THE TORSIONFREE PART OF THE ZIEGLER SPECTRUM OF ORDERS OVER DEDEKIND DOMAINS

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ABSTRACT. We study the R-torsionfree part of the Ziegler spectrum of an order Λ over a Dedekind domain R. We underline and comment on the role of lattices over Λ. We describe the torsionfree part of the spectrum when Λ is of finite lattice representation type.

1. INTRODUCTION

In his posthumous paper [13] Gena Puninski made substantial progress in the description of the Cohen-Macaulay part of the Ziegler spectrum over Cohen-Macaulay rings.

Gena also raised a similar question for torsionfree modules over orders. In fact this topic has been investigated in just few cases. The most advanced (see [14]) deals with the integral group ring $\mathbb{Z}G$, where G is Klein's four group; note that $\mathbb{Z}G$ is of infinite lattice representation type. On the other hand a general theoretical analysis over group rings RG, where R is a Dedekind domain of characteristic 0 and G is a finite group, had been previously developed in [10]. Motivations came from the interest in the model theory of abelian-by-finite groups. Recall that a group H is said to be abelian-by-finite if and only if H admits an abelian normal subgroup N of finite index. Let G denote the quotient group H/N. So N inherits a natural structure of module over the group ring $\mathbb{Z}G$ and most model theory of H as a group is given by that of N as a module (see [9]).

In this note we plan to extend the approach of [10] to orders over Dedekind domains. Thus let us first introduce this setting. We start with a Dedekind domain R that is not a field. Let Q be the field of fractions of R. Now let A be a finite dimensional Q-algebra. We will sometimes assume A *semisimple*, and even *separable* (i.e. A remains semisimple when extending scalars). An R-*order* in A is a subring Λ of A such that the centre of Λ contains R, Λ is finitely generated as an R-module and $Q \cdot \Lambda = A$. For instance the group ring RG, with G a finite group, is an R-order in the group algebra $A = QG$.

We are interested in (say right) modules over such an order Λ, in particular in R-torsionfree Λ-modules. Recall that a Λ-module M is R-*torsionfree* if for all $0 \neq m \in M$ and $0 \neq r \in R$ we have $mr \neq 0$. Finitely generated R-torsionfree Λ-modules are known as Λ-*lattices*. Over a Dedekind domain R, Λ-lattices can be equivalently introduced as Λ-modules finitely generated

and projective over R, or also as direct summands (still over R) of R^n for some positive integer n.

Observe that, if M is any Λ-module, then $\text{Tor } M := \{m \in M \mid mr = 0 \text{ for some } r \in R \setminus \{0\}\}$ is a Λ-submodule of M and the quotient module $M/\text{Tor } M$ is R-torsionfree.

Let us fix some further notation. For Λ an order over a Dedekind domain R,

- Tf_Λ is the category of all R-torsionfree (right) Λ-modules,
- Latt_Λ is the category of (right) Λ-lattices.

Moreover L_Λ is the first order language of Λ-modules, and in this language T_Λ^{tf} is the first order theory of R-torsionfree Λ-modules, so of Tf_Λ.

Note that Tf_Λ is the smallest *definable* subcategory of the category of all Λ-modules containing Latt_Λ.

For every positive integer n, pp_Λ^n denotes the lattice of pp-formulas with n free variables of L_Λ (warning: here the word *lattice* has a different meaning, that of an ordered structure, see below). In detail, pp_Λ^n is the quotient set of these pp-formulas with respect to the logical equivalence relation (in the theory of Λ-modules). The lattice structure is given by the partial order relation determined by logical implication (modulo the same theory). Then meet corresponds to the conjunction of pp-formulas, and join to their sum $+$. If one identifies pp-formulas in n variables equivalent in the first order theory of some given Λ-module M, then one forms another lattice $\text{pp}_\Lambda^n(M)$ – a quotient lattice of pp_Λ^n. The same can be done starting from a class of Λ-modules instead of a single M. For instance, one builds in this way $\text{pp}_\Lambda^n(\text{Tf}_\Lambda)$.

We will denote the binary relation in these lattices by $\leq$ (with the usual meaning for $<$). When necessary, a subscript will specify to which lattice we refer. For instance, we write $\leq_{\text{pp}_\Lambda^1(M)}$ when dealing with pp-formulas in 1 free variable with respect to the first order theory of a module M. Likewise $[\,,\,]$ will denote a closed interval in a lattice, with possible use of subscripts to say which lattice we deal with, as before. Similar conventions will regard open or half closed intervals.

The m-dimension of these lattices $L = \text{pp}_\Lambda^n, \text{pp}_\Lambda^n(\text{Tf}_\Lambda), \ldots$ is defined as follows, see [11] and [12] for details. Construct a sequence of lattices L^α (with α an ordinal) collapsing at each successor step intervals of finite length. For instance, in the basic step two pp-formulas φ and ψ are identified if and only if the closed interval $[\varphi \wedge \psi, \varphi + \psi]$ is of finite length. Then the m-dimension of L, m-dim L, is defined as the smallest ordinal α such that L^α is the one-point lattice, if such an ordinal exists, and ∞ (or undefined) otherwise.

Let us come back to illustrate the aim of the paper. As said, we consider R-torsionfree modules over an order Λ over a Dedekind domain R. Let Zg_Λ denote the whole (right) Ziegler spectrum of Λ, that is, the topological space of (isomorphism classes of) indecomposable pure injective Λ-modules. A basis of open sets of the topology is given by $(\varphi/\psi) = \{N \in \text{Zg}_\Lambda : \varphi \wedge \psi <_{\text{pp}_\Lambda^1(N)} \varphi\}$ where φ and ψ range over pp_Λ^1. We are interested in the subset of Zg_Λ formed by R-torsionfree indecomposable pure injective Λ-modules. Notice that this is a closed set, as the complement of

the union of $(xr = 0/x = 0)$ where r ranges over the non zero elements of R. Let Zg_Λ^{tf} denote it. Observe that since Zg_Λ is compact and Zg_Λ^{tf} is a closed subset of Zg_Λ, Zg_Λ^{tf} is also compact.

One may wonder what is the role of Λ-lattices in this framework, for instance if indecomposable Λ-lattices are pure injective, so points of Zg_Λ^{tf}. We cannot expect that in general, but we will see in the next section that the answer is positive at least over complete discrete valuation domains. Apart from this, we will also discuss the relevance of lattices in the R-torsionfree part of the spectrum, just as [10] did over group rings.

Here is a more detailed plan of this paper. In § 2 we prove some first results on lattices, and above all that, when R is a complete discrete valuation domain and A is separable, they are isolated points of Zg_Λ^{tf}, dense in the whole space Zg_Λ^{tf}. In § 3 we provide a description of the torsionfree part of the Ziegler spectrum of an order Λ over a Dedekind domain R in a semisimple Q-algebra, extending that over group rings in [10]. We also investigate the m-dimension of $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ in that section. Applications of the (classical) Maranda theorem to our setting will be treated in § 4. The final section 5 considers orders of finite lattice representation type and provides a complete description of their Ziegler spectrum, confirming a conjecture of Gena Puninski. As an application, it is shown that the theory of integral group rings $\mathbb{Z}G$ torsionfree over $\mathbb{Z}$ (with G a cyclic group of order p or p^2 for some prime p) is decidable, which positively answers questions in [9].

We assume some familiarity with model theory of modules, as treated in [11], [12] and [19].

Finally let us call again the reader's attention to the fact that, as this introduction already witnesses, the word *lattice* denotes in this paper two different concepts: lattice as a module, and as a partially ordered set. Indeed the same is true of *order*, that can be meant in the usual sense but also as a ring. We hope this coincidence will not cause any misunderstanding and the meaning of any occurrence of *lattice* or *order* will always be clear.

2. The role of lattices

We mainly devote this section to some first results on lattices. We keep R, Λ, and so on, in agreement with the introduction.

Since R is hereditary and noetherian, Λ-lattices are closed under submodules. On the contrary, quotients of lattices need not be lattices, but the category of Λ-lattices does have pseudo-kernels.

Fact 2.1. *Every lattice M over an R-order Λ decomposes as a finite direct sum of indecomposable lattices.*

Proof. Let $M = L \oplus N$. As tensor products preserve direct sums, QM decomposes as $QL \oplus QN$. Thus if both L and N are non zero then $\dim QL, \dim QN < \dim QM$. $\qquad\square$

This decomposition may not be unique. In other words the category of lattices over an order may not be Krull-Schmidt (see [4, p. 768]. But this is true over complete discrete valuation domains (see [4, (30.6), p. 620]).

Proposition 2.2. *Let R be a complete discrete valuation domain and Λ be an R-order. Then every Λ-lattice L is pure injective and the set of indecomposable Λ-lattices is dense in Zg_Λ^{tf}.*

Proof. Since R is a discrete valuation domain and L is finitely generated and torsionfree over R, as an R-module, L is isomorphic to R^n. Since R is complete, it is linearly compact as a module over itself, see [12, Sect. 4.2.2.] for the definition of linear compactness. Since the class of linearly compact modules is closed under extensions [12, 4.2.10], R^n is linearly compact. Since all pp-definable subgroups of L as a Λ-module are R-submodules, L is algebraically compact over Λ.

If M is an R-torsionfree Λ-module, then it is a direct limit of its finitely generated submodules, which are lattices. Then M is in the closure of these lattices. $\qquad\square$

When $Q\Lambda$ is separable and R is complete, the category of lattices has almost split sequences (see [16], [17], [1]). We will use this to show that every indecomposable Λ-lattice is isolated.

The following result may have its own interest and indeed will be used also later.

Let $\varphi \in \mathrm{pp}_\Lambda^n$ and $(M, \mathbf{m})$, $\mathbf{m} \in M^n$, a free realisation of φ [12, 1.2.2 p. 23]. Look at the pp-type of $\mathbf{m} + \mathrm{Tor}\, M$ in $M/\mathrm{Tor}\, M$ and take a pp-formula $\overline\varphi \in \mathrm{pp}_\Lambda^n$ generating this pp-type.

Lemma 2.3. *The map $\varphi \mapsto \overline\varphi$ defines a $+$-semi-lattice homomorphism from pp_Λ^n to pp_Λ^n such that $\overline\varphi \le \varphi$ for every φ.*

Moreover, if N is an R-torsionfree Λ-module then $\varphi(N) = \overline\varphi(N)$.

The partially ordered set $\{\overline\varphi \mid \varphi \in \mathrm{pp}_\Lambda^n\}$ is isomorphic to $\mathrm{pp}_\Lambda^n(\mathrm{Tf}_\Lambda)$ and hence is a lattice.

Proof. Let $(M, \mathbf{m})$ be a free realisation of $\varphi \in \mathrm{pp}_\Lambda^n$. Since there is a homomorphism from M to $M/\mathrm{Tor}\, M$ sending $\mathbf{m}$ to $\mathbf{m} + \mathrm{Tor}\, M$, $\overline\varphi \le \varphi$.

