# LORENTZIAN MANIFOLDS WITH SHEARFREE CONGRUENCES AND KÄHLER-SASAKI GEOMETRY 

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Dedicated to the memory of Alexandre Mikhailovich Vinogradov


#### Abstract

We study Lorentzian manifolds $(M, g)$ of dimension $n \geq 4$, equipped with a maximally twisting shearfree null vector field p , for which the leaf space $S=M /\{\exp t \mathrm{p}\}$ is a smooth manifold. If $n=2 k$, the quotient $S=M /\{\exp t \mathrm{p}\}$ is naturally equipped with a subconformal structure of contact type and, in the most interesting cases, it is a regular Sasaki manifold projecting onto a quantisable Kähler manifold of real dimension $2 k-2$. Going backwards through this line of ideas, for any quantisable Kähler manifold with associated Sasaki manifold $S$, we give the local description of all Lorentzian metrics $g$ on the total spaces $M$ of $A$-bundles $\pi: M \rightarrow S, A=S^{1}, \mathbb{R}$, such that the generator of the group action is a maximally twisting shearfree $g$-null vector field p . We also prove that on any such Lorentzian manifold $(M, g)$ there exists a non-trivial generalized electromagnetic plane wave having p as propagating direction field, a result that can be considered as a generalization of the classical 4-dimensional Robinson Theorem. We finally construct a 2-parametric family of Einstein metrics on a trivial bundle $M=\mathbb{R} \times S$ for any prescribed value of the Einstein constant. If $\operatorname{dim} M=4$, the Ricci flat metrics obtained in this way are the well-known Taub-NUT metrics.


## 1. Introduction

In this paper we study Lorentzian manifolds $(M, g)$ of dimension $n \geq 4$, equipped with a maximally twisting shearfree null vector field p , for which the leaf space $S=M /\{\exp t \mathrm{p}\}$ is a smooth manifold and $\pi: M \rightarrow S$ is a principal bundle with one-dimensional structure group $A=\exp (t \mathrm{p})$ isomorphic to $\mathbb{R}$ or $S^{1}$. In case $n=2 k$, the quotient $S=M / A$ is odd-dimensional and naturally equipped with a contact distribution with a subconformal structure. In the most interesting situations, such subconformal structure comes from a regular Sasaki structure on $S$ and the manifold $S$ is the total space of an $A^{\prime}$-bundle $\pi^{\prime}: S \rightarrow N, A^{\prime}=\mathbb{R}, S^{1}$, over a quantisable Kähler manifold $N$. Our main results concern such Lorentzian manifolds fibering over regular Sasaki manifolds. More precisely, for any given quantisable Kähler manifold $N$ with associated Sasaki bundle $\pi^{\prime}: S \rightarrow N$, we give the local description of the Lorentzian metrics $g$ on the principle $A$-bundles $\pi: M \rightarrow S$ over $S$, for which the generator p of the $A$-action is a maximally twisting shearfree $g$ null vector field. We also prove a generalization of the classical 4-dimensional Robinson Theorem, namely for any such Lorentzian manifold we prove the existence of a non-trivial

[^0]generalized electromagnetic plane wave, having p as the propagating direction field of the wave. We finally construct a 2 -parametric family of Einstein metrics on any trivial bundle of the form $M=\mathbb{R} \times S$ for any prescribed value of the Einstein constant. If $\operatorname{dim} M=4$, the Ricci flat metrics obtained in this way are the well-known Taub-NUT metrics.

Let $(M, g)$ be a 4 -dimensional Lorentzian space-time and $F$ a 2 -form representing an electromagnetic plane wave, i. e. an harmonic decomposable 2-form $F=\vartheta \wedge \mathrm{e}^{*}$, determined by a null 1 -form $\vartheta$ and a $g$-orthogonal space-like 1 -form $\mathrm{e}^{*}$. Any such electromagnetic field is associated with a flag structure, namely the pair of nested distributions $\mathcal{K} \subset \mathcal{W}$, given by the spaces $\mathcal{K}_{x}=\operatorname{ker} F_{x} \cap \operatorname{ker}(* F)_{x}$ and $\mathcal{W}_{x}=\operatorname{ker} \vartheta_{x}$, respectively. These distribution have the following physical interpretations: $\mathcal{K}$ is the null 1 -dimensional distribution giving the propagating directions of the wave and $\mathcal{W}$ is the codimension one distribution generated by the wave fronts and the propagating directions. Note that a nested pair $\mathcal{K} \subset \mathcal{W}$, with $\mathcal{K}$ null and 1-dimensional and $\mathcal{W}$ of codimension one, is the flag structure of an electromagnetic plane wav e only if the integral lines of $\mathcal{K}$ constitute a geodesic shearfree congruence, that is a family of curves $\gamma(t)$ tangent to $\mathcal{K}$ and such that

$$
\mathcal{L}_{\dot{\gamma}} \mathcal{W} \subset \mathcal{W} \quad \text { and } \quad \mathcal{L}_{\dot{\gamma}} g=f g+\vartheta \vee \eta \quad \text { with } \quad \vartheta:=g(\dot{\gamma}, \cdot)
$$

for some function $f$ and a 1 -form $\eta$. This fact has a famous converse, the Robinson Theorem ([11,22]): any geodesic shearfree congruence of a real analytic Lorentzian 4-manifold ( $M, g$ ) locally coincides with the family of propagation lines of a non-trivial electromagnetic plane wave.

