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Propagation of regularity for Monge-Ampère exhaustions and Kobayashi metrics

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Abstract
We prove that if a smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$, $n \geq 2$, admits at least one Monge-Ampère exhaustion smooth up to the boundary (i.e., a plurisubharmonic exhaustion $\tau : \overline{D} \to [0, 1]$, which is $\mathscr{C}^\infty$ at all points except possibly at the unique minimum point x and with $u := \log \tau$ satisfying the homogeneous complex Monge-Ampère equation), then there exists a bounded open neighborhood $\mathscr{U} \subset D$ of the minimum point x, such that for each $y \in \mathscr{U}$ there exists a Monge-Ampère exhaustion with minimum at y. This yields that for each such domain D, the restriction to the subdomain $\mathscr{U} \subset D$ of the Kobayashi pseudo-metric κ_D is a smooth Finsler metric for $\mathscr{U}$ and each pluricomplex Green function of D with pole at a point $y \in \mathscr{U}$ is of class $\mathscr{C}^\infty$. The boundary of the maximal open subset having all such properties is also explicitly characterized. The result is a direct consequence of a general theorem on abstract complex manifolds with boundary, with Monge-Ampère exhaustions of regularity $\mathscr{C}^k$ for some $k \geq 5$. In fact, analogues of the above properties hold for each bounded strongly pseudoconvex complete circular domain with boundary of such weaker regularity.

Keywords Monge-Ampère equations · Pluricomplex Green functions · Manifolds of circular type · Kobayashi metric · Deformations of complex structures

Mathematics Subject Classification 32Q45 · 32U35 · 32G05 · 32W20

1 Introduction

Motivated by Lempert's results on smoothly bounded convex domains and the several following developments, the second author introduced in [24] the notion of *Lempert manifold*, that is a complex manifold (M, J) such that:

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(i) the Kobayashi pseudo-metric κ of M is a smooth strongly pseudoconvex Finsler metric, i.e.,

(a) It is a strictly positive and $\mathscr{C}^\infty$ on $TM^o = TM\backslash\{Zero\ section\}$;
(b) $\kappa(\lambda v) = |\lambda|\kappa(v)$ for any $\lambda \in \mathbb{C}^*$ and $v \in TM^o$;
(c) The set of tangent vectors $\mathscr{I}_x = \{v \in T_xM : \kappa(v) < 1\}$ is a strongly pseudoconvex domain of $\mathbb{C}^n \simeq T_xM$ for each $x \in M$;

(ii) all exponentials $\exp_x : T_xM \to M$, $x \in M$, are diffeomorphisms and the distance function d, determined by κ, is complete.
(iii) for each $w \in T_x^{\mathbb{C}}M^o$, the 2-dimensional real submanifold $\exp(\mathbb{C}w)$ of M is a complex curve, which is totally geodesic and with induced metric of constant holomorphic curvature -4.

This definition was designed to capture most of the properties of the smoothly bounded strictly convex domains. Indeed, by [1,2,12,13,18], *each such domain is a Lempert manifold*. It is also quite remarkable that, in contrast with the fact that on each smoothly bounded strictly convex domain (and each domain biholomorphic to one of them) the Kobayashi pseudo-metric κ has so many and so nice properties, up to now no other explicit class of domains, for which κ is smooth on the whole TM^o, is known.

A closely related notion introduced by the first author in [19] is the class of *manifolds of circular type* which naturally includes all smoothly bounded strictly convex domains and all smoothly bounded strongly pseudoconvex circular domains. Manifolds of circular type are characterized by the Because of the uniqueness of the exhaustion $\tau : M \to [0, 1)$, which is smooth at all points except possibly at the minimal set $\{\tau = 0\}$ and with $u := \log \tau$ satisfying the homogeneous complex Monge-Ampère equation. The minimal set always consists of only one point x_o, called *center*, and the function $\kappa = \kappa^{(\tau)} : T_{x_o}M\backslash\{0\} \to [0, +\infty)$ for $(x_o, v) \in T_{x_o}M$, defined by

$$\kappa(x_o, v) := \lim_{t_o \to 0} \frac{\mathrm{d}}{\mathrm{d}t} \sqrt{\tau(\gamma(t))}\Big|_{t=t_o}, \tag{1.1}$$

where γ is any smooth curve with $\gamma(0) = x_o$, $\dot\gamma(0) = v$, coincides with the Kobayashi metric of M at the point x_o.

Apparently strictly convex domains and strongly pseudoconvex circular domains are manifolds of circular type for unrelated reasons. Indeed, for a smoothly bounded strictly convex domain $\Omega \subset \mathbb{C}^n$, for *any* $x_o \in \Omega$, the exhaustion τ defined by $\tau(y) = (\tanh \delta(y, x_o))^2$, with $\delta(y, x_o)$ the Kobayashi distance from y to x_o, is a Monge-Ampère exhaustion centered at x_o. This is a consequence of the very special properties of the Kobayashi metric and distance and their peculiar relation with the pluricomplex Green potential, discovered by Lempert for this class of domains. On the other hand, a smoothly bounded, strongly pseudoconvex complete circular domain $D \subset \mathbb{C}^n$ is of circular type because its squared Minkowski function μ_D^2 is a Monge-Ampère exhaustion centered at 0. This is a simple consequence of the symmetry properties of such domains, namely of the S^1 action that defines them and has the origin as the unique fixed point. While it is known that the pluricomplex Green potential exists at every point and it is in general only $C^{1,1}$, there is no simple reason why such a domain should admit a Monge-Ampère exhaustion as defined above, centered at some other point but the origin. Indeed, this would imply strong regularity and the least possible degeneracy of the Levi form of the pluricomplex Green potential.

Important steps toward a deeper understanding of the geometry of the set of centers of domains of circular type are provided by the results of Pang ([18]). Indeed, Pang detected

a special condition on holomorphic stationary disks, called *coercitivity property*, which is invariant under biholomorphisms (in contrast with the strict convexity condition for domains), it characterizes isolated local extremal disks and, most importantly, *it is stable under small deformations*. As an important application of such results and especially of the quoted stability of coercitivity property, Pang managed to prove that *any smoothly bounded strongly pseudoconvex complete circular domain D admits an open subset $\mathscr{U} \subset D$ with the property that each $y \in \mathscr{U}$ is the center of an appropriate smooth Monge-Ampère exhaustion*.

In this paper, we show that the existence of such "clouds" of centers is actually a general property, provided that the considered domain admits at least one Monge-Ampère exhaustion τ, which is smoothly extendible up to the boundary. More precisely, we prove the following

Theorem 1.1 *Let $D \subset \mathbb{C}^n$ be a smoothly bounded strongly pseudoconvex domain, admitting a Monge-Ampère exhaustion $\tau : D \to [0, 1)$, centered at $x_o \in D$, which is smoothly extendible up to the boundary. Then there exists an open neighborhood $\mathscr{U} \subset D$ of x_o, such that for each $x \in \mathscr{U}$ there exists a Monge-Ampère exhaustion $\tau^{(x)} : M \to [0, 1)$, smoothly extendible up to the boundary and centered at x. The family of exhaustions $\tau^{(x)}$, considered as collections of maps parameterized by the uniquely associated centers $x \in \mathscr{U}$, depends smoothly on such parameter.*

Further, in the cases in which the maximal connected neighborhood $\mathscr{U}^{(\max)}$ of x_o with the above property is strictly smaller than D, the intersection $\partial\mathscr{U}^{(\max)} \cap f(\Delta)$ with each stationary disk $f(\Delta) \subset D$ through x_o coincides with the set of points, where an appropriate matrix valued function is singular (see Proposition 5.4 below for details)

In this result, the "maximality" of $\mathscr{U}^{(\max)}$ means that for each point $y \in \partial\mathscr{U}^{(\max)} \subset \overline{D}$ there is no Monge-Ampère exhaustion centered at y that is smooth up to the boundary. Actually, as mentioned in the statement, our proof gives also a constructive method to determine the maximal open set $\mathscr{U}^{(\max)}$ and, in particular, its boundary. It thus provides a biholomorphically invariant characterization of the domains for which $D = \mathscr{U}^{(\max)}$, i.e., for the domains for which each point is the center of a smooth Monge-Ampère exhaustion, hence of domains, on which the Kobayashi metric is a smooth complex Finsler metric.

In Theorem 1.1, the smoothness properties are deliberately not specified. Indeed whenever the boundary of the domain D and the Monge-Ampère exhaustion τ are either $\mathscr{C}^k$, for some $k \geq 5$, or $\mathscr{C}^\infty$ or $\mathscr{C}^\omega$, then our result yields that the exhaustions $\tau^{(x)}$, $x \in \mathscr{U}^{(\max)}$, are at least $\mathscr{C}^{k-3,\alpha}$, $\alpha \in (0, 1)$, in the first case, or $\mathscr{C}^\infty$ or $\mathscr{C}^\omega$ respectively in the other cases, up to the boundary.

The above theorem has the noticeable consequence that *for any smooth domain of circular type $D \subset \mathbb{C}^n$ admitting at least one Monge-Ampère exhaustion that is smooth up to the boundary, the Kobayashi pseudo-metric κ is a strongly pseudoconvex Finsler metric on a (possibly non-connected) open subdomain of D*.

Of course, all this can be rephrased in terms of the pluricomplex Green functions and, in particular, gives new information on their regularity. In fact, if $\tau^{(x)} : D \to [0, 1)$ is a Monge-Ampère exhaustion, with center x and smooth up to the boundary, the associated function

$$u^{(x)} = \log \tau^{(x)} : \overline{D} \longrightarrow [-\infty, 0] \tag{1.2}$$

is the pluricomplex Green function for $\overline{D}$ with pole at x, it is of class C^∞ and satisfies the homogeneous complex Monge-Ampère equation on $D\backslash\{x\}$ with the least possible degeneracy, i.e., the annihilator of the form $\partial\bar{\partial}u^{(x)}$ has rank 1 at every point of $D\backslash\{x\}$. Conversely, if the pluricomplex Green function $u^{(x)}$ with pole at x is of class C^∞ and satisfies the homogeneous complex Monge-Ampère equation on $D\backslash\{x\}$ with the least possible degeneracy, then $\tau^{(x)} = \exp(u^{(x)})$ is a Monge-Ampère exhaustion centered at x.

By general results on hyperconvex manifolds ([4,5,9]), it is known that for each point x of a smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$ there exists a unique pluricomplex Green function $u^{(x)}$ with pole at x. Its regularity is at least $\mathscr{C}^{1,1}$ up to the boundary but, in general, of not higher regularity. Our result implies the following curious phenomenon of propagation of regularity: *if a smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$ admits at least one pluricomplex Green function $u^{(x)}$ of class $\mathscr{C}^\infty$, then necessarily* all *other pluricomplex Green functions $u^{(x')}$ with poles at the points x' in an appropriate subdomain $\mathscr{U}^{\max} \subset D$, are of class $\mathscr{C}^\infty$.*

While, by Lempert's results ([12]), it is known that on a smoothly bounded *strictly convex* domain $D \subset \mathbb{C}^n$ the pluricomplex Green function $u^{(x)}$ with pole at x is of class $\mathscr{C}^\infty$ for each $x \in D$, there are *no explicit examples* of smoothly bounded strongly pseudoconvex domain $D \subset \mathbb{C}^n$ with a point x_o for which pluricomplex Green function $u^{(x_o)}$ is $\mathscr{C}^\infty$ and such that the surrounding "cloud" $\mathscr{U}^{(\max)} \subset D$, made of the points x for which also their pluricomplex Green functions $u^{(x)}$ are $\mathscr{C}^\infty$, is strictly smaller than in D. On the other hand, in Proposition 5.3, we provide specific (closed) necessary conditions for the "cloud" $\mathscr{U}^{(\max)}$ to be properly contained in D. At the end of the paper, we discuss how such conditions can be used to construct explicit examples with $\mathscr{U}^{(\max)} \subsetneq D$ and to determine biholomorphically invariant sufficient conditions for propagations of regularity to the whole domain to occur.

Theorem 1.1 is obtained as a consequence of a more general result (Theorem 2.8) concerning abstract complex manifolds with boundary, equipped with Monge-Ampère exhaustions of regularity $\mathscr{C}^k$, $k \geq 5$. Such result shows that the above phenomena of propagation of regularity hold also for the $\mathscr{C}^k$-analogues of domains of circular type, in particular for strongly pseudoconvex circular domains with boundaries of such regularity. We remark that the assumption $k \geq 5$ is rather technical and determined by the special approach we exploit, namely the use of the so-called *manifolds in normal form*. It is reasonable to expect that such lower bound for k is not sharp and that the propagation phenomena for Monge-Ampère exhaustions and pluricomplex Green functions should occur in full generality also if $k \geq 3$. This is actually true if the considered complex manifold is already a manifold in normal form (Theorem 2.7). We also notice that Theorem 2.8 holds also in the C^∞-smooth category and that its C^ω version follows from ancillary results needed as part of its proof.

The structure of the paper is the following. In §2, we introduce the new notion of complex manifolds with boundary of circular type of class $\mathscr{C}^{k,\alpha}$, a minor modification of the original definition of manifolds of circular type, and we state the main results of our paper. In §3, we study one-parameter families of Monge-Ampère exhaustions of a manifold of circular type in normal form, with centers at the point of a given segment of the manifold. We then prove the existence of a bijection between such families and the so-called *abstract fundamental pairs*. These are one-parameter families of pairs, formed by nonstandard complex structures and vector fields satisfying appropriate conditions. Section 4 is dedicated to a detailed study of the vector fields occurring in the abstract fundamental pairs, and in Sect. 5 it is given a crucial result on the existence of abstract fundamental pairs and the proof of the main theorem.

2 Manifolds of circular type

2.1 Complex manifolds with boundary of class $\mathscr{C}^{k,\alpha}$

Given $k \in \mathbb{N} \cup \{0, \infty\}$ and $\alpha \in [0, 1]$, a (real or complex) tensor field T on a (real) smooth manifold with boundary $\overline{M} = M \cup \partial M$ is said to be *of class $\mathscr{C}^{k,\alpha}$ on $\overline{M}$* if all components

of T in any system of coordinates of the structure of the manifold with boundary $\overline{M}$ are of class $\mathscr{C}^k$ and with α-Hölder continuous k-th order derivatives up to the boundary.

In this paper, we need to exploit in many ways the relation provided by the Newlander–Nirenberg Theorem between the notions of *complex manifold M of dimension n*, i.e., a topological $2n$-dimensional manifold equipped with a complete atlas $\mathscr{A}$ of $\mathbb{C}^n$-valued homeomorphisms between open sets of M and of $\mathbb{C}^n$ with holomorphic overlaps, and of *integrable complex structure of class $\mathscr{C}^{k,\alpha}$*, $k \geq 1$, $0 < \alpha < 1$, on a *real* $2n$-manifold M, which is a tensor field J of type $(1, 1)$ of class $\mathscr{C}^{k,\alpha}$ with $J_x^2 = -\operatorname{Id}_{T_xM}$ at each point x and vanishing Nijenhuis tensor N_J i.e., such that for all vector fields X, Y on M

$$N_J(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y] = 0.$$

A $\mathscr{C}^{k,\alpha}$-complex structure J on M induces the direct sum decompositions $T_x^{\mathbb{C}}M = T_x^{1,0}M \oplus T_x^{0,1}M$ of the complexified tangent spaces into $\pm i$-eigenspaces of the linear maps $J_x : T_xM \to T_xM$. If $T^{1,0}M \subset T^{\mathbb{C}}M$ is the complex bundle with fiber $T_x^{1,0}M$, $x \in M$, a *set of J-holomorphic coordinates* is any $\mathscr{C}^1$-map $F = (F^1, \ldots, F^n) : \mathscr{U} \subset M \longrightarrow \mathbb{C}^n$, which is homeomorphic onto its image and satisfies, for any choice of local generators $\{X_i\}_{1\leq i\leq n}$ of $T_x^{1,0}M$:

$$\overline{X_i}(F^j) = 0 \qquad \text{for any } 1 \leq i, j \leq n. \tag{2.1}$$

By the Newlander–Nirenberg Theorem ([10,14,16,17,25]), if J is $\mathscr{C}^{k,\alpha}$ with $k \geq 1$ and $0 < \alpha < 1$, then there exists an atlas $\mathscr{A}_J$ of J-holomorphic coordinates $F = (F^i)$ over M, which makes $(M, \mathscr{A}_J)$ a complex manifold. Moreover, all $\mathbb{C}$-valued functions F^i, that are components of some set of J-holomorphic coordinates, are of class at least $\mathscr{C}^{k+1,\alpha}$ relatively to any set of real coordinates of the original real manifold structure of M.

Conversely, if $(M, \mathscr{A})$ is a complex manifold of dimension n (hence, a manifold of real dimension $2n$), the real tensor field J having the form $J = i\frac{\partial}{\partial z^j} \otimes dz^j - i\frac{\partial}{\partial \overline{z}^j} \otimes d\overline{z}^j$ in any chart (z^i) of the complex atlas $\mathscr{A}$, is easily seen to be a complex structure, for which the charts of $\mathscr{A}$ are J-holomorphic coordinates. Such a tensor field J is of class $\mathscr{C}^\omega$ relatively to the charts in $\mathscr{A}$ and, hence, of class $\mathscr{C}^{k,\alpha}$ relatively to any other atlas $\mathscr{A}'$ of real coordinates that overlaps with those of $\mathscr{A}$ in a $\mathscr{C}^{k+1,\alpha}$ fashion.

For $k \geq 1$ and $\alpha \in (0, 1)$, if M is a real $2n$-manifold and J a complex structure of class $\mathscr{C}^{k,\alpha}$ on M, we call the pair (M, J) a *complex manifold of class $\mathscr{C}^{k+1,\alpha}$* and denote by $\mathscr{A}_J$ the $\mathscr{C}^{k+1,\alpha}$ atlas of J-holomorphic coordinates.

Let us now consider a convenient analogue of real manifolds with boundary.

Definition 2.1 Let $\overline{M} = M \cup \partial M$ be a real $2n$-manifold with boundary. A *complex structure on $\overline{M}$ of class $\mathscr{C}^{k,\alpha}$, with $k \geq 1$, $\alpha \in (0, 1)$*, is a triple $(J, \mathscr{D}, J^{\mathscr{D}})$ where J is a complex structure on M of class $\mathscr{C}^{k,\alpha}$, $\mathscr{D}$ is a $\mathscr{C}^\infty$ codimension one distribution on ∂M, smoothly extendible to a $2(n-1)$-dimensional distribution on a tubular neighborhood $\mathscr{U} \subset \overline{M}$ of ∂M and $J^{\mathscr{D}}$ is a tensor field of type $(1, 1)$ on the distribution $\mathscr{D}$ of ∂M, with components of class $\mathscr{C}^{k,\alpha}$ in any smooth frame field for the spaces $\mathscr{D}_x$, $x \in \partial M$, of the distribution $\mathscr{D}$ subject to the following conditions:

(i) the smooth extension of $\mathscr{D}$ on a tubular neighborhood $\mathscr{U} \subset \overline{M}$ of ∂M, can be taken to be J-invariant;
(ii) the restrictions $J|_{\mathscr{D}_x}$, $x \in \mathscr{U} \setminus \partial M$, together with the tensors $J_y^{\mathscr{D}}$, $y \in \partial M$, form a $(1, 1)$-tensor field on the smooth extension of $\mathscr{D}$ in (i), with components of bounded $\mathscr{C}^{k,\alpha}$-norm in any smooth frame field for the spaces of $\mathscr{D}$.

A manifold with boundary $\overline{M}$, endowed with a triple $(J, \mathscr{D}, J^{\mathscr{D}})$, is briefly denoted by the pair $(\overline{M}, J)$. We may also write $J^{\mathscr{D}} = J|_{\mathscr{D}_{\partial M}}$ and we call $(\mathscr{D}, J^{\mathscr{D}})$ the *CR structure of the boundary of* $(\overline{M}, J)$. In fact, by boundary regularity assumptions, the pair $(\mathscr{D}, J^{\mathscr{D}})$ is an integrable CR structure on ∂M. We say that the boundary ∂M is *strongly pseudoconvex* if the boundary CR structure has strictly positive Levi forms at all points.

Note that if $(N, \mathbb{J})$ is a complex manifold of class $\mathscr{C}^{k,\alpha}$, with $k \geq 1$ and $\alpha \in (0, 1)$, any relatively compact strongly pseudoconvex domain $D \subset N$ with smooth boundary is a manifold with boundary, equipped with a complex structure $(J, \mathscr{D}, J^{\mathscr{D}})$ of class $\mathscr{C}^{k,\alpha}$. Indeed, Definition 2.1 is designed to capture precisely the properties of such domains.

2.2 Monge-Ampère exhaustions

Definition 2.2 Let $(\overline{M} = M \cup \partial M, J)$ be a complex manifold of class $\mathscr{C}^\infty$ with a strongly pseudoconvex boundary ∂M. Given $k \geq 2$ and $\alpha \in (0, 1)$, we call *Monge-Ampère exhaustion for* $\overline{M}$ *of class* $\mathscr{C}^{k,\alpha}$ a continuous exhaustion $\tau : \overline{M} \to [0, 1]$, with $\partial M = \{x \in M \,:\, \tau(x) = 1\}$, such that:

(i) The level set $\{\tau = 0\} \subset M$ consists of a single point x_o, called *center*, and the pull back $p^*(\tau)$ of τ on the blowup $p : \widetilde{\overline{M}} \to \overline{M}$ at x_o, is of class $\mathscr{C}^{k,\alpha}$;

(ii) On the complement $M\backslash\{x_o\} = \{0 < \tau < 1\}$ of the center, the exhaustion τ is a solution to the differential problem

$$\begin{cases} 2i\,\partial\overline{\partial}\tau = dd^c\tau > 0\,, \\ 2i\,\partial\overline{\partial}\log\tau = dd^c\log\tau \geq 0\,, \\ (dd^c\log\tau)^n = 0 \ \text{ (Monge-Ampère Equation)}\ ; \end{cases} \tag{2.2}$$

(iii) In some (hence, in *any*) system of complex coordinates $z = (z^i)$ centered at x_o, the exhaustion τ has a logarithmic singularity at x_o, i.e.,

$$\log\tau(z) = \log\|z\| + O(1)\,.$$

Basic examples of Monge-Ampère exhaustions of class $\mathscr{C}^{k,\alpha}$ are given by the Minkowski functionals μ_D of the strongly pseudoconvex complete circular domains $D \subset \mathbb{C}^n$ with boundary of class $\mathscr{C}^{k,\alpha}$, $k \geq 2$, $\alpha \in (0, 1)$. In fact, by definition, μ_D is the function

$$\mu_D : \mathbb{C}^n \longrightarrow [0, +\infty)\,, \qquad \mu_D(z) = \begin{cases} 0 & \text{if } z = 0\,, \\ 1/t_z & \text{if } z \neq 0\,, \end{cases}$$

where $t_z = \sup\{t \in \mathbb{R} : tz \in D\}$, so that $D = \{\mu_D < 1\}$ because D is balanced. Being D strongly pseudoconvex, $\tau := \mu_D^2|_D$ is a Monge-Ampère exhaustion of class $\mathscr{C}^{k,\alpha}$, centered at $x_o = 0$.

Other crucial examples are given by (reparametrizations of) the Kobayashi distance functions of the strictly convex domains in $\mathbb{C}^n$. Indeed, by the results of Lempert in [12], if $D \subset \mathbb{C}^n$ is a strictly convex domain with a $\mathscr{C}^{k+2,\alpha}$-boundary, $k \geq 2$, each point $x_o \in D$ is the center of a Monge-Ampère exhaustion $\tau^{(x_o)} : \overline{D} \longrightarrow [0, 1]$ of class $\mathscr{C}^{k,\alpha}$, where $\tau^{(x_o)}(x)$ is the squared hyperbolic tangent of the Kobayashi distance between x and x_o. The same is true for strictly linearly convex domains in $\mathbb{C}^n$ in the C^∞ and the C^ω case ([13]).

Let $\tau : \overline{M} \to [0, 1]$ be a Monge-Ampère $\mathscr{C}^{k,\alpha}$-exhaustion with center x_o and

$$\kappa : T_{x_o}M \simeq \mathbb{C}^n \longrightarrow \mathbb{R}_{\geq 0}\,, \qquad \kappa(v) := \lim_{t_o \to 0} \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\tau(\gamma_t)}\bigg|_{t=t_o}\,, \tag{2.3}$$

where γ_t is any smooth curve with $\gamma_0 = x_o$ and $\dot\gamma_0 = v$. Such κ is well defined and coincides with the Kobayashi infinitesimal metric of M at x_o ([21]), it is of class $\mathscr{C}^{k,\alpha}$ on $T_{x_o}M\backslash\{0\}$ and satisfies $\kappa(\lambda v) = |\lambda|\kappa(v)$ for any $\lambda \in \mathbb{C}$ so that the *(closed) indicatrix at x_o of* $((M, J), \tau)$ defined by

$$\overline{\mathscr{I}} := \{v \in T_{x_o}M : \kappa(v) \le 1\} \subset T_{x_o}M \simeq \mathbb{C}^n \,, \tag{2.4}$$

is a complete circular domain.

Let $p : \widetilde{\overline{\mathscr{I}}} \to \overline{\mathscr{I}}$ and $p' : \widetilde{\overline{M}} \to \overline{M}$ be the blowups of $\overline{\mathscr{I}}$ and $\overline{M}$ at 0 and x_o, respectively. A word by word repetition of the arguments in [20], where it was assumed τ to be of class $\mathscr{C}^\infty$, shows that *if the Monge-Ampère exhaustion τ is of class $\mathscr{C}^{k,\alpha}$, for some $k \ge 3$ and $\alpha \in (0, 1)$, there exists a unique $\mathscr{C}^{k-2,\alpha}$-diffeomorphism* $\Psi : \widetilde{\overline{\mathscr{I}}} \to \widetilde{\overline{M}}$ satisfying the following three conditions:

(i) $\Psi|_{p^{-1}(0)} = \mathrm{Id}_{p'^{-1}(0)}$;
(ii) For each $t \in (0, 1]$ the map

$$\Psi^{(t)} : \partial\mathscr{I} \to \widetilde{\overline{M}} \,, \qquad \Psi^{(t)}([v], z) := \Psi([v], tz) \,,$$

is a diffeomorphism between $\partial\mathscr{I}$ and $\{\tau = t^2\}$ mapping the real distribution of the CR structure of $\partial\mathscr{I}$ into the real distribution of the CR structure of $\{\tau = t^2\}$;
(iii) For each $([v], z) \in \partial\mathscr{I}$, the map

$$\widetilde{f}^{([v],z)} : \overline{\Delta} \longrightarrow \widetilde{\overline{M}} \,, \qquad \widetilde{f}^{([v],z)}(\zeta) := \Psi([v], \zeta z) \,,$$

is proper holomorphic and injective and $\widetilde{f}^{([v],z)}(\Delta\backslash\{0\})$ is an integral leaf of the distribution $\mathscr{Z}$, given by the spaces $\mathscr{Z}_x = \ker dd^c\tau_x$.

Such a map Ψ is called *circular representation of* $(\overline{M}, J)$ determined by τ.

The proof shows also that this circular representation Ψ is such that:

(a) The projection onto $\overline{\mathscr{I}}$ of the pulled-back exhaustion $\tau \circ \Psi : \widetilde{\overline{\mathscr{I}}} \to [0, 1]$ coincides with the Kobayashi infinitesimal metric κ and is of class $\mathscr{C}^{k-2,\alpha}$ on $\overline{\mathscr{I}}\backslash\{0\}$;
(b) If $k \ge 4$, the pulled-back complex structure $\widetilde{J}' = \Psi^*(J)$ on $\widetilde{\overline{\mathscr{I}}}$ is integrable and of class $\mathscr{C}^{k-3,\alpha}$, i.e., $(\widetilde{\overline{\mathscr{I}}}, \widetilde{J}')$ is a $\mathscr{C}^{k-2,\alpha}$ complex manifold.