Suppose that $\psi \le \varphi$ in pp_Λ^n. Let $(N, \mathbf{n})$ be a free realisation of ψ. Since $\psi \le \varphi$ there is a homomorphism $f : M \to N$ with $f(\mathbf{m}) = \mathbf{n}$. Let $\overline f : M/\mathrm{Tor}\, M \to N/\mathrm{Tor}\, N$ be the homomorphism induced by f. Then $\overline f(\mathbf{m} + \mathrm{Tor}\, M) = \mathbf{n} + \mathrm{Tor}\, N$. Thus $\overline\psi \le \overline\varphi$. This also shows that the map sending φ to $\overline\varphi$ is well-defined.

We now just have to observe that for all $\varphi, \psi \in \mathrm{pp}_\Lambda^n$, $\overline{\varphi + \psi} = \overline\varphi + \overline\psi$. This is true because if $(M, \mathbf{m})$, $(N, \mathbf{n})$ are free realisations of φ, ψ respectively, then, see [12, 1.2.27], $(M \oplus N, \mathbf{m} + \mathbf{n})$ is a free realisation of $\varphi + \psi$ and $\mathrm{Tor}(M \oplus N) = \mathrm{Tor}\, M \oplus \mathrm{Tor}\, N$.

Now suppose that $\varphi \in \mathrm{pp}_\Lambda^n$ is freely realised by $(M, \mathbf{m})$ and N is an R-torsionfree Λ-module. Suppose $\mathbf{n} \in \varphi(N)$. There exists $f : M \to N$ such that $f(\mathbf{m}) = \mathbf{n}$. Since N is R-torsionfree, $\mathrm{Tor}\, M \subseteq \ker f$. Thus the homomorphism $\overline f : M/\mathrm{Tor}\, M \to N$ induced by f satisfies $\overline f(\mathbf{m} + \mathrm{Tor}\, M) = \mathbf{n}$. Hence $\mathbf{n} \in \overline\varphi(N)$. $\qquad\square$

We now provide a detailed proof that the indecomposable Λ-lattices are isolated in Zg_Λ^{tf}, when R is complete and $Q\Lambda$ is separable, following that of the analogous result for Artin algebras.

Lemma 2.4. *Let R be a complete discrete valuation domain with field of fractions Q, Λ be an R-order in a finite dimensional separable Q-algebra A. Then all indecomposable Λ-lattices are isolated in Zg_Λ^{tf}.*

Proof. As said, the category of Λ-lattices has left almost split morphisms (see [1, 2.1], for instance). That is, for all indecomposable lattices N there exists a homomorphism of lattices $f : N \to E$ such that f is a non split monomorphism and for any Λ-homomorphism of lattices $h : N \to X$ which is a non split monomorphism, there exists $\lambda \in \mathrm{Hom}(E, X)$ such that $f\lambda = h$.

Pick $\mathbf{n} \in N$ a generating tuple for N. Let φ generate the pp-type of $\mathbf{n}$ and ψ generate the pp-type of $f(\mathbf{n})$. We first show that $N \in (\varphi/\psi)$. By definition $\mathbf{n} \in \varphi(N)$. Suppose, for a contradiction, that $\mathbf{n} \in \psi(N)$. Then there exists $g : E \to N$ sending $f(\mathbf{n})$ to $\mathbf{n}$. Since $\mathbf{n}$ is a generating tuple for N, $gf = \mathrm{Id}_N$. But this contradicts our assumption that f is not split. Thus $N \in (\varphi/\psi)$.

Now we take any $\sigma \in \mathrm{pp}_\Lambda^n(\mathrm{Tf}_\Lambda)$ and we claim that, if $\sigma < \varphi$ then $\sigma \le \psi$. Since we are working modulo the theory of R-torsionfree Λ-modules, we may replace σ by $\overline{\sigma}$ (see Lemma 2.3). Let $M \in \mathrm{Latt}_\Lambda$ and $\mathbf{m} \in M$ be such that $(M, \mathbf{m})$ is a free realisation of $\overline{\sigma}$. Thus there is a homomorphism $h : N \to M$ such that $h(\mathbf{n}) = \mathbf{m}$. So either h is a split monomorphism or there exists $\lambda \in \mathrm{Hom}(E, X)$ such that $f\lambda = h$. If h is a split monomorphism then the pp-type of $\mathbf{n}$ is equal to the pp-type of $f(\mathbf{n}) = \mathbf{m}$, so $\sigma = \varphi$. In the second case, $\sigma \le \psi$. Thus $\sigma \le_{\mathrm{Tf}_\Lambda} \psi$.

Therefore φ/ψ is a Tf_Λ-minimal pair. Hence (φ/ψ) isolates N in Zg_Λ^{tf}. $\square$

Next let us deal with the closed points of Zg_Λ^{tf}. We start by proving an auxiliary result.

Lemma 2.5. *For all $N \in \mathrm{Tf}_\Lambda$ and $\varphi \in \mathrm{pp}_\Lambda^1$, $Q\varphi(N) = \varphi(QN)$.*

Proof. Let $\varphi(x) \doteq \exists\mathbf{y}\ (x, \mathbf{y})T_\varphi = 0$ where T_φ is a matrix of a suitable size with entries in Λ. Since N is a submodule of QN, $\varphi(N) \subseteq \varphi(QN)$. All pp-definable subsets of QN are Q-vector subspaces, so $Q\varphi(N) \subseteq \varphi(QN)$.

Now suppose that $m \in \varphi(QN)$. There exists $\mathbf{l} = (l_1, \ldots, l_k)$ in QN such that $(m, \mathbf{l})T_\varphi = 0$. Let $c \in R$ be such that $mc, l_1c, \ldots l_kc \in N$. Then $(mc, \mathbf{l}c)T_\varphi = 0$. So $mc \in \varphi(N)$. Thus $m \in Q\varphi(N)$. Therefore $Q\varphi(N) = \varphi(QN)$. $\square$

When R is a complete noetherian valuation domain, we are now able to describe the closure of a Λ-lattice.

Proposition 2.6. *Let R be a complete discrete valuation domain and Λ an order over R. Let π denote a generator of the maximal ideal of R. If N is an indecomposable Λ-lattice and M is in the (Ziegler) closure of N but is not equal to N, then M is a direct summand of QN. In particular M is a closed point and $\mathrm{pp}_\Lambda^1(M)$ is of finite length.*

Proof. Let M be in the closure of N but not equal to N. Suppose that $M \in (\varphi/\psi)$. We aim to show that $\varphi(QN) \supsetneq \psi(QN)$.

Since $M \in (\varphi/\psi)$ and M is in the closure of N, $N \in (\varphi/\psi)$. If $\varphi(N)/\psi(N)$ were finite length as an R-module then the interval $[\psi, \varphi]_N \subseteq \mathrm{pp}^1_\Lambda(N)$ would be finite length. Let $\varphi =: \varphi_0 \geq \varphi_1 \geq \ldots \geq \varphi_{n+1} := \psi$ be such that $[\varphi_{i+1}, \varphi_i]_N$ is simple. Then $(\varphi/\psi) = \bigcup_{i=0}^n (\varphi_i/\varphi_{i+1})$. Since $[\varphi_{i+1}, \varphi_i]_N$ is simple, by [19, 7.10], $(\varphi_i/\varphi_{i+1})$ isolates N in its closure and hence (φ/ψ) isolates N in its closure. Therefore $\varphi(N)/\psi(N)$ is infinite length as an R-module.

Since R is noetherian and N is finitely generated as an R-module, $\varphi(N)/\psi(N)$ is finitely generated as an R-module. Thus $\varphi(N)/\psi(N)$ is isomorphic to $R^n \oplus T$ where T is a finitely generated torsion R-module. Since $\varphi(N)/\psi(N)$ is infinite length as an R-module, $n \geq 1$. Thus there exists $m \in \varphi(N)$ such that $m\pi^l \notin \psi(N)$ for all $l \in \mathbb{N}$. By Lemma 2.5, $m \in \varphi(QN)$ and $m \notin \psi(QN)$.

Since N is a lattice, QN is finite dimensional. Let $L_1, \ldots, L_m$ be the indecomposable summands of QN. If $\varphi(QN) \supsetneq \psi(QN)$ then $L_i \in (\varphi/\psi)$ for some $1 \leq i \leq m$. Thus if $M \in (\varphi/\psi)$ then $L_i \in (\varphi/\psi)$ for some $1 \leq i \leq m$. So M is in the closure of $\{L_1, \ldots, L_m\}$. Since each L_i is a closed point, M is in the closure of $\{L_1, \ldots, L_m\}$ if and only if $M = L_i$ for some $1 \leq i \leq m$. $\qquad\square$

Recall that the *support* of a Λ-module M, $\mathrm{Supp}(M)$, is the set of indecomposable pure injective Λ-modules N such that for all pp-pairs φ/ψ, $\varphi(M) = \psi(M)$ implies $\varphi(N) = \psi(N)$. In particular, if $M \in \mathrm{Zg}_\Lambda$ then $\mathrm{Supp}(M)$ is the closure of M in Zg_Λ.

Lemma 2.7. *If $N \in \mathrm{Zg}_\Lambda^{tf}$ and $S \in \mathrm{Supp}(QN)$ then S is in the closure of N. In particular, if $N \in \mathrm{Zg}_\Lambda^{tf}$ and $S \in \mathrm{Zg}_A$ is a direct summand of QN, then S is in the closure of N.*

Proof. Suppose that $\varphi \geq \psi \in \mathrm{pp}^1_\Lambda$. By 2.5, if $\varphi(N) = \psi(N)$ then $\varphi(QN) = Q\varphi(N) = Q\psi(N) = \psi(QN)$. By definition, if $S \in \mathrm{Supp}(QN)$ then $\varphi(QN) = \psi(QN)$ implies $\varphi(S) = \psi(S)$. Thus S is in the closure of N. $\qquad\square$

Corollary 2.8. *If $N \in \mathrm{Zg}_\Lambda^{tf}$ is a closed point then $N \in \mathrm{Zg}_A$ and N is a closed point in Zg_A.*

We include in this section some further useful remarks. For every R-module M, let $\mathrm{Sub}_R(M)$ be the lattice of R-submodules of M.

Lemma 2.9. *Let R be a Dedekind domain and M a torsionfree finitely generated module over R. Then $\mathrm{Sub}_R(M)$ has m-dimension 1.*

Proof. Since M is torsionfree, R is a submodule of M and hence $\mathrm{Sub}_R(M)$ is not of finite length. Thus m-dim $\mathrm{Sub}_R(M) \neq 0$.

Since M is finitely generated and torsionfree, M is a direct summand of R^n for some positive integer n. It follows that, if m-dim $\mathrm{Sub}_R(R^n) \leq 1$ for all $n \in \mathbb{N}$, then m-dim $\mathrm{Sub}_R(M) \leq 1$ and hence m-dim $\mathrm{Sub}_R M = 1$.

On the other hand, since R^n can be filtered as a finite chain of submodules with quotients isomorphic to R, m-dim $\mathrm{Sub}_R(R^n) = $ m-dim $\mathrm{Sub}_R(R)$.

