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A short note on coproducts of Abelian pro-Lie groups

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Abstract
The notion of *conditional coproduct* of a family of abelian pro-Lie groups in the category of abelian pro-Lie groups is introduced. It is shown that the cartesian product of an arbitrary family of abelian pro-Lie groups can be characterized by the universal property of the *conditional coproduct*.

Keywords Pro-Lie groups · Coproducts

Mathematics Subject Classification 22E20 · 22A05

In [3] the second author and Morris have provided criteria when the (co)product of a family of locally compact abelian groups exists. For profinite abelian groups the product of any family of profinite groups exists and agrees with the cartesian product. The latter, namely $\prod_{i\in I} A_i$, for any family $(A_i)_{i\in I}$ of profinite abelian groups, as has shown Neukirch in [5], has the universal property resembling that of a coproduct (direct sum) in the category of (discrete) abelian groups. In the present note we present a version of his result, valid for cartesian products of a family of abelian pro-Lie groups. For formulating our result, we need to adapt the concepts, originally introduced for

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classes of profinite groups by Neukirch in [5] (see also [6, D.3]) to the category of abelian pro-Lie groups. We shall use additive notation, in particular, 0 denotes the identity element. Our notations and the terminology are standard in topological group theory and follow [2, 6].

Definition 1 Let $(A_j)_{\in J}$ be a family of topological groups, H a topological group, and $\mathbb{F}$ a family of continuous homomorphisms $\phi_j\colon A_j \to H$. We say that $\mathbb{F}$ is *convergent*, if for every neighborhood U of the identity of H the set $J_U := \{j \in J : \phi_j(A_j) \not\subseteq U\}$ is finite.

Example 2 For any family $(A_j)_{j\in J}$ of topological groups let $H = \prod_{j\in J} A_j$ be the cartesian product with the product topology. Then the family $\mathbb{F} = (\tau_j)_{j\in J}$ of canonical embeddings $\tau_j\colon A_j \to H$, given by $\tau_j(a) = (a_l)_{l\in J}$ where $a_l = a$ if $l = j$ and else $a_l = 0$, *is convergent*. This is immediate from the fact that given an open subgroup N of P, then by the very definition of the cartesian product, almost all factors $\tau_j(A_j)$ must be subgroups of N.

We define the *conditional product* by means of a universal property, resembling the one of the coproduct (direct sum) of abelian discrete groups:

Definition 3 For a convergent family $\tau_j\colon A_j \to G$, $j \in J$ of morphisms in a category $\mathcal{A}$ of topological groups we call G a *conditional coproduct* of $(A_j)_{j\in J}$ if for every convergent family of morphisms $\psi_j\colon A_j \to H$ in $\mathcal{A}$ there is a unique morphism $\omega\colon G \to H$ such that $\psi_j = \omega \circ \tau_j$ for all $j \in J$.

We have the following Theorem:

Theorem 4 *In the category of abelian pro-Lie groups, the conditional coproduct of a family $(A_j)_{j\in J}$ of abelian pro-Lie groups is the cartesian product $P := \prod_{j\in J} A_j$ together with the canonical embeddings $\tau_j\colon A_j \to P$.*

The next result is well-known, see e.g. [4, 1. Theorem (g), p. 68]:

Lemma 5 *Let $f : X \to Y$ be a continuous map and $A \subseteq X$. Then*

$$\overline{f(\bar{A})} = \overline{f(A)}.$$

Conditional coproducts are unique in the following sense:

Proposition 1 *If G and G' are conditional coproducts of a family $(A_j)_{j\in J}$ of topological groups in a category $\mathcal{A}$ for the convergent families $\tau_j\colon A_j \to G$ and $\tau'_j\colon A_j \to G'$, $j \in J$ of morphisms in $\mathcal{A}$, then there is an isomorphism $\lambda\colon G \to G'$ such that $\tau'_j = \lambda \circ \tau_j$ for all $j \in J$.*

Proof Letting in Definition 3 G' play the role of H and the $\tau'_j : A_j \to G'$ play the role of the ψ_j we may conclude the existence of $\omega : P \to P'$ such that

$$(\forall j \in J)\ \ \tau'_j = \omega \circ \tau_j. \tag{1}$$

Likewise, letting G play the role of H and τ_j the role of ψ_j we obtain

$$(\forall j \in J)\ \ \tau_j = \omega' \circ \tau'_j. \tag{2}$$

Equations (1) and (2) together imply that

$$(\forall j \in J)\omega \circ \omega' \restriction_{\tau_j(A_j)} = \mathrm{id}_{\tau_j(A_j)} \text{ and } \omega' \circ \omega \restriction_{\tau'_j(A_j)} = \mathrm{id}_{\tau'_j(A_j)}. \tag{3}$$