By classical results on blowups and Remmert reductions, the complex manifold $(\widetilde{\overline{\mathscr{I}}}, \widetilde{J}')$ has a blow-down, which can be naturally identified with the manifold with boundary $\overline{\mathscr{I}}$, equipped with an appropriate nonstandard atlas $\mathscr{A}$ of complex charts. Denoting by J' the tensor field, for which $\mathscr{A}$ is the atlas of J'-holomorphic coordinates, we conclude that *the $\mathscr{C}^{k-2,\alpha}$-diffeomorphism $\Psi : \widetilde{\overline{\mathscr{I}}} \to \widetilde{\overline{M}}$ determines a (J', J)-biholomorphism $\Psi : (\overline{\mathscr{I}}, J') \longrightarrow (\overline{M}, J)$ and induces the $\mathscr{C}^{k-2,\alpha}$ Monge-Ampère exhaustion $\kappa := \tau \circ \Psi$ on $(\overline{\mathscr{I}}, J')$*.

We stress that, despite of the fact that the tensor field J' on $\overline{\mathscr{I}}$ has smooth components in each chart of the atlas $\mathscr{A}$, such tensor field has in general non-smooth components in the standard coordinates of $\mathscr{I} \subset \mathbb{C}^n$. Indeed, they are in most cases not even differentiable at 0. Nonetheless, J' has the same regularity of $\widetilde{J}'$ at the points of $\overline{\mathscr{I}}\backslash\{0\} = \widetilde{\overline{\mathscr{I}}}\backslash p^{-1}(0)$—we refer to [23] for a more detailed discussion of all this. (Here and in all what follows, for any blowup we tacitly identify the complementary region of the exceptional divisor with its image in the blow down.)

2.3 Manifolds of circular type with boundary

The discussion of the previous section leads to the following notion.

Definition 2.3 Given $k \geq 2$, $\alpha \in (0,1)$, we call *manifold of circular type with boundary of class* $\mathscr{C}^{k,\alpha}$ (or, simply, $\mathscr{C}^{k,\alpha}$*-manifold of circular type*) a pair $((\overline{M}, J), \tau)$ formed by

- a complex manifold $(\overline{M}, J)$ of class $\mathscr{C}^{k,\alpha}$ with strongly pseudoconvex boundary, diffeomorphic to the closed unit ball $\overline{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ of $\mathbb{C}^n$;
- a Monge-Ampère exhaustion $\tau : \overline{M} \longrightarrow [0,1]$ of class $\mathscr{C}^{k,\alpha}$ in each set of J-holomorphic coordinates.

The main example of a $\mathscr{C}^\infty$-manifold of circular type to keep in mind is the pair $((\overline{\mathbb{B}}^n, J_{\rm st}), \tau_o))$, where $\overline{\mathbb{B}}^n \subset \mathbb{C}^n$ is the closed unit ball centered at 0, equipped with the standard complex structure $J_{\rm st}$ of $\mathbb{C}^n$ and the standard Monge-Ampère exhaustion $\tau_o(x) = \|x\|^2$ where $\|\cdot\|$ is the Euclidean norm of $\mathbb{C}^n$. This basic example comes with two special distributions in the tangent space $T(\mathbb{B}^n\backslash\{0\})$, which have direct analogues in any other manifold of circular type. The first is the $J_{\rm st}$-invariant distribution $\mathscr{Z} \subset T(\mathbb{B}^n\backslash\{0\})$ of the spaces

$$\mathscr{Z}_x := \{v \in T_x\mathbb{B}^n : (\partial\bar\partial \log \tau_o)(v, \cdot) = 0\} = Span_{\mathbb{C}}\left(z^i \frac{\partial}{\partial z^i}\Big|_x\right). \tag{2.5}$$

The second is the distribution $\mathscr{H}$ of the complementary subspaces

$$\mathscr{H}_x = (\mathscr{Z}_x)^\perp = \{v \in T_x\mathbb{B}^n : < v, \mathscr{Z}_x >= 0\}, \tag{2.6}$$

where $< \cdot, \cdot >$ denotes the standard Euclidean inner product of $\mathbb{C}^n$.

These distributions are not defined at 0, but they both admit smooth extensions at all points of the exceptional divisor of the blowup $p : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{B}^n$ at 0. Note also that $\mathscr{Z}$ is integrable, with integral leaves given by the straight disks through 0. On the contrary, $\mathscr{H}$ is not integrable. In fact, the restriction of $\mathscr{H}$ to each sphere $S_r := \{\|x\| = r\}$, $0 < r \leq 1$, coincides with the contact distribution underlying the CR structure of such sphere. Both distributions $\mathscr{Z}$ and $\mathscr{H}$ are $J_{\rm st}$-invariant, so that the complex distributions $\mathscr{Z}^{10}$, $\mathscr{Z}^{01} \subset \mathscr{Z}^{\mathbb{C}}$ and $\mathscr{H}^{10}$, $\mathscr{H}^{01} \subset \mathscr{H}^{\mathbb{C}}$, of the $(+i)$- and $(-i)$-eigenspaces of $J_{\rm st}$, are well defined at each point of $\widetilde{\mathbb{B}}^n$. The complex distributions $\mathscr{H}^{10}$, $\mathscr{H}^{01}$ admit smooth extensions up to the boundary, where they determine the standard CR structure of $\partial\mathbb{B}^n$.

We now introduce a crucial class of deformations of $((\overline{\mathbb{B}}^n, J_{\rm st}), \tau_o)$.

Definition 2.4 Let $k \geq 2$ and $\alpha \in (0,1)$. We call *L-complex structure of class* $\mathscr{C}^{k-1,\alpha}$ a complex structure $\widetilde{J}$ of class $\mathscr{C}^{k-1,\alpha}$ on the blowup $\widetilde{\overline{\mathbb{B}}}^n$ of $(\overline{\mathbb{B}}^n, J_{\rm st})$ at the origin, satisfying the following conditions:

(i) it leaves invariant all spaces of the distributions $\mathscr{Z}$ and $\mathscr{H}$;
(ii) $\widetilde{J}|_{\mathscr{Z}} = J_{\rm st}|_{\mathscr{Z}}$;
(iii) there exists a homotopy $\widetilde{J}_t$, $t \in [0,1]$, of class $\mathscr{C}^{k-1,\alpha}$ in the parameter t, between $\widetilde{J}_{t=0} = J_{\rm st}$ and $\widetilde{J}_{t=1} = \widetilde{J}$, such that each $\widetilde{J}_t$ is a complex structures of class $\mathscr{C}^{k-1,\alpha}$ satisfying (i) and (ii).

If $\widetilde{J}$ is an L-complex structure of class $\mathscr{C}^{k-1,\alpha}$ on $\widetilde{\overline{\mathbb{B}}}^n$, let J be the (nonstandard) complex structure on $\overline{\mathbb{B}}^n$ that makes $(\overline{\mathbb{B}}^n, J)$ the blow-down of the complex manifold with boundary $(\widetilde{\overline{\mathbb{B}}}^n, \widetilde{J})$ (It exists by the above-mentioned facts on Remmert reductions.) By the results in [23] and the following remark, each pair of the form $((\overline{\mathbb{B}}^n, J), \tau_o)$, with J coming from an L-complex structure $\widetilde{J}$ as above, is a $\mathscr{C}^{k,\alpha}$-manifold of circular type, called *in normal form*.

Remark 2.5 The fact that τ_o is of class $\mathscr{C}^{k,\alpha}$ in the charts of $(\overline{\mathbb{B}}^n\backslash\{0\}, J)$ can be easily checked as follows. By construction, the components in standard coordinates of the complex structure J are of class $\mathscr{C}^{k-1,\alpha}$ at all points of $\overline{\mathbb{B}}^n\backslash\{0\} = \widetilde{\overline{\mathbb{B}}}^n\backslash p^{-1}(0)$. This implies that each chart of J-holomorphic coordinates of $(\overline{\mathbb{B}}^n\backslash\{0\}, J)$ overlaps in a $\mathscr{C}^{k,\alpha}$ way with the standard coordinates (see § 2.1). Being $\tau_o = \|\cdot\|^2$ of class $\mathscr{C}^\infty$ in the standard coordinates outside the origin, it must be of class $\mathscr{C}^{k,\alpha}$ when it is expressed in J-holomorphic coordinates.

The interest for the manifolds in normal forms comes from the fact that, for $k \geq 4$, any $\mathscr{C}^{k,\alpha}$-manifold of circular type is biholomorphic with a $\mathscr{C}^{k-2,\alpha}$-manifold of circular type in normal form. This property was proven in [22] just for the case of $\mathscr{C}^\infty$-manifolds of circular type, but a careful check of all arguments in that paper shows that they actually go through for each $\mathscr{C}^{k,\alpha}$-manifold of circular type, provided that $k \geq 4$. More precisely:

(1) By the remarks in Sect. 2.2, if $((\overline{M}, J), \tau)$ is a $\mathscr{C}^{k,\alpha}$-manifold of circular type with $k \geq 4$, $\alpha \in (0, 1)$, the circular representation determines a (J, J')-biholomorphism from $(\overline{M}, J)$ into the $\mathscr{C}^{k-2,\alpha}$-manifold of circular type $((\overline{\mathscr{I}}, J'), \kappa)$, given by the closed indicatrix $\overline{\mathscr{I}}$ of the Kobayashi infinitesimal metric κ at the center x_o, an appropriate complex structure J' of class $\mathscr{C}^{k-3,\alpha}$ on $\overline{\mathscr{I}}$ and the exhaustion κ, which is $\mathscr{C}^{k,\alpha}$ in the complex charts of $(\overline{\mathscr{I}}, J')$.

(2) Since $dd^c\kappa$ is of class $\mathscr{C}^{k-2,\alpha}$ and $k \geq 4$, all arguments of the proofs of Moser's Theorem in [15] and of Lemma 3.5 and Thm. 3.4 in [22] remain valid and yield to the existence of:

(i) A smooth family of $\mathscr{C}^{k-2,\alpha}$ diffeomorphisms $\Phi_t : \widetilde{\mathbb{C}^n} \to \widetilde{\mathbb{C}^n}$ from the blowup $\widetilde{\mathbb{C}^n}$ of $\mathbb{C}^n$ at 0 into itself, satisfying all conditions of Lemma 3.5 in [22];
(ii) A smooth isotopy of L-complex structures $J_t = \Phi_{t*}(J_{\rm st})$ of class $\mathscr{C}^{k-3,\alpha}$ between $J_0 = J_{\rm st}$ and a (nonstandard) complex structure $J_1 = J''$ that makes $((\widetilde{\overline{\mathbb{B}}}^n, J''), \tau_o)$ biholomorphic to the blowup $(\widetilde{\overline{\mathscr{I}}}, J')$ of $(\overline{\mathscr{I}}, J')$ at the origin. Such a complex structure J'' projects onto a (nonstandard) complex structure J'' on $\overline{\mathbb{B}}^n$ that makes $((\overline{\mathbb{B}}^n, J''), \tau_o)$ a $\mathscr{C}^{k-2,\alpha}$-manifold of circular type in normal form, biholomorphic to $((\overline{\mathscr{I}}, J'), \kappa^2)$.

Combining (1) and (2) (as it is done in [22], Thm. 3.4), one gets

Theorem 2.6 *Let $k \geq 4$, $\alpha \in (0, 1)$ and $((\overline{M}, J), \tau)$ be a $\mathscr{C}^{k,\alpha}$-manifold of circular type with center x_o. Then, there is a (J, J')-biholomorphism $\Phi : (\overline{M}, J) \to (\overline{\mathbb{B}}^n, J')$ from $((\overline{M}, J), \tau)$ to a $\mathscr{C}^{k-2,\alpha}$-manifold in normal form $((\overline{\mathbb{B}}^n, J'), \tau_o)$ with the following properties:*

(a) $\Phi(x_o) = 0$ and $\tau = \tau_o \circ \Phi = \Phi^*(\tau_o)$;
(b) Φ maps the integral leaves of the distribution on $M\backslash\{x_o\}$

$$\mathscr{Z}^{J,\tau} := \bigcup_{x\in M\backslash\{x_o\}} \mathscr{Z}^{J,\tau}_x \,, \quad \mathscr{Z}^{J,\tau}_x := \{v \in T_xM \,:\, dd^c_J \log\tau(v,\cdot) = 0\} \tag{2.7}$$

into the integral leaves of the distribution $\mathscr{Z}$ of $\mathbb{B}^n$, that is into the straight disks through the origin of $\mathbb{B}^n$.

A biholomorphism $\Phi : (\overline{M}, J) \longrightarrow (\overline{\mathbb{B}}^n, J')$, mapping a $\mathscr{C}^{k,\alpha}$-manifold of circular type into one in normal form, satisfying (a) and (b) of the above theorem, is called *normalizing map*.

2.4 Statement of the main result

All results of this paper are consequence of the following theorem on manifolds in normal form, the proof of which is divided in the remaining three sections.

Theorem 2.7 *Let $((\overline{\mathbb{B}}^n, J), \tau_o)$ be a manifold of circular type in normal form of class $\mathscr{C}^k$ with $k \geq 3$. Then, for each $v \in \mathbb{C}^n$ with $\|v\| = 1$ there exists a value $\lambda_v \in (0, 1]$ such that:*

(a) For each $\lambda \in (0, \lambda_v)$ and the corresponding parameterized segment $x_t := t\lambda v$ between 0 and $x_1 = \lambda v$, there is a one-parameter family of Monge-Ampère exhaustions $\tau^{(x_t)} : \overline{\mathbb{B}}^n \longrightarrow [0, 1]$, with $t \in [0, 1]$, centered at x_t and of class $\mathscr{C}^{k-1,\alpha}$ for each $\alpha \in (0, 1)$. The restrictions of the $\tau^{(x_t)}$, $t \in I \subset [0, 1]$, on any open subset $\mathscr{V} \subset D$ not containing the centers x_t, $t \in I$, gives a real function on $\mathscr{V} \times I$ which is of class $\mathscr{C}^{k-1,\alpha}$ in all of its arguments, i.e., also with respect to t.

(b) If $\lambda_v < 1$, there is no Monge-Ampère exhaustion of class $\mathscr{C}^{k-1,\alpha}$ for any $\alpha \in (0, 1)$, which is centered at the point $\widehat{x} = \lambda_v v$.

The set $\mathscr{S} := \{z \in \overline{\mathbb{B}}^n \ : \ z = \lambda_v v \ , \ \|v\| = 1\}$ is the boundary of an open neighborhood $\mathscr{U} \subset \overline{\mathbb{B}}^n$ of 0 and is the projection onto $\overline{\mathbb{B}}^n \setminus p^{-1}(0)$ of the intersection of two submanifolds, uniquely determined by the complex structure J, of the bundle $\mathscr{H}^{\mathbb{C}*} \otimes \mathscr{H}^{\mathbb{C}}|_{\widetilde{\overline{\mathbb{B}}^n} \setminus p^{-1}(0)} \to \widetilde{\overline{\mathbb{B}}^n} \setminus p^{-1}(0)$.

Due to Theorem 2.7, this immediately implies our main result, which is

Theorem 2.8 *Let $k \geq 5$ and $((\overline{M}, J), \tau)$ be a $\mathscr{C}^k$-manifold of circular type. Then there exists an open neighborhood M' of the center x_o of τ with the property that for each $x \in M'$ there exists a Monge-Ampère exhaustion $\tau^{(x)} : \overline{M} \longrightarrow [0, 1]$, of class $\mathscr{C}^{k-3,\alpha}$ for any $\alpha \in (0, 1)$, with center x. The dependence of the exhaustions $\tau^{(x)}$ on the center x is also of class $\mathscr{C}^{k-3,\alpha}$.*

By uniqueness of the Monge-Ampère exhaustions with given centers, Theorem 2.8 shows that if a manifold with boundary of circular type $\overline{M}$ admits a Monge-Ampère exhaustion of class $\mathscr{C}^\infty$, hence of class $\mathscr{C}^k$ for each k, it is equipped with $\mathscr{C}^\infty$ Monge-Ampère exhaustions, centered at all points of an appropriate open subset M'. In fact, as we shall point out later on, the determination of the maximal open set M' with the properties described in Theorem 2.8 is independent of k as long as $k \geq 5$. Furthermore some of the steps in the proof of Theorem 2.7 show that its $\mathscr{C}^\omega$ version holds. Thus, we also have:

Theorem 2.9 *Let $((\overline{M}, J), \tau)$ be a $\mathscr{C}^\infty$-manifold (resp. $\mathscr{C}^\omega$-manifold) of circular type. Then there exists an open neighborhood M' of the center x_o of τ with the property that for each $x \in M'$ there exists a Monge-Ampère exhaustion $\tau^{(x)} : \overline{M} \longrightarrow [0, 1]$, of class $\mathscr{C}^\infty$ (resp. $\mathscr{C}^\omega$), with center x. The dependence of the exhaustions $\tau^{(x)}$ on the center x is also of class $\mathscr{C}^\infty$ (resp. $\mathscr{C}^\omega$).*

Direct consequences of this are Theorem 1.1 and the properties of pluricomplex Green functions, which have been discussed in the Introduction.

3 One-parameter families of Monge-Ampère exhaustions

3.1 Quasi-diffeomorphisms and quasi-regular vector fields

Let $k \geq 2$, $\alpha \in (0, 1)$ and assume that $(\overline{M}, J)$ and $(\overline{M'}, J')$ are two complex n-manifolds with boundary of class $\mathscr{C}^{k,\alpha}$ and let

$$\pi : \widetilde{\overline{M}} \longrightarrow \overline{M} \, , \qquad \pi' : \widetilde{\overline{M}}' \longrightarrow \overline{M'}$$

be the blowups of $\overline{M}$ and $\overline{M'}$ at some fixed interior points x, x' with *exceptional divisors* $\pi^{-1}(x) \subset \widetilde{\overline{M}}$ and $\pi'^{-1}(x') \subset \widetilde{\overline{M}}'$ both biholomorphic to $\mathbb{C}P^{n-1}$.

Definition 3.1 A $\mathscr{C}^{k,\alpha}$-diffeomorphism $\widetilde{F} : \widetilde{\overline{M}} \to \widetilde{\overline{M'}}$ is called *tame* if it maps diffeomorphically the exceptional divisor of $\widetilde{\overline{M}}$ onto the exceptional divisor of $\widetilde{\overline{M'}}$, so that $\widetilde{F}$ induces a homeomorphism $F : \overline{M} \to \overline{M'}$, mapping x into x', which is of class $\mathscr{C}^{k,\alpha}$ on $\overline{M}\backslash\{x\}$. We call any such homeomorphism a *quasi-diffeomorphism between* $\overline{M}$, $\overline{M'}$ *of class* $\mathscr{C}^{k,\alpha}$, *pointed at* x. The associated tame diffeomorphism $\widetilde{F}$, inducing F, is called *tame lift of* F.

Among the various reasonable infinitesimal counterparts of the notion of quasi-diffeomorphism, the one that better fits with our purposes (and which we formally introduce at the end of this section) is rooted in the following observations. Let $x_t : [0, 1] \longrightarrow M = \overline{M}\backslash\partial M$ be a $\mathscr{C}^2$ curve entirely included in the interior M. For each $t \in [0, 1]$, consider the blowup $\pi_t : \widetilde{\overline{M}}_t \longrightarrow \overline{M}$ of $\overline{M}$ at x_t. A one-parameter family $F_t : \overline{M} \longrightarrow \overline{M}$ of quasi-diffeomorphisms of class $\mathscr{C}^{k,\alpha}$ pointed at the x_t induced by a family of tame diffeomorphisms $\widetilde{F}_t : \widetilde{\overline{M}}_t \longrightarrow \widetilde{\overline{M}}_1$ of class $\mathscr{C}^{1,\beta}$, $\beta > 0$, in the coordinates of $\bigcup_{t\in[0,1]} \widetilde{\overline{M}}_t \simeq \widetilde{\overline{M}}_1 \times [0, 1]$, is called $\mathscr{C}^{1,\beta}$*-family of* $\mathscr{C}^{k,\alpha}$ *quasi-diffeomorphisms, pointed at the points* x_t.

For any such family, at a fixed $t \in [0, 1]$, we may consider the vector field X_t, defined at the points $x \in M\backslash\{x_1\}$ by

$$X_t\big|_x := \left.\frac{\mathrm{d}F_{t+s}(F_t^{-1}(x))}{\mathrm{d}s}\right|_{s=0} . \tag{3.1}$$

It is a vector field of class $\mathscr{C}^{k,\alpha}$ on $\overline{M}\backslash\{x_1 = F_t(x_t)\}$, whose restriction to ∂M is always tangent to the boundary and of class $\mathscr{C}^{k,\alpha}$. Moreover, if we extend X_t at the point x_1 by setting (for the existence of this derivative, see the argument below)

$$X_t\big|_{x_1} := \left.\frac{\mathrm{d}F_{t+s}(x_t)}{\mathrm{d}s}\right|_{s=0} , \tag{3.2}$$

we get a vector field over the whole $\overline{M}$ with the following property: *for each sequence* $y_k \to x_1$, *with* $y_k \in \overline{M}\backslash\{F_t(x_s), s \in [0, 1]\}$,

$$\lim_{k\to\infty} X_t\big|_{y_k} = X_t\big|_{x_1} .$$

Indeed, this can be checked as follows. At each point y_k, the vector $X_t\big|_{y_k}$ is the tangent vector at $s = 0$ of the curve $\eta_s^{(t|y_k)} := F_{t+s}(F_t^{-1}(y_k))$ with s in some fixed small interval $[-\varepsilon, \varepsilon]$. Note that, by the assumption on the sequence y_k, for each $s \in [-\varepsilon, \varepsilon]$, the point $\eta_s^{(t|y_k)}$ satisfies

$$F_{t+s}^{-1}(\eta_s^{(t|y_k)}) = F_t^{-1}(y_k) \neq x_{t+s} \qquad \Longrightarrow \qquad \eta_s^{(t|y_k)} \neq F_{t+s}(x_{t+s}) = x_1 .$$

This implies that each curve $\eta_s^{(t|y_k)}$ admits a unique $\mathscr{C}^{1,\beta}$ lifted curve $\widetilde{\eta}_s^{(t|y_k)}$ on the blowup $\widetilde{\overline{M}}_1$. Moreover, by the regularity assumptions on the family of diffeomorphisms F_t, when $y_k \to x_1$, the lifted curves $\widetilde{\eta}_s^{(t|y_k)}$ tend uniformly in $\mathscr{C}^1$-norm to a curve $\widetilde{\eta}$ in $\widetilde{\overline{M}}_1$. Such a limit curve projects onto the $\mathscr{C}^{1,\beta}$ curve

$$\eta_s = F_{t+s}(F_t^{-1}(x_1)) = F_{t+s}(x_t) .$$

This implies that the vectors $X_t\big|_{y_k}$ (that are the tangent vectors at $s = 0$ of the curves $\eta_s^{(t|y_k)}$) tend to the tangent vector of η_s at $s = 0$, that is to (3.2).

All this motivates the following

Definition 3.2 Let $(\overline{M}, J)$ be a complex manifold of dimension n with boundary of class $\mathscr{C}^{k,\alpha}$ and $x \in M = \overline{M} \backslash \partial M$. A vector field X on $\overline{M}$ is called *quasi-regular of class* $\mathscr{C}^{k,\alpha}$, *pointed at* x, if:

(i) It is of class $\mathscr{C}^{k,\alpha}$ on $\overline{M}\backslash\{x\}$ and tangent to ∂M at boundary points;
(ii) There is an open and dense subset of $\mathscr{U} \subset \overline{M}\backslash\{x\}$, such that $\lim_{k\to\infty} X|_{y_k} = X|_x$ for each sequence $y_k \in \mathscr{U}$ converging to x.

3.2 Curves of centers and families of quasi-diffeomorphisms

Consider now a manifold of circular type in normal form $((\overline{\mathbb{B}}^n, J), \tau_o)$ of class $\mathscr{C}^{k,\alpha}$, with $k \geq 1, \alpha \in (0, 1)$. Let also $v \in \mathbb{B}^n\backslash\{0\}$ and x_t the straight curve

$$x_t = tv : [0, 1] \longrightarrow \overline{\mathbb{B}}^n \,. \tag{3.3}$$

If $(\overline{\mathbb{B}}^n, J)$ admits Monge-Ampère exhaustions τ_t of class $\mathscr{C}^{k,\alpha}$, $k \geq 4$, $\alpha \in (0, 1)$, centered at the points x_t, then it also has a one-parameter family of normalizing maps $\Phi_t : ((\overline{\mathbb{B}}^n, J), \tau_t) \longrightarrow ((\overline{\mathbb{B}}^n, J_t), \tau_o)$ for appropriate nonstandard complex structures J_t. On the base of this, we consider the following notion.

Definition 3.3 Let $k \geq 2$ and $\alpha \in (0, 1)$. We call *curve of Monge-Ampère quasi-diffeomorphisms of class* $\mathscr{C}^{k,\alpha}$, *guided by the curve* (3.3), a $\mathscr{C}^{k,\alpha}$-family of quasi-diffeomorphisms Φ_t of $(\overline{\mathbb{B}}^n, J)$, each of them pointed at $x_t = tv, t \in [0, 1]$, with the property that $\Phi_t(x_t) = 0$ for each t and satisfying the following condition: each pushed-forward complex structure $\widetilde{J}_t = \Phi_{t*}(\widetilde{J})$ of the complex structure $\widetilde{J}$ of the blowup $\pi_t : \widetilde{\overline{\mathbb{B}}}^n_t \to \overline{\mathbb{B}}^n$ at the point x_t, is an L-complex structure, so that its projected complex structure J_t determines a normal form $((\widetilde{\overline{\mathbb{B}}}^n, J_t), \tau_o)$.

We have the following

Proposition 3.4 *If Φ_t is a curve of Monge-Ampère quasi-diffeomorphisms of class $\mathscr{C}^{k,\alpha}$, guided by* (3.3)*, then for each $t \in [0, 1]$ the function*

$$\tau_t : \overline{\mathbb{B}}^n \longrightarrow [0, 1] \,, \qquad \tau_t(x) := \begin{cases} 0 & \text{if } x = x_t \\ (\tau_o \circ \Phi_t)(x) & \text{otherwise} \end{cases}$$

is a Monge-Ampère exhaustion of $(\overline{\mathbb{B}}^n, J)$ of class $\mathscr{C}^{k,\alpha}$, centered at x_t.