Let J be a non zero ideal of R. Then R/J is of finite length. Thus the interval $[J, R]$ in $\mathrm{Sub}_R(R)$ is of finite length. So m-dim $\mathrm{Sub}_R(R) = 1$. $\qquad\square$

Corollary 2.10. *Let R be a Dedekind domain, Λ an order over R and M a Λ-lattice. Then $\mathrm{pp}^1_\Lambda(M)$ has m-dimension 1.*

Proof. Since all pp-definable subgroups of M are R-submodules, $\mathrm{pp}^1_\Lambda(M)$ is a sublattice of $\mathrm{Sub}_R(M)$. Thus m-dim $\mathrm{pp}^1_\Lambda(M) \leq$ m-dim$\mathrm{Sub}_R(M) = 1$. Since M is R-torsionfree and not R-divisible, there exist $c \in R \backslash \{0\}$ and $m \in M$ such that $m \notin Mc$. Since M is R-torsionfree, $mc^n \notin Mc^{n+1}$. Thus $Mc^{n+1} \subsetneq Mc^n$ for all $n \in \mathbb{N}$. Thus $pp^1_\Lambda(M)$ is not of finite length, whence its m-dimension cannot be 0. $\qquad\square$

3. The torsionfree part of the Ziegler spectrum

In this section we extend the main results of [10], about the torsionfree part of the Ziegler spectrum of a group ring RG, with R a Dedekind domain of characteristic 0 and G a finite group, to arbitrary orders Λ over a Dedekind domain R in a *semisimple Q-algebra* $A = Q\Lambda$. We also investigate the m-dimension of $\mathrm{pp}^1_\Lambda(\mathrm{Tf}_\Lambda)$ and the Cantor-Bendixson rank of Zg^{tf}_Λ in both this framework and the more general setting where A is not assumed to be semisimple.

If P is a prime ideal of R then we write Λ_P for the central localisation of Λ at P and $\widehat{\Lambda_P}$ for its completion at P. Note that Λ_P is an R_P-order in A and $\widehat{\Lambda_P}$ is an $\widehat{R_P}$-order.

We start in the general setting. The endomorphism ring of every indecomposable pure injective module N, $\mathrm{End}(N)$, is local. Let $P(N)$ denote its maximal ideal. When N is R-torsionfree, R embeds in a natural way into $\mathrm{End}(N)$. Moreover $P(N) \cap R$ is a prime ideal of R. Thus, every indecomposable pure injective Λ-module is a module over Λ_P for some prime and even maximal ideal P of R. The homomorphism $\Lambda \to \Lambda_P$ is an epimorphism and hence restriction of scalars induces an embedding of Zg_{Λ_P} into Zg_Λ whose image is a closed subset. This embedding restricts to an embedding of $\mathrm{Zg}^{tf}_{\Lambda_P}$ into Zg^{tf}_Λ and again the image is a closed subset. Identifying $\mathrm{Zg}^{tf}_{\Lambda_P}$ with the set of $N \in \mathrm{Zg}^{tf}_\Lambda$ such that $P(N) \cap R \subseteq P$, we may write

$$\mathrm{Zg}^{tf}_\Lambda = \bigcup_P \mathrm{Zg}^{tf}_{\Lambda_P}$$

where P ranges over maximal ideals of R. Since R has Krull dimension 1, if P, P' are distinct maximal ideals of R then $\mathrm{Zg}^{tf}_{\Lambda_P} \cap \mathrm{Zg}^{tf}_{\Lambda_{P'}} = \mathrm{Zg}_A$.

This description of the space is not particularly useful for computing the Cantor-Bendixson rank of Zg^{tf}_Λ because if T is a topological space, X is a closed subset of T and $p \in X$ then the Cantor-Bendixson rank of p as a point in X may strictly less than the Cantor-Bendixson rank of p as a point in T. Thus we now work to give a more useful description.

Since R is noetherian, every (maximal) ideal P of R is finitely generated, whence there is a pp-formula of L_R defining in any R-module M just MP: if $\mathbf{r} = (r_1, \ldots, r_l)$ is a generating tuple of P, it suffices to take $\exists y_1 \ldots \exists y_l \, (x = y_1 r_1 + \ldots + y_l r_l)$. Let $P \mid x$ denote this formula. For instance, when R is a discrete valuation domain and π is a generator of its maximal ideal P, then the formula is $\pi \mid x$, that is, $\exists y(x = y\pi)$.

If $N \in \mathrm{Zg}_{\Lambda_{P'}}^{tf}$ and $P \neq P'$ then $NP = N$ since some element of P is not in P', that is, some element of P acts invertibly on N.

Now suppose that $N \in \mathrm{Zg}_{\Lambda_P}^{tf}$ and $N \notin (x = x/P|x)$. Let $(r_1, \ldots, r_l)$ still denote a tuple of generators of P. Since R_P is a valuation domain, there exists $1 \leq j \leq l$ such that $r_i \in r_j R_P$ for all $1 \leq i \leq l$. Put $r = r_j$ and for all i write $r_i = r\, c_i/a_i$ with $c_i, a_i \in R$ and $a_i \notin P$. Then $r c_i = r_i a_i$ for all i. Set $a = \prod_{1 \leq h \leq l} a_h$. Then $a \in R \setminus P$. Multiply the i-th equation above by $\prod_{h \neq i} a_h$ and get for every i a new equation $r b_i = r_i a$ for a suitable $b_i \in R$. It follows that $NP = Nr$. Hence the fact that $NP = N$ implies that r acts invertibly on N. So $P(N) \cap R \subsetneq P$. Therefore $P(N) \cap R$, as a prime ideal of R, coincides with 0. So N is R-divisible i.e. $N \in \mathrm{Zg}_A$. Thus we have shown that

$$\mathrm{Zg}_\Lambda^{tf} = \bigcup_P ((x = x/P \mid x) \cap \mathrm{Zg}_\Lambda^{tf}) \cup \mathrm{Zg}_A$$

where P ranges over maximal ideals of R.

As promised, we now generalise the main results [10] to orders in semisimple algebras. A large part of the proof is the same as over group rings, but adaptions are sometimes necessary.

Theorem 3.1. *Let R be a Dedekind domain with field of fractions Q, and Λ an R-order in a semisimple Q-algebra A. If $N \in \mathrm{Zg}_\Lambda^{tf}$, then either*

- *N is a simple A-module, or*
- *there is some maximal ideal P of R such that $N \in \mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$ and N is R_P-reduced.*

Moreover, if $N \in \mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$ is R_P-reduced then $N \in \mathrm{Zg}_\Lambda^{tf}$.

Here N being R_P-reduced means that $\cap_{i=0}^\infty NP^i = 0$. Recall that $\widehat{\Lambda_P}$ is an order over $\widehat{R_P}$ in $\widehat{A} = \widehat{Q} \otimes_Q A$ where $\widehat{Q}$ denotes the field of fractions of $\widehat{R_P}$.

Proof. We follow the proof of [10, Theorem 2.1, pp. 1127-1130]. For simplicity we divide our argument in several steps. Let N be an indecomposable pure injective R-module.

Step 1. For some maximal ideal P of R, N is a module over Λ_P.

This step has already been covered in the discussion preceding this theorem. Let π denote a generator of the maximal ideal PR_P of R_P.

Step 2. Any Λ_P-module divisible and torsionfree over R_P is injective over Λ_P.

The proof is the same as [10, Claim 2, p. 1128].

Step 3. N, as an R_P-torsionfree module over Λ_P, decomposes over Λ_P as $N' \oplus N''$ where N'' is R_P-divisible (hence an A-module) and N' is R_P-reduced, i.e. $\cap_{i=1}^\infty N' P^i = \cap_{i=1}^\infty N' \pi^i = 0$.

To prove this claim, first we put

$$N'' := \{m \in N \mid \pi^n | m \text{ for all } n \in \mathbb{N}\}.$$

Take $n \in \mathbb{N}$, $m, m' \in N''$ and $r \in \Lambda$. Since $m, m' \in N''$ there exists $a, a' \in N$ such that $m = a\pi^n$ and $m' = a'\pi^n$. Thus $mr + m' = a\pi^n r + a'\pi^n = (ar + a')\pi^n$ because π is central. Thus

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N'' is a submodule of N. Since N'' is R_P-divisible by definition, it is injective by Step 2 and thus a direct summand of N.

Let N' be a complement of N'' in N. If $m \in \cap_{i=1}^{\infty} N'\pi^i$ then $\pi^n | m$ for all $n \in \mathbb{N}$ and thus $m \in N''$. So $m = 0$.

This concludes Step 3.

As N is indecomposable, either

   (a) N is an A-module, or

   (b) N is R_P-reduced.

In the former case N must be a simple A-module. So let us turn to (b). We assume from now on that N is R_P-reduced.

Step 4. Every R_P-reduced pure injective Λ_P-module M can be equipped with a $\widehat{\Lambda_P}$-module structure, and M remains pure injective over $\widehat{\Lambda_P}$.

To see this, one proceeds exactly as in [10, pp. 1128-1129]. Suppose $r \in \widehat{\Lambda_P}$ and $m \in M$. For each $i \in \mathbb{N}$, let $r_i \in \Lambda_P$ satisfy $\pi^i | r - r_i$. For each $i \in \mathbb{N}$, look at the equation $x - mr_i = y_i\pi^i$. When i ranges over $\mathbb{N}$, this set of equations is finitely solvable and so, since M is pure injective, solvable in M. Let $n, n' \in M$ be such that $\pi^i | n - mr_i$ and $\pi^i | n' - mr_i$ for all i. Then $n - n' \in \cap_{i=1}^{\infty} M\pi^i = 0$.

Define mr to be the unique element $n \in M$ such that $\pi^i | n - mr_i$ for all i. Note that this definition of mr does not depend on the particular choice of r_i above. If $r, s \in \widehat{\Lambda_P}$ then mr is the unique element m_1 in M such that $\pi^i | m_1 - mr_i$ and ms is the unique element m_2 in M such that $\pi^i | m_2 - ms_i$. Thus $\pi^i | m_1 + m_2 - m(r_i + s_i)$ for all $i \in \mathbb{N}$. Hence $m_1 + m_2 = m(r + s)$.

That M is pure injective as a $\widehat{\Lambda_P}$-module is a consequence of [10, Lemma p. 1129].

Conversely, independently of the assumption that N is R_P reduced, every pure injective $\widehat{\Lambda_P}$-module N remains pure injective after restricting it over Λ_P. This is simply because Λ_P is a subring of its P-adic completion. For the same reason any decomposable module over $\widehat{\Lambda_P}$ is decomposable over Λ_P. On the other hand the following holds.

Step 5. If N is an R_P-reduced indecomposable pure injective $\widehat{\Lambda_P}$-module, then N is indecomposable as a Λ-module.

This is explained in [10, Remark 1, p. 1130].

