bundle is canonical in the sense that it is preserved by any CR automorphism. More precisely, in case $f : M \rightarrow M$ is a diffeomorphism in $\text{Aut}(M, \mathcal{D}, J)$, the corresponding push-forward map $f_* : TM \rightarrow TM$ induces, in a standard way, a Lie algebras isomorphism $f_*|_x : \mathfrak{m}(x) \rightarrow \mathfrak{m}(f(x))$ between the fundamental algebras at the points x and $f(x)$, for each $x \in M$. This allows to consider the map

$$\widehat{f}(\xi_x) := \xi_x \circ (f_*|_x)^{-1} : \mathfrak{m}(f(x)) \rightarrow \mathfrak{m},$$

which transforms each isomorphism $\xi_x : \mathfrak{m}(x) \rightarrow \mathfrak{m}$ of the fiber $P^0|_x$ into a corresponding isomorphism $\widehat{f}(\xi_x) : \mathfrak{m}(f(x)) \rightarrow \mathfrak{m}$ of the fiber $P^0|_{f(x)}$. This defines a principle bundle automorphism $\widehat{f} : P^0 \rightarrow P^0$, which projects onto the diffeomorphism $f : M \rightarrow M$ and it is uniquely determined by such projection. Such

an automorphism \widehat{f} is called *the canonical lift of $f \in \text{Aut}(M, \mathcal{D}, J)$ to P^0* . Taking in consideration flows, the canonical correspondence $f \rightarrow \widehat{f}$ naturally induces a canonical correspondence between the vector fields $X \in \text{aut}(M, \mathcal{D}, J)$ and appropriate vector fields \widehat{X} on P^0 , which we call *canonical lifts* of the infinitesimal CR transformations.

2.2. Prolongations of CR Tanaka algebras and CR geometries of finite order.

2.2.1. *Prolongations of Tanaka algebras.* Given a Tanaka algebra $\mathfrak{m} + \mathfrak{g}^0$, its (full) prolongation is the graded Lie algebra

$$(\mathfrak{m} + \mathfrak{g}^0)^\infty := \mathfrak{m}^{-\mu} + \dots + \mathfrak{m}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 + \dots ,$$

in which the positively graded spaces \mathfrak{g}^ℓ with $1 \leq \ell < \infty$ and the Lie brackets are inductively defined as follows. Given an integer $\ell \geq 1$, assume that all spaces \mathfrak{g}^i , $0 \leq i \leq \ell - 1$, and all brackets $[X, Y]$ between homogeneous elements $X \in \sum_{i=0}^{\ell-1} \mathfrak{g}^i$ and homogeneous element $Y \in \mathfrak{m}$ have been defined. Then, set

$$\begin{aligned} \mathfrak{g}^\ell := \{ X \in \sum_{p=-\mu}^{-1} \text{Hom}(\mathfrak{m}^{-p}, \widetilde{\mathfrak{g}}^{-p+\ell}) : \\ [X(Y), Z] + [Y, X(Z)] = X([Y, Z]) \quad \text{for each } Y, Z \in \mathfrak{m} \} , \end{aligned} \quad (2.3)$$

where $\widetilde{\mathfrak{g}}^i$ stands for \mathfrak{m}^i in case $i \leq -1$, and for \mathfrak{g}^i in case $i \geq 0$. Define also brackets between homogeneous elements of the form $X^\ell \in \mathfrak{g}^\ell$, $Y^{-k} \in \mathfrak{m}^{-k}$ by

$$[X^\ell, Y^{-k}] := X^\ell(Y^{-k}) \in \widetilde{\mathfrak{g}}^{-k+\ell} . \quad (2.4)$$

In [14], Tanaka proved that there exists a unique way to define a graded Lie algebra structure on the direct sum $(\mathfrak{m} + \mathfrak{g}^0)^\infty = \mathfrak{m} + \mathfrak{g}^0 + \sum_{i=1}^{\infty} \mathfrak{g}^i$, whose Lie brackets coincide with the original brackets between pairs in $\mathfrak{m} + \mathfrak{g}^0$ and is equal to (2.4) for each k and ℓ . Note that, by construction, *if there is k_o such that $\mathfrak{g}^{k_o+1} = \{0\}$, then $\mathfrak{g}^{k_o+\ell} = \{0\}$ for all $\ell \geq 1$.*

2.2.2. *Tanaka's towers of CR manifolds and lifts of CR automorphisms.* Let (M, \mathcal{D}, J) be a regular CR manifold of type (\mathfrak{m}, J) . According to Tanaka's theory of differential systems ([14, 1]), there exists a sequence of canonically associated bundles (sometimes called *Tanaka's tower*)

$$\dots \longrightarrow P^{k+1} \xrightarrow{\pi_{k+1}} P^k \xrightarrow{\pi_k} P^{k-1} \xrightarrow{\pi_{k-1}} \dots \longrightarrow P^1 \xrightarrow{\pi_1} P^0 \xrightarrow{\pi} M ,$$

where:

- (i) $\pi : P^0 \rightarrow M$ is the above introduced principal bundle with the structure group $G^0 = \text{Aut}(\mathfrak{m}, J)$;
- (ii) each bundle $\pi_k : P^k \rightarrow P^{k-1}$ is a principal bundle with abelian structure group G^k with $\text{Lie}(G^k) \simeq \mathfrak{g}^k$

and for which the following holds: *If there exists k_o such that $\mathfrak{g}^{k_o+1} = \{0\}$, then:*

- (1) *the CR automorphisms of (M, \mathcal{D}, J) form a Lie group of dimension $\dim \text{Aut}(M, \mathcal{D}, J) \leq \dim P^{k_o} = \dim(\mathfrak{m} + \mathfrak{g}^0 + \dots + \mathfrak{g}^{k_o})$;*

- (2) there exists a canonical map which associates to each $X \in \text{aut}(M, \mathcal{D}, J)$ a unique element \widehat{X}^{k_o} in a certain Lie algebra \mathfrak{p}^{k_o} of vector fields of P^{k_o} , whose flows are local diffeomorphisms preserving an appropriate absolute parallelism on P^{k_o} ; the correspondence $X \mapsto \widehat{X}^{k_o}$ is a Lie algebra isomorphism between $\text{aut}(M, \mathcal{D}, J)$ and \mathfrak{p}^{k_o} and generalises the above described lifting map $X \rightarrow \widehat{X}$, which transforms infinitesimal CR automorphisms X and vector fields on P^0 ;
- (3) an element $Y \in \mathfrak{p}^{k_o} (\simeq \text{aut}(M, \mathcal{D}, J))$ is identically vanishing if and only if it vanishes at a single point $y \in P^{k_o}$.

If there is $k_o \geq 0$ such that $\mathfrak{g}^{k_o+1} = \{0\}$ holds, the CR structures of type (\mathfrak{m}, J) are called *of finite order* and the integer k_o is called the *order* of their geometry.

A (locally) homogeneous CR manifold (M, \mathcal{D}, J) of type (\mathfrak{m}, J) ⁽¹⁾ and with geometry of finite order k_o is called a *maximally homogeneous model* (resp. *maximally homogeneous local model*) of type (\mathfrak{m}, J) if $\text{Aut}(M, \mathcal{D}, J)$ (resp. $\text{aut}(M, \mathcal{D}, J)$) has dimension equal to the dimension of $\mathfrak{m} + \mathfrak{g}^0 + \dots + \mathfrak{g}^{k_o}$.

By Prop. 10.7 in [14], if a fundamental CR algebra (\mathfrak{m}, J) corresponds to CR structures of order k_o , then *the Lie algebra of infinitesimal CR transformation $\text{aut}(G^{\mathfrak{m}}, \mathcal{D}^{\mathfrak{m}}, J^{\mathfrak{m}})$ of the standard CR model $G^{\mathfrak{m}}$ is isomorphic to $\mathfrak{m} + \mathfrak{g}^0 + \dots + \mathfrak{g}^{k_o}$. In particular, any standard CR model associated with structures of finite order is a maximally homogeneous local model for that class of geometric structures.* Note that, under nondegeneracy conditions, a converse of this property have been proved in [8]. We also remark that, as it was pointed out above, each such standard model can be realised as a regularly embedded submanifold of some \mathbb{C}^N .

The most familiar examples of CR manifolds of finite order are probably the real hypersurfaces M of \mathbb{C}^N with strictly positive Levi forms: their geometry is of order $k_o = 2$ and the unit sphere $S^{2N-1} = \text{SU}_N/\text{U}_{N-1} \times \mathbb{Z}_2$ is a maximally homogenous model.

We finally remark that, due to the above properties (2) and (3) of Tanaka's towers, if the geometry of a CR manifold has order k_o , then an infinitesimal CR transformation X of such manifold is zero if and only if the corresponding lifted vector field X^{k_o} is equal to 0 at some point $y \in P^{k_o}$ (hence, equal to 0 at all points of P^{k_o}).

2.2.3. CR geometries of order 0 and first order jets of automorphisms. In the very particular case of a CR manifold (M, \mathcal{D}, J) with CR geometry of order $k_o = 0$, the Lie algebra isomorphism between $\text{aut}(M, \mathcal{D}, J)$ and the Lie algebra \mathfrak{p}^0 described in the above point (2) of Tanaka's towers is nothing but the canonical lifting map $X \rightarrow \widehat{X}$ from infinitesimal CR automorphisms and vector fields on P^0 , as described in §2.1.2. From this and the above property (3), it follows that two infinitesimal CR automorphisms X, X' coincide if and only if the lifted map $(\widehat{X - X'})$ vanishes at just one point $y \in P^0$ (and, hence, at all point of P^0). An explicit check shows that this occurs if and only if there exists $x \in M$ such that the first order jet $j^1(X)|_x$ coincides with $j^1(X')|_x$.

¹By "locally homogeneous CR manifold" we mean that the Lie algebra $\text{aut}(M, \mathcal{D}, J)$ of infinitesimal CR transformations generates local actions that are transitive on open sets of the manifold.

Summing up, we have that *for manifolds with CR geometries of order 0, the infinitesimal automorphisms are uniquely determined by their first order jets at a single point.*

This property has immediate counterparts for higher order CR geometries, as for instance the Levi nondegenerate real hypersurfaces. For such different kind of CR manifolds, the infinitesimal automorphisms are uniquely determined by jets of orders strictly larger than one. Due to this, if one looks for analogies and/or differences between 0-th order CR geometries and higher order ones, the former looks like with much fewer degrees of freedom than the latter. Such phenomenon is sometimes mentioned as the *rigidity* feature of 0-th order geometries (see e.g. [4]).

2.3. Complexifications of CR Tanaka algebras and prolongations.

2.3.1. Complexifications of Tanaka algebras. Let \mathfrak{m} be a *real* fundamental algebra and denote by $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} + i\mathfrak{m}$ its associated complexified fundamental algebra. Note that the real subspace $\mathfrak{m} \subset \mathfrak{m}^{\mathbb{C}}$ can be characterised as the fixed point set $(\mathfrak{m}^{\mathbb{C}})^{\tau}$ of an appropriate anti-involution τ of $\mathfrak{m}^{\mathbb{C}}$, namely of the conjugation $\tau(X + iY) := \overline{X + iY} = X - iY$. This yields to the following efficient characterisation of \mathbb{C} -linear extensions of real linear maps, whose proof is basically a straightforward consequence of the definitions:

(*) *A map $L \in \text{Hom}_{\mathbb{C}}(\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}})$ is the \mathbb{C} -linear extension of an \mathbb{R} -linear map in $\text{Hom}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{m})$ if and only if $L(W) = L(\overline{W})$ for all $W = X + iY \in \mathfrak{m}^{\mathbb{C}}$.*

Consider now a Tanaka algebra $\mathfrak{m} + \mathfrak{g}^0$ over \mathbb{R} . In this case the natural map $\iota : \text{Hom}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{m}) \rightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}})$, sending each \mathbb{R} -linear map L into its unique \mathbb{C} -linear extension, induces a natural injective Lie algebras homomorphism

$$j^0 := \iota|_{\mathfrak{g}^0} : \mathfrak{g}^0 \subset \text{aut}(\mathfrak{m}) \longrightarrow \text{aut}(\mathfrak{m}^{\mathbb{C}}).$$

The complex subalgebra $\text{Span}_{\mathbb{C}}(j^0(\mathfrak{g}^0)) = \mathfrak{g}^0 + ij^0(\mathfrak{g}^0)$ of $\text{aut}(\mathfrak{m}^{\mathbb{C}})$ is clearly isomorphic to $(\mathfrak{g}^0)^{\mathbb{C}}$, thus, for simplicity of notation, we constantly identify those two. Note that the *complex* vector space $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}}$ has a natural structure of *complex* Tanaka structure, associated with the fundamental algebra $\mathfrak{m}^{\mathbb{C}}$, and we call it *complexification* of $\mathfrak{m} + \mathfrak{g}^0$.