Set

$$Q := \langle \tau(A_j) : j \in J \rangle \ \text{ and } \ Q' := \langle \tau'(A_j) : j \in J \rangle. \tag{4}$$

It is an immediate consequence of Eq. (3) that

$$\omega \circ \omega' \restriction_{Q'} = \mathrm{id}_{Q'} \ \text{ and } \ \omega' \circ \omega \restriction_{Q} = \mathrm{id}_{Q}. \tag{5}$$

We apply Lemma 5 to $\omega : G \to G'$ and $A = Q$ and, noting that $\omega(Q) = Q'$, obtain the equality

$$\overline{Q'} = \overline{\omega(Q)} = \overline{\omega(G)}. \tag{6}$$

Next consider the family of morphisms $\chi_j : A_j \to P'/\overline{Q'}$, for $j \in J$, where χ_j sends A_j to $\{0\}$. Certainly the family $(\chi_j)_{j \in J}$ is convergent. Hence the universal property of P' renders a unique morphism $\omega'' : G' \to G'/\overline{Q'}$ such that

$$(\forall j \in J)\ \ \chi_j = \tau'_j \circ \omega''.$$

The unique candidate for ω'' must therefore map all of P to $\{0\}$. Taking for ω'' the quotient map from P' onto $G'/\overline{Q'}$ also satifies the commutativity relations and hence as claimed, $\omega : G \to G'$ is surjective. Therefore we have from the definitions of Q and Q' in Eqs. (4) and (6) that

$$G = \overline{Q} \ \text{ and } \ G' = \overline{Q'}.$$

Now Eq. (5) shows that

$$\omega \circ \omega' = \mathrm{id}_{G'} \ \text{ and } \ \omega' \circ \omega = \mathrm{id}_G$$

so that indeed $\omega' : G' \to G$ is an isomorphism.

We conclude the proof by remarking that λ can be taken to be ω. □

We note that for profinite groups the *conditional coproduct* agrees with the *free pro-$\mathcal{C}$ product* for $\mathcal{C}$ the variety of abelian profinite groups, see [5, 6].

We recall that for the cartesian product $P = \prod_{j \in J} A_j$ of a family of topological groups A_j there is a unique family $\tau_j : A_j \to P$ of continuous homomorphisms defined as $\tau_j(a_j) = (x_i)_{i\in J}$ where $x_i = 1$ for $i \neq j$ and $x_j = a_j$. These homomorphisms are sometimes called *canonical embeddings*.

Let $\mathcal{A}$ be the category of topological abelian pro-Lie groups (i.e. groups which are projective limits of Lie groups: see [1, pp. 160ff. and Chapter 5]). Each pro-Lie group G has a filter basis $\mathcal{N}(G)$ of closed normal subgroups such that G/N is a Lie group, and $G \cong \lim_{N\in\mathcal{N}(G)} G/N$. (See e.g. [1, p.160, Definition A.])

Recall that every locally compact abelian group is a pro-Lie group, every almost connected locally compact group is a pro-Lie group. Trivially, then, every profinite group is a pro-Lie group. Every cartesian product $P = \prod_{j\in J} A_j$ of pro-Lie groups A_j is itself a pro-Lie group.

Lemma 6 *Let H be a pro-Lie group and $\mathbb{F}$ be a convergent family of morphisms $\psi_j : A_j \to H$. Then, for each $N \in \mathcal{N}(H)$, the set $\{j \in J : \psi_j(A_j) \nsubseteq N\}$ is finite.*

Proof Let $N \in \mathcal{N}(H)$. The Lie group H/N has an identity neighborhood V in which $\{0\} = N/N$ is the only subgroup of H/N. Now let $p\colon H \to H/N$ be the quotient morphism and set $U = p^{-1}(V)$.

Therefore $\psi_j(A_j) \nsubseteq N$ implies $\psi_j(A_j) \nsubseteq U$. However the set of j satisfying this condition is finite by Definition 1 applied to the conditional coproduct of the family $(A_j)_{j\in J}$. This completes the proof of the Lemma. □

Proof of Theorem 4 The uniqueness, up to isomorphism, of the conditional coproduct follows from Proposition 1.