$\square$

The above theorem has shown that, when A is semisimple, R_P-reduced R-torsionfree indecomposable pure injective modules are the same over Λ_P and over $\widehat{\Lambda_P}$. Moreover, the set of R_P-reduced R-torsionfree indecomposable pure injective modules are exactly those modules in the open set $(x = x/P \mid x)$. For this reason we will sometimes write $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf+red}$ for this open set.

Note that any module $N \in \mathrm{Zg}_{\Lambda}^{tf}$ that can be regarded as a Λ_P-module but does not belong to $(x = x/P \mid x)$, that is, satisfies $N = PN$, is an A-module. Furthermore any two different sets $(x = x/P \mid x) \cap \mathrm{Zg}_{\Lambda}^{tf}$ are disjoint from each other. Thus Theorem 3.1 asserts that $\mathrm{Zg}_{\Lambda}^{tf}$ is the disjoint union of Zg_A and of the various $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf+red}$ where P ranges over maximal ideals of R.

Now let us deal with the topology. Notice that for every maximal ideal P of R the embedding of Λ into $\widehat{\Lambda_P}$ induces an inclusion of the R_P-reduced part of $(x = x/P \mid x) \cap \mathrm{Zg}_\Lambda^{tf} = \mathrm{Zg}_{\widehat{\Lambda_P}}^{tf+red}$ into Zg_Λ^{tf}. By the same argument given in [10, Theorem 2.2] over group rings, this inclusion is homeomorphic:

Theorem 3.2. *Suppose that $A = Q\Lambda$ and $\widehat{A} = \widehat{Q}\Lambda$ (where $\widehat{Q}$ is the field of fractions of $\widehat{R_P}$) are semisimple. For every maximal ideal P of R, $(x = x/P \mid x) \cap \mathrm{Zg}_\Lambda^{tf} = \mathrm{Zg}_{\widehat{\Lambda_P}}^{tf+red}$ has the same topology whether viewed as a subspace of $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$ or of Zg_Λ^{tf}.*

Here we give a different proof of a slightly stronger claim.

First of all, observe that every pp-formula α of L_{Λ_P}, $\alpha \doteq \exists \mathbf{y}(\mathbf{x}S = \mathbf{y}T)$, with S, T matrices of suitable sizes with entries in $\Lambda_P = R_P\Lambda$, can be translated into a pp-formula α' of L_Λ equivalent to α in all R-torsionfree Λ_P-modules. To build α', calculate the product r of all multiplicative inverses of scalars of R occurring in the entries of S and T. Then $r \in R \setminus P$, in particular $r \neq 0$. Now multiply the previous scalars by r and get α' as required, as $\exists \mathbf{y}(\mathbf{x}rS = \mathbf{y}rT)$. In fact the entries of rS and rT are in Λ. The torsionfree condition guarantees the equivalence to α. That is, for every R-torsionfree Λ_P-module M and $\mathbf{m}, \mathbf{n}$ in M, $r(\mathbf{m}S - \mathbf{n}T) = 0$ if and only if $\mathbf{m}S - \mathbf{n}T = 0$.

Thus we have to compare Λ_P and $\widehat{\Lambda_P}$. We may now assume that R is a discrete valuation domain and π is a generator of its maximal ideal P; Q is still the field of fraction of R, A a finite dimensional Q-algebra, Λ an order over R in A, $\widehat{\Lambda}$ its π-adic completion. We also assume both A and $\widehat{A}$ semisimple, which is true, in particular when A is separable. Under these conditions we prove the following.

Proposition 3.3. *Suppose that R is a discrete valuation domain whose maximal ideal is generated by π and that both A and $\widehat{A}$ are semisimple. The closed intervals $[\pi \mid x, x = x]_{\mathrm{Tf}_\Lambda}$ and $[\pi \mid x, x = x]_{\mathrm{Tf}_{\widehat{\Lambda}}}$ are isomorphic as lattices. Moreover the Ziegler open sets $(x = x/\pi \mid x)$ in Zg_Λ^{tf} and $(x = x/\pi \mid x)$ in $\mathrm{Zg}_{\widehat{\Lambda}}^{tf}$ are homeomorphic.*

Proof. For $1 \leq j \leq n$ and $1 \leq k \leq l$, let $s_j, r_{jk} \in \widehat{\Lambda}$, and let φ be the pp-formula

$$\exists\, y_1 \exists y_2 \ldots \exists y_l \bigwedge_{j=1}^n (xs_j + \sum_{k=1}^l y_k r_{jk} = 0).$$

Further suppose that $\pi | x \leq \varphi$.

For $i \in \mathbb{N}$, let $s_j^i, r_{jk}^i \in \Lambda$ be such that $\pi^i \mid s_j^i - s_j$ and $\pi^i \mid r_{jk}^i - r_{jk}$. For each $i \in \mathbb{N}$, let φ_i be the pp-formula

$$\exists\, y_1 \exists y_2 \ldots \exists y_l \bigwedge_{j=1}^n \pi^i | (xs_j^i + \sum_{k=1}^l y_k r_{jk}^i).$$

Clearly, $\varphi \leq \varphi_i$ and $\varphi_i \geq \varphi_{i+1}$ for each $i \in \mathbb{N}$.

We now show that for all indecomposable pure injective $\widehat{\Lambda}$-modules N, $\bigcap_{i \in \mathbb{N}} \varphi_i(N) = \varphi(N)$.

Since $\widehat{A}$ is semisimple, every indecomposable pure injective $\widehat{R}$-torsionfree $\widehat{\Lambda}$-module N is either R_P-reduced or $\widehat{R}$-divisible. In the latter case, since $\varphi_i \geq \varphi \geq \pi|x$, $\varphi_i(N) = \varphi(N) = N$. Hence assume that N is reduced. Suppose that $m \in \varphi_i(N)$ for all $i \in \mathbb{N}$. Then the infinite system of linear equations

$$ms_j^i + \sum_{k=1}^{l} y_k r_{jk}^i = z_j^i \pi^i$$

where $i \in \mathbb{N}$, $1 \leq j \leq n$ and $1 \leq k \leq l$, is finitely solvable. Consequently, since N is pure injective, it is solvable say with $y_k = a_k \in N$. So for each $1 \leq j \leq n$, $ms_j + \sum_{k=1}^{l} a_k r_{jk}^i \in N\pi^i$ for all $i \in \mathbb{N}$. Thus, since N is reduced, $ms_j + \sum_{k=1}^{l} a_k r_{jk} = 0$. Thus $m \in \varphi(N)$.

We now show that there exists an $i \in \mathbb{N}$ such that $\varphi_i = \varphi$. Suppose that $\varphi < \varphi_i$ for all $i \in \mathbb{N}$ with respect to the theory of $\widehat{R}$-torsionfree $\widehat{\Lambda}$-modules. Let F be the filter generated by $\{\varphi_i \mid i \in \mathbb{N}\}$ and I be the ideal generated by φ. Since $F \cap I$ is empty, by [11, 4.33], we can construct an irreducible pp-type p such that $\varphi_i \in p$ for all $i \in \mathbb{N}$ and $\varphi \notin p$. Now, $N(p)$, the pure injective hull of p, has an element m such that $m \in \varphi_i(N(p))$ for all $i \in \mathbb{N}$ but $m \notin \varphi(N(p))$. This contradicts the fact that $\varphi(N) = \bigcap_{i \in \mathbb{N}} \varphi_i(N)$ for all indecomposable pure injective $\widehat{R}$-torsionfree $\widehat{\Lambda}$-modules N. Thus $\varphi_i = \varphi$ for some $i \in \mathbb{N}$.

Restriction of scalars gives a lattice homomorphism from $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ to $\mathrm{pp}_{\widehat{\Lambda}}^1(\mathrm{Tf}_{\widehat{\Lambda}})$. We have shown that the restriction of this map to $[\pi \mid x, x = x]_{\mathrm{Tf}_R(\Lambda)}$ has image $[\pi \mid x, x = x]_{\mathrm{Tf}_{\widehat{\Lambda}}}$. Suppose that $\varphi > \psi \geq \pi|x$ in $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$. There exists an indecomposable pure injective Λ-module N such that $\varphi(N) \supsetneq \psi(N)$. Since $N\pi \neq N$, this implies that N is R_P-reduced and hence N can be equipped with the structure of a $\widehat{\Lambda}$-module. Hence $\varphi > \psi$ as pp-formulas in $\mathrm{pp}_{\widehat{\Lambda}}^1(\mathrm{Tf}_{\widehat{\Lambda}})$. Thus we have shown that $[\pi \mid x, x = x]_{\mathrm{Tf}_\Lambda}$ and $[\pi \mid x, x = x]_{\mathrm{Tf}_{\widehat{\Lambda}}}$ are isomorphic as lattices.

Since both A and $\widehat{A}$ are semisimple, the open sets $(x = x/\pi|x) \subseteq \mathrm{Zg}_\Lambda^{tf}$ and $(x = x/\pi|x) \subseteq \mathrm{Zg}_{\widehat{\Lambda}}^{tf}$ contain exactly the reduced indecomposable pure injective modules. Thus restriction of scalars gives a bijection from $(x = x/\pi \mid x) \subseteq \mathrm{Zg}_{\widehat{\Lambda}}^{tf}$ to $(x = x/\pi \mid x) \subseteq \mathrm{Zg}_\Lambda^{tf}$. Since the sets of the form (φ/ψ) for $\pi \mid x \leq \psi < \varphi$ in $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ (respectively $\pi \mid x \leq \psi < \varphi$ in $\mathrm{pp}_{\widehat{\Lambda}}^1(\mathrm{Tf}_{\widehat{\Lambda}})$) give a basis for $(x = x/\pi \mid x) \subseteq \mathrm{Zg}_\Lambda^{tf}$ (respectively $(x = x/\pi \mid x) \subseteq \mathrm{Zg}_{\widehat{\Lambda}}^{tf}$), this bijection is a homeomorphism. $\square$

Proposition 3.4. *Let A be semisimple. Then Zg_Λ^{tf} has Cantor-Bendixson rank if and only if Zg_A and each $\mathrm{Zg}_{\widehat{\Lambda}_P}^{tf+red}$, with P a maximal ideal of R, have Cantor-Bendixson rank.*

The proof is similar to that at the end of the next Lemma 3.5. So we ask the reader to wait for that lemma, and a few lines, to see its details.

Next let us deal with the m-dimension of $\mathrm{pp}_\Lambda^1(\mathrm{Tf})$. Recall the connection between m-dimension and Cantor-Bendixson rank [12, Corollary 5.3.60], at least under the *isolation condition*. The latter requires that for every closed subset $\mathcal{C}$ of Zg_Λ (indeed, in our case, of Zg_Λ^{tf}) and every isolated point N of $\mathcal{C}$, there is a pp-pair φ/ψ which is minimal such that $(\varphi/\psi) \cap \mathcal{C} = \{N\}$ (see [12, 5.3.2]).

We first prove a general statement which we were not able to find elsewhere in the literature.