The natural injections $\mathfrak{m} \hookrightarrow \mathfrak{m}^{\mathbb{C}}$, $\mathfrak{g}^0 \hookrightarrow (\mathfrak{g}^0)^{\mathbb{C}}$, $\mathfrak{g}^i \hookrightarrow ((\mathfrak{g}^0)^{\mathbb{C}})^i$, $i \geq 1$ (the latter determined by taking \mathbb{C} -linear extensions of the homomorphisms in the \mathfrak{g}^i) combine and determine an injective Lie algebras homomorphism

$$j^{\infty} : (\mathfrak{m} + \mathfrak{g}^0)^{\infty} \longrightarrow (\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}})^{\infty}.$$

In this paper, we are particularly interested in the injection

$$j^1 := j^{\infty}|_{\mathfrak{g}^1} : \mathfrak{g}^1 \subset (\mathfrak{m} + \mathfrak{g}^0)^{\infty} \hookrightarrow \widehat{\mathfrak{g}}^1 := ((\mathfrak{g}^0)^{\mathbb{C}})^1 \subset (\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}})^{\infty}.$$

The following technical lemma gives a characterisation of the image of j^1 .

Lemma 2.2. *The subspace $j^1(\mathfrak{g}^1)$ of $\widehat{\mathfrak{g}}^1$ consists of the \mathbb{C} -linear maps $L \in \text{Hom}_{\mathbb{C}}(\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}})$ satisfying (2.3) for $\ell = 1$ and such that*

$$\overline{L(W)(Z)} = L(\overline{W})(\overline{Z}) \quad \text{for all } W, Z \in \mathfrak{m}^{\mathbb{C}}. \quad (2.5)$$

Proof. Note that a map $L \in \widehat{\mathfrak{g}}^1$ is the \mathbb{C} -linear extension of a linear map in $\mathfrak{g}^1 \subset \text{Hom}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{m} + \mathfrak{g}^0)$ if and only if it satisfies the following two conditions:

- a) The restriction $L|_{\mathfrak{m}}$ maps any element $X \in \mathfrak{m}^{-1}$ in $j^0(\mathfrak{g}^0) \subset (\mathfrak{g}^0)^{\mathbb{C}}$;
- b) it satisfies the condition

$$[L(Y^{-1}), Z^{-p}] + [Y^{-1}, L(Z^{-p})] = L([Y^{-1}, Z^{-p}])$$

for any $Y^{-1} \in \mathfrak{m}^{-1}$, $Z^{-p} \in \mathfrak{m}^{-p}$, $-p = -\mu, \dots, -1$.

On the first hand, by definition of the space $\widehat{\mathfrak{g}}^1$ and the fact that \mathfrak{m} is generated by \mathfrak{m}^{-1} , we have that (a) implies (b), which can be consequently neglected. On the other hand, the elements of $j^0(\mathfrak{g}^0)$ (i.e. the \mathbb{C} -linear extensions of elements in \mathfrak{g}^0) are characterised by the above property (*). Hence (a) holds if and only if

$$\overline{L(X)(Z)} = L(X)(\overline{Z}) \quad \text{for each } X \in \mathfrak{m}, Z \in \mathfrak{m}^{\mathbb{C}}. \quad (2.6)$$

This is equivalent to say that, for all $W = X + iY$, $Z \in \mathfrak{m}^{\mathbb{C}}$,

$$\overline{L(W)(Z)} = \overline{L(X)(Z)} + \overline{L(iY)(Z)} = \overline{L(X)(Z)} - i\overline{L(Y)(Z)} \stackrel{(2.6)}{=} L(\overline{W})(\overline{Z}). \quad \square$$

2.3.2. Complexifications of CR Tanaka algebras. Let (\mathfrak{m}, J) be a fundamental CR algebra and denote by $\mathfrak{m} + \mathfrak{g}^0$, $\mathfrak{g}^0 = \text{aut}(\mathfrak{m}, J)$, the associated CR Tanaka algebra. Using the notation of previous subsection, we call *complexified fundamental CR algebra* the pair $(\mathfrak{m}^{\mathbb{C}}, J)$ and we call *complexified CR Tanaka algebra of (\mathfrak{m}, J)* the complexified Tanaka algebra $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}}$.

We recall that the \mathbb{C} -linear extension of the linear map $J : \mathfrak{m}^{-1} \rightarrow \mathfrak{m}^{-1}$ on the complexification $(\mathfrak{m}^{-1})^{\mathbb{C}}$ has exactly two eigenvalues, $+i$ and $-i$, and two associated eigenspaces, \mathfrak{m}^{10} , $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$, called *holomorphic* and *anti-holomorphic* subspaces. We also remind that, conversely, for each direct sum decomposition $(\mathfrak{m}^{-1})^{\mathbb{C}} = \mathfrak{m}' + \mathfrak{m}''$ with $\mathfrak{m}'' = \overline{\mathfrak{m}'}$, there exists a unique complex structure J on \mathfrak{m}^{-1} for which the holomorphic and anti-holomorphic subspaces are precisely $\mathfrak{m}^{10} = \mathfrak{m}'$ and $\mathfrak{m}^{01} = \mathfrak{m}''$. Finally, we observe that the complex structure $J : \mathfrak{m}^{-1} \rightarrow \mathfrak{m}^{-1}$ satisfies the integrability condition (2.1) if and only if the corresponding holomorphic and anti-holomorphic subspaces $\mathfrak{m}^{10}, \mathfrak{m}^{01} \subset \mathfrak{m}^{\mathbb{C}}$ satisfy the conditions

$$[\mathfrak{m}^{10}, \mathfrak{m}^{10}] = 0 \quad \text{and} \quad [\mathfrak{m}^{01}, \mathfrak{m}^{01}] \left(= \overline{[\mathfrak{m}^{10}, \mathfrak{m}^{10}]} \right) = 0. \quad (2.7)$$

Due to this, we have a natural one-to-one correspondence between fundamental CR algebras (\mathfrak{m}, J) and pairs $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{10})$, formed by a complexified fundamental algebra $\mathfrak{m}^{\mathbb{C}}$ and a subspace $\mathfrak{m}^{10} \subset (\mathfrak{m}^{-1})^{\mathbb{C}}$, satisfying (2.7) and such that the direct sum decomposition $(\mathfrak{m}^{-1})^{\mathbb{C}} = \mathfrak{m}^{10} + \overline{\mathfrak{m}^{10}}$ holds.

From this and characterisation (*) of \mathbb{C} -linear extensions, we get that the \mathbb{C} -linear extensions of the derivations $L \in \text{aut}(\mathfrak{m}, J)$ are determined by the following

Lemma 2.3. *An element $L \in \text{aut}(\mathfrak{m}^{\mathbb{C}})$ is the \mathbb{C} -linear extension of an element in $\mathfrak{g}^0 = \text{aut}(\mathfrak{m}, J)$ if and only if $\overline{L(X)} = L(\overline{X})$ for any $X \in \mathfrak{m}^{\mathbb{C}}$ and $L(\mathfrak{m}^{10}) \subset \mathfrak{m}^{10}$.*

3. TOTALLY NONDEGENERATE CR MANIFOLDS

3.1. Universal fundamental CR algebras. Given a (real or complex) n -dimensional vector space V and an integer $\mu \geq 1$, the following result by Tanaka gives a unified presentation of all fundamental algebras \mathfrak{n} of depth μ for which $\mathfrak{n}^{-1} \simeq V$.

Theorem 3.1. [14, §3] *For any finite-dimensional vector space V over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and any integer $\mu \geq 1$, there exists a fundamental algebra \mathfrak{U}_V of depth μ , unique up to isomorphisms, satisfying the following two properties:*

- i) $\mathfrak{U}_V^{-1} = V$;
- ii) *for any fundamental algebra \mathfrak{n} of depth μ with $\mathfrak{n}^{-1} \simeq V$, there exists a natural surjective Lie algebra homomorphism $\varphi_{(\mathfrak{n})} : \mathfrak{U}_V \rightarrow \mathfrak{n}$, so that $\mathfrak{n} \simeq \mathfrak{U}_V/\mathfrak{i}$, with $\mathfrak{i} = \ker \varphi_{(\mathfrak{n})}$.*

In this statement, the ‘‘naturalness’’ of the homomorphism $\varphi_{(\mathfrak{n})}$ has the following meaning: if $\psi : \mathfrak{n} \rightarrow \mathfrak{n}'$ is a Lie algebra homomorphism between two fundamental algebras of same depth and same space of degree -1 , and if $\varphi_{(\mathfrak{n})} : \mathfrak{U}_V \rightarrow \mathfrak{n}$ and $\varphi_{(\mathfrak{n}')} : \mathfrak{U}_V \rightarrow \mathfrak{n}'$ are the associated surjective homomorphisms, then $\varphi_{(\mathfrak{n}')} = \psi \circ \varphi_{(\mathfrak{n})}$ up to composition with isomorphisms.

The graded Lie algebra \mathfrak{U}_V of Theorem 3.1 is called *universal fundamental* (or *free*) *Lie algebra of depth μ generated by V* . An explicit construction of \mathfrak{U}_V is given in Tanaka’s proof of Theorem 3.1 in [14, §3] (see also [6, 9, 15]).

Consider now a *real* vector space V with a complex structure $J : V \rightarrow V$ and let $V^{\mathbb{C}} = V^{10} + V^{01}$ be the natural decomposition of $V^{\mathbb{C}}$ as the direct sum of J -eigenspaces. Further, given $\mu \geq 1$, consider the *complex* universal fundamental algebra $\mathfrak{U}_{V^{\mathbb{C}}}$ of depth μ generated by $V^{\mathbb{C}}$ and denote by $\mathfrak{i}_{10}, \mathfrak{i}_{01}$ the ideals of $\mathfrak{U}_{V^{\mathbb{C}}}$, generated by the -2 -degree spaces $[V^{10}, V^{10}]$, $[V^{01}, V^{01}]$, respectively. Finally, let \mathfrak{U}_J be the quotient algebra

$$\mathfrak{U}_J := \mathfrak{U}_{V^{\mathbb{C}}}/(\mathfrak{i}_{10} + \mathfrak{i}_{01}), \quad (3.8)$$

which, by construction, is a complex fundamental algebra.

Following the same steps of the proof in [14] of Theorem 3.1, the following CR analogue to that result can be immediately derived.

Proposition 3.2. *Given a $2n$ -dimensional real vector space V with complex structure J and an integer $\mu \geq 1$, the algebra \mathfrak{U}_J defined in (3.8) is a complexified fundamental CR algebra with $((\mathfrak{m}^{-1})^{\mathbb{C}}, J) = (V^{\mathbb{C}}, J)$.*

Moreover, for any other complexification $\mathfrak{m}'^{\mathbb{C}}$ of a fundamental CR algebra \mathfrak{m}' of depth μ with $((\mathfrak{m}'^{-1})^{\mathbb{C}}, J') \simeq (V^{\mathbb{C}}, J)$, there is a natural surjective Lie algebra homomorphism $\varphi_{(\mathfrak{m}'^{\mathbb{C}})} : \mathfrak{U}_J \rightarrow \mathfrak{m}'^{\mathbb{C}}$, so that $\mathfrak{m}'^{\mathbb{C}}$ is isomorphic to the quotient $\mathfrak{U}_J/\mathfrak{i}$ with $\mathfrak{i} := \ker \varphi_{(\mathfrak{m}'^{\mathbb{C}})}$.

Motivated by this, we call \mathfrak{U}_J the *universal fundamental CR algebra of depth μ generated by $(V^{\mathbb{C}}, J)$* . Note that, since any complexified fundamental CR algebra $\mathfrak{m}^{\mathbb{C}}$ with $(\mathfrak{m}^{\mathbb{C}})^{-1} = V^{\mathbb{C}}$ is isomorphic to a quotient of \mathfrak{U}_J by an ideal $\mathfrak{i} \subset \sum_{i \geq 2} \mathfrak{U}_J^{-i}$, we have that \mathfrak{U}_J can be characterised as *the complexified fundamental CR algebra*,

reaching the maximal values for the dimensions $n_k = \dim_{\mathbb{C}} \mathfrak{U}_J^{-k}$, $k \geq 2$, amongst all complexified fundamental CR algebras having the same complex space $V^{\mathbb{C}} := \mathfrak{U}_J^{-1}$ as subspace of degree -1 .

3.2. Totally nondegenerate CR manifolds. We are now ready to give the definition of the Lie algebras, which are the objects of study of this paper. Let $\mathfrak{m} + \mathfrak{g}^0$ be a CR Tanaka algebra of depth μ , with complex structure $J : \mathfrak{m}^{-1} \rightarrow \mathfrak{m}^{-1}$, and denote by $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}}$ the corresponding complexified CR Tanaka algebra.

Definition 3.3. A (real) CR Tanaka algebra $\mathfrak{m} + \mathfrak{g}^0$ of depth $\mu \geq 3$ is called *totally nondegenerate* if the complexification of \mathfrak{m} has the form $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J / \mathfrak{i}^{-\mu}$ for an ideal $\mathfrak{i}^{-\mu}$, entirely included in the lowest degree subspace $\mathfrak{U}_J^{-\mu}$ of the universal fundamental CR algebra \mathfrak{U}_J .

By the above characterisation of the universal fundamental CR algebras, the totally nondegenerate CR Tanaka algebras can be also described as the fundamental CR algebras, whose complexified algebras $\mathfrak{m}^{\mathbb{C}}$ reach the maximal values for the dimensions $n_k = \dim_{\mathbb{C}} (\mathfrak{m}^{\mathbb{C}})^{-k}$ for all $k \geq 2$ with the only possible exception of $k = \mu$: in fact, the subspace $(\mathfrak{m}^{\mathbb{C}})^{-\mu}$ of lowest degree might be a non-trivial quotient of \mathfrak{U}_J^{-k} .