Proof By the properties of τ_o, each exhaustion τ_t satisfies (i) and (iii) of Definition 2.2. Hence, we only need to show that each τ_t satisfies also (ii) on $(\overline{\mathbb{B}}^n, J)$. This is in turn equivalent to prove that $\tau_o = \tau_t \circ \Phi_t^{-1}$ satisfies (2.2) on the complex manifold $(\overline{\mathbb{B}}^n, J_t)$. For this, we need to the following lemma. □

Lemma 3.5 *If J is the complex structure of a normal form $((\overline{\mathbb{B}}^n, J), \tau_o)$ then $dd^c_J \tau_o = dd^c_{J_{\mathrm{st}}} \tau_o$ at all points different from* 0*.*

Proof Let $\widetilde{J}$ be the L-complex structure on $\widetilde{\overline{\mathbb{B}}}^n$ which induces the complex structure J on $\overline{\mathbb{B}}^n$. We recall that $\widetilde{J}$ preserves both distributions $\mathscr{Z}$, $\mathscr{H}$ and that $J|_{\mathscr{Z}} = \widetilde{J}|_{\mathscr{Z}} = J_{\mathrm{st}}|_{\mathscr{Z}}$. Hence, if we denote by $(\cdot)^{\mathscr{Z}}$ the natural projection of each tangent space $T_x\mathbb{B}^n = \mathscr{Z}_x \oplus \mathscr{H}_x$ onto the subspace $\mathscr{Z}_x$, we have that

$$\begin{aligned} dd^c_J\tau_o(X,Y) &= -X(JY(\tau_o)) + Y(JX(\tau_o)) + J([X,Y])(\tau_o) \\ &= -X(J(Y)^{\mathscr{Z}}(\tau_o)) + Y(J(X)^{\mathscr{Z}}(\tau_o)) + J([X,Y]^{\mathscr{Z}})(\tau_o) \\ &= -X(J_{\mathrm{st}}(Y)^{\mathscr{Z}}(\tau_o)) + Y(J_{\mathrm{st}}(X)^{\mathscr{Z}}(\tau_o)) + J_{\mathrm{st}}([X,Y]^{\mathscr{Z}})(\tau_o) \\ &= dd^c_{J_{\mathrm{st}}}\tau_o(X,Y). \end{aligned}$$

□

By this lemma, the 2-form $dd^c_{J_t}\tau_o$ is nowhere degenerate and hence it is positive definite on $(\overline{\mathbb{B}}^n\backslash\{0\}, J_t)$, since J_t is isotopic to J_{st}. Similarly, one has that $dd^c_{J_t}\log\tau_o \geq 0$ and that $dd^c_{J_t}\log\tau_o$ satisfies the Monge-Ampère equation on $\overline{\mathbb{B}}^n\backslash\{0\}$. Thus all conditions of (2.2) are satisfied. □

Proposition 3.6 *If Φ_t is a curve of Monge-Ampère quasi-diffeomorphisms of class $\mathscr{C}^{k,\alpha}$, $k \geq 2$, $\alpha \in (0,1)$, the corresponding family of complex structures $J_t := \Phi_{t*}(J)$ is such that, for each $t \in [0,1]$, on $\overline{\mathbb{B}}^n\backslash\{0\}$ one has*

$$\frac{\mathrm{d}J_t}{\mathrm{d}t} = -\mathscr{L}_{X_t}J_t \qquad \text{where } X_t \text{ is defined by} \quad X_t|_x := \left.\frac{\mathrm{d}\Phi_s}{\mathrm{d}s}\right|_{(\Phi_t^{-1}(x),t)}. \tag{3.4}$$

Proof First of all, consider the one-parameter family of quasi-regular vector fields Y_t, defined at each point $x \in \overline{\mathbb{B}}^n\backslash\{x_t\}$ by

$$Y_t|_x := \left.\frac{\mathrm{d}\Phi_s^{-1}}{\mathrm{d}s}\right|_{(\Phi_t(x),t)}.$$

Note that, since $\Phi_s \circ \Phi_s^{-1} = \mathrm{Id}_{\overline{\mathbb{B}}^n}$ for each s, by taking the derivatives of both sides with respect to s at a fixed x, we have

$$0 = \left.\frac{\mathrm{d}\Phi_s}{\mathrm{d}s}\right|_{(\Phi_t^{-1}(x),t)} + \Phi_{t*}\left(\left.\frac{\mathrm{d}\Phi_s^{-1}}{\mathrm{d}s}\right|_{(x,t)}\right) = X_t|_x + \Phi_{t*}\left(Y_t|_{\Phi_t^{-1}(x)}\right),$$

from which we infer that, for each given t and x, $Y_t|_x$ and $X_t|_x$ are related by

$$X_t|_x = -\Phi_{t*}(Y_t)|_x\,. \tag{3.5}$$

We now recall that each complex structure J_t, $t \in [0,1]$, is well defined and of class $\mathscr{C}^{k',\alpha}$, with $k' = k-1 \geq 1$, $\alpha \in (0,1)$. Actually, by the considered regularity assumptions at $t=0$ and $t=1$, we may assume that J_t is well defined and $\mathscr{C}^{k',\alpha}$ for each t in a slightly larger open interval of $[0,1]$, say $(-\varepsilon, 1+\varepsilon)$. Let us pick a fixed value $t \in [0,1]$ and, for each s close to t, consider the the maps $F_s := \Phi_s \circ \Phi_t^{-1}$, $F_s^{-1} := \Phi_t \circ \Phi_s^{-1}$. By the definition of J_s, we have

$$J_s = \Phi_{s*}(J) = F_{s*}(\Phi_{t*}(J)) = F_{s*}(J_t)\,. \tag{3.6}$$

We also remark that, from (3.5),

$$\begin{aligned} \left.\frac{\mathrm{d}F_s}{\mathrm{d}s}\right|_{(F_t^{-1}(x),t)} &= \left.\frac{\mathrm{d}\Phi_s}{\mathrm{d}s}\right|_{(\Phi_t^{-1}(x),t)} = X_t|_x\,, \\ \left.\frac{\mathrm{d}F_s^{-1}}{\mathrm{d}s}\right|_{(x,t)} &= \Phi_{t*}\left(\left.\frac{\mathrm{d}\Phi_s^{-1}}{\mathrm{d}s}\right|_{(x,t)}\right) = \Phi_{t*}\left(Y_t|_{\Phi^{-1}(x)}\right) = \Phi_{t*}(Y_t)|_x = -X_t|_x\,. \end{aligned} \tag{3.7}$$

Consider now a (local) system of real coordinates $\xi = (x^i)$ on an open set, on which the tensor field J_s, the vector field Y_s and the map Φ_s are of class $\mathscr{C}^{k',\alpha}$ for all s in a small open interval $(t-\delta, t+\delta)$, entirely included in $(-\varepsilon, 1+\varepsilon)$. In these coordinates, J_t, X_t and the maps F_s, F_s^{-1}, have the form

$$\begin{aligned} &J_t = J_{tj}^{\ i} \frac{\partial}{\partial x^i} \otimes dx^j \ , \qquad X_t = X_t^i \frac{\partial}{\partial x^i} \ , \\ &F_s := \big(F_s^1(x^1, \dots, x^{2n}), \dots, F_s^{2n}(x^1, \dots, x^{2n})\big) \ , \\ &F_s^{-1} = \big((F_s^{-1})^1(x^1, \dots, x^{2n}), \dots, (F_s^{-1})^{2n}(x^1, \dots, x^{2n})\big) \ . \end{aligned}$$

We now observe that, by (3.6) and the classical coordinate expressions of push-forwards, the components $J_{sj}^{\ i}$ of $J_s|_x$ have the form

$$J_{sj}^{\ i}|_x = \left.\frac{\partial F_s^i}{\partial x^\ell}\right|_{F_s^{-1}(x)} \left.\frac{\partial (F_s^{-1})^m}{\partial x^j}\right|_x J_{tm}^{\ \ell}|_{F_s^{-1}(x)} \ . \tag{3.8}$$

So, differentiating (3.8) with respect to s at $s = t$, from (3.7) we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}(J_{tj}^{\ i}|_x)\right|_t = \left.\frac{\partial X_t^i}{\partial x^\ell}\right|_x J_{tj}^{\ \ell}|_x - \left.\frac{\partial X_t^m}{\partial x^j}\right|_x J_{tm}^{\ \ i}|_x - \left.\frac{\partial J_{tj}^{\ i}}{\partial x^r}\right|_x X_t^r|_x \ . \tag{3.9}$$

As the right hand side is the coordinate expression of $-\mathscr{L}_{X_t} J_t|_x$, the claim follows. □

The one-parameter family of pairs (J_t, X_t), given by the complex structures $J_t = \Phi_{t*}(J)$ and the vector fields (3.4), is called *fundamental pair of the Monge-Ampère quasi-diffeomorphisms* Φ_t. Note that, if we extend each vector field X_t at 0 by setting

$$X_t|_0 := \left.\frac{\mathrm{d}\Phi_{t+s}(x_t)}{\mathrm{d}s}\right|_{s=0} \ , \qquad x_t = tv \ , \tag{3.10}$$

then *each X_t is a quasi-$\mathscr{C}^{k,\alpha}$-regular, pointed at* 0 (Definition 3.2).

3.3 Abstract fundamental pairs and associated curves of exhaustions

Motivated by the correspondence of the previous section between curves of Monge-Ampère quasi-diffeomorphisms and families of pairs (J_t, X_t) of complex structures and quasi-regular vector fields, we now introduce the following

Definition 3.7 Let $((\overline{\mathbb{B}}^n, J), \tau_o)$ be a manifold in normal form of class $\mathscr{C}^{k,\alpha}, k \geq 2, \alpha \in (0,1)$, and a one-parameter family of vectors $v_t \in \mathbb{C}^n \setminus \{0\}$, of class $\mathscr{C}^{1,\beta}$, $\beta \in (0,1)$ in the parameter $t \in [0,1]$. We call *abstract fundamental pair guided by* v_t a pair of isotopies (J_t, X_t), of class at least $\mathscr{C}^{1,\beta}$, $\beta > 0$, in the parameter $t \in [0,1]$, where

(1) J_t is an isotopy of complex structure of class $\mathscr{C}^{k,\alpha}$ of a manifold in normal form $((\overline{\mathbb{B}}^n, J_t), \tau_o)$,
(2) X_t is an isotopy of quasi-regular vector fields X_t on $\overline{\mathbb{B}}^n$ of class $\mathscr{C}^{k,\alpha}$, pointed at 0, with $X_t|_0 = v_t$,

satisfying the differential condition

$$\frac{\mathrm{d}J_t}{\mathrm{d}t} = -\mathscr{L}_{X_t} J_t \qquad \text{with initial condition } J|_0 = J \ . \tag{3.11}$$

From the discussion of § 3.2, it is clear that the fundamental pair (J_t, X_t) of a curve Φ_t of Monge-Ampère quasi-diffeomorphisms satisfies all conditions of the above definition. Our main goal now is to show that a converse is also true, namely that each abstract fundamental pair satisfying appropriate conditions is the fundamental pair of a curve of Monge-Ampère quasi-diffeomorphisms and generates a one-parameter family of Monge-Ampère exhaustions, with centers along a fixed straight line.

The proof of this crucial result requires a few preliminaries. For each $x \in \mathbb{B}^n$, we fix once and for all an automorphism $\mathbb{F}_x \in \mathrm{Aut}\,(\overline{\mathbb{B}}^n)$ of the standard closed ball $(\overline{\mathbb{B}}^n, J_{\mathrm{st}})$, mapping x into 0. We select such a distinguished automorphism $\mathbb{F}_x$ in such a way that $\mathbb{F}_0 = \mathrm{Id}_{\overline{\mathbb{B}}^n}$ and that $\mathbb{F}_x$ depends smoothly on the point x. For each vector $0 \neq v \in \mathbb{B}^n$, we consider the points x_t of the parameterized straight line $x_t = tv$, and for each $t \in [0, 1]$, we set

$$\mathbb{F}_t := \mathbb{F}_{tv} \qquad \text{and} \qquad \mathbf{v}_t := \frac{\mathrm{d}\mathbb{F}_{t+s}(x_t)}{\mathrm{d}s}\bigg|_{s=0} \in T_0\mathbb{B}^n \,. \tag{3.12}$$

Finally, for each $t \in [0, 1]$, we consider the blowup $\pi_t : \widetilde{\overline{\mathbb{B}}}^n_t \to \overline{\mathbb{B}}^n$ of $\overline{\mathbb{B}}^n$ at the point x_t and the lifted map $\widetilde{\mathbb{F}}_t : \widetilde{\overline{\mathbb{B}}}^n_t \to \widetilde{\overline{\mathbb{B}}}^n$ of the automorphism $\mathbb{F}_t$ between $\widetilde{\overline{\mathbb{B}}}^n_t$ and the blowup $\widetilde{\overline{\mathbb{B}}}^n$ of the unit ball at the origin. By assumptions, each map $\widetilde{\mathbb{F}}_t$ transforms the exceptional divisor of $\widetilde{\overline{\mathbb{B}}}^n_t$ (which projects onto x_t) onto the exceptional divisor of $\widetilde{\overline{\mathbb{B}}}^n$ (which projects onto 0). All blowups $\widetilde{\overline{\mathbb{B}}}^n_t$ of the complex manifold $(\overline{\mathbb{B}}^n, J)$ are diffeomorphic one to the other and, in particular, to $\widetilde{\overline{\mathbb{B}}}^n_{t=0} = \widetilde{\overline{\mathbb{B}}}^n$. On the other hand, the complex structure of $\widetilde{\overline{\mathbb{B}}}^n$ is of class $\mathscr{C}^{k,\alpha}$ with respect to the standard coordinates of $\widetilde{\overline{\mathbb{B}}}^n$ and this implies that the complex coordinates of $\widetilde{\overline{\mathbb{B}}}^n$ are of class $\mathscr{C}^{k+1,\alpha}$ with respect to the standard one. Thus each $\widetilde{\overline{\mathbb{B}}}^n_t$ is $\mathscr{C}^{k+1,\alpha}$-diffeomorphic to the standard blowup $\widetilde{\overline{\mathbb{B}}}^n$ and, consequently, that each map $\mathbb{F}_t$ is a quasi-regular diffeomorphisms of class $\mathscr{C}^{k+1,\alpha}$ for the complex manifold $(\overline{\mathbb{B}}^n, J)$, pointed at x_t.

This settled, we may now state the following important lemma.

Lemma 3.8 *Let $0 \neq v \in \mathbb{B}^n$ and $\mathbf{v}_t \in \mathbb{C}^n$, $t \in [0, 1]$, the one-parameter family of vectors, defined in (3.12). If (J_t, X_t) is an abstract fundamental pair of class $\mathscr{C}^{k,\alpha}$, $k \geq 2$, $\alpha \in (0, 1)$, of $((\overline{\mathbb{B}}^n, J), \tau_o)$, with $X_t|_0 = \mathbf{v}_t$ for all t, then there exists a curve of Monge-Ampère quasi-diffeomorphisms Φ_t of class $\mathscr{C}^{k,\alpha}$, guided by the curve $x_t = tv$, $t \in [0, 1]$, of which (J_t, X_t) is the associated fundamental pair.*

Proof We observe that any curve of Monge-Ampère quasi-diffeomorphisms of class $\mathscr{C}^{k,\alpha}$, guided by the straight curve $x_t = tv$, $t \in [0, 1]$, can be expressed as a composition of the form $\Phi_t = \Phi^o_t \circ \mathbb{F}_t$, where Φ^o_t is a quasi-regular $\mathscr{C}^{k,\alpha}$ diffeomorphism, pointed at 0 and mapping 0 into itself. Hence, the curve of quasi-diffeomorphisms, of which we need to prove the existence, corresponds to a one-parameter family of tame diffeomorphisms of the blowup at the origin $\widetilde{\Phi}^o_t : \widetilde{\overline{\mathbb{B}}}^n \longrightarrow \widetilde{\overline{\mathbb{B}}}^n$, whose associated quasi-regular maps Φ^o_t are solutions to the problem

$$\frac{\mathrm{d}\Phi^o_s \circ \mathbb{F}_s(x)}{\mathrm{d}s}\bigg|_{s=t} = X_t|_{\Phi^o_t(\mathbb{F}_t(x))} \,, \; x \in \overline{\mathbb{B}}^n \backslash \{0\} \,, \quad \text{with } \Phi^o_0 = \mathrm{Id} \,, \tag{3.13}$$

$$(\Phi^o_t \circ \mathbb{F}_t)_*(J) = J_t \,, \tag{3.14}$$

$$\frac{\mathrm{d}\Phi^o_s \circ \mathbb{F}_t(x_t)}{\mathrm{d}s}\bigg|_{s=0} = \mathbf{v}_t \,. \tag{3.15}$$

Being X_t quasi-regular with $X_t|_0 = \mathbf{v}_t$, by continuity, each one-parameter family of quasi-regular maps Φ_t^o that solve (3.13) necessarily satisfies also (3.15), proving that the latter is redundant. Moreover, if Φ_t^o is a solution to (3.13), then, by Proposition 3.6, the fundamental pair of the curve of quasi-diffeomorphisms $\Phi_t^o \circ \mathbb{F}_t$ is a solution to the same differential problem satisfied by J_t with the same initial condition. By uniqueness of such solution, we get that also (3.14) is necessarily satisfied. In conclusion we only need to show the existence of a one-parameter family of $\mathscr{C}^{k,\alpha}$ quasi-diffeomorphisms satisfying the differential problem (3.13).

In turn, this is equivalent to the problem on $(\overline{\mathbb{B}}^n\backslash\{0\}) \times [0, 1]$,

$$\frac{\mathrm{d}\Phi_s^o(y)}{\mathrm{d}s}\bigg|_{s=t} + \frac{\mathrm{d}\mathbb{F}_s(\mathbb{F}_t^{-1}(y))}{\mathrm{d}s}\bigg|_{s=t} \cdot \Phi_t^o\bigg|_y = X_t|_{\Phi_t^o(y)}\,, \qquad \Phi_0^o = \mathrm{Id} \tag{3.16}$$

(here and below, for simplifying some formulas, we sometimes use the notation $V \cdot f$ for the directional derivative $V(f)$ of a function f along a vector field V). Let the maps Φ_t^o be the values at t of a map $\Phi^o : [0, 1] \times \overline{\mathbb{B}}^n\backslash\{0\} \longrightarrow \overline{\mathbb{B}}^n$ and V, W denote the vector fields

$$V|_{(t,y)} := \frac{\partial}{\partial t} + \frac{\mathrm{d}\mathbb{F}_s(\mathbb{F}_t^{-1}(y))}{\mathrm{d}s}\bigg|_{s=t}\,, \qquad W_{(t,y)} := X_t|_{\Phi_t^o(y)}$$

taking values in $T([0, 1] \times \overline{\mathbb{B}}^n\backslash\{0\}) = \mathbb{R} + T(\overline{\mathbb{B}}^n\backslash\{0\})$ and $T(\overline{\mathbb{B}}^n\backslash\{0\})$, respectively. In this way, (3.16) can be equivalently formulated as

$$V \cdot \Phi^o|_{(t,y)} - W|_{(t,y)} = 0\,, \qquad \Phi^o(0, y) = y\,. \tag{3.17}$$

Now, a map $\Phi^o : [0, 1] \times \overline{\mathbb{B}}^n\backslash\{0\} \longrightarrow \overline{\mathbb{B}}^n$ is a solution to this problem if and only if its graph

$$S = \{\, (t, y, x) \;:\; x = \Phi^o(t, y) \,\} \subset \Big([0, 1] \times \overline{\mathbb{B}}^n\backslash\{0\}\Big) \times \overline{\mathbb{B}}^n$$

is a $(2n + 1)$-dimensional real manifold, tangent at all points to the vector field

$$T_{(t,y,x)} := \frac{\partial}{\partial t} + W^i|_{(t,y)} \frac{\partial}{\partial y^i} + \Big(V \cdot \Phi^{o\,j}|_{(t,y)}\Big) \frac{\partial}{\partial x^j}\,.$$

The required graph S can be directly obtained by taking the union

$$S = \bigcup_{s\in[0,1]} \Psi_s^T(S_0)\,, \qquad \Psi_s^T = \text{flow of } T$$

of the images $\Psi_s^T(S_0)$, under the diffeomorphisms of the flow of T, of the transversal $2n$-submanifold

$$S_0 = \{t, y, x) \in ([0, 1] \times \overline{\mathbb{B}}^n\backslash\{0\}) \times \overline{\mathbb{B}}^n \;:\; t = 0\,,\; x = y\,\}\,.$$

By classical properties of flows, each diffeomorphism Ψ_s^T is of class $\mathscr{C}^{k,\alpha}$, so that S is the graph of the unique solution of (3.16) of class $\mathscr{C}^{k,\alpha}$ on $\overline{\mathbb{B}}^n\backslash\{0\}$.

By the regularity assumptions on the maps $\mathbb{F}_t$ and the limiting behavior of X_t at $x \to 0$, we see that each map $\Phi_t^o = \Phi^o(t, \cdot)$ has a unique extension to a map $\widetilde{\Phi}_t^o : \widetilde{\overline{\mathbb{B}}}^n \to \widetilde{\overline{\mathbb{B}}}^n$, mapping the exceptional divisor into itself. On the other hand, since $\Phi_t = \Phi_t^o \circ \mathbb{F}_t$ is a (J, J_t)-holomorphic map of $\mathbb{B}^n\backslash\{0\}$, the unique continuous extension at x_t is necessarily (J, J_t)-holomorphic also at x_t, i.e., it is a holomorphic maps in the usual sense in J- and J_t-holomorphic coordinates. The regularity assumptions on the lifts of J, J_t at the levels of the blow ups, imply that the continuous extension of $\widetilde{\Phi}_t^o$ is of class $\mathscr{C}^{k,\alpha}$ on $\widetilde{\overline{\mathbb{B}}}^n$ for each t.

In order to conclude the proof, it remains to show that each map $\widetilde{\Phi}_t = \widetilde{\Phi}_t^o \circ \widetilde{\mathbb{F}}_t$ is a $\mathscr{C}^{k,\alpha}$-diffeomorphism or, equivalently, that each Jacobian $J(\widetilde{\Phi}_t)|_y$ is invertible at all $y \in \widetilde{\overline{\mathbb{B}}}^n$. In fact, this would directly imply that the induced maps $\Phi_t : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ constitute a one-parameter family of Monge-Ampère quasi-diffeomorphism of class $\mathscr{C}^{k,\alpha}$, guided by the curve $x_t = tv$.

Note that, having proven that each $\Phi_t(x)$ is well defined and smooth at each point $(t, y) \in [0, 1] \times \widetilde{\overline{\mathbb{B}}}^n$, all Jacobians $J(\Phi_t)|_y$ are bounded for each such (t, y). Hence, in order to prove what we need, we simply have to show that all these Jacobians have nonvanishing determinants. Since this is surely true for the points $(t, y) \in \{0\} \times \widetilde{\overline{\mathbb{B}}}^n$ and thus, by compactness of $\widetilde{\overline{\mathbb{B}}}^n$, for all points of $[0, \varepsilon) \times \widetilde{\overline{\mathbb{B}}}^n$, with ε sufficiently small, the claim is proven if the set

$$A := \{t \in [0, 1] \ : \ \text{for each } s \in [0, t), \ J(\widetilde{\Phi}_s^o)|_y \text{ is invertible at each } y \in \widetilde{\overline{\mathbb{B}}}^n \}$$

coincides with $[0, 1]$. By the above remark and the semi-continuity of the rank, A is non-empty and open and it remains to prove that A is closed, i.e., that for each sequence $t_k \in A$ converging to $t_\infty := \sup A$, one has that $\lim_{k\to\infty} \left|\det J(\widetilde{\Phi}_{t_k})|_y\right| > 0$ for all $y \in \widetilde{\overline{\mathbb{B}}}^n$ or, equivalently,

$$\lim_{k\to\infty} \left|\det J(\widetilde{\Phi}_{t_k}^{-1})|_y\right| < \infty \qquad \text{for each } y \in \widetilde{\overline{\mathbb{B}}}^n \ . \tag{3.18}$$

Since each quasi-diffeomorphism $\widetilde{\Phi}_{t_k} = \widetilde{\Phi}_{t_k}^o \circ \widetilde{\mathbb{F}}_{t_k}$, $t \in A$, satisfies (3.14), for each straight disk $\overline{\Delta^{(c)}} = \{ \zeta c \, , \zeta \in \overline{\Delta}\} \subset \overline{\mathbb{B}}^n$, $c = (c^a) \neq 0$, the restriction $\widetilde{\Phi}_{t_k}^{-1}|_{\overline{\Delta^{(c)}}}$ maps properly and biholomorphically $\overline{\Delta^{(c)}}$ into $\overline{\mathbb{B}}^n$. Therefore, on each such disk, the sequence of holomorphic maps $\widetilde{\Phi}_{t_k}^{-1}|_{\overline{\Delta^{(c)}}}$ converges uniformly to a proper holomorphic map $f_\infty : \overline{\Delta^{(c)}} \to \widetilde{\overline{\mathbb{B}}}^n$ passing through the point $\widetilde{\Phi}_{t_\infty}^{-1}(0) = t_\infty v \in \mathbb{B}^n$. This implies that for each $y \in \overline{\Delta^{(c)}}$, the restrictions of the push-forward $\widetilde{\Phi}_{t_k *}^{-1}|_y$ to the tangent space $T_y\Delta^{(c)}$ have uniformly bounded components. Since the tangent spaces $T_y\Delta^{(c)}$ coincides with the spaces $\mathscr{Z}_y$ of the distribution $\mathscr{Z}$, we conclude that the restrictions $\widetilde{\Phi}_{t_k *}^{-1}|_{\mathscr{Z}}$ are uniformly bounded and that the proof of (3.18) reduces to check that the restricted linear map $\widetilde{\Phi}_{t_k *}^{-1}|_{\mathscr{H}_y}$, $y \in \widetilde{\overline{\mathbb{B}}}^n$, have uniformly bounded components.

To prove this, we first focus on the limit behavior of the linear maps $\widetilde{\Phi}_{t_k *}^{-1}|_{\mathscr{H}_y}$ at the points $y \in \partial\mathbb{B}^n$. The exhaustion $\tau_\infty = \tau_o \circ \Phi_{t_\infty}$ is a regular defining functions for $(\overline{\mathbb{B}}^n, J)$, i.e., with $d\tau_\infty \neq 0$ at all points of $\partial\mathbb{B}^n$. This can be checked by simply noticing that the derivatives of τ_∞ at boundary points along vectors that are tangent to the limit disks $\overline{\Delta^{(c)\infty}} := \lim_{k\to\infty} \widetilde{\Phi}_{t_k}^{-1}(\overline{\Delta^{(c)}})$ (which are transversal to $\partial\mathbb{B}^n$ by Hopf Lemma) are equal to the derivatives of τ_o at boundary points along vectors that are tangent to the radial disks $\overline{\Delta(c)}$ and are therefore nonvanishing. Being τ_∞ a regular defining function, the Levi forms $dd_J^c\tau_\infty(\cdot, J\cdot)|_{\mathscr{H}_y \times \mathscr{H}_y}$ and $dd_J^c\tau_\infty(\cdot, J\cdot)|_{\mathscr{H}_y \times \mathscr{H}_y}$ at each $y \in \partial\mathbb{B}^n$ are multiples one of the other by the values at y of a nowhere vanishing continuous function. This implies that the limit (and bounded) 2-form

$$dd_J^c\tau_\infty(\cdot, J\cdot)|_{\partial\mathbb{B}^n} = \lim_{k\to\infty} \Phi_{t_k}^* \left(dd_{J_{t_k}}^c \tau_o(\cdot, J_{t_k}\cdot)\right)\Big|_{\partial\mathbb{B}^n} = \Phi_{t_\infty}^* \left(dd_{J_{t_\infty}}^c \tau_o(\cdot, J_{t_\infty}\cdot)\right)\Big|_{\partial\mathbb{B}^n}$$

is strictly positive at all points of $\partial\mathbb{B}^n$ and that $\ker \Phi_{t_\infty *}|_{\mathscr{H}_y} = \{0\}$ for each $y \in \partial\mathbb{B}^n$. Thus that the linear maps $\widetilde{\Phi}_{t_k *}^{-1}|_{\mathscr{H}_y}$, $y \in \partial\mathbb{B}^n$, have uniformly bounded limits and, consequently, uniformly bounded components. We now show that indeed the coordinate components of all linear maps $\widetilde{\Phi}_{t_k *}^{-1}|_{\mathscr{H}_y}$, $y \in \widetilde{\overline{\mathbb{B}}}^n$, are uniformly bounded. Consider at first a fixed

closed radial disk $\overline{\Delta^{(c_o)}}$, the associated lifted disk $\widetilde{\overline{\Delta^{(c_o)}}}$ in $\widetilde{\overline{\mathbb{B}}}^n$, the corresponding limit disk $\overline{\Delta^{(c_o)\infty}} := \lim_{k\to\infty} \widetilde{\Phi}_{t_k}^{-1}(\overline{\Delta^{(c_o)}})$ in $\overline{\mathbb{B}}^n$ and, finally, the corresponding lifted closed disk $\widetilde{\overline{\Delta^{(c_o)\infty}}}$ in the blowup $\widetilde{\overline{\mathbb{B}}}^n$. Since the lifted disks $\widetilde{\overline{\Delta^{(c)}}}$ form a regular foliation of $\widetilde{\overline{\mathbb{B}}}^n$, we may consider one-parameter families of maps $f_i^{(s)} : \Delta \to \widetilde{\overline{\mathbb{B}}}^n$, $i = 1, \ldots, n-1$, with the following properties:

(i) Each $f_i^{(s)}$ is a J-holomorphic parametrization of a radial disk $\widetilde{\overline{\Delta^{(c(s))}}}$ (recall that any such map is also J_t holomorphic for each $t \in [0, 1]$, by the properties of the L-complex structures J_t);

(ii) The map $f^{(0)}$ is the standard holomorphic parametrization of $\widetilde{\overline{\Delta^{(c_o)}}}$;

(iii) The vector fields Y_i, defined at the points of the disk $\Delta^{(c_o)} = f^{(0)}(\Delta)$, by the formula

$$Y_i(\zeta) := \left.\frac{\mathrm{d} f_i^{(s)}(\zeta)}{\mathrm{d}s}\right|_{s=0},$$

have J_{t_∞}-holomorphic components $Y_i^{(10)\infty}$ that are linearly independent and generate the J_{t_∞}-holomorphic space $\mathscr{H}^{10}_{J_{t_\infty}}|_{y(\zeta)} \subset \mathscr{H}^{\mathbb{C}}|_{y(\zeta)}$ at all points of the disk $f_i^{(0)}(\zeta) \in \Delta^{(c_o)}$.