Lemma 3.5. *Let S be an arbitrary ring, X a closed subset of Zg_S and $\{\varphi_i/\psi_i \mid i \in I\}$ a set of pp-pairs. Then $\mathrm{pp}_S^1(X)$ has m-dimension if and only if $[\psi_i, \varphi_i]_X$ has m-dimension for all $i \in I$ and $\mathrm{pp}_S^1(X \backslash \bigcup_{i \in I} (\varphi_i/\psi_i))$ has m-dimension.*

Proof. If $\mathrm{pp}_S^1(X)$ has m-dimension then $\mathrm{pp}_S^1(X \backslash \bigcup_{i \in I} (\varphi_i/\psi_i))$ has m-dimension since it is a quotient of $\mathrm{pp}_S^1(X)$ and for all $i \in I$, $[\psi_i, \varphi_i]_X$ has m-dimension because $[\psi_i, \varphi_i]_X$ is a sublattice of $\mathrm{pp}_S^n(X)$ for some $n \in \mathbb{N}$.

Now suppose that $[\psi_i, \varphi_i]_X$ has m-dimension for all $i \in I$ and $\mathrm{pp}_S^1(X \backslash \bigcup_{i \in I} (\varphi_i/\psi_i))$ has m-dimension. We first show that for all $N \in X$, N has an N-minimal pair, i.e. $\mathrm{pp}_S^1(N)$ has a simple interval.

Firstly, suppose that $N \notin (\varphi_i/\psi_i)$ for all $i \in I$. Then $\mathrm{cl}(N) \subseteq X \backslash \bigcup_{i \in I} (\varphi_i/\psi_i)$ where cl denotes closure. Thus $\mathrm{pp}_S^1(\mathrm{cl}(N)) = \mathrm{pp}_S^1(N)$ has m-dimension and hence N has an N-minimal pair.

Now suppose that $N \in (\varphi_i/\psi_i)$ for some $i \in I$. Since $[\psi_i, \varphi_i]_X$ has m-dimension, so does $[\psi_i, \varphi_i]_N$. Thus $[\psi_i, \varphi_i]_N$ contains a simple interval. So N has an N-minimal pair.

Therefore, by [12, 5.3.16], the isolation condition holds for X. So, by [12, 5.3.60], X has Cantor-Bendixson rank if and only if $\mathrm{pp}_S^1(X)$ has m-dimension. We now show that all points in X have Cantor-Bendixson rank.

Let $\alpha > \text{m-dim}[\psi_i, \varphi_i]_X$ for all $i \in I$. Then, by [12, 5.3.59], $\varphi_i(M) = \psi_i(M)$ for all $M \in X^{(\alpha)}$. Therefore $X \backslash \bigcup_{i \in I} (\varphi_i/\psi_i) \supseteq X^{(\alpha)}$. Since $\mathrm{pp}_S^1(X \backslash \bigcup_{i \in I} (\varphi_i/\psi_i))$ has m-dimension, so does $\mathrm{pp}_S^1(X^{(\alpha)})$. So $X^{(\alpha)}$ has Cantor-Bendixson rank and hence, X also has Cantor-Bendixson rank. $\qquad\square$

Corollary 3.6. *The lattice $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ has m-dimension if and only if, for all maximal ideals P of R, $[P \mid x, x = x]_{\mathrm{Tf}_\Lambda}$ has m-dimension and $\mathrm{pp}_{Q\Lambda}^1$ has m-dimension.*

Proof. Note that $\mathrm{Zg}_\Lambda^{tf} \backslash \bigcup_P (x = x/P \mid x)$ just consists of the R-divisible modules i.e. of modules over $Q \otimes \Lambda$. $\qquad\square$

The following is an easy consequence of the beginning of the proof of 3.5.

Remark 3.7. *m-dim $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda) \geq$ m-dim $[P \mid x, x = x]_{\mathrm{Tf}_\Lambda}$ for every maximal ideal P of R, and m-dim $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda) \geq$ m-dim pp_A^1.*

Specialising to the case where $A = Q\Lambda$ is semisimple we get the following.

Lemma 3.8. *Suppose that $A = Q\Lambda$ is semisimple and $[P \mid x, x = x]_{\mathrm{Tf}_\Lambda}$ has m-dimension for every maximal ideal P of R. Let α be the least ordinal such that $\alpha > $ m-dim$[P|x, x = x]_{\mathrm{Tf}_\Lambda}$ for all maximal ideals P of R. Then m-dim$\,\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda) = \alpha$.*

Proof. Since A is semisimple, pp_A^1 has m-dimension zero. So by 3.6, $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ has m-dimension. Thus the m-dimension of $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ is equal to the Cantor-Bendixson rank of Zg_Λ^{tf}.

Note that $X^{(\alpha)} \cap (x = x/P|x) = \emptyset$ for all maximal ideals P of R. Thus $\mathrm{Zg}_{Q\Lambda} \supseteq X^{(\alpha)}$. Since A is semisimple, all points in Zg_A are isolated. Thus $X^{(\alpha+1)} = \emptyset$. Thus we just need to show

that $X^{(\alpha)} \neq \emptyset$. We deal with the cases where α is a successor ordinal and α is a limit ordinal separately.

Suppose $\alpha = \beta + 1$. There exists a maximal ideal P of R such that $\beta = \text{m-dim}[P|x, x = x]$. Thus $X^{(\beta)} \cap (x = x/P|x) \neq \emptyset$. Take $N \in X^{(\beta)} \cap (x = x/P|x)$ and let L be an indecomposable direct summand of QN. By Lemma 2.7, L is in the closure of N. Thus $L \in X^{(\alpha)}$.

Now suppose α is a limit ordinal. For all $\beta < \alpha$, there exists a maximal ideal P of R such that $(x = x/P|x) \cap X^{(\beta)}$ is non-empty. Thus $X^{(\beta)} \neq \emptyset$ for all $\beta < \alpha$. Since Zg_Λ^{tf} is compact, $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$ is non-empty. $\qquad\square$

Corollary 3.9. *Suppose that $A = Q\Lambda$ is semisimple and $\text{pp}^1_{\Lambda_P}(\text{Tf}_{\Lambda_P})$ has m-dimension for every maximal ideal P of R. Then the m-dimension of $\text{pp}^1_\Lambda(\text{Tf}_\Lambda)$ is equal to the supremum of the m-dimensions of $\text{pp}^1_{\Lambda_P}(\text{Tf}_{\Lambda_P})$ where P ranges over maximal ideals of R. Moreover, if $A = Q\Lambda$ is separable, the m-dimension of $\text{pp}^1_\Lambda(\text{Tf}_\Lambda)$ is equal to the supremum of the m-dimensions of $\text{pp}^1_{\widehat{\Lambda_P}}(\text{Tf}_{\widehat{\Lambda_P}})$ where P ranges over maximal ideals of R.*

We will come back to m-dimension and Cantor-Bendixson rank after dealing with orders of finite lattice representation type. We will show, in 5.1, that if Λ has finite lattice representation type then the m-dimension of $\text{pp}^1_\Lambda(\text{Tf})$ is 1. This will allow us, in 5.2, to improve the above corollary when $Q\Lambda$ is separable.

4. Applications of Maranda's Theorem

Let us open a short parenthesis on Maranda's Theorem [4, § 30A]. Following [4, 30.12], we assume throughout this section that R is a discrete valuation domain, π is a generator of its maximal ideal P, Q is the field of fractions of R, A is a finite dimensional *separable* Q-algebra and Λ is an R-order in A.

We will deal with the quotient ring $\Lambda/\pi^k\Lambda$, often abbreviated as Λ_k, for every positive integer k. Similarly, for every Λ-module M, M_k will denote the quotient module $M/\pi^k M$. We will write $\mathbf{x}$ for tuples of variables and likewise, as in the previous sections, $\mathbf{m}$ for tuples of elements in a module.

There is a non negative integer, and hence a minimal non negative integer k_0 such that $\pi^{k_0} \text{Ext}^1(L, N) = 0$ for all Λ-lattices L and N (see [4, p. 624]). For instance, when $\Lambda = RG$ for some finite group G, then k_0 is the largest integer such that $|G| \in \pi^{k_0} R$. Note that, since Λ is noetherian, $\text{Ext}^1(L, -)$ is a finitely presented functor from Λ-modules to abelian groups (see [12, 10.2.35]). Hence $\pi^{k_0} \text{Ext}^1(L, -)$ is also a finitely presented functor. As Tf_Λ is the smallest definable subcategory of Λ-modules containing all Λ-lattices, it follows that $\pi^{k_0} \text{Ext}^1(L, N) = 0$ for every Λ-lattice L and $N \in \text{Tf}_\Lambda$.

Maranda's Theorem [4, Theorem 30.14] says that, under the previous assumptions, if k is any integer $> k_0$, then any two Λ-lattices M, N are isomorphic over Λ if and only if M_k and N_k are isomorphic over Λ_k. Moreover, even decomposability lifts from M_k to M [4, Theorem 30.19].

A generalization of these results to pure injective Λ-modules is given in a parallel paper [5], where the following is shown.

Theorem 4.1. *Let N, N' be R-reduced R-torsionfree pure injective Λ-modules. If $N_k \simeq N'_k$ for some integer $k > k_0$, then $N \simeq N'$. Moreover, if N is indecomposable over Λ, then the same is true of N_k over Λ_k for every $k > k_0$.*

On the other hand, let us also mention, again from [5]:

Theorem 4.2. *There exists a module N over $\widehat{\mathbb{Z}_2 C(2)}^2$ such that N is torsionfree and reduced over $\mathbb{Z}_2$, N_k is pure injective for all positive integer k but N is not pure-injective.*

We propose here some applications of the classical Maranda Theorem to our framework. If $f : M \to N$ is a morphism of Λ-modules, then we will write $\overline{f}$ for the induced homomorphism from M_k to N_k (k a positive integer). The following lemma is implicit in the proof of Maranda Theorem in [4].

Lemma 4.3. *Let L be a Λ-lattice and N an R-torsionfree Λ-module. If $k > k_0$ then for all $g \in \operatorname{Hom}_{\Lambda_k}(L_k, N_k)$ there exists $f \in \operatorname{Hom}_\Lambda(L, N)$ such that for all $m \in L$, $\pi^{k-k_0} + \Lambda\pi^k \,|\, (f(m) + \pi^k M) - g(m + \pi^k M)$.*

Definition 4.4. Assume $k > k_0$. Let φ be a pp-formula of L_Λ in l free variables $\mathbf{x} = (x_1, \ldots, x_l)$. Suppose that $(M, \mathbf{m})$ is a free realisation of φ, where M is R-torsionfree, so a Λ-lattice, and that $\varphi \geq \bigwedge_{i=1}^l \pi^{k-k_0} \,|\, x_i$. Then let φ_k be the generator of the pp-type of $\mathbf{m} + \pi^k M \in M_k$.

Thus φ_k is a pp-formula of L_{Λ_k} with l free variables.