Note that, by this property, if $\mathfrak{m} + \mathfrak{g}^0$ is totally nondegenerate, then for each fixed $X_o^{-1} \in \mathfrak{m}^{-1}$, all linear maps

$$\Lambda^{(k)} : \mathfrak{m}^{-k} \rightarrow \mathfrak{m}^{-k-1}, \quad \Lambda^{(k)}(Y^{-k}) := [X_o^{-1}, Y^{-k}], \quad 1 \leq k \leq \mu - 1, \quad (3.9)$$

have maximal possible rank or, equivalently, trivial kernels. This can be considered as a generalisation of the classical *Levi nondegeneracy condition* for CR structures and motivates the name “totally nondegenerate” for such fundamental CR algebras.

In correspondence with totally nondegenerate CR algebras we have the following

Definition 3.4. A regular CR manifold (M, \mathcal{D}, J) of type (\mathfrak{m}, J) is called *totally nondegenerate* if the associated Tanaka CR algebra $\mathfrak{m} + \mathfrak{g}^0$ is totally nondegenerate.

As we mentioned in the Introduction, in [2] Beloshapka introduced a special class of embedded CR manifolds in \mathbb{C}^N , called *model surfaces*, each of them being totally nondegenerate. He proved that, for each model surface M , the Lie algebra $\text{aut}(M, \mathcal{D}, J)$ of infinitesimal automorphisms surely includes a subalgebra, which generates a locally transitive action on M and is isomorphic to the fundamental Lie algebra \mathfrak{m} of such homogenous CR manifold. This means that the group $G^{\mathfrak{m}}$, which acts simply transitively on the standard CR model (see §2.1.2), has a simply transitive action on the (universal covering space of the) model surface with fundamental Lie algebra \mathfrak{m} . By a classical property of Lie groups, for each fixed $o \in M$ and each linear map $\varphi : T_e G^{\mathfrak{m}} (\simeq \mathfrak{m}) \rightarrow T_o M$, there exists a $G^{\mathfrak{m}}$ -equivariant covering map $f^{(\varphi)} : G^{\mathfrak{m}} \rightarrow M$ such that $f_*^{(\varphi)}|_e = \varphi$. Hence, if φ is a linear isomorphism between \mathfrak{m} and $T_o M$ that gives also an isomorphism between the CR structures at $e \in G^{\mathfrak{m}}$ and $o \in M$, respectively, using the $G^{\mathfrak{m}}$ -equivariance of $f^{(\varphi)}$, one can directly see that $f^{(\varphi)}$ maps the whole CR structure of $G^{\mathfrak{m}}$ onto the CR structure M . This means that, up to a covering, $G^{\mathfrak{m}}$ and the corresponding Beloshapka’s model surface M are CR equivalent.

We conclude observing that, by Corollary 3 in [14, §11], the geometry of totally nondegenerate CR manifolds is of finite order. This immediately implies that for any Beloshapka's model (M, \mathcal{D}, J) , the Lie algebra $\text{aut}(M, \mathcal{D}, J)$ of infinitesimal CR automorphisms is isomorphic to the prolongation $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ of the Tanaka algebra $\mathfrak{m} + \mathfrak{g}^0$, $\mathfrak{g}^0 = \text{aut}(\mathfrak{m}, J)$ ([14, Prop.10.7]).

3.3. Statement of the main result.

Theorem 3.5. *For any totally nondegenerate CR algebra (\mathfrak{m}, J) of depth $\mu > 3$, the space \mathfrak{g}^1 in the prolongation $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ of its Tanaka algebra is trivial. Consequently, the CR geometry of any totally nondegenerate CR manifolds is of order 0.*

As an immediate corollary of this result, by the remarks in §3.2, we have that for any Beloshapka's model surface of depth $\mu \geq 4$ (which we already observed to be locally CR equivalent to a standard model $(G^m, \mathcal{D}^m, J^m)$) the Lie algebra of infinitesimal CR automorphisms is isomorphic to $\mathfrak{m} + \mathfrak{g}^0$ with $\mathfrak{g}^0 = \text{aut}(\mathfrak{m}, J)$. This proves the so-called ‘‘maximum conjecture’’ of [3] for all cases with $\mu \geq 4$.

4. LINEAR INDEPENDENT SETS IN TOTALLY NONDEGENERATE CR ALGEBRAS

Let \mathfrak{U}_V be the universal fundamental algebra of depth μ , generated by the n -dimensional vector space $V = \mathfrak{U}_V^{-1}$. For the sake of simplicity, in this section we call *degree* of a homogeneous element of \mathfrak{U}_V the *opposite* of the integer which gives its degree as element of the negatively graded Lie \mathfrak{U}_V . So, for instance, in this section the elements in \mathfrak{U}_V^{-1} are said to be of *degree* $+1$, those in \mathfrak{U}_V^{-2} are of *degree* $+2$, etc.

4.1. Hall bases adapted to universal fundamental CR algebras.

4.1.1. *Hall bases of universal fundamental Lie algebras.* Given a (ordered) basis $\mathcal{B} = (E_1, \dots, E_n)$ for V , we call \mathcal{B} -*monomials of degree* 1 the elements $E_i \in \mathcal{B} \subset \mathfrak{U}_V^{-1}$ and, for $k \geq 2$, we call \mathcal{B} -*monomials of degree* k all elements $X \in \mathfrak{U}_V^{-k}$ that are equal to Lie brackets $X = [Y, Z]$ between two \mathcal{B} -monomials of lower degree. We may also say that the \mathcal{B} -monomials are the homogeneous elements of \mathfrak{U}_V that can be obtained by taking iterated Lie brackets between elements of the form $\text{ad}_{E_{i_1}} \circ \text{ad}_{E_{i_2}} \circ \dots \circ \text{ad}_{E_{i_{r-1}}}(E_{i_r})$. This latter kind of \mathcal{B} -monomials are called *elementary \mathcal{B} -monomials*.

A *Hall basis* $\mathcal{H}^{\mathcal{B}}$ associated with \mathcal{B} is an (ordered) basis for \mathfrak{U}_V , containing the \mathcal{B} -monomials selected as follows. For each degree $k \geq 1$, let us denote by $\mathcal{H}^{\mathcal{B}(k)}$ the subset of $\mathcal{H}^{\mathcal{B}}$ consisting of all its elements of degree k and denote by \prec the total order relation between such elements, which makes it an *ordered* basis. The sets $\mathcal{H}^{\mathcal{B}(k)}$ and the order relation are defined inductively by the following steps.

Step 1. The subset $\mathcal{H}^{\mathcal{B}(1)}$ coincides with the set of all \mathcal{B} -monomials of degree 1, i.e. all elements of $\mathcal{B} \subset \mathfrak{U}_V^{-1}$. The order relation \prec between elements in $\mathcal{H}^{\mathcal{B}(1)}$ is set equal to the order of \mathcal{B} .

Step k_o . Suppose that all sets $\mathcal{H}^{\mathcal{B}(\ell)}$, $1 \leq \ell \leq k_o - 1$, have been defined and that the order \prec between elements in the union $\bigcup_{1 \leq \ell \leq k_o - 1} \mathcal{H}^{\mathcal{B}(\ell)}$ has been fixed in such a way that if $\text{degree}(X) < \text{degree}(Y)$, then $X \prec Y$. Then the set $\mathcal{H}^{\mathcal{B}(k_o)}$ is defined as the collection of all \mathcal{B} -monomials of degree k_o of the form $[U, V]$ with $U, V \in \mathcal{H}^{\mathcal{B}}$ and of degree lower than k_o , satisfying the following two conditions: (i) $V \prec U$; (ii) if $U = [U_1, U_2]$ for some $U_i \in \mathcal{H}^{\mathcal{B}}$, then $U_2 \preceq V$. The ordering relation is also extended to the larger set $\bigcup_{1 \leq \ell \leq k_o} \mathcal{H}^{\mathcal{B}(\ell)}$ by setting: (a) for all $Y \in \mathcal{H}^{\mathcal{B}(k_o)}$ and all X of degree less than k_o , it is set $X \prec Y$; (b) if X, Y are distinct, but both in $\mathcal{H}^{\mathcal{B}(k_o)}$, then it is set either $X \prec Y$ or $X \succ Y$, according to an arbitrarily chosen order on the set of \mathcal{B} -monomials of degree k_o .

The ordered set $\mathcal{H}^{\mathcal{B}} = \bigcup_{1 \leq \ell \leq \mu} \mathcal{H}^{\mathcal{B}(\ell)}$ is proven to be a basis for \mathfrak{U}_V . The proof of this, given in [6], is based on the properties of an explicit procedure for each $\mathbb{X} \in \mathfrak{U}_V$, which transforms an expansion $\mathbb{X} = \sum_{i=1}^N a_i X_i$ in terms of \mathcal{B} -monomials X_i into a possibly new expansion $\mathbb{X} = \sum_{\ell=1}^{N'} a'_\ell X'_\ell$ in terms of \mathcal{B} -monomials, all belonging to $\mathcal{H}^{\mathcal{B}}$. Such final expansion is proven to be independent from the original expansion $\mathbb{X} = \sum_{i=1}^N a_i X_i$ of \mathbb{X} .

For our purposes, it is convenient to shortly review such a procedure. It is defined by induction on the degree k of \mathbb{X} . For $k = 1$, it simply consists in leaving the expansion $\mathbb{X} = \sum_{i=1}^N a_i X_i$ unchanged. Indeed, all \mathcal{B} -monomials of degree 1 are in $\mathcal{H}^{\mathcal{B}}$ and there is no need of changing the expansion. Assume now that the procedure has been explicitly determined for each elements of degree $1 \leq k \leq k_o$ and assume $\text{degree}(\mathbb{X}) = k_o + 1$. In this case the final expansion $\mathbb{X} = \sum_{\ell=1}^{N'} a'_\ell X'_\ell$ is reached by iteratively applying the following three steps:

First Step. Since each \mathcal{B} -monomial X_i has degree $k_o + 1 \geq 2$, it has the form $[U_i, V_i]$ for some monomials U_i, V_i of lower degree and for which the previously defined procedure allows to uniquely express them as $U_i := \sum_j a_{ij} U_j$ and $V_i := \sum_k b_{ik} U_k$ for some elements $U_j, U_k \in \mathcal{H}^{\mathcal{B}}$. In particular, we may re-write \mathbb{X} in the form $\mathbb{X} = \sum_{i,j,k} a_{ij} b_{ik} [U_j, U_k]$ with all $U_j, U_k \in \mathcal{H}^{\mathcal{B}}$.

Second Step. In the expansion $\mathbb{X} = \sum_{i,j,k} a_{ij} b_{ik} [U_j, U_k]$, replace each \mathcal{B} -monomial $[U_j, U_k]$ with $U_j \prec U_k$ by the expression $-[U_k, U_j]$.

Third Step. In the expansion $\mathbb{X} = \sum_{i,j,k} a'_{ij} b'_{ik} [U_j, U_k]$ which is reached after *Second Step*, consider all terms $[U_j, U_k]$ for which U_j has the form $U_j := [U'_m, U'_n]$ for some $U'_m, U'_n \in \mathcal{H}^{\mathcal{B}}$; in case $U'_n \preceq U_k$, the \mathcal{B} -monomial $[U_j, U_k] = [[U'_m, U'_n], U_k]$ is left unchanged, otherwise it is replaced by the expression $[[U'_m, U_k], U'_n] - [[U'_n, U_k], U'_m]$.

As we already mentioned, by applying First Step, Second Step and Third Step and then First Step, Second Step and Third Step again and so on, after a finite number of iterations, one reaches an expansion $\mathbb{X} = \sum_{\ell=1}^{N'} X'_\ell$, in which all X'_ℓ are in $\mathcal{H}^{\mathcal{B}}$ and which is stable under each of the above three steps (see [6] for a detailed proof of this property).

4.1.2. *Hall bases adapted to a universal fundamental CR algebra.* Let now $(V^{\mathbb{C}}, J)$ be the complexification of a real vector space V , with complex structure J , and $\mathfrak{U}_J = \mathfrak{U}_{V^{\mathbb{C}}}/\mathfrak{i}$, $\mathfrak{i} := \mathfrak{i}_{10} + \mathfrak{i}_{01}$, the universal CR fundamental algebra generated by $(V^{\mathbb{C}}, J)$. We recall that \mathfrak{i}_{10} and \mathfrak{i}_{01} are the ideals of $\mathfrak{U}_{V^{\mathbb{C}}}$ generated by the subspaces $[V^{10}, V^{10}]$ and $[V^{01}, V^{01}]$, respectively.

A basis $\mathcal{B} = (E_i)_{1 \leq i \leq 2n}$ for $V^{\mathbb{C}} = \mathfrak{U}_{V^{\mathbb{C}}}^{-1}$ is called *adapted to the ideal* \mathfrak{i}_{10} (resp. *to the ideal* \mathfrak{i}_{01}) if the first n elements E_1, \dots, E_n are all in V^{10} (resp. all in V^{01}). A basis \mathcal{B} for $V^{\mathbb{C}}$ is called *compatible with* \mathfrak{i} if there are two new orders on \mathcal{B} , the first making \mathcal{B} adapted to \mathfrak{i}_{10} , the second making it adapted to \mathfrak{i}_{01} .