Note that, since the complex structures J_{t_k} converge to the complex structure J_{t_∞} for $k \to \infty$, condition (iii) implies also that, for all sufficiently large k, the J_{t_k}-holomorphic components $Y_i^{(10)k} \in \mathscr{H}^{10}_{J_{t_k}}$ of the vector fields Y_i are linearly independent and generate the spaces $\mathscr{H}^{10}_{J_{t_k}}|_y$ at each point $y \in \Delta^{(c)}$.

By construction, the components in J-holomorphic coordinates of the restricted linear maps $\widetilde{\Phi}^{-1}_{t_k *}|_{\mathscr{H}_y}$ at the points of the disk $y \in \Phi_{t_k}^{-1}(\Delta^{(c_o)})$, are uniformly bounded if and only if the sequences of vector fields $\widehat{Y}_i^{(10)k} := \widetilde{\Phi}^{-1}_{t_k *}(Y_i^{(10)k})$, defined at the points of the disks $\widetilde{\Phi}_{t_k}^{-1}(\widetilde{\overline{\Delta^{(c_o)}}})$, have uniformly bounded components at all points of their domains. To prove this claim, we first observe that the components of these vector fields in J-holomorphic coordinates are necessarily holomorphic functions. Indeed, the components of the vector fields $Y_i^{(10)k}$, which take values at the points of $\Delta^{(c_o)}$, are actually derivatives with respect to the parameter s of the expressions in J_{t_k}-holomorphic coordinates of the J_{t_k}-holomorphic disks $f_i^{(s)}$ of $(\widetilde{\overline{\mathbb{B}}}^n, J_{t_k})$, which are in fact J_t-holomorphic disks for any t, as we remarked. Since $J = \widetilde{\Phi}^{-1}_{t_k *}(J_{t_k})$, this shows that the components in J-holomorphic coordinates of the vector fields $\widehat{Y}_i^{(10)k} = \widetilde{\Phi}^{-1}_{t_k *}(Y_i^{(10)k})$ are derivatives in s of coordinated expressions of the J-holomorphic disks $\widetilde{\Phi}_{t_k}^{-1} \circ f_i^{(s)}$. From this the holomorphicity of the components of the vector fields $\widehat{Y}_i^{(10)k}$ follows immediately.

Observe that the limits for $k \to \infty$ of the restricted vector fields $\widehat{Y}_i^{(10)k}|_{\partial\mathbb{B}^n}$ coincide with the vector fields $Y_i^{(10)\infty}|_{\partial\mathbb{B}^n} = \lim_{k\to\infty} \Phi^{-1}_{t_k *}|_{\partial\mathbb{B}^n}(Y_i^{(10)k}|_{\partial\Delta})$, which are well-defined and bounded because of previous discussion on the limit behavior of the $\Phi^{-1}_{t_k *}|_y$, $y \in \partial\mathbb{B}^n$. This remark guarantees the existence of uniform upper bounds for the components of the restricted vector fields $\widehat{Y}_i^{(10)k}|_{\partial\mathbb{B}^n}$. The holomorphic dependence on $\zeta \in \Delta$ of the components of the vector fields $\widehat{Y}_i^{(10)k}$ implies the existence of uniform bounds for such components at *all* points of the disk $\Phi_{t_k}^{-1}(\Delta^{(c_o)})$, proving the desired claim.

Since the above arguments imply that the derived upper bounds depend continuously on the parameter c_o of the radial disk $\Delta^{(c_o)}$, the compactness of $\widetilde{\widetilde{\mathbb{B}}}^n$ yields the existence of a uniform bound for all components in J-holomorphic coordinates of all linear maps $\widetilde{\Phi}_{t_k*}^{-1}|_{\mathscr{H}_y}$, $y \in \widetilde{\widetilde{\mathbb{B}}}^n$. This concludes the proof. □

Remark 3.9 Proposition 3.4 and Lemma 3.8 imply that if there is an abstract $\mathscr{C}^{k,\alpha}$ fundamental pair (J_t, X_t) with $X_t|_0 = \mathbf{v}_t$, $k \geq 2$, $\alpha \in (0, 1)$, then there is also a one-parameter family of Monge-Ampère exhaustions τ_t of class $\mathscr{C}^{k,\alpha}$, whose centers are at the points of the curve $x_t = tv$, $t \in [0, 1]$. Due to this, the proof of Theorem 2.7 reduces to determine the exact set of vectors $0 \neq v \in \mathbb{B}^n$ for which there exists an associated fundamental pair (J_t, X_t) with $X_t|_0 = tv$.

4 Special vector fields of manifolds in normal form

The main purpose of this section is to prove that any vector field X_t, in an abstract fundamental pair (J_t, X_t), is subjected to very strong constraints, forcing X_t to be in a very small class of quasi-regular vector fields, called *special vector fields*. We shall need convenient complex coordinates for $\mathbb{C}^n$.

4.1 Generalized polar coordinates

Let $\xi : \mathbb{C}P^{n-1}\backslash\{z^n = 0\} \to \mathbb{C}^{n-1}$ be the standard affine coordinates

$$\xi([z^1 : \cdots : z^n]) := \left(w^1 := \frac{z^1}{z^n}, \ldots, w^{n-1} := \frac{z^{n-1}}{z^n}\right)$$

and define $\varphi : \mathbb{C}^{n-1} \times S^1 \longrightarrow S^{2n-1}\backslash\{z^n = 0\}$ by

$$\varphi(w^1, \ldots, w^{n-1}, e^{i\theta}) = e^{i\theta} \frac{1}{\sqrt{\sum_{\ell=1}^{n-1} |w^\ell|^2 + 1}} \left(w^1, \ldots, w^{n-1}, 1\right).$$

This is a (real) diffeomorphism onto $S^{2n-1}\backslash\{z^n = 0\}$ with the useful property that, for each $(w^i) \in \mathbb{C}^n$ and $\vartheta \in \mathbb{R}$, the corresponding point $\varphi(w^i, e^{i\vartheta})$ is in the complex line $[w^1 : \cdots : w^{n-1} : 1] = \xi^{-1}(w^1, \ldots, w^{n-1})$. If $\|z\|$ denotes the euclidean norm of z and $\eta : \mathbb{C}^n\backslash\{z^n = 0\} \longrightarrow \mathbb{C}^n$ is defined by

$$\eta(z^1, \ldots, z^n) := \left(w^1 := \frac{z^1}{z^n}, \ldots, w^{n-1} := \frac{z^{n-1}}{z^n}, \zeta := \|z\| \frac{z^n}{|z^n|}\right), \tag{4.1}$$

one has $\eta|_{\mathbb{C}^{n-1}\times S^1} = \varphi^{-1}$. This implies that η is a (real) diffeomorphism from $\mathbb{C}^n\backslash\{z^n = 0\}$ onto its image, whose inverse map is given by

$$\eta^{-1}(w^i, \zeta) := \frac{\zeta}{\sqrt{\sum_{\ell=1}^{n-1} |w^\ell|^2 + 1}} \left(w^1, \ldots, w^{n-1}, 1\right).$$

We call η a *generalized system of polar coordinates on* $\mathbb{C}^n\backslash\{z^n = 0\}$. Note that a similar construction can be performed if we replace $\{z^n = 0\}$ by any other affine hyperplane $\pi \subset \mathbb{C}^n$ through the origin. The corresponding system of (generalized) polar coordinates on $\mathbb{C}^n\backslash\pi$

will be denoted by $\eta_{(\pi)}$. In case π is a coordinate hyperplane $\pi_i := \{z^i = 0\}$, we simply denote it by $\eta_{(i)}$.

For later purposes, it is convenient to have the explicit expressions of the coordinate vector fields $\zeta \frac{\partial}{\partial \zeta}$ and $\frac{\partial}{\partial w^\alpha}$, $1 \le \alpha \le n-1$, of the polar coordinates $\eta = (w, \zeta)$, in terms of the standard coordinate vector fields $\frac{\partial}{\partial z^j}$, $\frac{\partial}{\partial \bar{z}^j}$ of $\mathbb{C}^n$. Using just the definitions, one can directly see that

$$\begin{aligned} \zeta \frac{\partial}{\partial \zeta} &= z^i \frac{\partial}{\partial z^i} \,, \\ \frac{\partial}{\partial w^\alpha} &= z^n \frac{\partial}{\partial z^\alpha} - \frac{1}{2} \frac{\overline{z}^\alpha z^n}{\|z\|^2} \left(z^i \frac{\partial}{\partial z^i} + \overline{z}^i \frac{\partial}{\partial \overline{z}^i} \right) \\ &= z^n \frac{\partial}{\partial z^\alpha} - \frac{1}{2} \frac{\overline{w}^\alpha}{1 + \sum_{\beta=1}^{n-1} |w^\beta|^2} \left(\zeta \frac{\partial}{\partial \zeta} + \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right) . \end{aligned} \tag{4.2}$$

4.2 Adapted polar frame fields

Consider a fixed system of polar coordinates, say $\eta_{(n)} = (\zeta, w^\alpha)$, and a circular domain $D \subset \mathbb{C}^n$ of class $\mathscr{C}^{k,\alpha}$ determined by the Minkowski functional $\mu = \mu_D$. By the homogeneity property of the Minkowski functionals, the expression of μ in polar coordinates has the form $\mu(\zeta, w^\alpha, \bar{\zeta}, \overline{w^\alpha}) = |\zeta| \rho(w^\alpha, \overline{w^\alpha})$ for some $\rho > 0$ of class $\mathscr{C}^{k,\alpha}$. Associated with the considered polar coordinates and the circular domain D, there is the set (Z, e_α) of complex vector fields defined by

$$\begin{aligned} Z &:= \zeta \frac{\partial}{\partial \zeta} \,, \\ e_\alpha = e_\alpha^{(\rho)} &:= \frac{\partial}{\partial w^\alpha} - \frac{\partial \rho^2}{\partial w^\alpha} \zeta \frac{\partial}{\partial \zeta} + \frac{1}{2} \frac{\overline{w^\alpha}}{1 + \sum_{\beta=1}^{n-1} |w^\beta|^2} \left(\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} - \zeta \frac{\partial}{\partial \zeta} \right) \\ &= \frac{\partial}{\partial w^\alpha} - \frac{\partial \log \rho^2}{\partial w^\alpha} \zeta \frac{\partial}{\partial \zeta} + \frac{1}{2} \frac{\partial \log \left(1 + \sum_{\beta=1}^{n-1} |w^\beta|^2 \right)}{\partial w^\alpha} \left(\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} - \zeta \frac{\partial}{\partial \zeta} \right) \\ &= z^n \frac{\partial}{\partial z^\alpha} - \left(\frac{\partial \rho^2}{\partial w^\alpha} + \frac{\overline{w^\alpha}}{1 + \sum_{\beta=1}^{n-1} |w^\beta|^2} \right) z^i \frac{\partial}{\partial z^i} \,. \end{aligned} \tag{4.3}$$

We call it *adapted polar frame field, associated with the circular domain D and the coordinates* (ζ, w^α). The fields of these frames satisfy the equations

$$Z(\mu^2) = \mu^2 \,, \qquad e_\alpha(\mu^2) = e_\alpha(|\zeta|^2 \rho^2) = 0 \,, \qquad J_{\mathrm{st}} e_\alpha = i e_\alpha \,, \tag{4.4}$$

where we denote by J_{st} the standard complex structure of $\mathbb{C}^n$. This means that $\mathrm{Re}(Z)$ and $\mathrm{Im}(Z)$ are generators for the distribution $\mathscr{Z}^{J_{\mathrm{st}}, \mu^2}$, defined in (2.7), of the manifold of circular type $(\overline{D}, J_{\mathrm{st}}, \tau = \mu^2)$ and that the vector fields $(\mathrm{Re}(e_\alpha), \mathrm{Im}(e_\alpha))$ are generators for the J_{st}-invariant distribution $\mathscr{H}^{J_{\mathrm{st}}, \mu^2}$, which is $dd^c_{J_{\mathrm{st}}} \tau$-orthogonal and a complementary distribution of $\mathscr{Z}^{J_{\mathrm{st}}, \mu^2}$. We recall that a normalizing map $\Phi : (\overline{D}, J_{\mathrm{st}}) \to (\overline{\mathbb{B}}^n, J)$ maps the distributions $\mathscr{Z}^{J_{\mathrm{st}}, \mu^2}$ and $\mathscr{H}^{J_{\mathrm{st}}, \mu^2}$ onto the distributions $\mathscr{Z}$ and $\mathscr{H}$ of $\overline{\mathbb{B}}^n$ discussed in §2.3. It is useful to explicitly express the Lie brackets between such generators:

$$[Z, e_\alpha] = 0, \quad [Z, \overline{e_\alpha}] = 0, \quad [e_\alpha, e_\beta] = 0, \quad [e_\alpha, \overline{e_\beta}] = g_{\alpha\bar{\beta}} \left(\zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right) , \tag{4.5}$$

where

$$g_{\alpha\bar\beta} := \frac{\delta_{\alpha\beta}(1+\sum_{\gamma=1}^{n-1}|w^\gamma|^2) - w^\beta\overline{w^\alpha}}{(1+\sum_{\gamma=1}^{n-1}|w^\gamma|^2)^2} + \frac{\partial^2\log\rho^2}{\partial w^\alpha\partial\overline{w^\beta}} = \frac{\partial^2}{\partial w^\alpha\partial w^{\bar\beta}}\left(\log\rho^2 + \log\left(1+\sum_{\gamma=1}^{n-1}|w^\gamma|^2\right)\right) . \tag{4.6}$$

The coefficients (4.6) are strictly related with the components of the 2-form $dd^c_{J_{\rm st}}\tau$, with $J_{\rm st}$ standard complex structure of $\mathbb{C}^n$. In fact, from (4.5),

$$dd^c_{J_{\rm st}}\tau(e_\alpha, e_{\bar\beta}) = J_{\rm st}[e_\alpha, e_{\bar\beta}](\tau) = 2i|\zeta|^2\rho^2\, g_{\alpha\bar\beta} = 2i\tau\, g_{\alpha\bar\beta} . \tag{4.7}$$

4.3 Special vector fields of a manifold in normal form

We now want to characterize the quasi-regular vector fields X, pointed at 0, of a manifold in normal form $((\overline{\mathbb{B}}^n, J), \tau_o)$, whose (local) flows preserve the restriction $J|_{\mathscr{Z}}$ of J to the distribution $\mathscr{Z}$. For stating our result, we need to fix some notation.

Consider the Kobayashi infinitesimal metric κ at the center $x_o = 0$ of $((\overline{\mathbb{B}}^n, J), \tau_o)$, the (closed) indicatrix $(\overline{\mathscr{I}}, J_{\rm st}) \subset T_0\overline{\mathbb{B}}^n \simeq \mathbb{C}^n$, determined by κ, and the corresponding circular representation $\Psi : \widetilde{\overline{\mathscr{I}}} \to \widetilde{\overline{\mathbb{B}}}^n$ (see § 2.2). Note that the standard coordinates of $\overline{\mathbb{B}}^n$ in general *do not overlap* in a $\mathscr{C}^{k,\alpha}$ fashion with the atlas of complex manifold structure of $(\overline{\mathbb{B}}^n, J)$. On the contrary, by the properties of the circular representations, any chart of the form $\eta_{(\pi)}\circ\Psi^{-1}$ with $\eta_{(\pi)} = (\zeta, w^\alpha)$ generalized polar coordinates on $\widetilde{\overline{\mathscr{I}}} \to \mathbb{C}^n$, does overlap in a $\mathscr{C}^{k-2,\alpha}$-way with the atlas of the complex manifold of $(\overline{\mathbb{B}}^n, J)$. We call this kind of coordinates *adapted polar coordinates* of $((\overline{\mathbb{B}}^n, J), \tau_o)$. The corresponding polar frame fields $(Z, e_\alpha = e_\alpha^{(\rho)})$ where ρ is such that $\kappa = |\zeta|^2\rho(w, \bar w)$, are called *associated adapted polar frame fields*. From now on, the only sets of coordinates we consider on $(\overline{\mathbb{B}}^n, J)$ are either adapted polar coordinates or coordinates of the form $\xi\circ\Psi^{-1}$, with $\xi = (z^i)$ standard coordinates of $\overline{\mathscr{I}} \subset \mathbb{C}^n$. We also denote by $J_{\rm st}$ the complex structure on $\widetilde{\overline{\mathbb{B}}}^n$ induced via $\Psi : \widetilde{\overline{\mathscr{I}}} \to \widetilde{\overline{\mathbb{B}}}^n$ from the standard complex structure of $\overline{\mathscr{I}} \subset \mathbb{C}^n$. *This complex structure should not be confused with the classical complex structure J_o of* $\overline{\mathbb{B}}^n$, where $\overline{\mathbb{B}}^n$ is considered as a domain of $\mathbb{C}^n$. In fact, the components in a set of adapted polar coordinates of the tensor field J_o are possibly not even continuous in $x_o = 0$, while the components of $J_{\rm st}$ in such coordinates are actually constant.

Each *real* vector field on $\overline{\mathbb{B}}^n\backslash\{0\}$ of class $\mathscr{C}^{k,\alpha}$, $k \geq 1$, $\alpha \in (0,1)$, admits a unique expansion of the form

$$X = X^0 Z + X^\alpha e_\alpha + \overline{X^0}\,\overline{Z} + \overline{X^\alpha}\,\overline{e_\alpha} \tag{4.8}$$

where Z, e_α are the complex vector fields (4.3) of some adapted polar frame field. In (4.8), the components X^0, X^α are functions of class $\mathscr{C}^{k,\alpha}$ of the polar coordinates on the open subset $\{\zeta \neq 0\}$.

Lemma 4.1 *Let $v = (v^i) \neq 0$ and X a quasi-regular vector field, pointed at* 0*, of class $\mathscr{C}^{k,\alpha}$ on $(\overline{\mathbb{B}}^n, J)$ and with $X|_0 := \mathrm{Re}(v^i\frac{\partial}{\partial z^i})$. If*

$$((\mathscr{L}_X J)|_{\mathscr{Z}})^{\mathscr{Z}} = 0 , \tag{4.9}$$

where $(\cdot)^{\mathscr{Z}} : T(\mathbb{B}^n\backslash\{0\} \to \mathscr{Z}$ is the projection onto $\mathscr{Z}$, then, for each domain $\mathscr{U}_{(i)}$ of affine coordinates $\xi : \mathscr{U}_{(i)} = \mathbb{C}P^{n-1}\backslash\{z^i = 0\} \to \mathbb{C}^{n-1}$, there exist two real functions

$\kappa, \rho : \mathscr{U}_{(i)} \to \mathbb{R}$ of class $\mathscr{C}^{k,\alpha}$, the first unconstrained, the second with $\rho > 0$ and $dd^c\rho > 0$ at all points, such that:
(1) The components $X^\alpha = X^\alpha(w, \bar{w}, \zeta, \bar{\zeta})$ are necessarily of the form

$$X^\alpha = \frac{Y^\alpha(w, \bar{w})}{\zeta} + \widetilde{Y}^\alpha(w, \bar{w}, \zeta, \bar{\zeta}) \tag{4.10}$$

with $Y^\alpha(w, \bar{w})$ defined by

$$Y^\alpha(w, \bar{w}) = (v^\alpha - v^n w^\alpha)\sqrt{\textstyle\sum_\gamma |w^\gamma|^2 + 1} \tag{4.11}$$

and $\widetilde{Y}^\alpha = \widetilde{Y}^\alpha(w, \bar{w}, \zeta, \bar{\zeta})$ functions such that, for each sequence $x_k \in \overline{\mathbb{B}}^n \backslash \{0\}$ converging to 0, the limits $\lim_{k\to\infty}(\widetilde{Y}^\alpha\zeta)|_{x_k}$ are 0;

(2) The component X^0 in (4.8) has necessarily the form

$$X^0 = \frac{Y^0(w, \bar{w})}{\zeta} + i\kappa(w, \bar{w}) - \overline{Y^0(w, \bar{w})}\zeta \ , \tag{4.12}$$

where Y^0 is the complex function on $\mathbb{C}P^{n-1}$

$$Y^0(w, \bar{w}) := v^n\sqrt{\textstyle\sum_\gamma |w^\gamma|^2 + 1} + Y^\alpha \frac{\partial\left(\log\rho^2 + \log(1 + \sum |w^\gamma|^2)\right)}{\partial w^\alpha}\bigg|_{(w,\bar{w})} . \tag{4.13}$$

Proof Condition (4.9) is actually equivalent to the pair of conditions

$$(\mathscr{L}_X J(Z))^{\mathscr{Z}} = (\mathscr{L}_X J(\overline{Z}))^{\mathscr{Z}} = 0 \, . \tag{4.14}$$

However, since X and J are real, these equations are one conjugate to the other, so that they are both satisfied if and only if just the second one holds. Using the fact that $J|_{\mathscr{Z}} = J_{\mathrm{st}}|_{\mathscr{Z}}$, this is in turn equivalent to

$$\begin{aligned} 0 &= [X, J\overline{Z}]^{\mathscr{Z}} - (J[X, \overline{Z}])^{\mathscr{Z}} \\ &= [X, J_{\mathrm{st}}\overline{Z}]^{\mathscr{Z}} - J_{\mathrm{st}}([X, \overline{Z}]^{\mathscr{Z}}) = -i[X, \overline{Z}]^{\mathscr{Z}} - J_{\mathrm{st}}[X, \overline{Z}]^{\mathscr{Z}} \ , \end{aligned} \tag{4.15}$$

meaning that $[X, \overline{Z}]^{\mathscr{Z}}$ is a complex vector field taking values in the anti-holomorphic distribution $T^{01}_{J_{\mathrm{st}}}(\overline{\mathbb{B}}^n \backslash \{0\})$ of the standard complex structure J_{st}. The Lie bracket $[X, \overline{Z}]$ can be easily computed using (4.5). One gets that $[X, \overline{Z}]^{\mathscr{Z}}$ is in $T^{01}_{J_{\mathrm{st}}}(\overline{\mathbb{B}}^n \backslash \{0\})$ if and only if

$$\overline{Z} \cdot X^0 = \bar{\zeta}\frac{\partial X^0}{\partial\bar{\zeta}} = 0 \, . \tag{4.16}$$

This yields that, along each straight disk $\Delta^{(c)} := \{w = c\}$, the complex vector field $X^{(c)} := (X^0 Z)|_{\Delta^{(c)}\backslash\{0\}}$ is a vector field of type $(1, 0)$, holomorphic in the polar coordinate ζ. Since we are also assuming that X is quasi-regular (hence with a continuous extension at 0), it follows that $X^{(c)} = X^0|_{\Delta^{(c)}}\zeta\frac{\partial}{\partial\zeta}|_{\Delta^{(c)}\backslash\{0\}}$ extends continuously at the origin and that the function $a := X^0\big|_{\Delta^{(c)}\backslash\{0\}}$ has at most one pole of order 1 at the origin. On the other hand, since the vector field X is tangent to the boundary at the points of $\partial\mathbb{B}^n$, the vector field $X^{(c)}$ is tangent to $\partial\Delta^{(c)}$ at all boundary points. This implies that

$$X^{(c)}|_{\partial\Delta^{(c)}} + \overline{X^{(c)}|_{\partial\Delta^{(c)}}} = s\left(iZ - i\overline{Z}\right)\bigg|_{\partial\Delta^{(c)}} \tag{4.17}$$

for some smooth real function $s : \partial\Delta^{(c)} \longrightarrow \mathbb{R}$. This is equivalent to require that $a{:=}X^0\big|_{\Delta^{(c)}\setminus\{0\}}$ is such that $a|_{\partial\Delta^{(c)}} = -\overline{a|_{\partial\Delta^{(c)}}}$. Very standard arguments show that the above two conditions imply that $a = \frac{\lambda}{\zeta} + i\kappa - \bar{\lambda}\zeta$ for some complex number λ and some real number κ which depend only on $c = (c^\alpha)$. From this we obtain that X^0 has the form (4.12) for some appropriate function $Y^0(w, \bar{w})$.

Consider now a sequence of points x_k in $\overline{\mathbb{B}}^n\setminus\{0\}$ converging to 0. With no loss of generality, we may assume that all points x_k are in the domain of the polar coordinates $\eta_{(n)} = (w, \zeta)$ and that their expressions in polar coordinates $x_k = (w_k^\alpha, \zeta_k)$ converge to an n-tuple $(w_o^\alpha, 0)$ with $0 \leq |w_o| < \infty$.