Note that φ_k is well defined. In fact, let $(M, \mathbf{m})$ and $(N, \mathbf{n})$ be free realisations of φ, with both M and N Λ-lattices. Then there exist Λ-module morphisms $f : M \to N$ and $g : N \to M$ such that $f(\mathbf{m}) = \mathbf{n}$ and $g(\mathbf{n}) = \mathbf{m}$. The homomorphisms $\overline{f} : M_k \to N_k$ and $\overline{g} : N_k \to M_k$ induced by f and g respectively are such that $\overline{f}(\mathbf{m} + \pi^k M) = \mathbf{n} + \pi^k N$ and $\overline{g}(\mathbf{n} + \pi^k N) = \mathbf{m} + \pi^k M$. Thus the pp-type of $\mathbf{n} + \pi^k N$ in N_k is equal to the pp-type of $\mathbf{m} + \pi^k M$ in M_k, which guarantees that the above pp-formula φ_k is well defined, as said.

Lemma 4.5. *Let $k > k_0$. Suppose that $\varphi \in \operatorname{pp}_\Lambda^l$ is freely realised in a Λ-lattice and $\varphi \geq \bigwedge_{i=1}^l \pi^{k-k_0}|x_i$. Then, for all R-torsionfree Λ-modules N and l-tuples $\mathbf{n} \in N$,*

$$\mathbf{n} \in \varphi(N) \text{ if and only if } \mathbf{n} + \pi^k N \in \varphi_k(N_k).$$

Proof. Let M be a Λ-lattice and suppose that φ is freely realised by $\mathbf{m} \in M$. Then, by definition, φ_k is freely realised by $\mathbf{m} + \pi^k M$ in M_k.

Let N be an R-torsionfree Λ-module. If $\mathbf{n} \in \varphi(N)$ then there exist a morphism $f : M \to N$ sending $\mathbf{m}$ to $\mathbf{n}$. Consequently $\overline{f} : M_k \to N_k$ sends $\mathbf{m} + \pi^k M$ to $\mathbf{n} + \pi^k N$. Therefore $\mathbf{n} + \pi^k N \in \varphi_k(N_k)$.

Now suppose that $\mathbf{n} + \pi^k N \in \varphi_k(N_k)$. Then there exists a morphism $f : M_k \to N_k$ sending $\mathbf{m} + \pi^k M$ to $\mathbf{n} + \pi^k N$. By 4.3, there exists $g \in \operatorname{Hom}_\Lambda(M, N)$ such that $g(m_i) - n_i \in \pi^{k-k_0} N$

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for all $i = 1, \ldots, l$. Thus $\mathbf{n}$ satisfies the pp-formula $\bigwedge_{i=1}^{l} \pi^{k-k_0} \mid x_i + \varphi(\mathbf{x})$. Since $k - k_0 \geq 1$ and $\varphi \geq \bigwedge_{i=1}^{l} \pi^{k-k_0} \mid x_i$, $\mathbf{n} \in \varphi(N)$. $\qquad\square$

Given $\varphi \in \mathrm{pp}_{\Lambda_k}^n$, we define $\varphi^* \in \mathrm{pp}_{\Lambda}^n$ such that for all $N \in \mathrm{Mod}\text{-}\Lambda$, $\mathbf{m} \in \varphi^*(N)$ if and only if $\mathbf{m} + \pi^k N \in \varphi(N_k)$. Suppose $\varphi \doteq \exists \mathbf{y} \ (\mathbf{xy})T = 0$ where T is an appropriately sized matrix with entries from Λ_k. Further suppose that $T_{ij} := t_{ij} + \Lambda\pi^k$ where $t_{ij} \in \Lambda$. Let T^* be the matrix with entries t_{ij} and let $\varphi^* := \exists \mathbf{y} \ \pi^k | (\mathbf{xy})T^*$. A quick computation shows that for all $N \in \mathrm{Mod}\text{-}\Lambda$, $\mathbf{m} \in \varphi^*(N)$ if and only if $\mathbf{m} + N\pi^k \in \varphi(N_k)$ as required. Using this property of φ^*, one can check that for all $N \in \mathrm{Mod}\text{-}\Lambda$, the map which sends $\varphi(N_k) \in \mathrm{pp}_{\Lambda_k}^n(N_k)$ to $\varphi^*(N) \in \mathrm{pp}_{\Lambda}^n(N)$ is a lattice homomorphism. Note that $(\pi^{k-k_0} + \Lambda\pi^k | \mathbf{x})^*$ is equivalent to $\pi^{k-k_0} | \mathbf{x}$.

The next proposition applies to arbitrary R-torsionfree Λ-modules. In its statement we write $\pi^{k-k_0} | \mathbf{x}$ to mean the pp-n-formula $\bigwedge_{i=1}^{n} \pi^{k-k_0} | x_i$. Recall that, for every pp-formula $\varphi(x) \in \mathrm{pp}_{\Lambda}^n$, $\overline{\varphi}(x)$ is the pp-formula associated to $\varphi(x)$ defined just before Lemma 2.3.

Proposition 4.6. *Let $k > k_0$. The map from the closed interval $[\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ of pp_{Λ}^n to the interval $[\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ of $\mathrm{pp}_{\Lambda_k}^n$ which sends any $\varphi \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ to $\overline{\varphi}_k \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ induces a lattice isomorphism from $[\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]_N$ to $[\pi^{k-k_0} + \pi^k\Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]_{N_k}$ for all $N \in \mathrm{Tf}_{\Lambda}$.*

In particular, this lattice isomorphism is inverse to the lattice homomorphism which sends $\psi(N_k) \in [\pi^{k-k_0} + \pi^k\Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]_{N_k}$ to $\psi^*(N) \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]_N$ for all $N \in \mathrm{Tf}_{\Lambda}$.

Proof. First note that $\pi^{k-k_0} | \mathbf{x}$ is freely realised by the n-tuple from Λ with all entries π^{k-k_0}. Thus $\pi^{k-k_0} | \mathbf{x} \leq \varphi$ implies $\pi^{k-k_0} | \mathbf{x} \leq \overline{\varphi}$. So $\overline{\varphi}_k$ is defined. Suppose $\pi^{k-k_0} | \mathbf{x} \leq \varphi$. Then $(\pi^{k-k_0}, \ldots, \pi^{k-k_0}) \in \varphi(\Lambda) = \overline{\varphi}(\Lambda)$. So $(\pi^{k-k_0} + \Lambda\pi^k, \ldots, \pi^{k-k_0} + \Lambda\pi^k) \in \overline{\varphi}_k(\Lambda_k)$. Hence $\pi^{k-k_0} + \Lambda\pi^k | \mathbf{x} \leq \overline{\varphi}_k$.

Recall that $(\pi^{k-k_0} + \Lambda\pi^k | \mathbf{x})^*$ is equivalent to $\pi^{k-k_0} | \mathbf{x}$. So $\psi(N_k) \mapsto \psi^*(N)$ defines a lattice homomorphism from $[\pi^{k-k_0} + \pi^k\Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]_{N_k}$ to $[\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]_N$ for all $N \in \mathrm{Tf}_{\Lambda}$.

Since, when it exists, the set-wise inverse of a lattice homomorphism is a lattice isomorphism, it is therefore enough to show that for all $N \in \mathrm{Tf}_{\Lambda}$, $\varphi \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ and $\psi \in [\pi^{k-k_0} + \pi^k\Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]$, $(\overline{\varphi}_k)^*(N) = \varphi(N)$ and $\overline{(\psi^*)}_k(N_k) = \psi(N_k)$. But this follows from Lemma 4.5 and the property of φ^* described just before this proposition. $\qquad\square$

5. Finite lattice representation type

In this final section we recover our largest setting and we deal with a Dedekind domain R which is not a field, with its field of fractions Q and with an R-order Λ in a finite dimensional Q-algebra A.

Recall that Λ is said to be of *finite lattice representation type* if it has only finitely many non isomorphic indecomposable lattices. Our aim is to obtain a complete description of $\mathrm{Zg}_{\Lambda}^{tf}$ when Λ is of finite lattice representation type.

But let us first concern ourselves with the m-dimension of $\mathrm{pp}_{\Lambda}^1(\mathrm{Tf}_{\Lambda})$ under the finite lattice representation type hypothesis.

Proposition 5.1. *Let Λ be an order over a Dedekind domain R. If Λ is finite lattice representation type then m-dim $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda) = 1$.*

Proof. Let $L_1, \ldots, L_n$ be a complete list of indecomposable Λ-lattices up to isomorphism. By Fact 2.1 and Proposition 2.2, the canonical surjection from $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ to $\mathrm{pp}_\Lambda^1(\oplus_{i=1}^n L_i)$ is an isomorphism. Since $\oplus_{i=1}^n L_i$ is a Λ-lattice, by Corollary 2.10, $\mathrm{pp}_\Lambda^1(\oplus_{i=1}^n L_i)$ has m-dimension 1. $\square$

Here is a first consequence of this proposition. Let $S(\Lambda)$ be the set of maximal ideals P of R such that Λ_P is not maximal. If $Q\Lambda$ is separable then $S(\Lambda)$ is finite, see [4, p. 642]. Moreover, see [15, 11.5], Λ_P is maximal if and only if $\widehat{\Lambda_P}$ is maximal. By [15, 18.1], if $\widehat{\Lambda_P}$ is maximal then $\widehat{\Lambda_P}$ is hereditary. So, by [15, 10.6], all $\widehat{\Lambda_P}$-lattices are projective. Finally, since that category of $\widehat{\Lambda_P}$-lattices is Krull-Schmidt, there are only finitely many indecomposable projective lattices and hence $\widehat{\Lambda_P}$ is finite lattice representation type.

Corollary 5.2. *Suppose that $Q\Lambda$ is separable and $S(P)$ non-empty. The m-dimension of $\mathrm{pp}_\Lambda^1(\mathrm{Tf}_\Lambda)$ is equal to $max_{P \in S(\Lambda)} \mathrm{pp}_{\widehat{\Lambda_P}}^1(\mathrm{Tf}_{\widehat{\Lambda_P}})$.*

Proof. By 5.1, if $P \notin S(\Lambda)$ then $\mathrm{pp}_{\widehat{\Lambda_P}}^1(\mathrm{Tf}_{\widehat{\Lambda_P}})$ has m-dimension 1. For any maximal ideal P of R, the m-dimension of $\mathrm{pp}_{\widehat{\Lambda_P}}^1(\mathrm{Tf}_{\widehat{\Lambda_P}})$ is greater than or equal to 1. $\square$

Note that, if $S(P)$ is empty, then the m-dimension is 1.

The following is a further consequence of Proposition 5.1: if Λ is of finite lattice representation type, then Zg_Λ^{tf} has the isolation property and so its Cantor-Bendixson rank is equal to m-dim $pp_\Lambda^1(\mathrm{Tf}_\Lambda)$, namely it is 1 (see [12, Proposition 5.3.17]). Let us confirm this conclusion in a different way, also providing the topological description of Zg_Λ^{tf} we promised before. For this we focus on completions of localizations of Λ at maximal ideals of R, so at complete discrete valuation domains. First an easy fact - a sort of converse of the main result we are going to prove.