Lemma 4.1. *If a basis $\mathcal{B} = (E_i)$ for $V^{\mathbb{C}} = \mathfrak{U}_{V^{\mathbb{C}}}^{-1}$ is adapted to \mathfrak{i}_{10} , then for any associated Hall basis $\mathcal{H}^{\mathcal{B}}$ of $\mathfrak{U}_{V^{\mathbb{C}}}$, one has that $\mathcal{H}^{\mathcal{B}(10)} := \mathcal{H}^{\mathcal{B}} \cap \mathfrak{i}_{10}$ is a basis, as a vector space, for the ideal \mathfrak{i}_{10} . A similar property holds for bases adapted to \mathfrak{i}_{01} .*

Proof. Let us denote by $(A_j)_{j=1}^N$ the (ordered) set of linearly independent elements in $\mathcal{H}^{\mathcal{B}(10)} := \mathcal{H}^{\mathcal{B}} \cap \mathfrak{i}_{10}$. We just need to show that, for each $2 \leq k \leq \mu$, the subset $\mathcal{H}^{\mathcal{B}(10|k)} := \mathcal{H}^{\mathcal{B}(10)} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-k}$ of elements A_j of degree k is actually a basis for the intersection $\mathfrak{i}_{10}^{-k} := \mathfrak{i}_{10} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-k}$. We prove this by induction on k .

For $k = 2$, the claim is true because the set $\mathcal{H}^{\mathcal{B}(10|2)}$ is not only linearly independent (it is a subset of a Hall basis) but it also contains all generators $\{[E_i, E_j], 1 \leq j < i \leq n\}$ of \mathfrak{i}_{10} . Suppose now that the claim has been proved for $2 \leq k \leq k_o$ and let us show that it holds also for $k = k_o + 1$. A non-trivial element $\mathbb{X} \in \mathfrak{i}_{10}^{-(k_o+1)}$ has the form $\mathbb{X} = \sum_s [Y_s, Z_s]$ for some $Y_s \in \mathfrak{i}_{10}$ and $Z_s \in \mathfrak{U}_{V^{\mathbb{C}}}$. The degrees of the Y_s and Z_s are less than $k_o + 1$. By the inductive hypothesis, each Y_s has the form $Y_s = \lambda_s^j A_j$ for monomials $A_j \in \bigcup_{\ell \geq 2}^{k_o} \mathcal{H}^{\mathcal{B}(10|\ell)}$, while each Z_s has the generic form $Z_s = \mu_s^r A_r + \nu_s^t B_t$ with $A_r \in \mathcal{H}^{\mathcal{B}(10)}$ and $B_t \in \mathcal{H}^{\mathcal{B}} \setminus \mathcal{H}^{\mathcal{B}(10)}$. Thus

$$\mathbb{X} = \sum_s \lambda_s^j \mu_s^r [A_j, A_r] + \sum_s \lambda_s^j \nu_s^t [A_j, B_t]. \quad (4.1)$$

The elements $[A_j, A_r], [A_j, B_t]$ appearing in (4.1) are all \mathcal{B} -monomials and all in \mathfrak{i}_{10} , but not necessarily they are all in $\mathcal{H}^{\mathcal{B}}$. On the other hand, we know that applying a finite number of times the sequence of the *First, Second and Third Steps* of Hall's procedure described above, one can pass from (4.1) to an equivalent expansion, in which appear only \mathcal{B} -monomials in $\mathcal{H}^{\mathcal{B}}$. We claim that

After each application of the triple "Second Step \rightarrow Third Step \rightarrow First Step" the expansion of \mathbb{X} is still a sum of \mathcal{B} -monomials of the form $[A_i, A_j], [A_i, B_j], [B_j, A_i]$.

To see this, let us start with a linear combination \mathbb{X} of \mathcal{B} -monomials of the form $[A_i, A_j], [A_i, B_j], [B_j, A_i]$ (as for instance, in (4.1)). The Second Step of Hall's algorithm can only possibly exchange A_i with A_j in each monomial of the form $[A_i, A_j]$ or exchange A_i with B_j in all others. Hence, after such step, the form of the expansion is as claimed. A more careful analysis is needed for the Third Step, in which one has to replace terms of the form $[[X, Y], Z]$, for some $[X, Y] = A_r$ and $Z = A_m, B_n$, or $[X, Y] = B_s$ and $Z = A_m$, in which $Z \prec Y$. We recall that these terms must be replaced by the equivalent sum $-[[Y, Z], X] + [X, Z], Y]$.

We want to show that, whenever such a replacement is needed, *each summand in $-[[Y, Z], X] + [X, Z], Y]$ is always equal to a Lie bracket $[W_1, W_2]$, in which either W_1 or W_2 is in \mathfrak{i}_{10} .*

To see this, consider at first the situations in which $Z = A_m \in \mathcal{H}^{\mathcal{B}(10)}$: in such cases, both brackets $[Y, Z]$ and $[X, Z]$ are in \mathfrak{i}_{10} and the desired property is true. Second, consider the case in which $Z = B_m \in \mathcal{H}^{\mathcal{B}} \setminus \mathcal{H}^{\mathcal{B}(10)}$ and, consequently, $[X, Y]$ is equal to some $A_r \in \mathcal{H}^{\mathcal{B}(10)}$ of degree $\tilde{k} \leq k_o$. Recall that we are assuming that $Z = B_m \prec Y$. In case $\tilde{k} = 2$, the relation $Z = B_m \prec Y$ might occur only if $k_o + 1 = 3$ and all three elements X, Y, B_m have degree 1. Since we are assuming that $A_r = [X, Y]$ is in $\mathfrak{i}_{10} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-2}$, this means that X, Y are amongst the first n elements of \mathcal{B} , i.e., in V^{10} , and $B_m \prec Y$ implies that also B_m is in V^{10} . We thus have that also $[X, B_m], [Y, B_m]$ are in \mathfrak{i}_{10} and the desired claim holds. It remains to consider the cases $3 \leq \tilde{k} \leq k_o$. For those, we recall that, by the iterative algorithm of construction of the monomials of the Hall basis, the bracket $[X, Y] = A_r$ is either a monomial of the form $[X, Y] = [A_m, A_n]$ or a monomial of the form $[X, Y] = [A_m, B_n]$ (or $[B_n, A_m]$). In both cases, each summand of $-[[Y, Z], X] + [X, Z], Y]$ has the desired form: indeed, the first has the form $[W_1, W_2]$ with $W_2 = X = A_m \in \mathfrak{i}_{10}$, while the second has the form $[W_1, W_2]$ with $W_1 = [X, Z] = [A_m, Z] \in \mathfrak{i}_{10}$ (for the alternative cases in which $[X, Y] = [B_n, A_m]$, one needs only to exchange the role of X with the one of Y).

Now that we know that, after the Third Step, we have a linear combination of brackets $[W_1, W_2]$ in which at least one W_i is in \mathfrak{i}_{10} , we can newly perform the First Step (i.e. express all elements W_1 and W_2 in terms of the elements of the Hall Basis) and obtain a new expansion of X , which is once again a linear combination of elements of the form $[A_i, A_j], [A_i, B_j]$ or $[B_j, A_i]$ as claimed. This concludes the proof of the claim.

As we mentioned above, after a finite number of iterations of these steps, the expansion of $\mathbb{X} \in \mathfrak{i}_{10}^{-(k_o+1)}$ in terms of the elements of the Hall basis $\mathcal{H}^{\mathcal{B}}$ is reached. But, by the above claim, such final expansion consists of elements of the form $[A_i, A_j], [A_i, B_j], [B_j, A_i]$, thus in \mathfrak{i}_{10} . So, $\mathfrak{i}_{10}^{-(k_o+1)}$ is generated by $\mathcal{H}^{\mathcal{B}(10|k_o+1)}$, which is therefore a basis for $\mathfrak{i}_{10}^{-(k_o+1)}$. \square

4.2. \mathcal{B} -types and decompositions of universal CR fundamental algebras. Let $\mathfrak{U}_{V^{\mathbb{C}}}$ be a universal fundamental algebra of depth μ and $\mathcal{B} = (E_i)_{i=1, \dots, 2n}$ a basis for $V^{\mathbb{C}} = \mathfrak{U}_{V^{\mathbb{C}}}^{-1}$. To each \mathcal{B} -monomial X , we assign inductively an element of \mathbb{N}^{2n} as follows. If X is a \mathcal{B} -monomial of degree 1, i.e. a vector $X = E_i$ for some $1 \leq i \leq 2n$, we define \mathcal{B} -type of X the $2n$ -tuple

$$\mathcal{B}\text{-type}(E_i) = (0, \dots, 0, \underset{i\text{-th entry}}{1}, 0, \dots, 0).$$

Suppose now that the type has been defined for all \mathcal{B} -monomials of degree less than or equal to $k_o \geq 1$. Then if X is a \mathcal{B} -monomial of degree $k_o + 1$, thus of the form $X = [Y, Z]$ for \mathcal{B} -monomials of lower degree, we set

$$\mathcal{B}\text{-type}(X) = \mathcal{B}\text{-type}(Y) + \mathcal{B}\text{-type}(Z).$$

Roughly speaking, the \mathcal{B} -type is the $2n$ -tuple, whose i -th entry gives the number of times the corresponding vector E_i is used in the construction of the \mathcal{B} -monomial. Due to this property, if two bases $\mathcal{B}, \mathcal{B}'$ have the same elements, but differ by their order, a \mathcal{B} -monomial X is also a \mathcal{B}' -monomial and the \mathcal{B} -type and \mathcal{B}' -type differ only by an appropriate permutation of the entries.

Given $I = (I_1, \dots, I_{2n}) \in \mathbb{N}^{2n}$, we denote $|I| = \sum_{j=1}^{2n} I_j$. In this way, if \mathcal{B} -type(X) = I , then we have $\text{degree}(X) = |I|$.

For any $I \in \mathbb{N}^{2n}$ with $0 < |I| \leq \mu$, we define

$$\mathfrak{U}_{V^{\mathbb{C}}}^{-I} := \text{Span}_{\mathbb{C}}\{ \text{monomials } X \in \mathfrak{U}_{V^{\mathbb{C}}} : \mathcal{B}\text{-type}(X) = I \}.$$

By the above described property of types, two bases $\mathcal{B}, \mathcal{B}'$ of $V^{\mathbb{C}}$, which differ only by the order of their elements, determine the same subspaces $\mathfrak{U}_{V^{\mathbb{C}}}^{-I}$, even if such space might be associated with two distinct $2n$ -tuples I and I' of natural numbers (differing only by a permutation of the entries). Note also that, for each $1 \leq k \leq \mu$, the vector subspace $\mathfrak{U}_{V^{\mathbb{C}}}^{-k}$ admits the direct sum decomposition

$$\mathfrak{U}_{V^{\mathbb{C}}}^{-k} = \bigoplus_{|I|=k} \mathfrak{U}_{V^{\mathbb{C}}}^{-I}.$$

Lemma 4.2. *Assume that the basis $\mathcal{B} = (E_i)$ for $V^{\mathbb{C}}$ is compatible with the ideal $\mathfrak{i} = \mathfrak{i}_{10} + \mathfrak{i}_{01}$ (see §4.1.2 for the definition) and, for each $I \in \mathbb{N}^{2n}$ with $|I| \leq \mu$, let*

$$\mathfrak{i}_{10}^{-I} := \mathfrak{i}_{10} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-I}, \quad \mathfrak{i}_{01}^{-I} := \mathfrak{i}_{01} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-I}.$$

Then, for each $1 \leq k \leq \mu$ and each $2n$ -tuple $I_o \in \mathbb{N}^{2n}$, we have

$$\mathfrak{i}_{10} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-k} = \bigoplus_{|I|=k} \mathfrak{i}_{10}^{-I}, \quad \mathfrak{i}_{01} \cap \mathfrak{U}_{V^{\mathbb{C}}}^{-k} = \bigoplus_{|I|=k} \mathfrak{i}_{01}^{-I} \quad (4.2)$$

$$\mathfrak{U}_{V^{\mathbb{C}}}^{-I_o} / \mathfrak{i} \simeq \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o} / \left(\mathfrak{i}_{10}^{-I_o} + \mathfrak{i}_{01}^{-I_o} \right). \quad (4.3)$$

In particular, the universal CR fundamental algebra $\mathfrak{U}_J = \mathfrak{U}_{V^{\mathbb{C}}} / \mathfrak{i}$ admits the following direct sum decomposition into vector subspaces

$$\mathfrak{U}_J \simeq \bigoplus_{k=1}^{\mu} \left(\bigoplus_{|I|=k} \mathfrak{U}_{V^{\mathbb{C}}}^{-I} / \left(\mathfrak{i}_{10}^{-I} + \mathfrak{i}_{01}^{-I} \right) \right). \quad (4.4)$$

Proof. Let $\mathcal{B}', \mathcal{B}''$ be two ordered bases for $V^{\mathbb{C}}$, made with the elements in \mathcal{B} and ordered so that \mathcal{B}' is adapted to \mathfrak{i}_{10} and \mathcal{B}'' is adapted to \mathfrak{i}_{01} . Let also $\mathcal{H}^{\mathcal{B}'}$ and $\mathcal{H}^{\mathcal{B}''}$ be two associated Hall bases, with the corresponding subsets $\mathcal{H}^{\mathcal{B}'(10)}, \mathcal{H}^{\mathcal{B}''(01)}$, which are bases for \mathfrak{i}^{10} and \mathfrak{i}^{01} by Lemma 4.1. Finally, for each \mathcal{B}' -type I' , denote by $\mathcal{H}^{\mathcal{B}'(10)I'}$ the subcollections of $\mathcal{H}^{\mathcal{B}'(10)}$ consisting of \mathcal{B}' -monomials of \mathcal{B}' -type I' . Similarly, denote by $\mathcal{H}^{\mathcal{B}''(01)I''}$ the subcollections of $\mathcal{H}^{\mathcal{B}''(01)}$ consisting of \mathcal{B}'' -monomials of \mathcal{B}'' -type I'' . Due to the fact that the basis $\mathcal{H}^{\mathcal{B}'(10)}$ of \mathfrak{i}_{10} (resp. $\mathcal{H}^{\mathcal{B}''(01)}$ of \mathfrak{i}_{01}) coincides with the union of the subsets $\mathcal{H}^{\mathcal{B}'(10)I'}$ (resp. $\mathcal{H}^{\mathcal{B}''(01)I''}$), the direct sum decompositions (4.2) follow immediately.