Thus, according to Definition 3, we need to show that given $H \in \mathcal{A}$ and a convergent family of morphisms $\psi_j : A_j \to H$ then there exists a unique morphism $\omega : P \to H$ with $\psi_j = \omega \circ \tau_j$ for all $j \in J$.

We note first that every $x \in P$ has a presentation

$$x = (\tau_j(a_j))_{j\in J} \tag{7}$$

for unique elements $a_j \in A_j$. Denote by $\mathcal{N}(H)$ the set of all closed subgroups of H such that H/N is a Lie group. It is a consequence of [1, Theorem 3.27] that $\mathcal{N}(H)$ is a filter basis of closed subgroups of H and that

$$H \cong \varprojlim_{N\in\mathcal{N}} H/N \tag{8}$$

algebraically and topologically.

Fix $N \in \mathcal{N}(H)$ and let let $J_N := \{j \in J : \psi_j(A_j) \not\leq N\}$. Then, by Lemma 6, the set J_N is finite and, taking the presentation Eq. (7) for $x \in P$ and $\tau_j(A_j) \leq N$ for all $j \notin J_N$ into account, we obtain a well-defined morphism $\omega_N : P \to H/N$ by letting

$$\omega_N(x) := \sum_{j\in J_N} \psi_j(a_j) + N. \tag{9}$$

For subgroups $M \leq N$ of H, both in $\mathcal{N}(H)$, let $\pi_{MN} : H/M \to H/N$ denote the canonical epimorphism.

For $M \le N$ one obtains from Eq. (9) the compatibility relation

$$\omega_N = \pi_{MN} \circ \omega_M, \tag{10}$$

as depicted in the following diagram:

$$\begin{tikzcd}
 & & & & P \arrow[llll, "\omega_N"'] \arrow[lll, "\omega_M"] \arrow[d, "\exists!\,\omega"] \\
\cdots & H/N & H/M \ldots \arrow[l, "\pi_{NM}"] & & H = \varprojlim_{N\in\mathcal{U}} H/N \arrow[ll]
\end{tikzcd}$$

Taking the relations in Eq. (10) into account we see that the universal property of the inverse limit $H = \varprojlim_{N\in\mathcal{U}} H/N$ implies the existence of a unique continuous homomorphism $\omega : P \to H$, which satisfies the desired relations

$$(\forall j \in J) \quad \psi_j = \omega \circ \tau_j. \tag{11}$$

□

Final comments

A *coproduct* of a family of objects in a category $\mathcal{A}$ is a *product* in the category obtained by reversing all arrows. Curiously, while products are usually considered simple concepts, coproducts are often tricky in many categories $\mathcal{A}$ other than the category of abelian groups. Therefore, in conclusion of this note, a few general comments may be in order.

One of the early surprises is that in the familiar *category of groups*, the coproduct of $\mathbb{Z}(2)$ and $\mathbb{Z}(3)$ is $\mathrm{PSL}(2, \mathbb{Z})$.

In any category $\mathcal{A}$ with a well-introduced dual category, such as *the category of locally compact abelian groups*, the coproduct $\coprod_j A_j$ of a family A_j, $j \in J$, is naturally isomorphic to the dual $\widehat{P}$ of $P := \prod_j \widehat{A_j}$, the product of its duals.

Even in special cases, such as the case of *compact* abelian groups A_j, the result is a complicated coproduct, since the character group of an infinite product of discrete abelian groups may be hard to deal with.

If $\mathcal{A}$ is *the category of profinite abelian groups*, then its dual is the category $\mathcal{T}$ of abelian torsion groups. The product in $\mathcal{T}$ of a family of torsion groups T_j is the torsion group $\mathrm{tor}(\prod_j T_j)$ of the cartesian product. So by the time we arrive at the coproduct of, say, an unbounded family of cyclic groups A_j in $\mathcal{A}$, we may have a complicated object $\coprod_{j\in J} A_j$ in our hands.

Therefore, any special situation may be welcome, where a coproduct is lucid–even when its scope of application may be restricted. An example of such a situation is our present *conditional coproduct* in the rather large yet reasonably well-understood category of abelian pro-Lie groups (see Chapter 5 of [1]). The authors encountered such a coproduct in a study of certain locally compact abelian p-groups. Our conditional coproduct covers a somewhat restricted supply of families of morphisms which we call "convergent". Here we encounter the rather extraordinary event that for each of such families *their conditional coproduct agrees with their cartesian product*. Classically, one is familiar with a situation of coproducts agreeing with products in the category of finite abelian groups which, after all, is rather special.

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