Proposition 5.3. *Let R be a complete discrete valuation domain and Λ an order over R. If Zg_Λ^{tf} has Cantor-Bendixson rank 1 then Λ is finite lattice representation type.*

Proof. Let π generate the maximal ideal of R. If $N \in (x = x/\pi \mid x)$ then N is not π-divisible and hence by Corollary 2.8 is not closed. Therefore $(x = x/\pi \mid x)$ contains only isolated points. Since $(x = x/\pi \mid x)$ is compact, it must be finite. Since all indecomposable Λ-lattices are pure injective and not π-divisible, Λ is of finite lattice representation type. $\square$

Next we provide the description of Zg_Λ^{tf} when Λ is an order over a complete discrete valuation domain.

Proposition 5.4. *Suppose that R is a complete discrete valuation domain with field of fractions Q, Λ is an order over R, $A = Q\Lambda$ is a semisimple Q-algebra and Λ is of finite lattice representation type. Then the set Zg_Λ^{tf} consists exactly of*

- *finitely many indecomposable lattices over Λ,*
- *finitely many simple A-modules.*

The indecomposable modules over A are closed points. If N is an indecomposable lattice then a simple A-module M is in the closure of N if and only if M is a direct summand of QN.

Proof. By Proposition 2.2, the set of indecomposable Λ-lattices is dense in Zg_Λ^{tf}. Thus if $N_1, \ldots, N_n$ are the indecomposable lattices over Λ, then Zg_Λ^{tf} coincides with the closure of $\{N_1, \ldots N_n\}$ and hence of the union of the closures of the N_i with $i = 1, \ldots, n$.

In Proposition 2.6 we showed that any point in the closure of N an indecomposable lattice which is not equal to N is a closed point. By Corollary 2.8, any closed point is an A-module. Thus we have shown that Zg_Λ^{tf} has exactly the points stated in the proposition.

The description of the topology follows from Lemmas 2.7 and 2.4. $\qquad\square$

Corollary 5.5. *Assume R and Λ as before, hence in particular Λ of finite lattice representation type. Let p be a non finitely generated indecomposable pp-type in the theory T_Λ^{tf} of R-torsionfree Λ-modules. Then p contains all divisibility formulas $\pi^k \mid x$ for k a positive integer.*

Proof. Any element of a simple A-module realizes all these formulas. $\qquad\square$

Note that Herzog and Puninskaya verified a similar result for torsionfree modules over 1-dimensional commutative noetherian local complete domains, see [6, Theorem 6.6].

Now let us come back to a Dedekind domain R and to an R-order Λ in a separable Q-algebra A. The following hold:

- ([4, Exercise 4.7 p. 99] there are only finitely many maximal ideals P of R such that $\widehat{\Lambda_P}$ is not a maximal order;
- ([4, Proposition 33.1] if $\widehat{\Lambda_P}$ is maximal, then it is of finite lattice representation type, and hence the topology of $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$ is that described in Proposition 5.1.

Recall that this topology is the same both in $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$ and $\mathrm{Zg}_{\Lambda_P}^{tf}$, when restricted to its R-reduced part (Theorem 3.2).

By the proof of [4, Theorem 33.2], if Λ is of finite lattice representation type then each non maximal Λ_P is.

On this basis it is easy to deduce:

Theorem 5.6. *Let R be a Dedekind domain and Λ an R-order in a separable Q-algebra A. Assume Λ of finite lattice representation type. Then the Cantor-Bendixson rank of Zg_Λ^{tf} is 1, and*
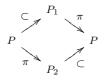
- *the isolated points are the indecomposable $\widehat{\Lambda_P}$-lattices, where P ranges over maximal ideals of R,*
- *the points of Cantor-Bendixson rank 1 are the simple A-modules.*

Let us give some examples illustrating the previous results. The first, over a complete discrete valuation domain R, was proposed by Gena Puninski. Indeed it is one of the last suggestions he left to us. So we like to mention it as a tribute to his memory.

Example 5.7. Let R be as just said, π be a generator of its maximal ideal. Let $\Lambda = \left(\begin{smallmatrix} R & R \\ \pi^2 R & R \end{smallmatrix} \right)$ (see [4, p. 779]. Also, let e_1, e_2 denote for simplicity the idempotents $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, respectively. It is well known that Λ has finite lattice representation type. In fact Λ is Gorenstein, i.e. projective modules $P_1 = e_1 \Lambda = \left(\begin{smallmatrix} R & R \\ 0 & 0 \end{smallmatrix} \right)$ (so basically (R, R)) and $P_2 = e_2 \Lambda = \left(\begin{smallmatrix} 0 & 0 \\ \pi^2 R & R \end{smallmatrix} \right)$ (hence (π^2, R)) are injective (in the category of lattices). The only remaining indecomposable lattice is $P = (\pi R, R)$ (note that $\left(\begin{smallmatrix} \pi R & R \\ 0 & 0 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ \pi R & R \end{smallmatrix} \right)$ are isomorphic as Λ-modules).

Hence a description of Zg_Λ^{tf} follows from Proposition 5.4. Anyway let us follow Gena's approach for the reasons we said.

First of all, note that, being Gorenstein, Λ admits a unique overorder $\Lambda' = \left(\begin{smallmatrix} R & R \\ \pi R & R \end{smallmatrix} \right)$ which is hereditary; and P is defined over Λ', i.e. Λ' is the ring of definable scalars of P. Furthermore the following is the AR-quiver of Λ

$$
\begin{array}{ccccc}
 & & P_1 & & \\
 & \overset{\subset}{\nearrow} & & \overset{\pi}{\searrow} & \\
P & & & & P \\
 & \underset{\pi}{\searrow} & & \underset{\subset}{\nearrow} & \\
 & & P_2 & &
\end{array}
$$

where π denotes the multiplication by π. From that we can see irreducible morphisms in the category of lattices and the unique almost split sequence:

$$0 \to P \xrightarrow{(i,\pi)} P_1 \oplus P_2 \xrightarrow{\left(\begin{smallmatrix} \pi \\ -i \end{smallmatrix} \right)} P \to 0 \,,$$
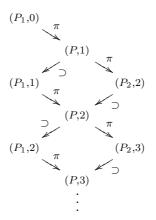
where i denotes inclusion. In detail the two intermediate morphisms act as follows:

- for all $a, b \in R$, (i, π) maps $(\pi a, b)$ to $((\pi a, b), (\pi^2 a, \pi b))$,
- for all $a', b', c', d' \in R$, $\left(\begin{smallmatrix} \pi \\ -i \end{smallmatrix} \right)$ sends $((a', b'), (\pi^2 c', d'))$ to $(\pi a' - \pi^2 c', \pi b' - d')$.

Let N be an indecomposable R-torsionfree pure injective Λ-module. First suppose that there exists $0 \neq n \in N e_1$. Hence look at pointed indecomposable lattices (M, m) such that $m \in M e_1$. Up to equivalence (of types realized by m) here is a complete list of them:
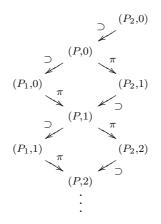
- $(P_1, (\pi^k, 0))$, $k \geq 0$,
- $(P_2, (\pi^l, 0))$, $l \geq 2$,
- $(P, (\pi^m, 0))$, $m \geq 1$.

Furthermore the following is the pattern of the module $(P_1, 0)$, i.e., the poset of morphisms from P_1 to indecomposable lattices (see [13] for a definition). Here we use an "exponential" notation: for instance (P_1, k) abbreviates $(P_1, (\pi^k, 0))$.

$$
\begin{array}{ccc}
(P_1,0) & & \\
 & \searrow^{\pi} & \\
 & (P,1) & \\
 \swarrow \;\; \supset & & \searrow^{\pi} \\
(P_1,1) & & (P_2,2) \\
 \searrow^{\pi} & & \swarrow \;\; \supset \\
 & (P,2) & \\
 \supset \;\; \swarrow & & \searrow^{\pi} \\
(P_1,2) & & (P_2,3) \\
 \searrow^{\pi} & & \swarrow \;\; \supset \\
 & (P,3) & \\
 & \vdots &
\end{array}
$$

We easily derive that there is a unique (critical over zero) indecomposable non finitely generated type p in the interval $[x = 0, e_1 \mid x]$ in $pp^1_\Lambda(\mathrm{Tf}_\Lambda)$. Furthermore p is realized by $(1,0) \in (Q,Q) = S$, the simple module over $A = M_2(Q)$. Thus in this case $N \cong P_i, P$ or $N \cong S$.

It remains to consider the case when there exists $0 \neq n \in Ne_2$. Again look at indecomposable pointed lattices (M,m) where $m \in Me_2$. They form the following pattern, where (P_1, k) now abbreviates $(P_1, (0, \pi^k))$, and so on.

$$
\begin{array}{ccc}
 & & (P_2,0) \\
 & \supset \;\; \swarrow & \\
 & (P,0) & \\
 \supset \;\; \swarrow & & \searrow^{\pi} \\
(P_1,0) & & (P_2,1) \\
 \searrow^{\pi} & & \swarrow \;\; \supset \\
 & (P,1) & \\
 \supset \;\; \swarrow & & \searrow^{\pi} \\
(P_1,1) & & (P_2,2) \\
 \searrow^{\pi} & & \swarrow \;\; \supset \\
 & (P,2) & \\
 & \vdots &
\end{array}
$$

Then there is a unique indecomposable non finitely generated type q in the interval $[x = 0, e_2 \mid x]$ which is realized as $(0,1) \in (Q,Q) = S$. Again we conclude that $N \cong P_i, P$ or $N \cong S$.

This completes the description of the R-torsionfree part of the Ziegler spectrum of our ring.

Example 5.8. The second example concerns an integral group ring $\mathbb{Z}\,C(p)$ with p a prime. This is of finite lattice representation type (see [4, 33A, in particular p. 690, and 34B]), whence Theorem 5.6 applies. Here is an explicit description of the torsionfree part of the Ziegler spectrum. Let g denote a generator of $C(p)$ and ζ_p a primitive p-th root of 1 in $\mathbb{C}$, so a root of the cyclotomic polynomial $\Phi_p(t) = t^{p-1} + t^{p-2} + \ldots + t + 1 \in \mathbb{Z}[t]$. Incidentally, let $\Phi_1(t) = t - 1$, whence $t^p - 1 = \Phi_1(t) \cdot \Phi_p(t)$. Also, let $e_1 = \frac{1}{p}\Phi_p(g)$ and $e_2 = 1 - e_1$ be the primitive idempotents of the algebra $\mathbb{Q}\,C(p)$. Thus the points of $\mathrm{Zg}^{tf}_{\mathbb{Z}\,C(p)}$ are the following.