To prove (4.3), let us denote by I'_o and I''_o the \mathcal{B}' -type and \mathcal{B}'' -type, respectively, of the monomials which have \mathcal{B} -type I_o . We recall that I'_o and I''_o can be derived from I_o by appropriate permutations of the entries and that $\mathfrak{U}_{V^{\mathbb{C}}}^{-I'_o} = \mathfrak{U}_{V^{\mathbb{C}}}^{-I''_o} = \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}$. So, using the direct sum decomposition (4.2), we get the isomorphisms

$$\mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{10} \simeq \mathfrak{U}_{V^{\mathbb{C}}}^{-I'_o}/\mathfrak{i}_{10}^{-I'_o} = \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{10}^{-I_o}, \quad \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{01} \simeq \mathfrak{U}_{V^{\mathbb{C}}}^{-I''_o}/\mathfrak{i}_{01}^{-I''_o} = \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{01}^{-I_o}.$$

By standard arguments, this implies

$$\begin{aligned} \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i} &= \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/(\mathfrak{i}_{10} + \mathfrak{i}_{01}) \simeq \\ &\simeq (\mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{10})/(\mathfrak{i}_{01}/\mathfrak{i}_{10}) \simeq (\mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{10}^{-I_o})/(\mathfrak{i}_{01}/\mathfrak{i}_{10}^{-I_o}) \simeq \\ &\simeq \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/(\mathfrak{i}_{10}^{-I_o} + \mathfrak{i}_{01}) \simeq (\mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{01})/(\mathfrak{i}_{10}^{-I_o}/\mathfrak{i}_{01}) \simeq \\ &\simeq (\mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/\mathfrak{i}_{01}^{-I_o})/(\mathfrak{i}_{10}^{-I_o}/\mathfrak{i}_{01}^{-I_o}) \simeq \mathfrak{U}_{V^{\mathbb{C}}}^{-I_o}/(\mathfrak{i}_{10}^{-I_o} + \mathfrak{i}_{01}^{-I_o}), \end{aligned}$$

i.e. (4.3).

The last claim is a direct consequence of (4.3). In fact, the isomorphisms (4.3) allow to construct an isomorphism between each space $\mathfrak{U}_{V^{\mathbb{C}}}^{-k}/\mathfrak{i} \subset \mathfrak{U}_J$ and the direct sum of all quotients $\mathfrak{U}_{V^{\mathbb{C}}}^{-I}/(\mathfrak{i}_{10}^{-I} + \mathfrak{i}_{01}^{-I})$ with $|I| = k$. \square

This lemma yields to the following

Corollary 4.3. *Let \mathcal{B} be a basis for $V^{\mathbb{C}}$, which is compatible with $\mathfrak{i} = \mathfrak{i}_{10} + \mathfrak{i}_{01}$. If $X_1, \dots, X_\ell \in \mathfrak{U}_{V^{\mathbb{C}}}$ are \mathcal{B} -monomials, whose \mathcal{B} -types are pairwise distinct and none of them is in \mathfrak{i} , the classes $\widehat{X}_i = X_i + \mathfrak{i}$ in $\mathfrak{U}_J = \mathfrak{U}_{V^{\mathbb{C}}}/\mathfrak{i}$ constitute a linearly independent set in \mathfrak{U}_J .*

Proof. For each i , let I_i be the \mathcal{B} -type of X_i . Since the types I_i are pairwise distinct, the classes \widehat{X}_i belong to the distinct subspaces $\mathfrak{U}_{V^{\mathbb{C}}}^{-I_i}/(\mathfrak{i}_{10}^{-I_i} + \mathfrak{i}_{01}^{-I_i})$ and are therefore linearly independent, because of the direct sum decomposition (4.4). \square

4.3. Linear independent sets in a totally nondegenerate CR algebra. We recall that a (real) CR Tanaka algebra

$$\mathfrak{m} + \mathfrak{g}^0 = \mathfrak{m}^{-\mu} + \mathfrak{m}^{-\mu+1} + \dots + \mathfrak{m}^{-1} + \mathfrak{g}^0$$

with complex structure J on \mathfrak{m}^{-1} is called *totally nondegenerate* if and only if $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J/\mathfrak{i}^{-\mu}$ for some ideal $\mathfrak{i}^{-\mu}$, which is entirely included in the subspace $\mathfrak{U}_J^{-\mu}$ of the universal fundamental CR algebra \mathfrak{U}_J of depth μ . This means that each homogeneous element of $\mathfrak{m}^{\mathbb{C}}$ of degree strictly less than μ can be identified with an element of the universal algebra \mathfrak{U}_J , while the elements in $(\mathfrak{m}^{-\mu})^{\mathbb{C}}$ are identifiable with equivalence classes of $\mathfrak{U}_J^{-\mu}/\mathfrak{i}^{-\mu}$.

On the other hand, the elements of the universal CR fundamental algebra \mathfrak{U}_J are in turn classes of the quotient $\mathfrak{U}_{V^{\mathbb{C}}}/(\mathfrak{i}_{10} + \mathfrak{i}_{01})$.

Since many of the following arguments are based on properties of the elements of the algebra $\mathfrak{U}_J = \mathfrak{U}_{V^{\mathbb{C}}}/(\mathfrak{i}_{10} + \mathfrak{i}_{01})$ and on the corresponding elements in $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J/\mathfrak{i}^{-\mu}$, for the sake of clarity, it is crucial to fix a notational convention, which allows to easily indicate if an element is in \mathfrak{U}_J or $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J/\mathfrak{i}^{-\mu}$. So, from now on,

- the elements of \mathfrak{U}_J are denoted by capital roman letters as X, Y, Z, \dots ,

- the corresponding images in $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J/\mathfrak{i}^{-\mu}$ are indicated by $X^{\mathfrak{m}}, Y^{\mathfrak{m}}$, etc.

By previous remarks, a homogeneous element $X \in \mathfrak{U}_J$ can be different from the corresponding $X^{\mathfrak{m}} \in \mathfrak{m}^{\mathbb{C}}$ if and only if $X \in \mathfrak{U}_J^{-\mu}$. Due to this, the elements of $\sum_{k=1}^{\mu-1} (\mathfrak{m}^{-k})^{\mathbb{C}}$ will be often tacitly identified with the corresponding elements in \mathfrak{U}_J and hence denoted by capital roman letters as X, Y, Z, \dots . The above special notation $X^{\mathfrak{m}}, Y^{\mathfrak{m}}, \dots$, for elements in $\mathfrak{m}^{\mathbb{C}}$ is going to be used very rarely, namely only when elements in $(\mathfrak{m}^{-\mu})^{\mathbb{C}}$ are considered and it is important to prevent misunderstandings.

Corollary 4.3 has the following immediate consequence

Corollary 4.4. *Let $\mathcal{B} = (E_i)$ be a basis for $V^{\mathbb{C}} = (\mathfrak{m}^{-1})^{\mathbb{C}} = \mathfrak{U}_{V^{\mathbb{C}}}^{-1}$, which is compatible with $\mathfrak{i} = \mathfrak{i}_{10} + \mathfrak{i}_{01}$. If $\tilde{X}_1, \dots, \tilde{X}_\ell \in \mathfrak{U}_{V^{\mathbb{C}}}$ are non-zero \mathcal{B} -monomials of degree strictly less than μ , with pairwise distinct \mathcal{B} -types and none of them in \mathfrak{i} , the corresponding projected elements $X_i (= X_i^{\mathfrak{m}}) \in \mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_{V^{\mathbb{C}}}/(\mathfrak{i}_{10} + \mathfrak{i}_{01} + \mathfrak{i}^{-\mu}) = \mathfrak{U}_J/\mathfrak{i}^{-\mu}$ constitute a linearly independent set in $\mathfrak{m}^{\mathbb{C}}$.*

5. PROOF OF THEOREM 3.5

5.1. Notational issues. Let $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}} = \mathfrak{U}_J/\mathfrak{i}^{-\mu} + (\mathfrak{g}^0)^{\mathbb{C}}$ be the complexification of a totally nondegenerate CR algebra. As above, we denote by X, Y, \dots elements in \mathfrak{U}_J and we tacitly identify them with the corresponding elements in $\mathfrak{m}^{\mathbb{C}}$ whenever they belong to $\sum_{k=1}^{\mu-1} \mathfrak{U}_J^{-k}$. Otherwise, their equivalence classes in $\mathfrak{m}^{\mathbb{C}}$ are denoted by $X^{\mathfrak{m}}, Y^{\mathfrak{m}}$, etc. We point out that any element in \mathfrak{g}^1 will be also constantly and tacitly identified with the corresponding element, obtained by \mathbb{C} -linearity extension, in the subspace $\mathfrak{j}(\mathfrak{g}^1) \subset \widehat{\mathfrak{g}}^1$ of the prolongation of the complexification $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}}$.

Given $X, Y \in \mathfrak{m}^{\mathbb{C}}$ and $L \in \mathfrak{g}^1 (= \mathfrak{j}(\mathfrak{g}^1))$, we set:

- $X^r \cdot Y := (\text{ad}_X)^r(Y)$ for each integer $r \geq 0$ (we assume $(\text{ad}_X)^0(Y) := Y$);
- $L_X := [L, X]$;
- $L_{X|Y} := [[L, X], Y]$.

According to such convention, for any $L \in \mathfrak{g}^1$, $E \in (\mathfrak{m}^{-1})^{\mathbb{C}}$ and $W \in (\mathfrak{m}^{-k})^{\mathbb{C}}$,

$$L_E \in (\mathfrak{g}^0)^{\mathbb{C}}, \quad L_{E|W} \in (\mathfrak{m}^{-k})^{\mathbb{C}}, \quad L_{E^r \cdot W} \in (\mathfrak{m}^{-k-r+1})^{\mathbb{C}}.$$

5.2. A basic property of the elements $L \in \mathfrak{g}^1$. Recall that, by the integrability condition on J and Lemma 2.3, for each $L \in \mathfrak{g}^1$ and for any $E, F \in \mathfrak{m}^{10}$ (so that their conjugate elements $\overline{E}, \overline{F}$ are in \mathfrak{m}^{01})

$$[E, F] = 0 \quad \text{and} \quad [\overline{E}, \overline{F}] = 0, \quad (5.5)$$

$$L_{E|F} \in \mathfrak{m}^{10} \quad \text{and} \quad L_{\overline{E}|\overline{F}} \in \mathfrak{m}^{10}, \quad (5.6)$$

$$L_{E|\overline{F}} = \overline{L_{\overline{E}|F}} \in \mathfrak{m}^{01} \quad \text{and} \quad L_{\overline{E}|F} = \overline{L_{E|\overline{F}}} \in \mathfrak{m}^{01}. \quad (5.7)$$

Using these basic identities, we get the next useful lemma, which can be considered as a CRversion of [15, Lemma 1].