We now consider the sequences of complex values $C^\alpha(x_k){:=}(X^\alpha\zeta)|_{x_k}$. Up to a subsequence, we may assume that the limits $C^\alpha = \lim_{k\to\infty} C_k^\alpha$ exist even if they might not be all finite. Then: (i) From (4.3) and using (4.1) to relate (z^j) and (w^α, ζ), we have that the sequence of vectors $X^\alpha e_\beta\big|_{x_k}$ converges to the vector in $T_0\mathbb{B}^n$ (here, some components might be equal to ∞)

$$\frac{C^\beta}{\sqrt{\sum|w_o^\gamma|^2+1}}\left(\delta_\beta^\alpha - w_o^\alpha\frac{\partial\log\rho^2}{\partial w^\beta}\Big|_{(w_o,\bar{w}_o)} - \frac{\overline{w}_o^\beta w_o^\alpha}{1+\sum|w_o^\gamma|^2}\right)\frac{\partial}{\partial z^\alpha}\Big|_0$$
$$-\frac{C^\beta}{\sqrt{\sum|w_o^\gamma|^2+1}}\left(\frac{\partial\log\rho^2}{\partial w^\beta}\Big|_{(w_o,\bar{w}_o)} + \frac{\overline{w}_o^\beta}{1+\sum|w_o^\gamma|^2}\right)\frac{\partial}{\partial z^n}\Big|_0. \tag{4.18}$$

(ii) Using once again (4.2) and the fact that X^0 has the form (4.12), the sequence of vectors $X^0 Z\big|_{x_k}$ converges to the vector in $T_0\mathbb{B}^n$

$$\frac{Y^0(w_o,\bar{w}_o)w_o^\alpha}{\sqrt{\sum|w_o^\gamma|^2+1}}\frac{\partial}{\partial z^\alpha}\Big|_0 + \frac{Y^0(w_o,\bar{w}_o)}{\sqrt{\sum|w_o^\gamma|^2+1}}\frac{\partial}{\partial z^n}\Big|_0. \tag{4.19}$$

The condition that X is continuous at 0 is equivalent to requiring that for any sequence x_k as above, the limit $\lim_{x_k\to 0} X|_{x_k}$ exists and does not depend on the choice of the sequence. From (i) and (ii), we infer that the vector

$$\frac{C^\beta\delta_\beta^\alpha}{\sqrt{\sum|w_o^\gamma|^2+1}}\frac{\partial}{\partial z^\alpha}\Big|_0$$
$$+\frac{w_o^\alpha}{\sqrt{\sum|w_o^\gamma|^2+1}}\left(Y^0(w_o,\bar{w}_o) - C^\beta\frac{\partial\log\rho^2}{\partial w^\beta}\Big|_{(w_o,\bar{w}_o)} - \frac{C^\beta\overline{w}_o^\beta}{1+\sum|w_o^\gamma|^2}\right)\frac{\partial}{\partial z^\alpha}\Big|_0$$
$$+\frac{1}{\sqrt{\sum|w_o^\gamma|^2+1}}\left(Y^0(w_o,\bar{w}_o) - C^\beta\frac{\partial\log\rho^2}{\partial w^\beta}\Big|_{(w_o,\bar{w}_o)} - \frac{C^\beta\overline{w}_o^\beta}{1+\sum|w_o^\gamma|^2}\right)\frac{\partial}{\partial z^n}\Big|_0 \tag{4.20}$$

is independent of w_o. Necessary and sufficient conditions for this are:

(a) there is a constant v^n such that

$$Y^0(w_o, \bar{w}_o) - C^\beta \frac{\partial \log \rho^2}{\partial w^\beta}\Big|_{(w_o,\bar{w}_o)} - \frac{C^\beta \overline{w}_o^\beta}{1+\sum |w_o^\gamma|^2}$$
$$= Y^0(w_o, \bar{w}_o) - C^\beta \frac{\partial \left(\log \rho^2 + \log(1+\sum |w_o^\gamma|^2)\right)}{\partial w^\beta}\Big|_{(w_o,\bar{w}_o)} = v^n \sqrt{\sum |w_o^\gamma|^2 + 1} \tag{4.21}$$

and, in particular, all limits C^α are finite;

(b) the vector (4.20), which can now be written as

$$\left(\frac{C^\alpha}{\sqrt{\sum |w_o^\gamma|^2 + 1}} + w_o^\beta v^n \right) \frac{\partial}{\partial z^\beta}\Big|_0 + v^n \frac{\partial}{\partial z^n}\Big|_0 ,$$

does not depend on w_o.

This last condition is equivalent to require that there exist constants v^β such that $C^\alpha = (v^\beta - v^n w_o^\beta)\sqrt{\sum |w_o^\gamma|^2 + 1}$. Replacing this into (4.21) and (4.18), claims (1) and (2) follow.

□

The above lemma motivates the following notion, which enters as crucial ingredient in the proof of our main result. In what follows, $\rho : \mathbb{C}P^{n-1} \to \mathbb{R}$ is always the positive function, determined by the Kobayashi metric $\kappa_{x=0}$ of $(\overline{\mathbb{B}}^n, J)$ and gives the Minkowski function of the indicatrix $\overline{\mathscr{I}_0} \subset T_0\overline{\mathbb{B}}^n = \mathbb{C}^n$.

Definition 4.2 Consider a vector $v = (v^i) \in \mathbb{C}^n$, a real valued function $\sigma : \mathbb{C}P^{n-1} \to \mathbb{R}$ of class $\mathscr{C}^{k,\alpha}$ and $n-1$ complex functions $\widetilde{Y}^\alpha : \overline{\mathbb{B}}^n \to \mathbb{C}$, which are of class $\mathscr{C}^{k,\alpha}$ on $\overline{\mathbb{B}}^n \backslash \{0\}$ and with $\lim_{k\to\infty}(\widetilde{Y}^\alpha \zeta)|_{x_k} = 0$ for any sequence $x_k \to 0$ in any set of adapted polar coordinates of $((\overline{\mathbb{B}}^n, J), \tau_o)$. We call *special vector field of* $(\overline{\mathbb{B}}^n, J)$ *determined by* v, σ *and* $\widetilde{Y}^\alpha$ the quasi-regular vector field of $(\overline{\mathbb{B}}^n, J)$, pointed at 0, whose expansion in terms of an adapted polar frame field (Z, e_α) has the form

$$X^{[\rho](v,\sigma,\widetilde{Y}^\alpha)} = \left(\frac{Y^0}{\zeta} + i\sigma - \overline{Y^0}\zeta\right) Z + \left(\frac{Y^\alpha}{\zeta} + \widetilde{Y}^\alpha\right) e_\alpha$$
$$+ \left(\frac{\overline{Y^0}}{\bar{\zeta}} - i\sigma - Y^0\bar{\zeta}\right)\overline{Z} + \left(\frac{\overline{Y^\alpha}}{\bar{\zeta}} + \overline{\widetilde{Y}^\alpha}\right)\overline{e_\alpha} \tag{4.22}$$

with Y^0 and Y^α as in (4.13) and (4.11), respectively.

From the proof of Lemma 4.1, the next corollary follows immediately.

Corollary 4.3 *A quasi-regular vector field of class* $\mathscr{C}^{k,\alpha}$, *pointed at* 0, *on* $(\overline{\mathbb{B}}^n, J)$, *with* $X|_0 = v$ *and satisfying* (4.9) *is a special vector field* $X = X^{[\rho](v,\sigma,\widetilde{Y}^\alpha)}$ *for some triple* $(v, \sigma, \widetilde{Y}^\alpha)$. *Conversely, for each choice of a triple* $(v, \sigma, \widetilde{Y}^\alpha)$, *the corresponding special vector field* $X^{[\rho](v,\sigma,\widetilde{Y}^\alpha)}$ *is a quasi-regular vector field of class* $\mathscr{C}^{k,\alpha}$ *pointed at* 0 *on* $(\overline{\mathbb{B}}^n, J)$, *with* $X^{[\rho](v,\sigma,\widetilde{Y}^\alpha)}|_0 = v$ *and satisfying* (4.9).

Moreover, we have the following technical remark.

Lemma 4.4 *In each set of polar coordinates (ζ, w^α), the functions Y^0, Y^α, defined in (4.13) and (4.11), satisfy the identities*

$$\begin{aligned} e_\beta\left(\frac{Y^0}{\zeta}-\overline{Y^0\zeta}\right) &= \frac{H_\beta}{\zeta}-\overline{\left(H_{\bar\beta}-Y^0 e_{\bar\beta}(\log\rho^2)\right)\zeta}, e_{\bar\beta}\left(\frac{Y^0}{\zeta}-\overline{Y^0\zeta}\right) \\ &= \frac{H_{\bar\beta}}{\zeta}-\overline{\left(H_\beta-Y^0 e_{\bar\beta}(\log\rho^2)\right)\zeta}\ , \end{aligned} \tag{4.23}$$

where, if $h_{\alpha\beta}:=\frac{\partial\left(\log\rho^2+\log(1+\sum|w_o^\gamma|^2)\right)}{\partial w^\alpha\partial w^\beta}$,

$$\begin{aligned} H_\beta &:= Y^\alpha\left(h_{\alpha\beta}+\frac{\partial\left(\log\rho^2+\log(1+\sum|w^\gamma|^2)\right)}{\partial w^\alpha}\frac{\partial\left(\rho^2+\log(1+\sum|w^\gamma|^2)\right)}{\partial w^\beta}\right) \\ H_{\bar\beta} &:= Y^\alpha g_{\alpha\bar\beta} \end{aligned} \tag{4.24}$$

Proof We first observe that

$$e_\beta(Y^\alpha)=-v^n\delta^\alpha_\beta\sqrt{1+\textstyle\sum_\gamma|w^\gamma|^2}+\frac{Y^\alpha\overline{w^\beta}}{2(1+\sum_\gamma|w^\gamma|^2)}\ ,\qquad e_{\bar\beta}(Y^\alpha)=\frac{Y^\alpha w^\beta}{2(1+\sum_\gamma|w^\gamma|^2)}\ .$$

Then

$$\begin{aligned} e_{\bar\beta}\left(\frac{Y^0}{\zeta}\right) &= \frac{1}{\zeta}e_{\bar\beta}\left(v^n\sqrt{\textstyle\sum_\gamma|w^\gamma|^2+1}+Y^\alpha\frac{\partial\left(\log r^2+\log(1+\sum|w^\gamma|^2)\right)}{\partial w^\alpha}\right) \\ &\quad -\frac{1}{2\zeta}\frac{\partial\log\left(1+\sum_\gamma|w^\gamma|^2\right)}{\partial\overline{w^\beta}}\left(v^n\sqrt{\textstyle\sum_\gamma|w^\gamma|^2+1}\right) \\ &\quad -\frac{1}{2\zeta}\frac{\partial\log\left(1+\sum_\gamma|w^\gamma|^2\right)}{\partial\overline{w^\beta}}\left(Y^\alpha\frac{\partial\left(\log r^2+\log(1+\sum|w^\gamma|^2)\right)}{\partial w^\alpha}\right) \\ &= \frac{1}{2\zeta}v^n\frac{w^\beta}{\sqrt{\sum_\gamma|w^\gamma|^2+1}}+\frac{1}{\zeta}\frac{Y^\alpha w^\beta}{2(1+\sum_\gamma|w^\gamma|^2)}\frac{\partial\left(\log r^2+\log(1+\sum|w^\gamma|^2)\right)}{\partial w^\alpha} \\ &\quad +\frac{Y^\alpha g_{\alpha\bar\beta}}{\zeta}-\frac{1}{2\zeta}v^n\frac{w^\beta}{\sqrt{\sum_\gamma|w^\gamma|^2+1}} \\ &\quad -\frac{1}{2\zeta}\frac{Y^\alpha w^\beta}{(1+\sum_\gamma|w^\gamma|^2)}\frac{\partial\left(\log r^2+\log(1+\sum|w^\gamma|^2)\right)}{\partial w^\alpha}=\frac{Y^\alpha g_{\alpha\bar\beta}}{\zeta}=\frac{H_{\bar\beta}}{\zeta}\ . \end{aligned}$$

Similarly one gets that $e_\beta\left(\frac{Y^0}{\zeta}\right)=\frac{H_\beta}{\zeta}$. From this and $e_\beta(\zeta\bar\zeta^2\rho^2)=e_{\bar\beta}(\zeta\bar\zeta^2\rho^2)=0$, one can also derive the expressions for $e_\beta(\overline{Y^0}\zeta)$, $e_{\bar\beta}(\overline{Y^0}\zeta)$ and get (4.23). □

5 The proof of the propagation of regularity theorem

5.1 The differential problem characterizing fundamental pairs

In this section, we show that the abstract fundamental pairs (J_t, X_t) of a manifold in normal form $((\overline{\mathbb{B}}^n, J), \tau_o)$ are precisely the solutions of an appropriate differential problem. By

Remark 3.9, this reduces our main result to the proof of the existence of solutions to such problem.

Proposition 5.1 *Let (J_t, X_t) be an abstract fundamental pair of class $\mathscr{C}^{k,\alpha}$ on a manifold in normal form $((\overline{\mathbb{B}}^n, J), \tau_o)$, guided by the curve $v_t = X_t|_0$, $t \in [0,1]$, and let (ζ, w^α) be a system of polar coordinates with associated polar frame field (Z, e_α). Then each X_t is a special vector field $X_t = X^{[\rho_t](v_t,\sigma_t,\widetilde{Y}_t^\alpha)}$, with components*

$$\begin{aligned}\widetilde{Y}_t^\alpha &= ig^{\alpha\bar\beta}e_{\bar\beta}(\sigma_t) - \frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\bar\delta} g^{\bar\delta\gamma}\left(\frac{H_{t\gamma}}{\zeta} + \overline{\frac{H_{t\bar\gamma}}{\zeta}}\right) - \frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\delta} g^{\delta\bar\gamma}\left(\frac{H_{t\bar\gamma}}{\zeta} + \overline{\frac{H_{t\gamma}}{\zeta}}\right)\\ &\quad + \frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\bar\delta} g^{\bar\delta\gamma}\left(\overline{\left(H_{t\bar\gamma} - Y_t^0 e_{\bar\beta}(\log\rho_t^2)\right)}\zeta + \left(H_{t\gamma} - Y_t^0 e_\gamma(\log\rho_t^2)\right)\bar\zeta\right)\\ &\quad + \frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\delta} g^{\delta\bar\gamma}\left(\overline{\left(H_{t\gamma} - Y_t^0 e_\gamma(\log\rho_t^2)\right)}\zeta + \left(H_{t\bar\gamma} - Y_t^0 e_{\bar\gamma}(\log\rho_t^2)\right)\bar\zeta\right)\,, \end{aligned} \tag{5.1}$$

where $(g^{\alpha\bar\beta})$ is the inverse matrix of (4.6), $H_{t\beta}$, $H_{t\bar\beta}$ are as in (4.24), $(J_t - J_{\mathrm{st}})^\gamma_{\bar\beta}$, $(J_t - J_{\mathrm{st}})^{\bar\gamma}_{\bar\beta}$ are the components of $(J_t - J_{\mathrm{st}})(e_{\bar\beta})$ in the frame $(e_\alpha, e_{\bar\alpha})$ and Y_t^α, Y_t^0 are as in (4.11), (4.13) with $v = v_t$ and ρ_t determined by the Kobayashi metric at 0 of $(\overline{\mathbb{B}}^n, J_t)$, as described above.

Conversely, if $X_t = X^{[\rho_t](v_t,\sigma_t,\widetilde{Y}_t^\alpha)}$, $t \in [0,1]$, is a one-parameter family of special vector fields, with $\widetilde{Y}_t^\alpha$ as in (5.1), and if J_t and ρ_t are one-parameter families of complex structures and positive real functions on $\mathbb{C}P^{n-1}$, which satisfy the differential problem given by (5.7) and

$$\frac{\mathrm{d}J_t}{\mathrm{d}t} = -\mathscr{L}_{X_t} J_t\,, \qquad J_0 = J\,, \tag{5.2}$$

then (J_t, X_t) is a fundamental pair guided by the curve v_t.

Proof Since (J_t, X_t) is a fundamental pair, the one-parameter family J_t consists of complex structures that are pushed-down onto $\overline{\mathbb{B}}^n$ of L-complex structures of $\widetilde{\overline{\mathbb{B}}}^n$. This implies that

$$J_t|_{\mathscr{Z}} = J_{\mathrm{st}}|_{\mathscr{Z}} \qquad \text{and} \qquad J_t(\mathscr{H}) \subset \mathscr{H} \quad \text{for each } t \in [0,T]\,.$$

Since $J_0 = J$, this is tantamount to say that

$$\left.\frac{\mathrm{d}J_t}{\mathrm{d}t}\right|_{\mathscr{Z}} = 0\,, \qquad \frac{\mathrm{d}J_t}{\mathrm{d}t}(\mathscr{H}) \subset \mathscr{H} \qquad \text{for all } t\,. \tag{5.3}$$

Let us focus on the first of these conditions. Combining it with the property $\mathscr{L}_{X_t} J_t = -\frac{\mathrm{d}J_t}{\mathrm{d}t}$, this is equivalent to $\mathscr{L}_{X_t} J_t(Z) = \mathscr{L}_{X_t} J_t(\overline{Z}) = 0$ at all points of $\overline{\mathbb{B}}^n\backslash\{0\}$. However, being X_t and J_t real, these equations are one conjugate to the other, so that we may consider just the second one. This is in turn equivalent to

$$0 = [X_t, J_t\overline{Z}] - J_t[X_t, \overline{Z}] = [X_t, J_{\mathrm{st}}\overline{Z}] - J_t[X_t, \overline{Z}] = -i[X_t, \overline{Z}] - J_t[X_t, \overline{Z}]\,, \tag{5.4}$$

meaning that $[X_t, \overline{Z}]$ is a complex vector field taking values in the J_t-anti-holomorphic distribution $T^{01}_{J_t}(\overline{\mathbb{B}}^n\backslash\{0\})$. Consider now the expansion of the vector fields X_t in terms of a polar frame field (Z, e_α), associated with a set of polar coordinates

$$X_t = X_t^0 Z + X_t^\alpha e_\alpha + \overline{X_t^0}\overline{Z} + \overline{X_t^\alpha} e_{\bar\alpha}\,.$$

Since $T^{01}_{J_t}(\overline{\mathbb{B}}^n\backslash\{0\})|_x = \mathscr{Z}^{01}_x \oplus \left(T^{01}_{J_t}(\overline{\mathbb{B}}^n\backslash\{0\})|_x \cap \mathscr{H}^{\mathbb{C}}_x\right)$, we have that $[X_t, \overline{Z}]$ is in $T^{01}_{J_t}(\overline{\mathbb{B}}^n\backslash\{0\})$ if and only if

$$\begin{aligned} &\overline{Z}(X^0_t) = 0\,, \\ &[\overline{Z}, X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}] = \overline{Z}(X^\alpha_t)e_\alpha + \overline{Z}(\overline{X^\alpha_t})\overline{e_\alpha} \in T^{01}_{J_t}(\overline{\mathbb{B}}^n\backslash\{0\}) \cap \mathscr{H}^{\mathbb{C}}\,. \end{aligned} \tag{5.5}$$

The first part of the proof of Lemma 4.1 shows that the first condition in (5.5) is equivalent to the condition $(\mathscr{L}_{X_t} J_t)^{\mathscr{Z}} = 0$. Hence Corollary 4.3 applies and, for each t, the vector field X_t is a special vector field $X_t = X^{[\rho_t](v_t,\sigma_t,\widetilde{Y}^\alpha_t)}$. In particular, X^0_t is equal to

$$X^0_t = \frac{Y^0_t(w,\bar{w})}{\zeta} + i\sigma_t(w,\bar{w}) - \overline{Y^0_t(w,\bar{w})}\zeta \tag{5.6}$$

with Y^0_t defined in (4.13).

Let us now consider the second condition in (5.5). Since the 2-form $-\frac{i}{2\tau}dd^c_{J_t}\tau_o|_{\mathscr{H}^{\mathbb{C}}\times\mathscr{H}^{\mathbb{C}}}$ is non-degenerate and J_t-Hermitian, it can be equivalently stated saying that for each $t \in [0,1]$,

$$-\frac{i}{2\tau}dd^c_{J_t}\tau_o([\overline{Z}, X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}], F)_x = -\frac{i}{2\tau}dd^c_{J_{\mathrm{st}}}\tau_o([\overline{Z}, X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}], F)_x = 0$$

for any complex vector field F in $\mathscr{H}^{01}_t = \mathscr{H}^{\mathbb{C}} \cap T^{01}_{J_t}\overline{\mathbb{B}}^n$. Since $\mathscr{L}_{\overline{Z}}(\frac{i}{2\tau}dd^c_{J_{\mathrm{st}}}\tau_o) = 0$, this is equivalent to

$$-\frac{i}{2\tau}dd^c_{J_{\mathrm{st}}}\tau_o(X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}, [\overline{Z}, F]) = 0 \qquad \text{for any } F \in \mathscr{H}^{01}_t\,. \tag{5.7}$$

This relation is actually an identity that is a consequence of the integrability of the J_t, a claim that can be directly checked using deformation tensors of complex structures ([3,22]). For reader's convenience, we give the details of such an argument at the end of § 5.2.1 below.

Let us now focus on the second part of (5.3), which we did not consider yet. Once again, since J_t and X_t are related by (3.11), this is the same of requiring that, for any $E \in \mathscr{H}$,

$$(\mathscr{L}_{X_t} J_t)(E) = [X_t, J_t E] - J_t[X_t, E] \in \mathscr{H}\,. \tag{5.8}$$

This is also the same of saying that for any $E \in \mathscr{H}$

$$d\tau_o([X_t, J_t E]) - d\tau_o(J_t[X_t, E]) = 0\,. \tag{5.9}$$

We now observe that

$$d\tau_o([X_t, J_t E]) = X_t\,(d\tau_o(J_t E)) - J_t E\,(d\tau_o(X_t)) = -\tau_o\left(J_t E(X^0_t) + J_t E(\overline{X^0_t})\right) \tag{5.10}$$

and

$$\begin{aligned} -d\tau_o(J_t[X_t, E]) &= -dd^c_{J_t}\tau_o(X_t, E) - X_t\,(d\tau_o(J_t E)) + E\,(d\tau_o(J_t X_t)) \\ &= -dd^c_{J_t}\tau_o(X_t, E) + i\tau_o\left(E(X^0_t) - E(\overline{X^0_t})\right) \\ &= -dd^c_{J_t}\tau_o(X^0_t Z + \overline{X^0_t}\overline{Z}, E) - dd^c_{J_t}\tau_o(X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}, E) + i\tau_o E(X^0_t - \overline{X^0_t}) \\ &= -dd^c_{J_t}\tau_o(X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}, E) + i\tau_o E(X^0_t - \overline{X^0_t}) \\ &\overset{\text{Lemma 3.5}}{=} -dd^c_{J_{\mathrm{st}}}\tau_o(X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}, E) + i\tau_o E(X^0_t - \overline{X^0_t}) \end{aligned} \tag{5.11}$$

From this and (5.10), it follows that (5.9) is equivalent to

$$\begin{aligned} dd^c_{J_{\rm st}}\tau_o(X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha}, E) &= i\tau_o\left((E + iJ_tE)(X^0_t) - (E - iJ_tE)(\overline{X^0_t})\right) \\ &= i\tau_o\left(d(X^0_t - \overline{X^0_t}) + id(X^0_t + \overline{X^0_t})\circ J_t\right)\big|_{\mathscr{H}}(E) \\ &= i\tau_o\left(d(X^0_t - \overline{X^0_t}) + id(X^0_t + \overline{X^0_t})\circ J_{\rm st}\right)\big|_{\mathscr{H}}(E) \\ &\quad - \tau_o d(X^0_t + \overline{X^0_t})\circ(J_t - J_{\rm st})\big|_{\mathscr{H}}(E)\,. \end{aligned} \tag{5.12}$$

Moreover, denoting by $(dd^c_{J_{\rm st}}\tau_o)^{-1}$ the $(2,0)$-type tensor field, with components given by the inverse matrix of the components of $dd^c_{J_{\rm st}}\tau_o$, (5.12) becomes

$$\begin{aligned} X^\alpha_t e_\alpha + \overline{X^\alpha_t e_\alpha} &= i\tau_o(dd^c_{J_{\rm st}}\tau_o)^{-1}\left(d(X^0_t - \overline{X^0_t}) + id(X^0_t + \overline{X^0_t})\circ J_{\rm st}\right)\big|_{\mathscr{H}} \\ &\quad - \tau_o(dd^c_{J_{\rm st}}\tau_o)^{-1}\left(d(X^0_t + \overline{X^0_t})\circ(J_t - J_{\rm st})\big|_{\mathscr{H}}\right)\,, \end{aligned} \tag{5.13}$$

so that, by (4.6) and (4.23),

$$\begin{aligned} X^\alpha_t &= g^{\alpha\bar\beta}\overline{e_\beta}(X^0_t) - \frac{i}{2}g^{\alpha\bar\beta}(J_t - J_{\rm st})^\gamma_{\bar\beta}e_\gamma(X^0_t + \overline{X^0_t}) \\ &\quad - \frac{i}{2}g^{\alpha\bar\beta}(J_t - J_{\rm st})^{\bar\gamma}_{\bar\beta}e_{\bar\gamma}(X^0_t + \overline{X^0_t}) \\ &= \frac{Y^\alpha_t}{\zeta} + ig^{\alpha\bar\beta}e_{\bar\beta}(\sigma_t) - \frac{i}{2}g^{\alpha\bar\beta}(J_t - J_{\rm st})^\gamma_{\bar\beta}\left(\frac{H_{t\gamma}}{\zeta} + \frac{\overline{H_{t\bar\gamma}}}{\bar\zeta}\right) \\ &\quad - \frac{i}{2}g^{\alpha\bar\beta}(J_t - J_{\rm st})^{\bar\gamma}_{\bar\beta}\left(\frac{H_{t\bar\gamma}}{\zeta} + \frac{\overline{H_{t\gamma}}}{\bar\zeta}\right) \\ &\quad + \frac{i}{2}g^{\alpha\bar\beta}(J_t - J_{\rm st})^\gamma_{\bar\beta}\left(\overline{\left(H_{t\bar\gamma} - Y^0_t e_{\bar\beta}(\log\rho^2)\right)}\zeta + \left(H_{t\gamma} - Y^0_t e_\gamma(\log\rho^2)\right)\bar\zeta\right) \\ &\quad + \frac{i}{2}g^{\alpha\bar\beta}(J_t - J_{\rm st})^{\bar\gamma}_{\bar\beta}\left(\overline{\left(H_{t\gamma} - Y^0_t e_\gamma(\log\rho^2)\right)}\zeta + \left(H_{t\bar\gamma} - Y^0_t e_{\bar\gamma}(\log\rho^2)\right)\bar\zeta\right)\,. \end{aligned} \tag{5.14}$$

It is now convenient to recall that, by Lemma 3.5, for any $X, Y \in \mathscr{H}$,

$$dd^c\tau_o(J_tX, Y) = dd^c_{J_t}\tau_o(J_tX, Y) = -dd^c_{J_t}\tau_o(X, J_tY) = -dd^c\tau_o(X, J_tY)\,.$$

This implies that $(J_t)^\gamma_\alpha g_{\gamma\bar\beta} = -(J_t)^{\bar\gamma}_\alpha g_{\alpha\bar\gamma}$ and $(J_t)^\gamma_{\bar\alpha}g_{\gamma\bar\beta} = -(J_t)^\gamma_{\bar\alpha}g_{\bar\alpha\gamma}$ and hence

$$g^{\alpha\bar\beta}(J_t)^\gamma_{\bar\beta} = -(J_t)^\alpha_{\bar\beta}g^{\bar\beta\gamma}\,, \qquad g^{\alpha\bar\beta}(J_t)^{\bar\gamma}_{\bar\beta} = -(J_t)^\alpha_{\bar\beta}g^{\beta\bar\gamma}\,.$$

Using this property, we see that (5.14) can be also written as

$$\begin{aligned} X^\alpha_t &= \frac{Y^\alpha_t}{\zeta} + ig^{\alpha\bar\beta}e_{\bar\beta}(\sigma_t) - \frac{i}{2}(J_t - J_{\rm st})^\alpha_{\bar\delta}g^{\bar\delta\gamma}\left(\frac{H_{t\gamma}}{\zeta} + \frac{\overline{H_{t\bar\gamma}}}{\bar\zeta}\right) \\ &\quad - \frac{i}{2}(J_t - J_{\rm st})^\alpha_\delta g^{\delta\bar\gamma}\left(\frac{H_{t\bar\gamma}}{\zeta} + \frac{\overline{H_{t\gamma}}}{\bar\zeta}\right) \\ &\quad + \frac{i}{2}(J_t - J_{\rm st})^\alpha_{\bar\delta}g^{\bar\delta\gamma}\left(\overline{\left(H_{t\bar\gamma} - Y^0_t e_{\bar\beta}(\log\rho^2)\right)}\zeta + \left(H_{t\gamma} - Y^0_t e_\gamma(\log\rho^2)\right)\bar\zeta\right) \\ &\quad + \frac{i}{2}(J_t - J_{\rm st})^\alpha_\delta g^{\delta\bar\gamma}\left(\overline{\left(H_{t\gamma} - Y^0_t e_\gamma(\log\rho^2)\right)}\zeta + \left(H_{t\bar\gamma} - Y^0_t e_{\bar\gamma}(\log\rho^2)\right)\bar\zeta\right)\,. \end{aligned}$$

Since X_t must be a special vector field, determined by v_t and some appropriate functions σ_t, $\widetilde{Y}_t^\alpha$, we see that the $\widetilde{Y}_t^\alpha$ must be equal to (5.1). Note also that such functions (5.1) automatically satisfy the condition $\lim_{k\to\infty}(\widetilde{Y}^\alpha\zeta)|_{x_k} = 0$ for each sequence $x_k \to 0$. To see this, it suffices to remember that $J_{\rm st}$ is the standard complex structure of the indicatrix $\overline{\mathscr{I}}_{x=0}$ of $((\overline{\mathbb{B}}^n, J_t), \tau_o)$ and coincides with J_t at $x = 0$. Hence both the limits $\lim_{k\to\infty}(J_t - J_{\rm st})^\alpha_\delta|_{x_k}$, $\lim_{k\to\infty}(J_t - J_{\rm st})^{\bar\alpha}_\delta|_{x_k}$ are surely 0.