- First of all, three isolated lattices over $\Lambda = \widehat{\mathbb{Z}_p} C(p)$, i.e., Λe_i, $i = 1, 2$, and Λ itself, in other words $\widehat{\mathbb{Z}_p}$, $\widehat{\mathbb{Z}_p}(\zeta_p)$ and $\widehat{\mathbb{Z}_p} C(p)$. In the first two cases g acts as the identity and the multiplication by ζ_p, respectively. Moreover $\widehat{\mathbb{Z}_p} C(p)$ corresponds to the pullback of $\widehat{\mathbb{Z}_p}_p$ and $\widehat{\mathbb{Z}_p}(\zeta_p)$ via the projections onto $\mathbb{Z}/p\mathbb{Z}$ sending 1 and ζ_p into $1 + p\mathbb{Z}$ (see [8]).

- Next, for every prime $q \neq p$, two more isolated points, $\widehat{\mathbb{Z}_q}$ and $\widehat{\mathbb{Z}_q}(\zeta_p)$ respectively, as now $\widehat{\mathbb{Z}_q} C(p)$ is their direct sum.

- Finally, two more points of Cantor-Bendixson rank 1, $\mathbb{Q}$ and $\mathbb{Q}(\zeta_p)$, as $\mathbb{Q} C(p)$ is again their direct sum.

The topology is also easy to describe.

- In fact $\Lambda e_1 = \widehat{\mathbb{Z}_p}$, $\Lambda e_2 = \widehat{\mathbb{Z}_p}(\zeta_p)$ and $\Lambda = \widehat{\mathbb{Z}_p} C(p)$ are the only points in the basic open set $(x = x/p \mid x)$, and indeed can be isolated, and separated from each other, as follows: Λ by $(\sum_{j<p} g^j \mid x/p \mid x)$ where $\sum_{j<p} g^j = pe_1$ (see [3, end of p. 57]) and Λe_1, Λe_2 by $(e_1 \mid x/pe_1 \mid x)$, $(e_2 \mid x/(1-\zeta_p)e_2 \mid x)$ respectively (see [3, proof of 3.3(a)]). Note that, properly speaking, e_1 and e_2 are not in $\mathbb{Z} C(p)$. However, as we are working in a $\mathbb{Z}$-torsionfree framework we can use here the simple trick of multiplying every involved scalar by p and expressing the previous open sets as $(pe_1 \mid x/p^2e_1 \mid x)$ and $(pe_2 \mid x/p(1-\zeta_p)e_2 \mid x)$.

- For every prime $q \neq p$, $\widehat{\mathbb{Z}_q}$ and $\widehat{\mathbb{Z}_q}(\zeta_p)$ are isolated from the other points by $(x = x/q \mid x)$ and indeed separated from each other, and hence isolated at all, by $((1 + g + \ldots + g^{p-1}) \mid x/q \mid x)$ and $((1 - g) \mid x/q \mid x)$, respectively.

- $\mathbb{Q}$ and $\mathbb{Q}(\zeta_p)$, that is, $\mathbb{Q}\Lambda e_1$ and $\mathbb{Q}\Lambda e_2$, are the points of Cantor-Bendixson rank 1 and at this level can be separated from each other, for instance, by $(x(1 - e_1)p = 0 / x = 0)$ and $(x(1 - e_2)p = 0 / x = 0)$, respectively.

Example 5.9. Finally let us deal with the integral group ring $\mathbb{Z} C(p^2)$ with p a prime. This is again a $\mathbb{Z}$-order of finite lattice representation type. A description of $\mathrm{Zg}^{tf}_{\mathbb{Z} C(p^2)}$, both points and topology, can be extracted from the classification of lattices over $\widehat{\mathbb{Z}_p} C(p^2)$ given in [4, 34C p. 730] in terms of extension groups, or in [7] in terms of pullbacks, or also in [3, § 4]. We follow this third approach. Let g still denote a generator of the group $C(p^2)$, e_1, e_2, e_3 be the primitive idempotents of the algebra $\mathbb{Q} C(p^2)$. Thus

$$e_1 = \frac{1}{p^2} \sum_{j<p^2} g^j = \frac{1}{p^2} \Phi_p(g) \Phi_p(g^p),$$

$$e_2 = \frac{1}{p^2}(p - \Phi_p(g)) \Phi_p(g^p), \quad e_3 = \frac{1}{p}(p - \Phi_p(g^p))$$

where $\Phi_p(t^p) = \Phi_{p^2}(t)$ is the cyclotomic polynomial of order p^2. Then the points of $\mathrm{Zg}^{tf}_{\mathbb{Z} C(p^2)}$ are the following.

- Let us start this time from simple $\mathbb{Q} C(p^2)$-modules, that is, from points of Cantor-Bendixson rank 1. They are $\mathbb{Q}$, $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_{p^2})$ where ζ_p and ζ_{p^2} are primitive roots of 1 of order p, p^2 respectively.

- When q is a prime different from p, $\widehat{\mathbb{Z}_q} C(p^2)$-lattices admit a similar description.
- Hence let us focus on $\Lambda = \widehat{\mathbb{Z}_p} C(p^2)$. Indecomposable lattices are now $4p + 1$. The first four are Λ itself and the Λe_i, $i = 1, 2, 3$, that is, $\widehat{\mathbb{Z}_p} C(p^2)$ and then $\widehat{\mathbb{Z}_p}$, $\widehat{\mathbb{Z}_p}(\zeta_p)$ and $\widehat{\mathbb{Z}_p}(\zeta_{p^2})$. In the three last cases g acts as the multiplication by 1, ζ_p and ζ_{p^2} respectively. The remaining $4p - 3$ correspond via the representation equivalence described in [3, §§ 3 and 4] to the indecomposable objects in the category of finite dimensional $\mathbb{Z}/p\mathbb{Z}$-representations of the directed Dynkin diagram D_{2p}. These can be viewed as tuples $(W, (W_{0h}^s)_{0<s<p, h=1,2}, W_1, W_2)$, where

  (1) W is a vector space over $\mathbb{Z}/p\mathbb{Z}$,

  (2) $W_0 = W_{01}^1$, W_1, W_2 are subspaces of W such that the sum of any two of them gives the whole W,

  (3) the W_{0h}^s form an increasing chain of subspaces $W_0 = W_{01}^1 \subseteq W_{02}^1 \subseteq W_{01}^2 \subseteq W_{02}^2 \subseteq \ldots \subseteq W_{01}^{p-1} \subseteq W_{02}^{p-1} = W$.

Moreover Butler's functor uniformly defines in a first order way every such representation in the associated lattice via quotients of pp-subgroups. Then it suffices for our purposes to list these indecomposable representations of D_{2p}. In fact they recursively determine the corresponding lattices as a sort of ID card. As said, they are $4p - 3$. In all of them the dimension of W is either 1 or 2. In the former case, i.e. in dimension 1, we meet

  a) 3 points where $W_0 = W$ (and hence $W_{0h}^s = W$ for every h and s) and the pair (W_1, W_2) is one among $(W, 0)$, $(0, W)$ and (W, W),

  b) $2p - 3$ points where $W_1 = W_2 = W$, $W_0 = 0$ and the W_{0h}^s are constantly 0 before some s and h ($h = 1$ when $s = 1$) and then become equal to W.

In the 2-dimensional case, we find $2p - 3$ additional points in which W_0, W_1, W_2 are 1-dimensional subspaces such that W is the sum of any two of them (and so the intersection of any two of them is 0), and the W_{0h}^s equal W_0 before some s and h ($h = 1$ when $s = 1$) and then coincide with W.

Next let us see the topology, so how to separate isolated points from each other. The case of primes $q \neq p$ can be handled as for $C(p)$, with slight complications. For instance the open sets isolating $\widehat{\mathbb{Z}_q}$, $\widehat{\mathbb{Z}_q}(\zeta_p)$ and $\widehat{\mathbb{Z}_q}(\zeta_{p^2})$ are now $((1+g+\ldots+g^{p-1}) \mid x \wedge (1+g^p+\ldots+g^{p(p-1)}) \mid x / q \mid x)$, $((1-g) \mid x \wedge (1 + g^p + \ldots + g^{p(p-1)}) \mid x / q \mid x)$, $((1-g) \mid x \wedge (1 + g + \ldots + g^{p-1}) \mid x / q \mid x)$ respectively. Also the analysis of simple $\mathbb{Q}C(p^2)$-modules is similar to that of $C(p)$.

Hence let us deal with $q = p$ and with indecomposable lattices over $\Lambda = \widehat{\mathbb{Z}_p} C(p^2)$, those in $(x = x / p \mid x)$.

The way to isolate Λ and the Λe_i ($i = 1, 2, 3$) is the same as for $C(p)$, by $(\sum_{j<p^2} g^j \mid x / p \mid x)$, $(e_1 \mid x / pe_1 \mid x)$, $(e_2 \mid x / (1 - \zeta_p)e_2 \mid x)$, $(e_3 \mid x / (1 - \zeta_{p^2})e_3 \mid x)$ respectively.

The further $4p - 3$ points are those in the open set $(p^2 \sum_{1 \leq i \leq 3} e_i \mid x / p^2 \mid x)$ (see the construction in [3, § 3]). To separate them from each other, we can look at the associated representations of D_{2p} as abelian structures in their own language, because these representations

are uniformly pp-definable without parameters in the corresponding lattices. Let us write for simplicity $x \in W_0$, $x \in W_1$ and so on to denote the formulas admitting this interpretation in any given representation. Thus

    a) the first 3 points are isolated by $(x \in W_0 \wedge x \in W_1 \,/\, x \in W_2)$, $(x \in W_0 \wedge x \in W_2 \,/\, x \in W_1)$ and $(x \in W_1 \wedge x \in W_2 \wedge x \in W_0 \,/\, x = 0)$,

    b) the following $2p - 3$ are isolated by $(x \in W_1 \wedge x \in W_2 \wedge x \in W_{01}^{s+1} \,/\, x \in W_{02}^{s})$ or $(x \in W_1 \wedge x \in W_2 \wedge x \in W_{02}^{s} \,/\, x \in W_{01}^{s})$ for the right s.

Similarly, the last $2p - 3$ are isolated by $(x \in W_{01}^{s+1} \,/\, x \in W_{02}^{s} + (x \in W_1 \wedge x \in W_2))$ or $(x \in W_{02}^{s} \,/\, x \in W_{01}^{s} + (x \in W_1 \wedge x \in W_2))$ for the right s.

On this basis, one easily deduces the following:

Theorem 5.10. *For every prime p, the first order theories of both $\mathbb{Z}$-torsionfree $\mathbb{Z}\,C(p)$-modules and $\mathbb{Z}$-torsionfree $\mathbb{Z}\,C(p^2)$-modules are decidable.*

In fact, the descriptions of the torsionfree part of the Ziegler spectrum of $\mathbb{Z}\,C(p)$ and $\mathbb{Z}\,C(p^2)$ fit with the conditions of the decidability criterion in [19, Theorem 9.4] and [11, Theorem 17.12]. In fact both an effective list of indecomposable pure injective modules N (possibly through the related representations of D_{2p}) and, for every such module, an effective list of basic open sets (φ/ψ) around N were already provided. Furthermore straightforward calculations recursively determine $\varphi(N)/\psi(N)$ from N, φ and ψ.

Notice that Theorem 5.10 positively solves expectations in the final lines of [9]. See also [18].

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