Lemma 5.1. *Given $E \in \mathfrak{m}^{10}$ and $L \in \mathfrak{g}^1$, for any other element $W \in \mathfrak{m}^{\mathbb{C}}$*

$$L_{E \cdot W} = L_{E|W} - L_{W|E}, \quad (5.8)$$

and, more generally, for $r \geq 2$,

$$L_{E^r \cdot W} = E^{r-1} \cdot (rL_{E|W} - L_{W|E}) + \frac{r(r-1)}{2} E^{r-2} \cdot (L_{E|E} \cdot W). \quad (5.9)$$

Proof. We first observe that, from (5.6) and (5.5)

$$[\text{ad}_{L_{E|E}}, \text{ad}_E] = \text{ad}_{[L_{E|E}, E]} = 0, \quad (5.10)$$

meaning that the adjoint operators $\text{ad}_{L_{E|E}}$ and ad_E commute. Using this, we claim that for any $\ell \geq 2$,

$$[L_E, E^\ell \cdot W] = \ell E^{\ell-1} \cdot (L_{E|E} \cdot W) + E^\ell \cdot L_{E|W}. \quad (5.11)$$

Indeed, when $\ell = 2$ this is true because

$$\begin{aligned} [L_E, [E, [E, W]]] &\stackrel{\text{Jacobi id.}}{=} L_{E|E} \cdot (E \cdot W) + E \cdot ([L_E, [E, W]]) \stackrel{(5.10) \ \& \ \text{Jacobi id.}}{=} \\ &= E \cdot (L_{E|E} \cdot W) + E \cdot (L_{E|E} \cdot W) + E^2 \cdot L_{E|W}. \end{aligned}$$

Assume now that (5.11) has been proved for $2 \leq \ell \leq \ell_o - 1$. Then by (5.10), since $\text{ad}_{L_{E|E}}$ and ad_E commute, by the inductive hypothesis,

$$\begin{aligned} [L_E, E^{\ell_o} \cdot W] &\stackrel{\text{Jacobi id.}}{=} L_{E|E} \cdot (E^{\ell_o-1} \cdot W) + E \cdot [L_E, E^{\ell_o-1} \cdot W] = \\ &= E^{\ell_o-1} \cdot (L_{E|E} \cdot W) + (\ell_o - 1) E^{\ell_o-1} \cdot (L_{E|E} \cdot W) + E^{\ell_o} \cdot L_{E|W} = \\ &= \ell_o E^{\ell_o-1} \cdot (L_{E|E} \cdot W) + E^{\ell_o} \cdot L_{E|W}. \end{aligned}$$

We can now prove the lemma. We first observe that (5.8) is nothing but the Jacobi identity. For (5.9), the case $r = 2$ is obtained by iterated uses of Jacobi identity. Indeed,

$$\begin{aligned} L_{E^2 \cdot W} &= [L, [E, [E, W]]] = [[L, E], [E, W]] + [E, [L, [E, W]]] = \\ &= [[[L, E], E], W] + [E, [[L, E], W]] + E \cdot L_{E \cdot W} \stackrel{(5.8)}{=} \\ &= 2E \cdot L_{E|W} - E \cdot L_{W|E} + L_{E|E} \cdot W. \end{aligned}$$

Assume now that (5.9) holds for $2 \leq r \leq r_o - 1$. Then, by (5.11) and induction

$$\begin{aligned} L_{E^{r_o} \cdot W} &= [L_E, E^{r_o-1} \cdot W] + E \cdot (L_{E^{r_o-1} \cdot W}) = \\ &= (r_o - 1) E^{r_o-2} \cdot (L_{E|E} \cdot W) + E^{r_o-1} \cdot L_{E|W} + E^{r_o-1} \cdot ((r_o - 1) L_{E|W} - L_{W|E}) + \\ &\quad + \frac{(r_o - 1)(r_o - 2)}{2} E^{r_o-2} \cdot (L_{E|E} \cdot W), \end{aligned}$$

which gives (5.9) for $r = r_o$. \square

5.3. A finer analysis of the properties of the elements $L \in \mathfrak{g}^1$.

Lemma 5.2. *If $\mu \geq 4$, then for each $L \in \mathfrak{g}^1$ and each $E \in \mathfrak{m}^{10}$, there exists $\lambda^{(E)} \in \mathbb{C}$ such that $L_{E|E} = \lambda^{(E)}E$.*

Proof. Without loss of generality, we can assume that $E \neq 0$. Let ρ be an eigenvalue of the \mathbb{C} -linear map $L_E|_{\mathfrak{m}^{01}} : \mathfrak{m}^{01} \rightarrow \mathfrak{m}^{01}$ and \overline{F}_o a ρ -eigenvector. Since $E^\mu \cdot \overline{F}_o$ has degree $\mu + 1$ and the depth of \mathfrak{m} is μ , we have that $E^\mu \cdot \overline{F}_o = 0$. Consequently, setting $\widehat{E} := L_{E|E}$, from (5.5), (5.6), (5.9) and the hypothesis $L_{E|\overline{F}_o} = \rho\overline{F}_o$,

$$0 = L_{E^\mu \cdot \overline{F}_o} = \rho\mu E^{\mu-1} \cdot \overline{F}_o + \frac{\mu(\mu-1)}{2} E^{\mu-2} \cdot (\widehat{E} \cdot \overline{F}_o). \quad (5.12)$$

Applying L to both sides and, once again, using (5.5), (5.6), (5.9) and $L_{E|\overline{F}_o} = \rho\overline{F}_o$ we get

$$\begin{aligned} 0 &= L_{(L_{E^\mu \cdot \overline{F}_o})} = \\ &= \rho^2 \mu(\mu-1) E^{\mu-2} \cdot \overline{F}_o + \rho \frac{\mu(\mu-1)(\mu-2)}{2} E^{\mu-3} \cdot (\widehat{E} \cdot \overline{F}_o) + \\ &+ \frac{\mu(\mu-1)(\mu-2)}{2} E^{\mu-3} \cdot L_{E|(\widehat{E} \cdot \overline{F}_o)} - \frac{\mu(\mu-1)}{2} E^{\mu-3} \cdot L_{(\widehat{E} \cdot \overline{F}_o)|E} + \\ &+ \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{4} E^{\mu-4} \cdot (\widehat{E} \cdot (\widehat{E} \cdot \overline{F}_o)) = \\ &= \rho^2 \mu(\mu-1) E^{\mu-2} \cdot \overline{F}_o + \rho \frac{\mu(\mu-1)(\mu-2)}{2} E^{\mu-3} \cdot (\widehat{E} \cdot \overline{F}_o) + \\ &+ \frac{\mu(\mu-1)(\mu-2)}{2} E^{\mu-3} \cdot (L_{E|\widehat{E}} \cdot \overline{F}_o) + \frac{\mu(\mu-1)(\mu-2)}{2} E^{\mu-3} \cdot (\widehat{E} \cdot L_{E|\overline{F}_o}) + \\ &+ \frac{\mu(\mu-1)}{2} E^{\mu-2} \cdot L_{\widehat{E} \cdot \overline{F}_o} + \\ &+ \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{4} E^{\mu-4} \cdot (\widehat{E} \cdot (\widehat{E} \cdot \overline{F}_o)). \quad (5.13) \end{aligned}$$

Suppose now that $\widehat{E} = L_{E|E}$ is not a multiple of E . We may therefore consider a basis $\mathcal{B} = (E_i, \overline{F}_j)$ for \mathfrak{m}^{-1} , in which: (a) (E_i) is a basis for \mathfrak{m}^{10} with $E_1 := E$, $E_2 = \widehat{E}$, (b) (\overline{F}_j) is a basis for \mathfrak{m}^{01} with $F_1 := \overline{F}_o$. Identifying the elements of this basis with the corresponding elements in $\mathfrak{U}_{\mathbb{V}\mathbb{C}}^{-1} \simeq (\mathfrak{m}^{-1})^{\mathbb{C}}$, we have that \mathcal{B} is compatible with the ideal $\mathfrak{i} = \mathfrak{i}_{10} + \mathfrak{i}_{01}$ (see §4.1.2). Moreover, since $\mu \geq 4$, the last summand in the right hand side of (5.13) is a non-trivial \mathcal{B} -monomial with \mathcal{B} -type

$$(\mu - 4, 2, 0, \dots, 0, 1, 0, \dots, 0).$$

On the other hand, expanding all other summands in (5.13) in terms of \mathcal{B} -monomials, we see that all terms in such expansion are \mathcal{B} -monomials of \mathcal{B} -type of the form

$$(\mu - 2, *, * \dots, *) \quad \text{or} \quad (\mu - 3, *, * \dots, *).$$

Hence, by Corollary 4.4, the last summand cannot be equal to a linear combination of the other summands and this contradicts (5.13). \square

Remark 5.3. Note that the statement of Lemma 5.2 holds also when $\mu = 3$ and either $\dim_{\mathbb{C}} \mathfrak{m}^{10} = 1$ or $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J$. Indeed, if $\dim_{\mathbb{C}} \mathfrak{m}^{10} = 1$, then $L_{E|E}$ is proportional to E simply because \mathfrak{m}^{10} admits no more than one linearly independent vector. In case $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J$, that is if $\mathfrak{i}^{-\mu} = \{0\}$, then (5.12) is satisfied in $\mathfrak{m}^{\mathbb{C}}$ if and only if it is satisfied in $\mathfrak{U}_J^{-\mu}$. By the same arguments of the lemma, based on comparison of \mathcal{B} -types of the summands, (5.12) is satisfied in \mathfrak{U}_J if and only if $\widehat{E} = L_{E|E} = -\frac{\rho(\mu-1)}{2}E$.

In all the following, we constantly and tacitly assume that $\mathfrak{m} + \mathfrak{g}^0$ is a totally nondegenerate CR Tanaka algebra of depth $\mu \geq 4$, so that Lemma 5.2 surely applies to such Tanaka algebra. However, note that, due to Remark 5.3, all results of this section are valid also in case the depth is $\mu = 3$ and either $\dim \mathfrak{m}^{10} = 1$ or $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_J$.

Lemma 5.4. For any $L \in \mathfrak{g}^1$, the map

$$\alpha_L : \mathfrak{m}^{10} \rightarrow \mathbb{C}, \quad \alpha_L(E) := \lambda^{(E)},$$

with $\lambda^{(E)}$ as in Lemma 5.2, is linear. Moreover, for any pair $E, E' \in \mathfrak{m}^{10}$

$$L_{E|E'} = \frac{1}{2}\alpha_L(E')E + \frac{1}{2}\alpha_L(E)E'. \quad (5.14)$$

Proof. Given $a \in \mathbb{C}$ and $E \in \mathfrak{m}^{10}$,

$$\lambda^{(aE)}aE = L_{aE|aE} = a^2L_{E|E} = a^2\lambda^{(E)}E.$$

This means that $\lambda^{(aE)} = a\lambda^{(E)}$ for each $a \in \mathbb{C}$. Thus, in order to prove that $\lambda^{(E)}$ is linear in E , it remains to prove that $\lambda^{(E+E')} = \lambda^{(E)} + \lambda^{(E')}$ for any pair of linearly independent vectors $E, E' \in \mathfrak{m}^{10}$. For this, we first note that, by (5.8) and (5.5) $0 = L_{E \cdot E'} = L_{E|E'} - L_{E'|E}$, meaning that $L_{\cdot|\cdot}$ is symmetric in its arguments if both of them are in \mathfrak{m}^{10} . Moreover,

$$\lambda^{(E+E')}(E + E') = L_{E+E'|E+E'} = \lambda^{(E)}E + 2L_{E|E'} + \lambda^{(E')}E',$$

which implies that

$$L_{E|E'} = \frac{1}{2}(\lambda^{(E+E')} - \lambda^{(E)})E + \frac{1}{2}(\lambda^{(E+E')} - \lambda^{(E')})E'. \quad (5.15)$$

Now, by bilinearity of $L_{\cdot|\cdot}$, we observe that

$$\begin{aligned} \frac{1}{2}(\lambda^{(2E+E')} - \lambda^{(2E)})2E + \frac{1}{2}(\lambda^{(2E+E')} - \lambda^{(E')})E' &= \\ &= L_{2E|E'} = L_{E|2E'} = \\ &= \frac{1}{2}(\lambda^{(E+2E')} - \lambda^{(E)})E + \frac{1}{2}(\lambda^{(E+2E')} - \lambda^{(2E')})2E'. \end{aligned}$$

Since E and E' are linearly independent and $\lambda^{(2E)} = 2\lambda^{(E)}$, we infer that

$$\begin{aligned} 2\lambda^{(2E+E')} - 3\lambda^{(E)} - \lambda^{(E+2E')} &= 0, \\ \lambda^{(2E+E')} - 2\lambda^{(E+2E')} + 3\lambda^{(E')} &= 0. \end{aligned}$$

Subtracting twice the second from the first equation, we get $-3\lambda^{(E)} + 3\lambda^{(E+2E')} - 6\lambda^{(E')} = 0$, i.e. $\lambda^{(E+2E')} = \lambda^{(E)} + \lambda^{(2E')}$, which proves the linearity of α_L . The last claim is a direct consequence of (5.15) and linearity of α_L . \square

Lemma 5.5. *Let $L \in \mathfrak{g}^1$ with associated linear map $\alpha = \alpha_L \in (\mathfrak{m}^{10})^*$. Then, for each $0 \neq E \in \mathfrak{m}^{10}$ and $0 \neq \overline{F} \in \mathfrak{m}^{01}$,*

$$(\text{ad}_{L_E})^2(\overline{F}) = -\alpha(E) \frac{2\mu - 3}{2} \text{ad}_{L_E}(\overline{F}) - \alpha(E)^2 \frac{(\mu - 1)(\mu - 2)}{4} \overline{F}. \quad (5.16)$$

Proof. Given \overline{F} , let ν be the first integer such that $E^\nu \cdot \overline{F} = 0$. By definition of totally nondegenerate CR algebras, ν is either $\mu - 1$ or μ . We also set $\overline{F}' := L_{E|\overline{F}} = \text{ad}_{L_E}(\overline{F})$ and, as usual, we identify $E, \overline{F}, \overline{F}'$ with the corresponding elements in $\mathfrak{U}_{V^{\mathbb{C}}}^{-1} = (\mathfrak{m}^{-1})^{\mathbb{C}} = V^{\mathbb{C}}$. Finally, we denote by $\mathcal{B} = (E_i, \overline{F}_j)$ a basis for $\mathfrak{U}_{V^{\mathbb{C}}}^{-1}$, compatible with $\mathfrak{i} = \mathfrak{i}_{10} + \mathfrak{i}_{01}$, with $E_1 := E$ and $\overline{F}_1 := \overline{F}$. In case \overline{F}' is not a multiple of \overline{F} , we also assume that $\overline{F}_2 = \overline{F}'$.