The last claim of the proposition is an immediate consequence of the fact that if X_t is a special vector field satisfying (5.1) and J_t satisfies the differential problem (5.2), the first part of the proof implies that the J_t are complex structures on $\overline{\mathbb{B}}^n$ induced by L-complex structures on $\widetilde{\overline{\mathbb{B}}}^n$. □

5.2 Proof of Theorem 2.7

The proof of Theorem 2.7 is a consequence of the following existence result on abstract fundamental pairs.

Lemma 5.2 *Let $((\overline{\mathbb{B}}^n, J), \tau_o)$ be a manifold of circular type in normal form of class $\mathscr{C}^k$, $k \geq 3$, and v^o_s, $s \in [0, 1]$, a fixed one-parameter family of vectors in $T_0\mathbb{B}^n$ of class $\mathscr{C}^k$ in s. Then there is a maximal value $s_o \in (0, 1]$ such that for all $0 < s < s_o$ there exists a fundamental pair (J_t, X_t) on $((\overline{\mathbb{B}}^n, J), \tau_o)$ of class $\mathscr{C}^{k-1,\alpha}$ for any $\alpha \in (0, 1)$, guided by the curve $v_t{:=}v^o_{st}$, $t \in [0, 1]$, in which X_t is the special vector field $X_t = X^{[\rho_t](v_t,\sigma_t,\widetilde{Y}^\alpha_t)}$ with $\sigma_t \equiv 0$, the $\widetilde{Y}^\alpha_t$ as in (5.1) and $\rho_t = \rho$ for each t. The dependence of X_t on t is at least $\mathscr{C}^{k-1,\alpha}$. In case the maximal value s_o is strictly less than 1, there exists no fundamental pair (J_t, X_t), $t \in [0, 1]$, guided by the curve $v_t{:=}v^o_{ts_o}$, $t \in [0, 1]$.*

Indeed, given a vector $v \in \mathbb{C}^n$ with $\|v\| = 1$, one can consider an associated curve of vectors $\mathbf{v}_s$, $s \in (0, 1)$, defined as in (3.12). By Theorem 2.6, Remark 3.9, Proposition 5.1 and Lemma 5.2, for each $\lambda \in (0, \lambda_o{:=}s_o)$, there is a one-parameter family of Monge-Ampère exhaustions $\tau^{(x_t)} : \overline{\mathbb{B}}^n \longrightarrow [0, 1]$, $t \in [0, 1]$, centered at the points of the straight segment $x_t{:=}t(\lambda v)$, each of class $\mathscr{C}^{k-1,\alpha}$ for all $\alpha \in (0, 1)$. The last claim of the lemma implies that, if $\lambda_o < 1$, then there is no one-parameter family of Monge-Ampère exhaustions centered at the points of the straight segment $x_t{:=}t(\lambda_o v)$ that is well defined also in $x_{t=1}$.

Proof Consider a set of polar coordinates $(\zeta, w^\alpha, \bar\zeta, \overline{w^\alpha})$, an adapted polar frame field $(Z, e_\alpha = e^{(\rho)}_\alpha)$ and the corresponding dual coframe field (Z^*, e^α), both determined by the function ρ which gives the Minkowski function of the indicatrix $\overline{\mathscr{I}}_{x=0}$ of $(\overline{\mathbb{B}}^n, J)$. All tensor fields J_t of an abstract fundamental pair have the form

$$J_t = iZ\otimes Z^* - i\overline{Z\otimes Z^*} + J_{t\alpha}^{\ \beta}e^\alpha\otimes e_\beta + J_{t\bar\alpha}^{\ \beta}\overline{e^\alpha}\otimes e_\beta + J_{t\alpha}^{\ \bar\beta}e^\alpha\otimes\overline{e_\beta} + J_{t\bar\alpha}^{\ \bar\beta}\overline{e^\alpha}\otimes\overline{e_\beta}\,, \tag{5.15}$$

for $\mathbb{C}$-valued $\mathscr{C}^{k-1,\alpha}$ functions $J_{tB}^{\ A}$ satisfying the reality conditions $J_{t\bar\alpha}^{\ \bar\beta} = \overline{J_{t\alpha}^{\ \beta}}$, $J_{t\alpha}^{\ \bar\beta} = \overline{J_{t\bar\alpha}^{\ \beta}}$. Hence, the lemma corresponds to the existence of complex functions $J_{t\alpha}^{\ \beta}$, $J_{t\bar\alpha}^{\ \beta}$, $t \in [0, 1]$, satisfying the required regularity conditions and such that tensors (5.15) verify:

(*a*) $J_0 = J$;
(*b*) $J_t^2|_x = -Id_{T_x\widetilde{\overline{\mathbb{B}}}^n}$ for each $t \in [0, 1]$ and $x \in \widetilde{\overline{\mathbb{B}}}^n$;
(*c*) the Nijenhuis tensor N_{J_t} is identically zero for each t;

(d) for each $t \in [0,1]$ and for any vector field $E \in \mathscr{H}$

$$\frac{\mathrm{d}J_t}{\mathrm{d}t}(Z) + (\mathscr{L}_{X^{[\rho](v_t,0,\widetilde{Y}^\alpha_t)}} J_t)(Z) = 0\,, \tag{5.16}$$

$$\frac{\mathrm{d}J_t}{\mathrm{d}t}(E) + (\mathscr{L}_{X^{[\rho](v_t,0,\widetilde{Y}^\alpha_t)}} J_t)(E) = 0\,. \tag{5.17}$$

Here, for Y^0_t, $H_{t\bar\gamma}$, $H_{t\gamma}$ as in (4.13), (4.24) for $v = v_t$,

$$\begin{aligned}\widetilde{Y}^\alpha_t = &-\frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\bar\delta} g^{\bar\delta\gamma}\left(\frac{H_{t\gamma}}{\zeta} + \overline{\frac{H_{t\bar\gamma}}{\bar\zeta}}\right) - \frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\delta} g^{\delta\bar\gamma}\left(\frac{H_{t\bar\gamma}}{\zeta} + \overline{\frac{H_{t\gamma}}{\bar\zeta}}\right)\\ &+\frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\bar\delta} g^{\bar\delta\gamma}\left(\overline{\left(H_{t\bar\gamma} - Y^0_t e_{\bar\gamma}(\log\rho^2)\right)}\zeta + \left(H_{t\gamma} - Y^0_t e_\gamma(\log\rho^2)\right)\bar\zeta\right)\\ &+\frac{i}{2}(J_t - J_{\mathrm{st}})^\alpha_{\delta} g^{\delta\bar\gamma}\left(\overline{\left(H_{t\gamma} - Y^0_t e_\gamma(\log\rho^2)\right)}\zeta + \left(H_{t\bar\gamma} - Y^0_t e_{\bar\gamma}(\log\rho^2)\right)\bar\zeta\right)\,.\end{aligned} \tag{5.18}$$

We now observe that (5.16) can be safely neglected, since $X^{[\rho](v_t,0,\widetilde{Y}^\alpha)}$ is a special vector field and hence, by the proof of Proposition 5.1, such equation is identically satisfied. It is also convenient to decompose the vector field $X_t := X^{[\rho](v_t,0,\widetilde{Y}^\alpha)}$ into the sum

$$X_t = \mathbb{X}_t + \mathbb{Y}_t \quad \text{where } \mathbb{Y}_t = \widetilde{Y}^\alpha_t e_\alpha + \overline{\widetilde{Y}^\alpha_t} e_{\bar\alpha} \text{ and}$$

$$\mathbb{X}_t = \left(\frac{Y^0_t}{\zeta} - \overline{Y^0_t}\zeta\right) Z + \frac{Y^\alpha_t}{\zeta} e_\alpha + \left(\overline{\frac{Y^0_t}{\bar\zeta}} - Y^0_t\bar\zeta\right)\overline{Z} + \frac{Y^{\bar\alpha}_t}{\bar\zeta} e_{\bar\alpha}\,,$$

so that(5.17)can be written as

$$\frac{\mathrm{d}J_t}{\mathrm{d}t}(E) + \mathscr{L}_{\mathbb{X}_t+\mathbb{Y}_t} J_t(E) = 0\,, \tag{5.19}$$

Note that $\mathbb{X}_t$ is independent of J_t, while $\mathbb{Y}_t$ depends in a linear way on $J_t - J_{\mathrm{st}}$.

Our proof of the existence of $\mathscr{C}^{k-1,\alpha}$ tensor fields J_t satisfying *(a)* - *(d)* will follow from the following three steps:

Step 1. Translate the problem into an equivalent one on the deformation tensors ϕ_t for the L-complex structures J_t (for the definition, see § 5.2.1 below).

Step 2. Prove that all conditions on the deformation tensors ϕ_t are satisfied if just the single equation on ϕ_t corresponding to (5.19) is verified.

Step 3. Prove the existence of solutions to the equation corresponding to (5.19) for $v_t = v^o_{\lambda t}$, $t \in [0,1]$, for all λ in an appropriate open interval $(0, \lambda_o)$.

According to this, after a short introduction to the theory deformation tensors, the presentation will be structured in three parts, one per each of such steps. After that, we discuss the $\mathscr{C}^\infty$ and $\mathscr{C}^\omega$ cases and make a short remark on examples.

5.2.1 An introduction to the deformation tensors of L-complex structures

Any tensor field J_t of the form (5.15) is determined by its restrictions to the distribution $\mathscr{H}$. If we also assume that it is an almost complex structure—that is, condition *(b)*—then it is actually determined by its associated J_t-antiholomorphic distributions $\mathscr{H}^{01}_t \subset \mathscr{H}^{\mathbb{C}}$. If the family J_t satisfies also *(a)*, *(c)* and *(d)*, then, by Lemma 3.8, it is a one-parameter family of L-complex structure, according to Definition 2.4.

Consider now the holomorphic and anti-holomorphic subdistributions $\mathscr{H}^{10}$, $\mathscr{H}^{01}=\overline{\mathscr{H}^{10}}$ of $\mathscr{H}^{\mathbb{C}}\subset T\widetilde{\widetilde{\mathbb{B}}}^n$, given by the standard complex structure $J_{\rm st}$ of the indicatrix $\overline{\mathscr{I}}_{x=0}$ of $(\overline{\mathbb{B}^n}, J_{t=0}=J)$ and the analogous distributions $\mathscr{H}^{10}_t$, $\mathscr{H}^{01}_t=\overline{\mathscr{H}^{10}_t}\subset\mathscr{H}^{\mathbb{C}}$, given by the $\pm i$-eigenspaces in $\mathscr{H}^{\mathbb{C}}$ of the almost complex structures J_t, $t\in[0,1]$. By the results in [3,22], each distribution $\mathscr{H}^{01}_t$ is determined by a unique tensor field ϕ_t in $\mathscr{H}^{01*}\otimes\mathscr{H}^{10}$, which allows to express all subspace $\mathscr{H}^{01}_t\big|_x\subset T_x\widetilde{\widetilde{\mathbb{B}}}^n$, $x\in\widetilde{\widetilde{\mathbb{B}}}^n$, in the form

$$\mathscr{H}^{01}_t\big|_x:=\{v\in\mathscr{H}^{\mathbb{C}}|_x\ :\ v=w+\phi_t(w)\text{ for some }w^{01}\in\mathscr{H}^{10}|_x\}\,.$$

Such ϕ_t is called *deformation tensor* of J_t.

Note that, in each set of adapted polar coordinates, the components of J_t in the complex frames field $(Z,\widehat{e}_{\tilde\alpha}:=e_\alpha+\overline{\phi_t(e_{\tilde\alpha})},\overline{Z},\widehat{e}_{\tilde\alpha}=e_{\tilde\alpha}+\phi_t(e_{\tilde\alpha}))$ are constant and all equal to $\pm i$ or 0. This means that the regularity of J_t with respect to the adapted polar coordinates is always the same of the regularity of ϕ_t in those coordinates.

The properties that J_t is an L-complex structure, i.e., it is integrable or, equivalently, it satisfies *(c)*, and that τ_o is Monge-Ampère exhaustion for $(\mathbb{B}^n,J_t)$, corresponds to the following conditions on ϕ_t, to be satisfied for all $0\neq X,Y\in\mathscr{H}^{01}$:

(A) for each $(t,y)\in[0,1]\times\widetilde{\widetilde{\mathbb{B}}}^n$, one has that

$$\ker\left(I_{T_x\widetilde{\widetilde{\mathbb{B}}}^n}-\overline{\phi}_t\circ\phi_t|_y\right)\qquad\text{is trivial}\,;\tag{5.20}$$

(B) $dd^c_{J_{\rm st}}\tau_o(\phi_t(X),\overline{\phi_t(X)})<dd^c_{J_{\rm st}}\tau_o(\overline{X},X)$ and $dd^c_{J_{\rm st}}\tau_o(\phi_t(X),Y)+dd^c_{J_{\rm st}}\tau_o(X,\phi_t(Y))=0$;

(C) $[\overline{Z},X+\phi_t(X)]\in\mathscr{H}^{01}_t$ or, equivalently,

$$\mathscr{L}_{\overline{Z}}\phi_t=0\,,\tag{5.21}$$

where Z is the generator of $\mathscr{Z}^{10}$ defined in (4.3);

(D) $[X+\phi_t(X),Y+\phi_t(Y)]\in\mathscr{H}^{01}_t$.

The geometrical interpretations of these conditions are the following.

(1) Condition (A) corresponds to the fact that the tensors $\phi_t\in\mathscr{H}^{01*}\otimes\mathscr{H}^{10}$ determine a direct sum decomposition $\mathscr{H}^{\mathbb{C}}=\mathscr{H}^{10}_t\oplus\mathscr{H}^{01}_t$ with

$$\mathscr{H}^{01}_t|_x:=\{v\in\mathscr{H}^{\mathbb{C}}|_x\ :\ v=w+\phi_t(w)\,,\,w^{01}\in\mathscr{H}^{10}|_x\}\text{ and }\ \mathscr{H}^{10}_t=\overline{\mathscr{H}^{01}_t}\tag{5.22}$$

(see, e.g., [6]). If it holds, the one-parameter family of tensors ϕ_t defines an associated family of (possibly non-integrable) complex structures J_t on the distribution $\mathscr{H}$.

(2) Condition (B) expresses the condition that each level set of τ_o is strongly pseudoconvex and hence that τ_o is a Monge-Ampère exhaustion for $(\overline{\mathbb{B}^n},J_t)$ if J_t is the complex structure determined by the direct sum decomposition (5.22). Note that (B) implies (A), but (A) does not imply (B).

(3) Conditions (C) and (D) are equivalent to the condition of integrability for the one-parameter family of complex structures J_t determined by the decomposition (5.22).

The above conditions are also sufficient in the following sense: *if a tensor field* $\phi_t\in\mathscr{H}^{01*}\otimes\mathscr{H}^{01}$ *satisfies* (A)–(D) (actually, (B)–(D) are enough, since (B) implies (A)), *then the corresponding distribution* $\mathscr{H}^{01}_t\subset\mathscr{H}^{\mathbb{C}}$ *uniquely determines an L-complex structure* J_t ([3,22]). Finally we remark that conditions (A)–(D) are meaningful under very mild differentiability assumptions: $\mathscr{C}^2$-smoothness on the data is sufficient.

We conclude this short review with the proof that the identity (5.7) is a consequence of the integrability of the complex structures J_t, as claimed in the proof of Proposition 5.1. Indeed, using adapted polar frames, we may write (5.7) in the form

$$dd^c_{J_{\rm st}}\tau_o\left(\frac{Y^\alpha_t}{\zeta}e_\alpha+\frac{\overline{Y^\alpha_t}}{\bar\zeta}e_{\bar\alpha}+\mathbb{Y}_t,[\overline{Z},e_{\bar\beta}+\phi_t(e_{\bar\beta})]\right)=0 \tag{5.23}$$

Since $\mathscr{L}_{\bar Z}\phi_t=0$ by (C), we have $[\bar Z,e_{\bar\alpha}+\phi_t(e_{\bar\alpha})]=0$ and (5.23) is satisfied.

5.2.2 Step 1: Translation into a problem on deformations tensors

By the previous discussion, our original problem translates into a problem on one-parameter families of $\mathscr{C}^{k-1,\alpha}$ deformation tensors ϕ_t satisfying (A)–(D) and the differential equations (5.19) with initial condition $\phi_0=\phi_J$, where ϕ_J is the deformation tensor that allows to express $J_{t=0}=J$ as deformation of the complex structure $J_{\rm st}=J_{\rm st}{}^{(\rho)}$ of the indicatrix of $(\overline{\mathbb{B}}^n,J)$ at $x=0$. Moreover, we claim that condition (B) is satisfied whenever all other conditions hold and thus it can be neglected. This is because, if the ϕ_t satisfy (A), (C), (D) and (5.19), then the family of integrable complex structures J_t, determined by means of the direct sum decomposition (5.22), surely satisfies all hypotheses of Lemma 3.8. This implies that the exhaustion τ_o is Monge-Ampère for $(\overline{\mathbb{B}^n},J_t)$ for each t and that also (B) holds.

In the remaining part of this section, we derive a formulation of (5.19) as an explicit condition on the ϕ_t. For this, recall that for each $E+\phi_t(E)\in\mathscr{H}^{01}_t$, we have that $J_t(E+\phi_t(E))=-i(E+\phi_t(E))$. Taking Lie derivatives of both sides with respect to the vector field $\frac{\mathrm{d}}{\mathrm{d}t}+X_t$ of $[0,1]\times\widetilde{\overline{\mathbb{B}}}^n$, we get

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(J_t)(E+\phi_t(E))=-i(I+iJ_t)\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(E+\phi_t(E))\,.$$

This means that (5.19) holds if and only if

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(E+\phi_t(E)|_x\in\ker(I+iJ_t)|_x$$

for each (t,x), i.e., if and only if

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(E+\phi_t(E))\in\mathscr{H}^{01}_t+\mathscr{Z}^{01}\qquad\text{for all } E+\phi_t(E)\in\mathscr{H}^{01}_t\,. \tag{5.24}$$

On the other hand, since it is the special vector field described in Proposition 5.1, the vector field X_t necessarily satisfies (5.8). This implies that $\mathscr{L}_{X_t}(E+\phi_t(E))$ is always in $\mathscr{H}^{\mathbb{C}}$ and that (5.24) can be replaced by the weaker condition

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(E+\phi_t(E))\in\mathscr{H}^{01}_t+\mathscr{Z}^{\mathbb{C}},\qquad\text{for all } E\in\mathscr{H}^{01}\,. \tag{5.25}$$

This condition can be stated as a system of p.d.e. as follows. Fix a (local) complex frame field $(e_{\bar\alpha})$ for $\mathscr{H}^{01}$ and for each t consider the uniquely associated complex frame fields $(\mathbb{E}_{t|\bar\alpha}=e_{\bar\alpha}+\phi_t(e_{\bar\alpha}))$ and $(\mathbb{E}_{t\alpha}=\overline{\mathbb{E}_{t|\bar\alpha}})$ for $\mathscr{H}^{01}_t$ and $\mathscr{H}^{10}_t$, respectively. Then, let $g^{(\phi_t)}\in\mathrm{Hom}(\mathscr{H}\times\mathscr{H},\mathbb{R})$ be the only J_t-invariant tensor field such that

$$g^{(\phi_t)}(\mathbb{E}_{t|\alpha},\mathbb{E}_{t|\bar\beta})=(I-\bar\phi_t\circ\phi_t)^\gamma_\alpha g_{\gamma\bar\beta}\,, \tag{5.26}$$

where $g_{\alpha\bar\beta}$ are defined in (4.6) and where $(\cdot)^\gamma_\alpha$, $(\cdot)^{\bar\gamma}_{\bar\alpha}$ denote the components of the considered $(1,1)$-tensors of $\mathscr{H}^{\mathbb{C}}$ in the frames $(e_\alpha,e_{\bar\alpha})$. Note that if ϕ_t satisfies (B) (hence, $\|\phi_t\|_{\mathscr{C}^0}<1$ for each $t\in[0,\varepsilon]$), then $g^{(\phi_t)}$ is non-degenerate on each space $\mathscr{H}_x$. Observe also that, by construction and J_t-invariance,

$$
\begin{aligned}
g^{(\phi_t)}(E_{t|\bar{\alpha}}, \mathbb{E}_{t|\bar{\beta}}) &= 0\,,\\
g^{(\phi_t)}(e_\alpha, \mathbb{E}_{t|\bar{\beta}}) &= g^{(\phi_t)}(\mathbb{E}_{t|\alpha}, \mathbb{E}_{t|\bar{\beta}}) - \phi_{t\alpha}^{\bar{\gamma}} g^{(\phi_t)}(e_{\bar{\gamma}}, \mathbb{E}_{t|\bar{\beta}})\\
&= g^{(\phi_t)}(\mathbb{E}_{t|\alpha}, \mathbb{E}_{t|\bar{\beta}}) - \bar{\phi}_{t\alpha}^{\bar{\gamma}} g^{(\phi_t)}(\mathbb{E}_{t|\bar{\gamma}}, \mathbb{E}_{t|\bar{\beta}}) + (\bar{\phi}_t \circ \phi_t)_\alpha^\gamma g^{(\phi_t)}(e_\gamma, \mathbb{E}_{t|\bar{\beta}})\\
&= g^{(\phi_t)}(\mathbb{E}_{t|\alpha}, \mathbb{E}_{t|\bar{\beta}}) + (\bar{\phi}_t \circ \phi_t)_\alpha^\gamma g^{(\phi_t)}(e_\gamma, \mathbb{E}_{t|\bar{\beta}})\,.
\end{aligned}
\tag{5.27}
$$

In particular, from the last equality we get that

$$
g^{(\phi_t)}(e_\alpha, \mathbb{E}_{t|\bar{\beta}}) = ((I - \bar{\phi}_t \circ \phi_t)^{-1})_\alpha^\gamma g^{(\phi_t)}(\mathbb{E}_{t|\gamma}, \mathbb{E}_{t|\bar{\beta}}) = g_{\alpha\bar{\beta}}\,. \tag{5.28}
$$

This identity will appear to be quite useful in the last part of the proof and it is the main motivation for considering the above definition for $g^{(\phi_t)}$.

Notice that using the tensor fields $g^{(\phi_t)}$, condition (5.25) becomes equivalent to the system of p.d.e.'s

$$
g^{(\phi_t)}(\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t} \mathbb{E}_{t|\bar{\alpha}}, \mathbb{E}_{t|\bar{\beta}}) = 0\,. \tag{5.29}
$$

We now recall that in (5.25) the vector field $X_t = \mathbb{X}_t + \mathbb{Y}_t$ *does depend* on the unknown ϕ_t, due to the fact that $\mathbb{Y}_t$ depends on $J_t - J_{\mathrm{st}}$. We need to make such dependence fully explicit. For this, we first recall that, for each t, a vector field Y in $\mathscr{H}^{\mathbb{C}}$ uniquely decomposes not only as a sum of holomorphic and anti-holomorphic components with respect to J_{st}, but also as a sum of holomorphic and anti-holomorphic components with respect to J_t. We denote such two distinct decompositions by $Y = Y^{10} + Y^{01} = Y_t^{10} + Y_t^{01}$. On the other hand, we know that the components Y_t^{10}, Y_t^{01} have the form

$$
Y_t^{10} = \widehat{Y}^{10(t)} + \overline{\phi_t}(\widehat{Y}^{10(t)})\,, \qquad Y_t^{01} = \widehat{Y}^{01(t)} + \phi_t(\widehat{Y}^{01(t)})
$$

for appropriate $\widehat{Y}^{10(t)} \in \mathscr{H}^{10}$ and $\widehat{Y}^{01(t)} \in \mathscr{H}^{01}$. A straightforward algebraic computation shows that such vectors are expressed in terms of Y^{10}, Y^{01} by

$$
\begin{aligned}
\widehat{Y}^{10(t)} &:= (I - \phi_t \circ \overline{\phi_t})^{-1} \left(Y^{10} - \phi_t(Y^{01})\right)\,,\\
\widehat{Y}^{01(t)} &:= (I - \bar{\phi}_t \circ \phi_t)^{-1} \left(Y^{01} - \overline{\phi_t}(Y^{10})\right)\,,
\end{aligned}
\tag{5.30}
$$

where each $\phi_t|_x \in \mathrm{Hom}(\mathscr{H}_x^{01}, \mathscr{H}_x^{10})$ is here considered in $\mathrm{Hom}(\mathscr{H}_x^{\mathbb{C}}, \mathscr{H}_x^{\mathbb{C}})$, acting trivially on $\mathscr{H}_x^{10}$. Note that, *due to condition (A), the linear operators* $(I - \bar{\phi}_t \circ \phi_t)$ *are invertible, so that the above expressions are meaningful*. It follows that the J_t-holomorphic and J_t-anti-holomorphic parts of the $e_\alpha \in \mathscr{H}^{10}$ are

$$
\begin{aligned}
e_\alpha = &\left\{(I - \phi_t \circ \overline{\phi_t})^{-1}(e_\alpha) + \overline{\phi_t}((I - \phi_t \circ \overline{\phi_t})^{-1}(e_\alpha))\right\}\\
&- \left\{(I - \bar{\phi}_t \circ \phi_t)^{-1}\left(\overline{\phi_t}(e_\alpha)\right) + \phi_t((I - \bar{\phi}_t \circ \phi_t)^{-1}\left(\overline{\phi_t}(e_\alpha)\right))\right\}\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_t(e_\alpha) = &\, i\left\{(I - \phi_t \circ \overline{\phi_t})^{-1}(e_\alpha) + \overline{\phi_t}((I - \phi_t \circ \overline{\phi_t})^{-1}(e_\alpha))\right\}\\
&+ i\left\{(I - \bar{\phi}_t \circ \phi_t)^{-1}\left(\overline{\phi_t}(e_\alpha)\right) + \phi_t((I - \bar{\phi}_t \circ \phi_t)^{-1}\left(\overline{\phi_t}(e_\alpha)\right))\right\}
\end{aligned}
$$

and $(J_t - J_{\mathrm{st}})(e_\alpha) = J_t(e_\alpha) - i e_\alpha$ is

$$
(J_t - J_{\mathrm{st}})(e_\alpha) = 2i\left\{(I - \bar{\phi}_t \circ \phi_t)^{-1}\left(\overline{\phi_t}(e_\alpha)\right) + \phi_t((I - \bar{\phi}_t \circ \phi_t)^{-1}\left(\overline{\phi_t}(e_\alpha)\right))\right\}\,.
$$

From the identity $\phi_t \circ (I - \bar{\phi}_t \circ \phi_t)^{-1} = (I - \phi_t \circ \bar{\phi}_t)^{-1} \circ \phi_t$ (it can be checked using the expansion $(I - \bar{\phi}_t \circ \phi_t)^{-1} = \sum_{r=0}^{\infty} (\bar{\phi}_t \circ \phi_t)^r$) we may also say that

$$(J_t - J_{\mathrm{st}})(e_\alpha) = 2i \left\{ \left((I - \bar{\phi}_t \circ \phi_t)^{-1} \flat \overline{\phi_t} \right)(e_\alpha) + \left((I - \phi_t \circ \bar{\phi}_t)^{-1} \flat \left(\phi_t \circ \overline{\phi_t} \right) \right)(e_\alpha) \right\},$$

from which we get the components of $J - J_{\mathrm{st}}$ w.r.t. $(e_\alpha, e_{\bar{\beta}})$. They form the matrix

$$\begin{pmatrix} (J_t - J_{\mathrm{st}})^\beta_\alpha = 2i ((I - \phi_t \circ \bar{\phi}_t)^{-1})^\beta_\gamma (\phi_t \circ \bar{\phi}_t)^\gamma_\alpha & (J_t - J_{\mathrm{st}})^{\bar{\beta}}_\alpha = 2i ((I - \bar{\phi}_t \circ \phi_t)^{-1})^{\bar{\beta}}_{\bar{\gamma}} \bar{\phi}_t{}^{\bar{\gamma}}_\alpha \\ (J_t - J_{\mathrm{st}})^{\bar{\beta}}_{\bar{\alpha}} = -2i ((I - \bar{\phi}_t \circ \phi_t)^{-1})^{\bar{\beta}}_{\bar{\gamma}} (\bar{\phi}_t \circ \phi_t)^{\bar{\gamma}}_{\bar{\alpha}} & (J_t - J_{\mathrm{st}})^\beta_{\bar{\alpha}} = -2i ((I - \phi_t \circ \bar{\phi}_t)^{-1})^\beta_\gamma \phi_t{}^\gamma_{\bar{\alpha}} \end{pmatrix}.$$

Thus, the explicit dependence of the $\widetilde{Y}^\alpha_t$ on the ϕ_t we are looking for is

$$\begin{aligned} \widetilde{Y}^\alpha_t = {} & ((I - \phi_t \circ \bar{\phi}_t)^{-1})^\alpha_\gamma (\phi_t \circ \bar{\phi}_t)^\gamma_\delta g^{\delta\bar{\eta}} \Big(\frac{H_{t\bar{\eta}}}{\zeta} + \frac{\overline{H_{t\eta}}}{\bar{\zeta}} \\ & - \overline{\left(H_{t\eta} - Y^0_t e_\eta (\log \rho^2) \right)} \zeta - \left(H_{t\bar{\eta}} - Y^0_t e_{\bar{\eta}} (\log \rho^2) \right) \bar{\zeta} \Big) \\ & + ((I - \phi_t \circ \bar{\phi}_t)^{-1})^\alpha_\gamma \phi_t{}^\gamma_{\bar{\delta}} g^{\bar{\delta}\eta} \Big(\frac{H_{t\eta}}{\zeta} + \frac{\overline{H_{t\bar{\eta}}}}{\bar{\zeta}} \\ & - \overline{\left(H_{t\bar{\eta}} - Y^0_t e_{\bar{\eta}} (\log \rho^2) \right)} \zeta - \left(H_{t\eta} - Y^0_t e_\eta (\log \rho^2) \right) \bar{\zeta} \Big). \end{aligned} \tag{5.31}$$

Recalling that $(I - \phi_t \circ \bar{\phi}_t)^{-1} \circ \phi_t = \phi_t \circ (I - \bar{\phi}_t \circ \phi_t)^{-1}$ and that

$$(I - \phi_t \circ \bar{\phi}_t)^{-1} \circ (\phi_t \circ \bar{\phi}_t) = (I - \phi_t \circ \bar{\phi}_t)^{-1} - I \,,$$

from (5.31) we see that $\mathbb{Y}_t = \widetilde{Y}^\alpha_t e_\alpha + \overline{Y^\alpha_t} e_{\bar{\alpha}}$ has the form

$$\mathbb{Y}_t = (\mathbb{Y}_{t(1)} + \phi(\mathbb{Y}_{t(1)})) + (\mathbb{Y}_{t(2)} + \phi(\mathbb{Y}_{t(2)})) + \mathbb{Y}'_t$$

where

$$\begin{aligned} \mathbb{Y}_{t(1)} := {} & ((I - \bar{\phi}_t \circ \phi_t)^{-1} \bar{\phi}_t)^{\bar{\beta}}_\delta g^{\delta\bar{\eta}} \Big(\frac{\overline{H_{t\eta}}}{\bar{\zeta}} + \frac{H_{t\bar{\eta}}}{\zeta} - \\ & - \left(H_{t\bar{\eta}} - Y^0_t e_{\bar{\eta}} (\log \rho^2) \right) \bar{\zeta} - \overline{\left(H_{t\eta} - Y^0_t e_\eta (\log \rho^2) \right)} \zeta \Big) e_{\bar{\beta}} \,, \\ \mathbb{Y}_{t(2)} := {} & ((I - \bar{\phi}_t \circ \phi_t)^{-1})^{\bar{\beta}}_{\bar{\delta}} g^{\bar{\delta}\eta} \Big(\frac{H_{t\eta}}{\zeta} + \frac{\overline{H_{t\bar{\eta}}}}{\bar{\zeta}} + \\ & - \overline{\left(H_{t\bar{\eta}} - Y^0_t e_{\bar{\eta}} (\log \rho^2) \right)} \zeta - \left(H_{t\eta} - Y^0_t e_\eta (\log \rho^2) \right) \bar{\zeta} \Big) e_{\bar{\beta}} \,, \\ \mathbb{Y}'_t := {} & - g^{\bar{\beta}\eta} \Big(\frac{H_{t\eta}}{\zeta} + \frac{\overline{H_{t\bar{\eta}}}}{\bar{\zeta}} - \overline{\left(H_{t\bar{\eta}} - Y^0_t e_{\bar{\eta}} (\log \rho^2) \right)} \zeta - \left(H_{t\eta} - Y^0_t e_\eta (\log \rho^2) \right) \bar{\zeta} \Big) e_{\bar{\beta}} \,. \end{aligned} \tag{5.32}$$

Since $\sum_{i=1}^2 (\mathbb{Y}_{t(i)} + \phi(\mathbb{Y}_{t(i)}))$ is in $\mathscr{H}^{01}_t$ for each t, if (D) holds, then also $\mathscr{L}_{\sum_{i=1}^2 (\mathbb{Y}_{t(i)} + \phi(\mathbb{Y}_{t(i)}))} (E + \phi_t(E))$ is in $\mathscr{H}^{01}_t$. This means that, due to the integrability condition (D), in the set of the three conditions, the equation (5.25) involving the (real) vector field $X_t = \mathbb{X}_t + \mathbb{Y}_t$ can be replaced by the same equation with the (complex) vector field $X'_t = \mathbb{X}_t + \mathbb{Y}'_t$ in place of X_t. The crucial advantage of such replacement comes from the fact that *the new vector field X'_t is totally independent of ϕ_t*.

Summing up, our problem is now reduced to proving *the existence of a one parameter family of tensor fields* $\phi_t \in \mathscr{H}^{01*} \otimes \mathscr{H}^{10}$ *of class* $\mathscr{C}^{k-1,\alpha}$ *on* $[0,1] \times \widetilde{\overline{\mathbb{B}}}^n$ *satisfying the initial condition* $\phi_{t=0} = \phi_J$, *the non-degeneracy condition* (5.20) *and the system of equations*

$$[\overline{Z}, E + \phi_t(E)] \in \mathscr{H}^{01}_t \,, \tag{5.33}$$

$$[E + \phi_t(E), F + \phi_t(F)] \in \mathscr{H}^{01}_t \,, \tag{5.34}$$

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t} + X'_t}(E + \phi_t(E)) \in \mathscr{H}^{01}_t + \mathscr{Z}^{\mathbb{C}} \tag{5.35}$$

for all $E + \phi_t(E) \in \mathscr{H}^{01}_t$. Note that (5.35) can be equivalently stated as (5.29) putting X'_t in the place of X_t.

5.2.3 Step 2: Reduction to a single equation

In this subsection, we prove that if ϕ_t is a $\mathscr{C}^{k-1,\alpha}$ solution of (5.35) with initial condition $\phi_{t=0} = \phi_J$, then (5.33) and (5.34) are automatically satisfied. In fact, we prove this claim only in the case of the $\mathscr{C}^{k-1,\alpha}$ solutions ϕ_t, which can be obtained as limits of sequences of real analytic solutions $\phi^{(n)}_t$. This weaker result is enough for our purposes, because the solutions, of which we prove the existence in the final step, are precisely of such a kind.

As usual, given a solution ϕ_t to (5.35), let J_t be the corresponding one parameter family of (possibly non-integrable) complex structures of the form (5.15), for which ϕ_t is the deformation tensor relatively to J_{st}. Moreover, for any $x \in \widetilde{\overline{\mathbb{B}}}^n$, consider a (locally defined) complex frame field $(Z, E_\alpha, \bar{Z}, E_{\bar{\alpha}} := \overline{E_\alpha})$ on a neighborhood of x, in which the $E_{\bar{\alpha}}$ are generators for the distribution $\mathscr{H}^{01}$. (For instance, we may consider an adapted polar frame field.) For each t, we denote by $(Z, \mathbb{E}_{t|\alpha}, \bar{Z}, \mathbb{E}_{t|\bar{\alpha}} = \overline{\mathbb{E}_{t|\alpha}})$ the associated complex frame field, where the vector fields $\mathbb{E}_{t|\bar{\alpha}} = E_{\bar{\alpha}} + \varphi_t(E_{\bar{\alpha}})$ generate the J_t-antiholomorphic distribution $\mathscr{H}^{01}_t$ and the $\mathbb{E}_{t|\alpha} := \overline{\mathbb{E}_{t|\bar{\alpha}}}$ the distribution $\mathscr{H}^{10}_t$. We also denote by $(Z^*, \mathbb{E}^\alpha_{t|}, \bar{Z}^*, \mathbb{E}^{\bar{\alpha}}_{t|})$ the dual coframes field of $(Z, \mathbb{E}_{t|\alpha}, \bar{Z}, \mathbb{E}_{t|\bar{\alpha}} = \overline{\mathbb{E}_{t|\alpha}})$ and we set

$$\mathbb{F}_{t|\bar{\alpha}\bar{\beta}} := [E_{\bar{\alpha}} + \phi_t(E_{\bar{\alpha}}), E_{\bar{\beta}} + \phi_t(E_{\bar{\beta}})] = [\mathbb{E}_{t|\bar{\alpha}}, \mathbb{E}_{t|\bar{\beta}}] \,.$$

Observe that, since they are tangent to the level sets of τ_o, they have the form

$$\mathbb{F}_{t|\bar{\alpha}\bar{\beta}} = f^\gamma_{t|\bar{\alpha}\bar{\beta}} \mathbb{E}_{t|\gamma} + f^{\bar{\gamma}}_{t|\bar{\alpha}\bar{\beta}} \mathbb{E}_{t|\bar{\gamma}} + f^0_{t|\bar{\alpha}\bar{\beta}} (Z - \bar{Z})$$

with $f^\gamma_{t|\bar{\alpha}\bar{\beta}} = \mathbb{E}^\gamma_{t|}(\mathbb{F}_{t|\bar{\alpha}\bar{\beta}})$, $f^{\bar{\gamma}}_{t|\bar{\alpha}\bar{\beta}} = \mathbb{E}^{\bar{\gamma}}_{t|}(\mathbb{F}_{t|\bar{\alpha}\bar{\beta}})$, $f^0_{t|\bar{\alpha}\bar{\beta}} = Z^*(\mathbb{F}_{t|\bar{\alpha}\bar{\beta}})$. Note also that, being J integrable, the functions $f^\gamma_{t|\bar{\alpha}\bar{\beta}}|_{t=0}$, $f^0_{t|\bar{\alpha}\bar{\beta}}|_{t=0}$ are zero, while (5.34) holds if and only if the functions $f^\gamma_{t|\bar{\alpha}\bar{\beta}}$, $f^0_{t|\bar{\alpha}\bar{\beta}}$ vanish identically for each $t \in [0,1]$.

Assume for the moment that the integrable complex structure $J_{t=0} = J$, the solution ϕ_t to (5.35) and the family v_t we consider are all real analytic, so that also the components of the complex vector field X'_t are real analytic. We want to show that, under these assumptions, the $f^\gamma_{t|\bar{\alpha}\bar{\beta}}$, $f^0_{t|\bar{\alpha}\bar{\beta}}$ are identically 0 and that (5.34) holds. For this, we need the following technical result.

Sublemma 5.3 *Assume that all components of the complex vector field* X'_t *on* $[0,1] \times \widetilde{\overline{\mathbb{B}}}^n$ *are real analytic and consider a system of partial differential equations on* $[0,1] \times \widetilde{\overline{\mathbb{B}}}^n$ *for an unknown* $U : [0,1] \times \widetilde{\overline{\mathbb{B}}}^n \to \mathbb{C}^N$ *having the form in sets of polar coordinates*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + X'_t\right)(U^a) = \mathscr{F}^a(t, \zeta, \bar{\zeta}, w^i, \overline{w^i}, U^b) \,, \qquad 1 \le a \le N \,, \tag{5.36}$$

with $\mathscr{F}^a$ real analytic in $(t, \zeta, \bar{\zeta}, w^i, \overline{w^i})$ and polynomial in U^b. Then, for each $t \in [0,1]$ and each real analytic $V : \widetilde{\mathbb{B}}^n \to \mathbb{C}^N$, there is a unique local real analytic solution U on $[t_o - \varepsilon, t_o + \varepsilon] \times \widetilde{\mathbb{B}}^n$ that satisfies (5.36) with initial condition $U|_{t=t_o} = V$. Furthermore, if $\mathscr{S}$ is a subset of $\mathscr{C}^\omega(\widetilde{\mathbb{B}}^n, \mathbb{C}^N)$, for which there exists an a priori *upper bound for the $\mathscr{C}^0$-norms of all solutions to (5.36) with initial conditions in $\mathscr{S}$, then for each $V \in \mathscr{S}$ there exists a unique real analytic solution $U : [0,1] \times \widetilde{\mathbb{B}}^n \to \mathbb{C}^N$ to system (5.36) with $U|_{t=0} = V$.*

Proof For each point $(t_o, x_o) \in [0,1] \times \widetilde{\mathbb{B}}^n$, consider a neighborhood $\mathscr{U} \subset [0,1] \times \widetilde{\mathbb{B}}^n$, on which we may consider a system of real coordinates (x^i), which allow to identify $\mathscr{U}$ with an open subset of $\mathbb{C}^{2n+1} \cap \{\mathrm{Im}(z^i) = 0\}$. Assume also that $\mathscr{U}$ is sufficiently small so that all restrictions to $\mathscr{U}$ of the components of X'_t and of the coefficients of the polynomials $\mathscr{F}^a(U, \overline{U})$ extend as holomorphic functions of some open neighborhood $\mathscr{W} \subset \mathbb{C}^{2n+1}$ of $\mathscr{U}$. In this way, the *complex* components X^i of the complex vector field $\frac{\mathrm{d}}{\mathrm{d}t} + X'_t$ on $[0,1] \times \widetilde{\mathbb{B}}^n$ can be taken as the restrictions to $\mathscr{U}$ of some holomorphic functions of the form $A^i + iB^j$ on $\mathscr{W}$. These complex functions are such that the A^i and B^j take only real values at the points of $\mathscr{U} \subset \mathscr{W} \cap \{\mathrm{Im}(z^i) = 0\}$. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t} + X'_t = \left(A^i \frac{\partial}{\partial x^i} + iB^j \frac{\partial}{\partial x^j} \right) \Big|_{\mathscr{U}} = \left(A^i \frac{\partial}{\partial z^i} + iB^j \frac{\partial}{\partial z^j} + A^i \frac{\partial}{\partial \bar{z}^i} + iB^j \frac{\partial}{\partial \bar{z}^j} \right) \Big|_{\mathscr{U}}.$$

Any local real analytic solution $U = (U^a)$ of (5.36) admits a holomorphic extension on an open neighborhood of $\mathscr{U} \subset \mathbb{R}^{2n+1}$ in $\mathbb{C}^{2n+1}$. Being holomorphic, such extension of U is solution to the system of differential equations

$$\widehat{X}(U^a) - F^\alpha_\beta(z^i, U^b) = 0 \qquad \text{with} \qquad \widehat{X} := A^i \frac{\partial}{\partial z^i} + iB^j \frac{\partial}{\partial z^j}\,. \tag{5.37}$$

The graphs in $\mathbb{C}^{2n+1} \times \mathbb{C}^N$ of the holomorphic solutions $U : \mathscr{W} \to \mathbb{C}^N$ to (5.36) coincide with the complex submanifolds of $\mathbb{C}^{2n+1} \times \mathbb{C}^N$ that are tangent to the holomorphic vector field

$$\mathbb{W} := \widehat{X}|_z + \mathscr{F}^a(z, U) \frac{\partial}{\partial U^a}\,. \tag{5.38}$$

These graphs can be constructed as in the classical method of real characteristics, namely by taking unions of complex holomorphic curves tangent to the holomorphic vector field $\mathbb{W}$ (they do exist by complex Frobenius Theorem—we call them *complex characteristics*), passing through the points of the graph of an initial condition $V : \{t_o\} \times \widetilde{\mathscr{W}} \subset \mathscr{W} \to \mathbb{C}^N$. This method of construction gives local solutions of (5.37). Restricting them to $\mathscr{U} = \mathscr{W} \cap \{\mathrm{Im}(z^i) = 0\}$, we get the desired local real analytic solutions of (5.36). By previous observations, any local real analytic solution of (5.36) can be obtained in this way and, by construction, any such solution is uniquely determined by its values at some level set at $t = t_o$ for some fixed $t_o \in [0,1]$. This implies the first claim of the sublemma.

For the second claim, observe that by compactness of $\widetilde{\mathbb{B}}^n$, the above construction allows to determine a unique real analytic solution U to (5.36) for each initial value $U_{t=0} = V \in \mathscr{S}$ and we may assume that U is defined on a set of the form $[0, t_o] \times \widetilde{\mathbb{B}}^n$ for some $t_o \in (0, T]$. If there exists an *a priori* bound for the $\mathscr{C}^0$ norm of such solution, the vector field (5.38) has components that are *a priori* bounded at the boundary points of the graph of the solution $U : [0, t_o] \times \widetilde{\mathbb{B}}^n \to \mathbb{C}^N$. This implies that such solution can be extended to a solution $U : [0, t_o + \varepsilon] \times \widetilde{\mathbb{B}}^n \to \mathbb{C}^N$ for some $\varepsilon > 0$. A standard open and closed argument yields the existence of a solution to (5.36) on $[0,1] \times \widetilde{\mathbb{B}}^n$. □

Let us now go back to our discussion on the functions $f^{\gamma}_{t|\bar\alpha\bar\beta}$, $f^{0}_{t|\bar\alpha\bar\beta}$, under the assumption of real analyticity of the data. We recall that we are assuming that ϕ_t is solution to (5.35), i.e., that

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}_{t|\bar\eta} = \sigma^{\bar\mu}_{t|\bar\eta}\mathbb{E}_{t|\bar\mu} + \sigma^{0}_{t|\bar\eta}Z + \sigma^{\bar 0}_{t|\bar\eta}\overline{Z} \tag{5.39}$$

for some complex functions $\sigma^{\bar\mu}_{t|\bar\eta}$, $\sigma^{0}_{t|\bar\eta}$, $\sigma^{\bar 0}_{t|\bar\eta}$. Note also that, by (5.5),

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}Z = \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\overline{Z} = 0\,. \tag{5.40}$$

Hence, by duality,

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}^{\eta}_t = \rho^{\eta}_{t|\mu}\mathbb{E}^{\mu}_t + \rho^{\eta}_{t|0}Z^* + \rho^{\eta}_{t|\bar 0}\overline{Z}^*\,, \qquad \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}Z^* = \rho^{0}_{t|\mu}\mathbb{E}^{\mu}_t + \rho^{0}_{t|\bar\mu}\mathbb{E}^{\bar\mu}_t \tag{5.41}$$

for appropriate complex functions $\rho^{\eta}_{t|\mu}$, $\rho^{\eta}_{t|0}$, $\rho^{\eta}_{t|\bar 0}$, $\rho^{0}_{t|\mu}$, $\rho^{0}_{t|\bar\mu}$.

We now claim that the functions $\sigma^{0}_{t|\bar\eta}$ and $\sigma^{\bar 0}_{t|\bar\eta}$ are actually linear combinations of the functions $f^{0}_{t|\bar\alpha\bar\beta}$ defined above. Indeed, we recall that the (complex) vector field X'_t differs from the (real) vector field X_t by the vector field $\sum_{i=1}^{2}(\mathbb{Y}_{t(i)} + \phi(\mathbb{Y}_{t(i)}) \in \mathscr{H}^{01}_t$. We may therefore write that $X'_t = X_t + \mathbb{Y}^{\bar\alpha}_t\mathbb{E}_{t|\bar\alpha}$ for some functions $\mathbb{Y}^{\bar\alpha}_t$ and

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}_{t|\bar\eta} = \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}\mathbb{E}_{t|\bar\eta} + \mathbb{Y}^{\bar\mu}_t\mathbb{F}_{t|\bar\nu\bar\eta} - \mathbb{E}_{t|\bar\eta}(\mathbb{Y}^{\bar\mu}_t)\mathbb{E}_{t|\bar\mu}\,. \tag{5.42}$$

Since X_t is a special vector field as described in Proposition 5.1, by (5.8), we have that $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}\mathbb{E}_{t|\bar\eta} \in \mathscr{H}^{\mathbb{C}}$. Comparing (5.42) with (5.39) and using the general structure of the vectors $\mathbb{F}_{t|\bar\alpha\bar\beta}$, we get that

$$\sigma^{0}_{t|\bar\eta} = \mathbb{Y}^{\bar\mu}_t f^{0}_{t|\bar\mu\bar\eta}\,, \qquad \sigma^{\bar 0}_{t|\bar\eta} = -\mathbb{Y}^{\bar\mu}_t f^{0}_{t|\bar\mu\bar\eta}\,, \tag{5.43}$$

i.e., $\sigma^{0}_{t|\bar\eta}$, $\sigma^{\bar 0}_{t|\bar\eta}$ are linear combinations of $f^{0}_{t|\bar\alpha\bar\beta}$, as claimed.

Now, combining (5.39), (5.40), (5.41) and (5.43), we get

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t}+X'_t\right)(f^{\gamma}_{t|\bar\alpha\bar\beta}) &= \rho^{\gamma}_{t|\mu}f^{\mu}_{t|\bar\alpha\bar\beta} + (\rho^{\gamma}_{t|0}-\rho^{\gamma}_{t|\bar 0})f^{0}_{t|\bar\alpha\bar\beta} + \mathbb{E}^{\gamma}_{t|}([\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}_{t|\bar\alpha}, \mathbb{E}_{t|\bar\beta}]) \\
&\quad + \mathbb{E}^{\gamma}_{t|}([\mathbb{E}_{t|\bar\alpha}, \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}_{t|\bar\beta}]) \\
&= \rho^{\gamma}_{t|\mu}f^{\mu}_{t|\bar\alpha\bar\beta} + (\rho^{\gamma}_{t|0}-\rho^{\gamma}_{t|\bar 0})f^{0}_{t|\bar\alpha\bar\beta} \\
&\quad + \sigma^{\bar\mu}_{t|\bar\alpha}f^{\gamma}_{t|\bar\mu\bar\beta} + \sigma^{\bar\mu}_{t|\bar\beta}f^{\gamma}_{t|\bar\alpha\bar\mu} + \mathbb{Y}^{\bar\mu}_t f^{0}_{t|\bar\mu\bar\alpha}(\mathbb{E}^{\gamma}_{t|}([Z,\mathbb{E}_{t|\bar\beta}]) - \mathbb{E}^{\gamma}_{t|}([\overline{Z},\mathbb{E}_{t|\bar\beta}])) \\
&\quad + \mathbb{Y}^{\bar\mu}_t f^{0}_{t|\bar\mu\bar\beta}(\mathbb{E}^{\gamma}_{t|}([\mathbb{E}_{t|\bar\alpha}, Z]) - \mathbb{E}^{\gamma}_{t|}([\mathbb{E}_{t|\bar\alpha}, \overline{Z}]))\,,
\end{aligned} \tag{5.44}$$

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t}+X'_t\right)(f^{0}_{t|\bar\alpha\bar\beta}) &= \rho^{0}_{t|\bar\mu}f^{\mu}_{t|\bar\alpha\bar\beta} + \rho^{0}_{t|\bar\mu}f^{\bar\mu}_{t|\bar\alpha\bar\beta} + Z^*([\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}_{t|\bar\alpha}, \mathbb{E}_{t|\bar\beta}]) \\
&\quad + Z^*([\mathbb{E}_{t|\bar\alpha}, \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}\mathbb{E}_{t|\bar\beta}]) \\
&= \rho^{0}_{t|\mu}f^{\mu}_{t|\bar\alpha\bar\beta} - f^{0}_{t|\bar\nu\bar\mu}\mathbb{Y}^{\bar\nu}_t f^{\bar\mu}_{t|\bar\alpha\bar\beta} \\
&\quad + \sigma^{\bar\mu}_{t|\bar\alpha}f^{0}_{t|\bar\mu\bar\beta} + \mathbb{Y}^{\bar\mu}_t f^{0}_{t|\bar\mu\bar\alpha}(Z^*([Z,\mathbb{E}_{t|\bar\beta}]) - Z^*([\overline{Z},\mathbb{E}_{t|\bar\beta}])) \\
&\quad + \sigma^{\bar\mu}_{t|\bar\beta}f^{0}_{t|\bar\alpha\bar\mu} + \mathbb{Y}^{\bar\mu}_t f^{0}_{t|\bar\mu\bar\beta}(Z^*([\mathbb{E}_{t|\bar\alpha}, Z]) - Z^*([\mathbb{E}_{t|\bar\alpha}, \overline{Z}]))\,.
\end{aligned} \tag{5.45}$$

This shows that the functions $f^0_{t|\bar\alpha\bar\beta}$ and $f^\gamma_{t|\bar\alpha\bar\beta}$ satisfy a system of the form (5.36). Note also that (5.44) and (5.45) admit the trivial solution for the initial value problem $f^\gamma_{t|\bar\alpha\bar\beta}\big|_{t=0} = f^0_{t|\bar\alpha\bar\beta}\big|_{t=0} = 0$. By the uniqueness of the local solutions, proved in Sublemma 5.3, this implies that, under the above real analyticity assumptions, the condition (5.34) is identically satisfied.