We now prove (5.16) by considering the two possibilities for ν .

Case 1: $\nu = \mu - 1$.

Since $E^{\mu-1} \cdot \overline{F} = 0$, from (5.5), (5.6), (5.9) and (5.14),

$$0 = L_{E^{\mu-1} \cdot \overline{F}} = (\mu - 1)E^{\mu-2} \cdot \overline{F}' + \alpha(E) \frac{(\mu - 1)(\mu - 2)}{2} E^{\mu-2} \cdot \overline{F}.$$

Since all terms of this equality have degree $\mu - 1$, they can be identified with corresponding elements in \mathfrak{U}_J . Considering the \mathcal{B} -type of the two \mathcal{B} -monomials involved, by Corollary 4.4 we have that the equality holds if and only if

$$\text{ad}_{L_E}(\overline{F}) = \overline{F}' = -\alpha(E) \frac{\mu - 2}{2} \overline{F}.$$

Replacing this into (5.16), we see that the equality is satisfied, proving the lemma in this case.

Case 2: $\nu = \mu$.

Since $E^\mu \cdot \overline{F} = 0$, using as usual (5.5), (5.6) and (5.9), we have

$$0 = L_{E^\mu \cdot \overline{F}} = \mu E^{\mu-1} \cdot \overline{F}' + \alpha(E) \frac{\mu(\mu - 1)}{2} E^{\mu-1} \cdot \overline{F}.$$

Dividing by μ , applying L to both sides and using once again (5.5), (5.6) and (5.9), we get

$$\begin{aligned} 0 &= (\mu - 1)E^{\mu-2} \cdot L_{E|\overline{F}'} + \alpha(E) \frac{(\mu - 1)(\mu - 2)}{2} E^{\mu-2} \cdot \overline{F}' + \\ &\quad + \alpha(E) \frac{(\mu - 1)^2}{2} E^{\mu-2} \cdot \overline{F}' + \alpha(E)^2 \frac{(\mu - 1)^2(\mu - 2)}{4} E^{\mu-2} \cdot \overline{F}. \end{aligned}$$

With the usual arguments based on \mathcal{B} -types, we get that this can be satisfied if and only if (5.16) holds. \square

Proposition 5.6. *Let $L \in \mathfrak{g}^1$ with associated linear map $\alpha = \alpha_L \in (\mathfrak{m}^{10})^*$. Then, for each $E \in \mathfrak{m}^{10}$, the restricted map $\text{ad}_{L_E}|_{\mathfrak{m}^{01}} : \mathfrak{m}^{01} \rightarrow \mathfrak{m}^{01}$ is diagonalisable, with at most two eigenvalues, whose values can be only*

$$\rho = -\alpha(E) \frac{\mu - 1}{2} \quad \text{or} \quad \rho = -\alpha(E) \frac{\mu - 2}{2}. \quad (5.17)$$

In particular, if $E \in \ker \alpha$, then $\text{ad}_{L_E}|_{\mathfrak{m}^{01}} = 0$.

Proof. Let ρ be an eigenvalue of $\text{ad}_{L_E}|_{\mathfrak{m}^{01}} : \mathfrak{m}^{01} \rightarrow \mathfrak{m}^{01}$ and $\overline{F}_o \neq 0$ an associated eigenvector. Then, by (5.16),

$$\begin{aligned} 0 &= \left(\rho^2 + \alpha(E) \frac{2\mu - 3}{2} \rho + \alpha(E)^2 \frac{(\mu - 1)(\mu - 2)}{4} \right) \overline{F}_o = \\ &= \left(\rho + \alpha(E) \frac{\mu - 1}{2} \right) \left(\rho + \alpha(E) \frac{\mu - 2}{2} \right) \overline{F}_o . \end{aligned}$$

This implies that (5.17) are the only possibilities for the eigenvalues.

We now claim that only the following two cases might occur:

- a) $\alpha(E) \neq 0$ and the linear map $\text{ad}_{L_E}|_{\mathfrak{m}^{01}}$ admits a basis for \mathfrak{m}^{01} made of eigenvectors;
- b) $\alpha(E) = 0$ and hence with all eigenvalues of $\text{ad}_{L_E}|_{\mathfrak{m}^{01}}$ equal to 0.

For checking this, assume that neither (a) nor (b) occurs, i.e. that $\alpha(E) \neq 0$ and that $\text{ad}_{L_E}|_{\mathfrak{m}^{01}}$ is not diagonalisable. This means that there is a basis $(\overline{F}_1, \dots, \overline{F}_n)$ for \mathfrak{m}^{01} , in which the associated matrix of $\text{ad}_{L_E}|_{\mathfrak{m}^{01}}$ is in Jordan canonical form with at least one Jordan block of order greater than or equal to 2. However, by (5.16), no Jordan block can be of order strictly larger than 2. We may therefore assume that the basis (\overline{F}_j) is such that

$$\text{ad}_{L_E}(\overline{F}_1) = \rho \overline{F}_1, \quad \text{ad}_{L_E}(\overline{F}_2) = \rho \overline{F}_2 + \overline{F}_1$$

where ρ is one of the two possibilities given in (5.17). Plugging this into (5.16) with $\overline{F} = \overline{F}_2$, we obtain that

$$\rho^2 \overline{F}_2 + 2\rho \overline{F}_1 = -\alpha(E) \rho \frac{2\mu - 3}{2} \overline{F}_2 - \alpha(E) \frac{2\mu - 3}{2} \overline{F}_1 - \alpha(E)^2 \frac{(\mu - 1)(\mu - 2)}{4} \overline{F}_2 .$$

Looking at the coefficients of \overline{F}_1 , we see that ρ and $\alpha(E)$ have to satisfy the relation

$$4\rho = -\alpha(E)(2\mu - 3) .$$

Since $\alpha(E) \neq 0$ and 4ρ is either $-\alpha(E)(2\mu - 2)$ or $-\alpha(E)(2\mu - 4)$, we get a contradiction in both cases. This shows that only (a) and (b) can occur.

To conclude the proof, it remains only to show that $\text{ad}_{L_E}|_{\mathfrak{m}^{01}}$ is diagonalisable also in case $\alpha(E) = 0$. Suppose not. Then, considering Jordan canonical forms as before, we may pick a basis $(\overline{F}_1, \dots, \overline{F}_n)$ for \mathfrak{m}^{01} such that

$$L_E \overline{F}_1 = 0, \quad L_E \overline{F}_2 = \overline{F}_1 . \quad (5.18)$$

We may also consider a basis (E_1, \dots, E_n) for \mathfrak{m}^{10} with $E_1 = E$ and the corresponding basis $\mathcal{B} = (E_i, \overline{F}_j)$ for $V^{\mathbb{C}} = (\mathfrak{m}^{-1})^{\mathbb{C}}$. We start with a couple of preliminary observations. We first notice that, by the Jacobi identity and by the property $[\mathfrak{m}^{01}, \mathfrak{m}^{01}] = \{0\}$, for any $1 \leq i, j \leq n$

$$\overline{F}_i \cdot E \cdot \overline{F}_j = [\overline{F}_i, [E, \overline{F}_j]] = [[\overline{F}_i, E], \overline{F}_j] = \overline{F}_j \cdot E \cdot \overline{F}_i . \quad (5.19)$$

Second, by $L_{E|E} = \alpha(E)E = 0$ and (5.9), we have that for $1 \leq i, j \leq 2$ and $1 \leq r$

$$\begin{aligned}
L_{E^r \cdot (\overline{F}_i \cdot E \cdot \overline{F}_j)} &= rE^{r-1} \cdot L_{E|\overline{F}_i \cdot E \cdot \overline{F}_j} - E^{r-1} \cdot L_{\overline{F}_i \cdot E \cdot \overline{F}_j|E} = \\
&= rE^{r-1} \cdot L_{E|\overline{F}_i \cdot E \cdot \overline{F}_j} + rE^{r-1} \cdot \overline{F}_i \cdot E \cdot L_{E|\overline{F}_j} - E^{r-1} \cdot L_{\overline{F}_i \cdot E \cdot \overline{F}_j|E} \stackrel{(5.18)}{=} \\
&= rE^{r-1} \cdot (\delta_{i2}\overline{F}_1) \cdot E \cdot \overline{F}_j + rE^{r-1} \cdot \overline{F}_i \cdot E \cdot (\delta_{j2}\overline{F}_1) - E^{r-1} \cdot L_{\overline{F}_i \cdot E \cdot \overline{F}_j|E} \stackrel{(5.19)}{=} \\
&= rE^{r-1} \cdot (\delta_{i2}\overline{F}_j + \delta_{j2}\overline{F}_i) \cdot E \cdot \overline{F}_1 - E^{r-1} \cdot L_{\overline{F}_i \cdot E \cdot \overline{F}_j|E} .
\end{aligned} \tag{5.20}$$

We now observe that the term $L_{\overline{F}_i \cdot E \cdot \overline{F}_j}$ is in $[\mathfrak{g}^1, (\mathfrak{m}^{-3})^{\mathbb{C}}] \subset (\mathfrak{m}^{-2})^{\mathbb{C}}$. Thus it has the form $L_{\overline{F}_i \cdot E \cdot \overline{F}_j} = \sum_{k,\ell} c_{(ij)}^{\ell k} E_{\ell} \cdot \overline{F}_k$ for some appropriate coefficients $c_{(ij)}^{\ell k}$ and

$$L_{E^r \cdot (\overline{F}_i \cdot E \cdot \overline{F}_j)} = rE^{r-1} \cdot (\delta_{i2}\overline{F}_j + \delta_{j2}\overline{F}_i) \cdot E \cdot \overline{F}_1 - \sum_{k,\ell} c_{(ij)}^{\ell k} E^{r-1} \cdot E_{\ell} \cdot \overline{F}_k \cdot E . \tag{5.21}$$

We are now ready to conclude. Since $E^{\mu-2} \cdot \overline{F}_2 \cdot E \cdot \overline{F}_2 = 0$ (it has degree $\mu + 1$),

$$0 = L_{E^{\mu-2} \cdot (\overline{F}_2 \cdot E \cdot \overline{F}_2)} \stackrel{(5.21)}{=} 2(\mu-2)E^{\mu-3} \cdot \overline{F}_2 \cdot E \cdot \overline{F}_1 - \sum_{k,\ell} c_{(22)}^{\ell k} E^{\mu-3} \cdot E_{\ell} \cdot \overline{F}_k \cdot E , \tag{5.22}$$

which implies the following relation between elements of degree μ

$$E^{\mu-3} \cdot \overline{F}_2 \cdot E \cdot \overline{F}_1 = \frac{1}{2(\mu-2)} \sum_{k,\ell} c_{(22)}^{\ell k} E^{\mu-3} \cdot E_{\ell} \cdot \overline{F}_k \cdot E .$$

If we now apply ad_L to both sides and use (5.21) once again, we obtain the following identity between homogeneous elements of degree $\mu - 1$:

$$\begin{aligned}
&E^{\mu-4} \cdot \overline{F}_1 \cdot E \cdot \overline{F}_1 = \\
&= \frac{1}{\mu-3} \sum_{k,\ell} c_{(21)}^{\ell k} E^{\mu-4} \cdot E_{\ell} \cdot \overline{F}_k \cdot E + \frac{1}{2(\mu-2)(\mu-3)} \sum_{k,\ell} c_{(22)}^{\ell k} L_{E^{\mu-3} \cdot E_{\ell} \cdot \overline{F}_k \cdot E} .
\end{aligned} \tag{5.23}$$

We now observe that the left hand side is an element in $\mathfrak{m}^{\mathbb{C}} = \mathfrak{U}_{V^{\mathbb{C}}} / (\mathfrak{i}_{10} + \mathfrak{i}_{01} + \mathfrak{i}^{-\mu})$, which is the projection of a non-zero \mathcal{B} -monomial in $\mathfrak{U}_{V^{\mathbb{C}}}$ whose \mathcal{B} -type is

$$\left(\underbrace{\mu-3, 0, \dots, 0}_{\text{entries corresp. to the } E_i}, \underbrace{2, 0, 0, \dots, 0}_{\text{entries corresp. to the } \overline{F}_k} \right) . \tag{5.24}$$

On the other hand, a straightforward check shows that the right hand side is a linear combination of projections of \mathcal{B} -monomials, whose \mathcal{B} -types might only have the form

$$\left(\underbrace{*, *, \dots, *, *}_{\text{entries corresp. to the } E_i}, \underbrace{0, \dots, 0, 1, 0, \dots, 0}_{\text{entries corresp. to the } \overline{F}_k} \right) .$$

Since these \mathcal{B} -types are surely different from (5.24), Corollary 4.4 implies that the left hand side and the right hand side of (5.23) are linearly independent and that the equality cannot be true. This contradiction concludes the proof. \square

Corollary 5.7. *Given $L \in \mathfrak{g}^1$, with associated linear map $\alpha = \alpha_L \in (\mathfrak{m}^{10})^*$, there exists a basis $(E_1, \dots, E_n, \overline{F}_1, \dots, \overline{F}_n)$ for $\mathfrak{m}^{\mathbb{C}}$, with $E_i \in \mathfrak{m}^{10}$, $\overline{F}_j \in \mathfrak{m}^{01}$, and an n -tuple of rational numbers ρ_i , $1 \leq i \leq n$, each of them equal either to $-\frac{\mu-1}{2}$ or $-\frac{\mu-2}{2}$, such that $E_1, \dots, E_{n-1} \in \ker \alpha$ and for any $E \in \mathfrak{m}^{10}$ and $1 \leq i, k \leq n$*

$$L_{E|\overline{F}_i} = -\alpha(E)\rho_i\overline{F}_i, \quad L_{E|E_k} = \frac{\delta_{kn}\alpha(E_n)}{2}E + \frac{\alpha(E)}{2}E_k. \quad (5.25)$$

In particular, $L \neq 0$ if and only if $\alpha \neq 0$.