Assume now that $J_{t=0} = J$ and that ϕ_t and v_t are not real analytic, but that nonetheless there is a sequence of real analytic curves $v_t^{(n)}$ converging in $\mathscr{C}^k$-norm to v_t, a sequence of real analytic complex structures $J^{(n)}$, converging in $\mathscr{C}^k$-norm to J, and a corresponding sequence of real analytic solutions $\phi^{(n)}$ to (5.35) with initial data $\phi^{(n)}_{t=0} = \phi_{J_n}$, converging in $\mathscr{C}^k$-norm to the solution ϕ_t. Note that, for each of these sequences, the convergence is also in $\mathscr{C}^{k-1,\alpha}$ for any $\alpha \in (0, 1)$ and that the $\mathscr{C}^{k-1,\alpha}$-norms of the elements of the sequence converge to the finite value of the $\mathscr{C}^{k-1,\alpha}$-norm of the limit. In this case, the associated sequences $f^{(n)\gamma}_{t|\bar\alpha\bar\beta}$, $f^{(n)0}_{t|\bar\alpha\bar\beta}$ converge in $\mathscr{C}^{k-2}$-norm to the functions $f^\gamma_{t|\bar\alpha\bar\beta}$, $f^0_{t|\bar\alpha\bar\beta}$ determined by ϕ_t, and their $\mathscr{C}^{k-3,\alpha}$-norms converge to finite values. Since:

(a) the equations satisfied by the $f^{(n)\gamma}_{t|\bar\alpha\bar\beta}$, $f^{(n)0}_{t|\bar\alpha\bar\beta}$ are linear,

(b) the coefficients of the systems have $\mathscr{C}^{k-3,\alpha}$-norms which tend to the $\mathscr{C}^{k-3,\alpha}$-norms of the coefficients of the equations determined by the ϕ_t,

(c) the initial data tend to the zero functions in $\mathscr{C}^{k-3,\alpha}$-norm,

we conclude that also the limit functions $f^\gamma_{t|\bar\alpha\bar\beta}$, $f^0_{t|\bar\alpha\bar\beta}$ are identically vanishing and that (5.34) holds also in this case.

It remains to prove that any solution ϕ_t to (5.35), with $\phi_{t=0} = \phi_J$, necessarily satisfies also the integrability condition (5.33). For doing this, we first observe that, since we just proved that (5.34) is surely satisfied, by the discussion of § 5.2.2, we also have that the (real) special vector field $X_t := X'_t + \mathbb{Y}_{t(1)} + \phi\left(\mathbb{Y}_{t(1)}\right) + \mathbb{Y}_{t(2)} + \phi\left(\mathbb{Y}_{t(2)}\right)$ is so that $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t} J_t = 0$. Now, let g_t be a one-parameter family of J_t-invariant Riemannian metrics on $\widetilde{\widetilde{\mathbb{B}}}^n$ such that $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t} g_t = 0$. Since we know that $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t} J_t = 0$, one can construct such a one-parameter family g_t by considering the flow Φ_t on $\mathbb{R} \times \widetilde{\widetilde{\mathbb{B}}}^n$, determined by the vector field $\frac{\mathrm{d}}{\mathrm{d}t} + X_t$, and impose that, for each pair vector fields $W, W' \in T(\{t\} \times \widetilde{\widetilde{\mathbb{B}}}^n)$,

$$g_t(W, W') := -\frac{i}{2\tau} dd^c \tau_J(\Phi_{-t*}(W), J\Phi_{-t*}(W')) \,.$$

The family of these Riemannian metrics is clearly invariant under the flow Φ_t, but it is also $J_t|_x$-invariant at each (t, x), since also the family J_t is invariant under that flow. Finally, we set

$$\mathbb{F}_{t|\bar 0\bar\beta} := [\bar Z, E_{\bar\beta} + \phi_t(E_{\bar\beta})] = [\bar Z, \mathbb{E}_{t|\bar\beta}] \,, \qquad f_{t|\bar 0\bar\beta\bar\gamma} := g_t(\mathbb{F}_{t|\bar 0\bar\beta}, \mathbb{E}_{t|\bar\gamma}) \,.$$

Note that all the vector fields $\mathbb{F}_{t|\bar 0\bar\beta}$ are in $\mathscr{H}^{\mathbb{C}}$ and that (5.33) holds if and only if $f_{t|\bar 0\bar\beta\bar\gamma} = 0$ for each t. Notice also that $f_{t|\bar 0\bar\beta\bar\gamma}|_{t=0} = 0$, being $J_{t=0} = J$ integrable.

Imitating the above arguments, our next goal is to prove that the functions $f_{t|0\beta\gamma}$ are constrained by a differential problem, which has the trivial map as unique solution. This would imply our desired result. For proving this, we recall that being ϕ_t solution to (5.35), it also satisfies (5.29) and, due to (5.4),

$$\mathscr{L}_{X_t} \overline{Z} \in \mathscr{H}^{01}_t \qquad \text{for each } t \in [0, 1] \,.$$

Hence, from $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}} Z = 0$, (5.35) and (5.34), we obtain

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(\mathbb{F}_{t|\bar{0}\bar{\beta}}) = [\mathscr{L}_{X_t}\overline{Z}, \mathbb{E}_{t|\bar{\beta}}] + [\overline{Z}, \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(\mathbb{E}_{t|\bar{\beta}}] \in \mathscr{H}^{01} + [\mathscr{Z}^{01}, \mathscr{H}^{01}_t]\,. \quad (5.46)$$

This means that these Lie derivatives can be written as linear combinations of the $\mathbb{E}_{t|\bar{\alpha}}$ and of the $\mathbb{F}_{t|0\beta}$. Since $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t} g_t = 0$, $\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}(\mathbb{E}_{t|\bar{\gamma}}) \subset \mathscr{H}^{01}_t$ and $g_t(\mathscr{H}^{01}_t, \mathscr{H}^{01}_t) = 0$, we get

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t} + X_t\right)(f_{t|\bar{0}\bar{\beta}\bar{\gamma}}) &= g_t(\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}\mathbb{F}_{t|\bar{0}\bar{\beta}}, \mathbb{E}_{t|\bar{\gamma}}) + g_t(\mathbb{F}_{t|\bar{0}\bar{\beta}}, \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X_t}\mathbb{E}_{t|\bar{\gamma}}) \\ &= C^{\bar{\varepsilon}}_{t|\bar{0}\bar{\beta}\bar{\gamma}} g_t(\mathbb{F}_{t|\bar{0}\bar{\varepsilon}}, \mathbb{E}_{t|\bar{\gamma}}) + D^{\bar{\delta}}_{t|\bar{\gamma}} g_t(\mathbb{F}_{t|\bar{0}\bar{\beta}}, \mathbb{E}_{t|\bar{\delta}}) \\ &= C^{\bar{\varepsilon}}_{t|\bar{0}\bar{\beta}\bar{\gamma}} f_{t|\bar{0}\bar{\varepsilon}\bar{\gamma}} + D^{\bar{\delta}}_{t|\bar{\gamma}} f_{t|\bar{0}\bar{\beta}\bar{\delta}}\end{aligned} \quad (5.47)$$

for appropriate complex functions $C^{\bar{\varepsilon}}_{t|\bar{0}\bar{\beta}\bar{\gamma}}$ and $D^{\bar{\delta}}_{t|\bar{\gamma}}$. Thus, along each integral curve of the $\mathscr{C}^{k-1,\alpha}$ vector field $\frac{\mathrm{d}}{\mathrm{d}t} + X_t$ of $[0,1] \times \widetilde{\widetilde{\mathbb{B}}}^n$, the functions $f_{t|\bar{0}\bar{\beta}\bar{\gamma}}$ satisfy a system of linear ordinary differential equations, which admit a unique solution for each given initial value. Since $f_{t|\bar{0}\bar{\beta}\bar{\gamma}}|_{t=0} = 0$ and the submanifold $\{0\} \times \widetilde{\widetilde{\mathbb{B}}}^n$ is transversal to $\frac{\mathrm{d}}{\mathrm{d}t} + X_t$ at all points, the functions $f_{t|\bar{0}\bar{\beta}\bar{\gamma}}$ must vanish along each of the above integral curves, hence at all points of $[0,1] \times \widetilde{\widetilde{\mathbb{B}}}^n$ as claimed.

5.2.4 Step 3: Existence of a solution to the equation (5.35)

Let us now fix some new notation. Given a one-parameter family of tensors $\phi_t \in \mathscr{H}^{01*} \otimes \mathscr{H}^{10}$ and a system of polar coordinates (ζ, w^α) on $\widetilde{\widetilde{\mathbb{B}}}^n$, we denote by $\phi^\beta_{t|\bar{\alpha}}$ the components of ϕ_t in the associated adapted polar frame field $(Z, e_\alpha, \bar{Z}, e_{\bar{\alpha}})$ and we write $\phi_t = \phi^\beta_{t|\bar{\alpha}} e_\beta \otimes e^{\bar{\alpha}}$. We also set $\mathbb{E}_{t|\bar{\alpha}} := e_{\bar{\alpha}} + \phi_t(e_{\bar{\alpha}})$ and, for simplicity, we use the convenient notation $< \cdot, \cdot >$ for the $(0,2)$-tensor field $g^{(\phi_t)}$ introduced in (5.26). In this notation, (5.35) (i.e., (5.29)) becomes

$$< \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}(e_{\bar{\alpha}} + \phi^\beta_{t|\bar{\alpha}} e_\beta), e_{\bar{\gamma}} + \phi^\delta_{t|\bar{\gamma}} e_\delta > = < \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'_t}(e_{\bar{\alpha}} + \phi^\beta_{t|\bar{\alpha}} e_\beta), \mathbb{E}_{t|\bar{\gamma}} > = 0\,.$$

Recalling that $< e_\beta, \mathbb{E}_{t|\bar{\gamma}} > = g_{\alpha\bar{\gamma}}$ (see (5.28)) this can be also written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + X'_t\right)(\phi^\beta_{t|\bar{\alpha}}) g_{\beta\bar{\gamma}} = - < [X'_t, e_{\bar{\alpha}}], \mathbb{E}_{t|\bar{\gamma}} > - \phi^\beta_{t|\bar{\alpha}} < [X'_t, e_\beta], \mathbb{E}_{t|\bar{\gamma}} > \,. \quad (5.48)$$

Multiplying both sides by the inverse matrix $(g^{\delta\bar{\gamma}})$ of $(g_{\alpha\bar{\beta}})$, we get that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + X'_t\right)\phi^\beta_{t|\bar{\alpha}} = F^\alpha_{\bar{\beta}}\left(t, x, X'_t, \phi^\delta_{t|\bar{\gamma}}, \overline{\phi^\delta_{t|\bar{\gamma}}}\right)\,, \quad (5.49)$$

for some appropriate real analytic functions $F^\alpha_{\bar{\beta}}$ of the points $(t,x) \in [0,1] \times \widetilde{\widetilde{\mathbb{B}}}^n$, of the components of X'_t and of the components of ϕ_t.

So, by the previous sections, our proof is reduced to show the following:

there exists a maximum value $s_o \in (0,1]$ such that for each $s \in (0, s_o)$ there is a one-parameter family ϕ_t of tensor fields in $\mathscr{H}^{01*} \otimes \mathscr{H}^{10}$, which is solution to (5.49) for $t \in [0,s]$ with initial condition $\phi_{t=0} = \phi_J$ and satisfies the nondegeneracy condition (equivalent to (5.20))

$$\det\left(\delta^\alpha_\beta - \phi^\alpha_{t|\bar{\gamma}}\overline{\phi^\gamma_{t|\bar{\beta}}}\right) \neq 0\,; \quad (5.50)$$

the value s_o is less then 1 only in case such solutions are such that $\lim_{t\to s_o} \det\left(\delta^\alpha_\beta - \phi^\alpha_{t|\bar\gamma}\overline{\phi^\gamma_{t|\bar\beta}}\right) = 0$.

In the case of real analytic data, the proof is reached in a direct way. Indeed,

Proposition 5.4 *Assume that J and v_t are real analytic and, consequently, that also ρ and ϕ_J are real analytic. Then there exists a maximum value $0 < s_o \leq 1$ such that the system of partial differential equations (5.49) with initial condition $\phi_{t=0} = \phi_J$ has a unique solution for all $t \in [0, s_o)$. The case $s_o < 1$ occurs only if*

$$\lim_{t\to s_o} \det\left(\delta^\alpha_\beta - \phi^\alpha_{t|\bar\gamma}\overline{\phi^\gamma_{t|\bar\beta}}\right) = 0 \tag{5.51}$$

Proof By Sublemma 5.3, there exists a unique local solution to (5.49) with initial condition $\phi_{t=0} = \phi_J$ on a compact set of the form $[0, \varepsilon] \times \widetilde{\widetilde{\mathbb{B}}}^n$. Since ϕ_J is a deformation tensor of a complex structure, then it satisfies (5.50). Thus, there exists a sufficiently small ε, such that the obtained solution satisfies also (5.50) on the whole $[0, \varepsilon] \times \widetilde{\widetilde{\mathbb{B}}}^n$. If we consider the subset $B \subset (0, 1]$

$$\begin{aligned} B = \big\{\, s \in (0,1] \;:\; &\text{there is a solution on } [0,s]\times\widetilde{\widetilde{\mathbb{B}}}^n \text{ satisfying} \\ &\text{the initial condition } \phi_{t=0} = \phi_J \text{ and (5.50)} \,\big\}, \end{aligned} \tag{5.52}$$

then $s_o = \sup B$ satisfies all requirements. By previous remark, $s_o > 0$. On the other hand, if $s_o < 1$ and the solutions on the intervals $[0, s]$ are such that $\lim_{t\to s_o} \det\left(\delta^\alpha_\beta - \phi^\alpha_{t|\bar\gamma}\overline{\phi^\gamma_{t|\bar\beta}}\right) \neq 0$, the same argument shows that such solution extends to a solution defined on a larger interval $[0, s_o + \varepsilon']$, contradicting the definition of s_o. This implies the last claim of the proposition. □

We now need to prove the existence of the maximal value $0 < s_o \leq 1$ also in case we need to determine $\mathscr{C}^{k-1,\alpha}$ solutions on intervals $[0, s] \subset [0, s_o)$, corresponding to (not real analytic) $\mathscr{C}^k$ initial datum ϕ_J. By considering the set B defined in (5.52), it suffices to show that $B \neq \emptyset$ and then setting $s_o = \sup B$. In other words, we only need to show the existence of solutions with $\mathscr{C}^k$ initial datum ϕ_J and satisfying (5.50) over some compact set of the form $[0, \varepsilon] \times \widetilde{\widetilde{\mathbb{B}}}^n$ for some sufficiently small $\varepsilon > 0$. The method of construction of the required solution will also show that, in case $s_o < 1$, then necessarily $\lim_{t\to s_o} \det\left(\delta^\alpha_\beta - \phi^\alpha_{t|\bar\gamma}\overline{\phi^\gamma_{t|\bar\beta}}\right) = 0$: otherwise, one might determine solutions on some larger interval $[0, s_o + \varepsilon']$, in contrast with the definition of s_o.

The strategy consists in determining the desired solution ϕ_t as limit of real analytic solutions $\phi_{(n)t}$ of (5.49), whose initial values $\psi_{(n)} := \phi_{(n)t=0}$ are real analytic and approximate the initial value ϕ_J. For simplicity, we first work under the additional assumption that ρ and v_t are real analytic. Consider a finite covering of $\widetilde{\widetilde{\mathbb{B}}}^n$ by compact sets K_i, $i = 1, \ldots, N$, with smooth boundaries, on each of which the initial data $\phi^\alpha_{J\bar\beta}|_{K_i}$ are approximated in $\mathscr{C}^k$-norm (thus, also in $\mathscr{C}^{k-1,\alpha}$-norm for each $\alpha \in (0, 1)$) by a sequence $\psi^\alpha_{(n)\bar\beta}$ of real analytic functions. Such approximating sequence $\psi^\alpha_{(n)\bar\beta}$ surely exists and it can be even assumed to be formed by rational polynomials (see, e.g., [8], Thm. 6.10).

Now, given a compact set $K = K_i$ and an integer n, let $\phi^\alpha_{(n)t|\bar\beta}$ be the unique solution to (5.49) and (5.50) with initial value $\phi^\alpha_{(n)t=0|\bar\beta} = \psi^\alpha_{(n)\bar\beta}$ on $\{0\} \times K$, defined on some compact set of the form $[0, t_n] \times K$, with t_n less than the maximal value $\lambda^{(n)}_o$ determined by Proposition

5.4. We may assume that t_n is the infimum of the values t', for which $\sup_{[0,t']\times K} \|\phi^{\alpha}_{(n)t|\bar\beta}\| \leq 1$. By the method of construction of solutions using complex characteristics described in the proof of Sublemma 5.3, such infimum t_n is bounded from below by some $t_o > 0$, independent of n and determined by

- the value $\sup_K \|\psi^{\alpha}_{(n)|\bar\beta}\|$, which we may assume to be less than or equal to $\sup_K \|\psi^{\alpha}_{\bar\beta}\| + \varepsilon_o < 1$ for some $\varepsilon_o > 0$ for all sufficiently large n;
- the sup of the components of the holomorphic vector field $\widehat{X}$, determined through (5.37) by the data in (5.49), taken over the compact set of $\mathbb{C}^{n+1} \times \mathbb{C}^N$, which is the cartesian product $[0, 1] \times K$ and the hypercube in $\mathbb{C}^N$ spanned by all possible values for $\phi^{\alpha}_{\bar\beta}$ with norm bounded above by 2.

We may therefore assume that all solutions $\phi^{\alpha}_{(n)\bar\beta}$ are defined on a compact set $[0, t_o] \times K$, with $t_o > 0$, and that they are equibounded by $\sup_{[0,t_o]\times K} \|\phi^{\alpha}_{(n)t|\bar\beta}\| \leq 1$.

We now want to show that the solutions $\phi^{\alpha}_{(n)t|\bar\beta}$ have also equibounded first derivatives, so that they are equicontinuous on $[0, t_o] \times K$. To check this, for all complex vector field e_α and $e_{\bar\alpha}$ in $\mathscr{H}^{\mathbb{C}}|_K$, consider the unique real analytic extensions $\widehat{e}_{t|\alpha}, \widehat{e}_{t|\bar\alpha}$ in $\mathscr{H}^{\mathbb{C}}|_{[0,1]\times K}$ satisfying the differential problem

$$\mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'}\widehat{e}_{t|\alpha} = \mathscr{L}_{\frac{\mathrm{d}}{\mathrm{d}t}+X'}\widehat{e}_{t|\bar\alpha} = 0\,, \qquad \widehat{e}_{t=0|\alpha} = e_\alpha\,,\ \widehat{e}_{t=0|\bar\alpha} = e_{\bar\alpha}\,. \tag{5.53}$$

Since these differential equations have the form (5.36), with right hand side linear in the unknowns $\widehat{e}_{t|\alpha}$, $\widehat{e}_{t|\bar\alpha}$, such extensions exist on some set of the form $[0, \varepsilon] \times K$, $\varepsilon > 0$ (Sublemma 5.3). They have also $\mathscr{C}^0$-norms that are uniformly bounded from above by the $\mathscr{C}^0$-norm of the initial values multiplied by some fixed constant. This is indeed a consequence of the fact that the differential equations along the complex characteristics are linear. It actually implies that the $\widehat{e}_{t|\alpha}, \widehat{e}_{t|\bar\alpha}$ can be extended over the whole $[0, 1] \times K$.

We may now observe that, differentiating both sides of (5.48) along the vector fields $\widehat{e}_{t|\alpha}$ and $\widehat{e}_{t|\bar\alpha}$ and using the property that they commute with $\frac{\mathrm{d}}{\mathrm{d}t} + X'$ by (5.53), the derivatives $\widehat{e}_{t|\alpha}(\phi^{\gamma}_{(n)t|\bar\beta})$ and $\widehat{e}_{t|\bar\alpha}(\phi^{\gamma}_{(n)t|\bar\beta})$ are solutions of a system of the form (5.36), in which the right hand side is linear in the unknowns. Using once again Sublemma 5.3 and the linearity of the equations along the complex characteristic, we conclude that such derivatives are well defined at all points of $[0, t_o] \times K$, with $\mathscr{C}^0$-norm bounded above by the $\mathscr{C}^0$-norms of their initial values times some fixed constant. Since the initial values $\psi^{\gamma}_{(n)\bar\beta}$ converge in $\mathscr{C}^{k-1,\alpha}$-norm to $\phi^{\gamma}_{J\bar\beta}$, with $k \geq 2$, we conclude that also the family of derivatives $\widehat{e}_{t|\alpha}(\phi^{\gamma}_{(n)t|\bar\beta})$ and $\widehat{e}_{t|\bar\alpha}(\phi^{\gamma}_{(n)t|\bar\beta})$ are equibounded. From this, the equicontinuity of the solutions $\phi^{\gamma}_{(n)t|\bar\beta}$ follows, as claimed.

As direct consequence of Ascoli-Arzelà Theorem, the sequence of solutions $\phi^{\gamma}_{(n)t|\bar\beta}$ converges in $\mathscr{C}^0$-norm to the components of the family of deformation tensors ϕ_t, with initial value $\phi_{t=0} = \phi_J$. An argument that is basically the same as the one used for proving equiboundedness of $\mathscr{C}^1$-norms shows that also all derivatives up to order $k-1$ of the solutions $\phi^{\gamma}_{(n)t|\bar\beta}$ are equibounded, proving in this way that the limit ϕ_t is at least of class $\mathscr{C}^{k-1}$. In particular, it follows that the limit is a solution to the first order differential problem (5.49).

However we claim that also the Hölder ratios of power α of the $(k-1)$-th order derivatives of the solutions $\phi^{\gamma}_{(n)t|\bar\beta}$ are equibounded. This can be checked as follows. First, observe that the Hölder ratios of the $(k-1)$-th derivatives, evaluated for pairs of points that are in the same complex characteristic, are equibounded because the restrictions to the complex

characteristic of such $(k-1)$-th derivatives are solutions of the above described linear system of ordinary differential equations with equibounded coefficients. This means that, in order to check the equiboundedness of the Hölder ratios, we may restrict to considering those evaluated at pair of points $(t, \widetilde{x})$, $(t', \widetilde{y}) \in [0, 1] \times K$, with the same coordinate t. Now, for each given pair of points $x, y \in K$, consider the Hölder ratio of the $(k-1)$-th derivatives, evaluated per each t at the pair of points $(t, x_t, \phi^{\gamma}_{t|\bar{\beta}}(x_t))$, $(t, y_t, \phi^{\gamma}_{t|\bar{\beta}}(y_t))$, belonging to the two complex characteristics originating from $(t = 0, x, \phi^{\gamma}_{0|\bar{\beta}}(x))$ and $(t = 0, y, \phi^{\gamma}_{0|\bar{\beta}}(y))$, respectively. Up to multiplication by a positive constant, such Hölder ratios are equibounded by the Hölder ratios at their uniquely associated pair of points at $t = 0$, from which the complex characteristics originate. Indeed, this is obtained by considering such Hölder ratios as functions of the (complex) variable t of the pair of characteristics. Using the differential equation satisfied by the $(k-1)$-th order derivatives of ϕ_t in order to express the first derivatives in t of the Hölder ratios, one can observe that they satisfy a simple linear system of ordinary differential equations, with coefficients, given by rational functions in which a) the denominators are bounded away from 0 by constants that are independent of ϕ_t, b) the numerators depend on the functions $\phi^{\gamma}_{(n)t|\beta}$ and their $(k-1)$-th derivatives, and are equibounded. This implies the claimed existence of common upper bounds determined by the Hölder ratio of the corresponding pair of points at $t = 0$.

Since the initial values converge to $\phi^{\gamma}_{J\bar{\beta}}$ in $\mathscr{C}^{k-1,\alpha}$-norm, we derive in this way the desired equiboundedness of the $\mathscr{C}^{k-1,\alpha}$-norms of the $\phi^{\gamma}_{(n)t|\bar{\beta}}$ and the property that the limit deformation tensor ϕ_t is actually of class $\mathscr{C}^{k-1,\alpha}$. Note that, by the results of previous section, such ϕ_t necessarily satisfies also the integrability conditions and the condition (B) of the deformation tensors of the L-complex structures.

By uniqueness of the limits of the sequences of solutions $\phi^{\gamma}_{(n)t|\bar{\beta}}$ on the intersections $K_i \cap K_j$ of two compact sets of the considered covering $\{K_i\}$, we also obtain that all solutions constructed in this way on the sets $[0, t_{oi}] \times K_i$ combine together and give a global solution over a compact set of the form $[0, \varepsilon] \times \overline{\overline{\mathbb{B}}}^n$, for some appropriate $\varepsilon > 0$, under the considered regularity assumptions.

It remains to prove the existence of $\mathscr{C}^{k-1,\alpha}$ solutions to (5.49) and (5.50) under the assumption that also the functions v_t and ρ are not real analytic, but just of class $\mathscr{C}^k$. The method is precisely the same as before with the only difference that, now, we have to approximate with real analytic functions on compact sets K_i not only the initial datum $\phi^{\gamma}_{J\bar{\beta}}|_{K_i}$ but also the $\mathscr{C}^k$ functions v_t and ρ. A diagonal argument implies that also in this case the $\mathscr{C}^{k-1,\alpha}$-norms of the solutions $\phi^{\gamma}_{(n)t|\bar{\beta}}$, determined by the initial values $\psi^{\gamma}_{(n)\bar{\beta}}$ and real analytic vector fields $X'_{(n)t}$, determined by the $v^{(n)}_t$ and ρ_n, are equibounded. This implies the needed existence of a $\mathscr{C}^{k-1,\alpha}$-solution also in this case. □

5.2.5 The $\mathscr{C}^{\infty}$ and the $\mathscr{C}^{\omega}$ cases and Theorem 2.9

We end by observing that our arguments show that Theorem 2.7 may be rephrased both in the $\mathscr{C}^{\infty}$ and the $\mathscr{C}^{\omega}$ versions so that also Theorem 2.9 follows. Indeed, as we already pointed out, the determination of the maximal value λ_v in Theorem 2.7 depends on the fulfillment of the conditions (A)–(D) of Sect. 5.2.1, which, as long as the data are at least of class $\mathscr{C}^2$, is independent on the degree of smoothness. Because of the uniqueness of the Green pluripotential with a given pole, the $\mathscr{C}^{\infty}$ version of Theorem 2.7 is obtained applying the statement Theorem 2.7 for all $k \geq 5$. Furthermore the steps of proof of the crucial Lemma

5.2 are provided first in the $\mathscr{C}^\omega$ case (see Sublemma 5.3 and Proposition 5.4) and therefore also the $\mathscr{C}^\omega$ version of Theorem 2.7 holds.

5.2.6 A final remark

As we already mentioned, the consideration of manifolds of circular type was motivated by the relevance of two main classes of examples: the smoothly bounded, strictly convex domains and the smoothly bounded strongly circular domains of $\mathbb{C}^n$. However the manifolds of circular type are described in abstract terms, an aspect that also allows explicit construction procedures. In fact, as we know, each manifold of circular type is biholomorphic to a manifold in normal form. This is nothing but the unit ball $\mathbb{B}^n$ equipped with a non-standard complex structure J, obtained by deforming the standard one along special directions (Definition 2.4). By [3,22,23], all such non-standard complex structures J are uniquely determined by their deformation tensors which, in turn, are parameterized in an explicit (although non-trivial) way by a single freely specifiable complex function h. By choosing appropriately such a function, one can construct manifolds of circular type $(\mathbb{B}^n, J)$ with desired properties. For instance, this idea has been successfully used in [23] to prove the existence of abstract manifolds of circular type with prescribed regularity properties at the center. The same technique can be adopted to generate many domains of circular type $(\overline{\mathbb{B}}^n, J)$ for which there are points that satisfy the (closed) condition (5.51), provided that such condition is translated, as we did in [23], into manageable differential equations for the deformation tensor of the complex structure. By our proof of Theorem 1.1, the existence of such points would imply that the "cloud" of centers $\mathscr{U}^{(\max)}$ is a proper subset of the constructed domain. Further, as in [23], such abstract examples would be embeddable as domains of $\mathbb{C}^n$ as long as their complex structures are suitably small deformations of the standard euclidean complex structure. Conversely, we also expect that a careful study of the mentioned condition (5.51) and of its translation as a condition on the deformation tensor would suggest new biholomorphically invariant sufficient conditions for the regularity propagation to occur globally. We plan to address these lines of research in the future.

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