Proof. By Proposition 5.6, the claim is trivial if $\alpha = 0$. Then, we may assume that $\alpha \in \text{Hom}(\mathfrak{m}^{10}, \mathbb{C})$ is non-trivial and we consider a basis (E_1, \dots, E_n) for \mathfrak{m}^{10} such that (E_1, \dots, E_{n-1}) is a basis for $\ker \alpha$ and $E_n \notin \ker \alpha$. By Proposition 5.6, the restricted adjoint action $\text{ad}_{L_{E_n}}|_{\mathfrak{m}^{01}}$ is diagonalisable and there exists a basis $(\overline{F}_1, \dots, \overline{F}_n)$ for \mathfrak{m}^{01} , made of eigenvectors whose associated eigenvalues have the form $-\alpha(E_n)\frac{\mu-1}{2}$ or $-\alpha(E_n)\frac{\mu-2}{2}$. The first identity in (5.25) follows by the facts that, for an arbitrary $E = \lambda^k E_k \in \mathfrak{m}^{10}$, one has that

$$\alpha(E) = \sum_{k=1}^{n-1} \lambda^k \alpha(E_k) + \lambda^n \alpha(E_n) = \lambda^n \alpha(E_n)$$

and that, setting ρ_i as in the statement, for each \overline{F}_i

$$L_{E|\overline{F}_i} = \sum_{k=1}^{n-1} \lambda^k L_{E_k|\overline{F}_i} + \lambda^n L_{E_n|\overline{F}_i} = -\lambda^n \alpha(E_n) \rho_i \overline{F}_i = -\alpha(E) \rho_i \overline{F}_i.$$

The second identity of (5.25) comes from (5.14). The last claim is a consequence of (5.25) and the fact that $L \neq 0$ if and only if there exists some $E \in \mathfrak{m}^{10}$ for which the element $L_E \in (\mathfrak{g}^0)^{\mathbb{C}}$ is non trivial. \square

5.4. Proof of Theorem 3.5. Let $\mathfrak{m} + \mathfrak{g}^0$ be a totally nondegenerate CR Tanaka algebra of depth $\mu \geq 4$ and $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}} = \mathfrak{A}_J/\mathfrak{i}^{-\mu} + (\mathfrak{g}^0)^{\mathbb{C}}$ be its complexification. To prove the theorem, it suffices to show that a contradiction arises if one assumes that there is $L \neq 0$ in \mathfrak{g}^1 (which, as usual, we identify with the corresponding element in $\mathfrak{j}(\mathfrak{g}^1) \subset \widehat{\mathfrak{g}}^1$ of the prolongation of $\mathfrak{m}^{\mathbb{C}} + (\mathfrak{g}^0)^{\mathbb{C}}$).

So, let us assume the existence of $0 \neq L \in \mathfrak{g}^1$ with associated non-trivial linear map $\alpha = \alpha_L \neq 0$. We may therefore consider a basis $\mathcal{B} = (E_k, \overline{F}_i)$ with the properties described in Corollary 5.7 and with $\alpha(E_n) \neq 0$. We may also assume that $E_n = F_k$ for some k . In fact, the considered basis $(\overline{F}_1, \dots, \overline{F}_n)$ for \mathfrak{m}^{01} can be actually chosen in such a way to be independent of E , a fact that can be checked as follows. Pick an element $E_o \in \mathfrak{m}^{10}$ such that $\alpha(E_o) = 1$ and for any other $E \in \mathfrak{m}^{10}$, set $c_E := \alpha(E)$. Since $E - c_E E_o \in \ker \alpha$, by Proposition 5.6, it follows that $\text{ad}_{L_{E - c_E E_o}}|_{\mathfrak{m}^{01}} = 0$. This implies that $\text{ad}_{L_E}|_{\mathfrak{m}^{01}} = c_E \text{ad}_{L_{E_o}}|_{\mathfrak{m}^{01}}$ and that a diagonalising basis $(\overline{F}_1, \dots, \overline{F}_n)$ for $\text{ad}_{L_{E_o}}|_{\mathfrak{m}^{01}}$ is necessarily a diagonalising basis for $\text{ad}_{L_E}|_{\mathfrak{m}^{01}}$ for any possible choice of E . Now, being such a basis fixed, for at least one element \overline{F}_k we must have $\alpha(F_k) \neq 0$ (otherwise, we would have that $L|_{\mathfrak{m}^{10}} = 0$ and this would imply $L = 0$). We may therefore assume that $E_n = F_k$ as claimed.

Hence, by possibly reordering the basis (\overline{F}_i) for \mathfrak{m}^{01} and by an appropriate rescaling, there is no loss of generality if we have $E_n = F_n$ and $\alpha(E_n) = 1$.

In what follows, for simplicity of notation, we set $E := E_n$, $\overline{E} := \overline{F}_n$ and $\rho := \rho_n$.

Our proof is crucially based on the following identity, which holds for each $r \geq 0$:

$$L_{E^r \cdot (\overline{E} \cdot (E \cdot \overline{E}))} = \frac{r(4\rho + r + 1)}{2} E^{r-1} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + (2\rho + 1) E^{r+1} \cdot \overline{E}. \quad (5.26)$$

In fact, recalling that $[E, E] = 0 = [\overline{E}, \overline{E}]$ and that $L_{E|E} = E$, $L_{E|\overline{E}} = \rho\overline{E}$, $L_{\overline{E}|E} = \overline{L}_{E|\overline{E}} = \rho E$, the identity (5.26) for $r = 0$ holds because

$$\begin{aligned} L_{\overline{E} \cdot (E \cdot \overline{E})} &= [L, [\overline{E}, [E, \overline{E}]]] = [[L, \overline{E}], [E, \overline{E}]] + [\overline{E}, [L, [E, \overline{E}]]] = \\ &= [[[L, \overline{E}], E], \overline{E}] + [E, [[L, \overline{E}], \overline{E}]] + [\overline{E}, [[L, E], \overline{E}]] + [\overline{E}, [E, [L, \overline{E}]]] = \\ &= L_{\overline{E}|E} \cdot \overline{E} + E \cdot L_{\overline{E}|\overline{E}} - L_{E|\overline{E}} \cdot \overline{E} + L_{\overline{E}|E} \cdot \overline{E} = (2\rho + 1) E \cdot \overline{E}. \end{aligned} \quad (5.27)$$

When $r = 1$, the identity follows from the fact that $L_E \in \mathfrak{g}^0$ (which means that L_E is a derivation of the Lie algebra $\mathfrak{m}^{\mathbb{C}}$) and from (5.27). Indeed, all this implies

$$\begin{aligned} L_{E \cdot (\overline{E} \cdot (E \cdot \overline{E}))} &= [L_E, \overline{E} \cdot (E \cdot \overline{E})] + [E, L_{\overline{E} \cdot (E \cdot \overline{E})}] = \\ &= L_{E|\overline{E}} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + \overline{E} \cdot (L_{E|E} \cdot \overline{E}) + \overline{E} \cdot (E \cdot L_{E|\overline{E}}) + (2\rho + 1) E^2 \cdot \overline{E} = \\ &= (2\rho + 1) (\overline{E} \cdot (E \cdot \overline{E}) + E^2 \cdot \overline{E}). \end{aligned} \quad (5.28)$$

Finally, when $r \geq 2$, by (5.9),

$$\begin{aligned} L_{E^r \cdot (\overline{E} \cdot (E \cdot \overline{E}))} &= \\ &= rE^{r-1} \cdot L_{E|\overline{E} \cdot (E \cdot \overline{E})} - E^{r-1} \cdot L_{\overline{E} \cdot (E \cdot \overline{E})|E} + \frac{r(r-1)}{2} E^{r-2} \cdot (L_{E|E} \cdot (\overline{E} \cdot (E \cdot \overline{E}))) = \\ &= rE^{r-1} \cdot (L_{E|\overline{E}} \cdot (E \cdot \overline{E})) + rE^{r-1} \cdot (\overline{E} \cdot (L_{E|E} \cdot \overline{E})) + rE^{r-1} \cdot (\overline{E} \cdot (E \cdot L_{E|\overline{E}})) + \\ &\quad + E^r \cdot L_{\overline{E} \cdot (E \cdot \overline{E})} + \frac{r(r-1)}{2} E^{r-1} \cdot (\overline{E} \cdot (E \cdot \overline{E})) = \\ &= (2\rho + 1)rE^{r-1} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + (2\rho + 1)E^{r+1} \cdot \overline{E} + \frac{r(r-1)}{2} E^{r-1} \cdot (\overline{E} \cdot (E \cdot \overline{E})), \end{aligned} \quad (5.29)$$

which gives (5.26) in all remaining cases.

Let us now consider the first integer ν such that $E^\nu \cdot (\overline{E} \cdot (E \cdot \overline{E})) = 0$. Since $\mathfrak{m} + \mathfrak{g}^0$ is totally nondegenerate, then $\nu = \mu - 3$ or $\mu - 2$. We consider these two cases separately and show that in both cases the desired contradiction arises.

Case 1: $\nu = \mu - 3$.

Since $E^{\mu-3} \cdot (\overline{E} \cdot (E \cdot \overline{E})) = 0$, from (5.26) we have

$$\begin{aligned} 0 &= L_{E^{\mu-3} \cdot (\overline{E} \cdot (E \cdot \overline{E}))} = \\ &= \frac{(\mu-3)(4\rho + \mu - 2)}{2} E^{\mu-4} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + (2\rho + 1) E^{\mu-2} \cdot \overline{E}. \end{aligned}$$

The two monomials in the right hand side are in $(\mathfrak{m}^{-(\mu-1)})^{\mathbb{C}} = \mathfrak{U}_J^{-(\mu-1)}$ and have different \mathcal{B} -types. Hence, they are linearly independent and since $\mu \geq 4$, this contradicts the fact that, for both possibilities $\rho = -\frac{\mu-1}{2}$ and $\rho = -\frac{\mu-2}{2}$, the coefficients of the linear combination in the right hand side are non-zero.

Case 2: $\nu = \mu - 2$.

From $E^{\mu-2} \cdot (\overline{E} \cdot (E \cdot \overline{E})) = 0$ and (5.26)

$$\begin{aligned} 0 &= L_{E^{\mu-2} \cdot (\overline{E} \cdot (E \cdot \overline{E}))} = \\ &= \frac{(\mu-2)(4\rho + \mu - 1)}{2} E^{\mu-3} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + (2\rho + 1) E^{\mu-1} \cdot \overline{E}. \end{aligned}$$

This time the right hand side is an element in $(\mathfrak{m}^{-\mu})^{\mathbb{C}} = \mathfrak{U}_J^{-\mu}/i^{-\mu}$ and we cannot claim that the two monomials appearing in such expression are linearly independent. However, if we apply L to this expression and use (5.26) and (5.9) we get

$$\begin{aligned} 0 &= \frac{(\mu-2)(4\rho + \mu - 1)}{2} \left(\frac{(\mu-3)(4\rho + \mu - 2)}{2} E^{\mu-4} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + \right. \\ &\quad \left. + (2\rho + 1) E^{\mu-2} \cdot \overline{E} \right) + \\ &\quad + (2\rho + 1) \left(\rho(\mu-1) E^{\mu-2} \cdot \overline{E} + \frac{(\mu-1)(\mu-2)}{2} E^{\mu-2} \cdot \overline{E} \right) = \\ &= \frac{(\mu-2)(\mu-3)(4\rho + \mu - 1)(4\rho + \mu - 2)}{2} E^{\mu-4} \cdot (\overline{E} \cdot (E \cdot \overline{E})) + \\ &\quad + c E^{\mu-2} \cdot \overline{E} \end{aligned}$$

with $c = (2\rho + 1) \left(\frac{(\mu-2)(4\rho + \mu - 1)}{2} + \rho(\mu-1) + \frac{(\mu-1)(\mu-2)}{2} \right)$. Now, both monomials in the right hand side are in $(\mathfrak{m}^{-(\mu-1)})^{\mathbb{C}} = \mathfrak{U}_J^{-(\mu-1)}$ and have different \mathcal{B} -types. Thus they are linearly independent and the coefficient $\frac{(\mu-2)(\mu-3)(4\rho + \mu - 1)(4\rho + \mu - 2)}{2}$ of $E^{\mu-4} \cdot (\overline{E} \cdot (E \cdot \overline{E}))$ should be 0. This is in contradiction with the fact that, being $\mu \geq 4$, such coefficient is nonzero for both possibilities $\rho = -\frac{\mu-1}{2}$ and $\rho = -\frac{\mu-2}{2}$.

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