An analogue of Robinson Theorem holds for a large class of Lorentzian manifolds ( $M, g$ ) of higher even dimension $2 k>4$, provided that the classical notion of electromagnetic plane wave is extended as in the following definition, inspired by Trautman's discussion in 30]. A generalised electromagnetic plane wave on $(M, k)$ is a harmonic $k$-form $F=\vartheta \wedge \alpha$, which is the wedge product of a null 1 -form $\vartheta$ and a $(k-1)$-form $\alpha$ with the property that the null vectors in $\mathcal{W}=\operatorname{ker} \vartheta$ are also in $\operatorname{ker} \alpha$. Any such generalised plane wave determines a flag structure, namely the pair $(\mathcal{K}:=\operatorname{ker} F \cap \operatorname{ker}(* F), \mathcal{W}:=\operatorname{ker} \vartheta)$. As in the 4 -dimensional case, a nested pair $\mathcal{K} \subset \mathcal{W}$ of distributions might occur as a flag structure of a generalised electromagnetic plane wave only if it satisfies certain conditions. These constraints are satisfied if the distributions $\mathcal{K}$ and $\mathcal{W}$ are determined by some geodesic shearfree congruence on $(M, g)$. Furthermore, an analogue of Robinson Theorem for generalised electromagnetic plane waves holds on any even dimensional Lorentzian manifolds of Käher-Sasaki type, a very large class of space-times which we present below.

These facts, together with their tight relations with sub-Riemannian, CR and Kähler geometries discussed further in this paper, motivate our interest for the Lorentzian manifolds equipped with (geodesic) shearfree congruences. We call them shearfree Lorentzian manifolds. Such manifolds are particularly relevant also because they provide a large family of examples of null $G$-structures, a class of structures on space-times that has recently received attention in the context of string and $M$-theory (see [21] and references therein). Note also that the properties of electromagnetic plane waves and the Robinson Theorem in higher dimension has recently been object of intensive investigations (see e.g. [6, 7, 15, 20, 26]).

In this paper we focus on a particular class of shearfree Lorentzian manifolds, the regular ones. They are manifolds on which the shearfree congruence consists of the orbits of a onedimensional group $A=\mathbb{R}$ or $S^{1}$ acting freely and properly. For the manifolds of this kind, the orbit space $S=M / A$ is a smooth manifold, which is naturally equipped with a subconformal structure $(\mathcal{D},[h])$, i.e. a pair given by a codimension one distribution $\mathcal{D} \subset T S$ and a positive definite conformal metric $[h]$ on such distribution. If in addition $M$ is even dimensional and the shearfree congruence satisfies the so-called twisting condition, the distribution $\mathcal{D}$ is contact and the subconformal structure is canonically associated with a field of complex structures $J_{x}: \mathcal{D}_{x} \rightarrow \mathcal{D}_{x}, x \in M$, that makes the triple $(S, \mathcal{D}, J)$ a strongly pseudoconvex almost CR manifold.

The geometry of the orbit space $S=M / A$ of the regular shearfree Lorentzian manifolds is reacher and more interesting in case the CR structure ( $\mathcal{D}, J$ ) on $S$ is integrable (that is, with identically vanishing Nijenhuis tensor) and there is a free proper action of a onedimensional group $A^{\prime}$ of diffeomorphisms preserving $(\mathcal{D}, J)$ and a contact form $\theta$. In fact, if these additional properties hold, the almost CR manifold $(S=M / A, \mathcal{D}, J)$ is a regular Sasaki manifold and the quotient $N=S / A^{\prime}=M /\left(A \cdot A^{\prime}\right)$ is naturally equipped with a Kähler metric. The regular shearfree Lorentzian manifolds of this kind are called of Kähler-Sasaki type. This is the type of space-times mentioned above, admitting several nontrivial generalised electromagnetic plane waves propagating along the lines of the shearfree congruence.

The first purpose of this paper is to discuss in detail each of the above mentioned relations between regular shearfree Lorentzian manifolds, strongly pseudoconvex CR manifolds, Sasaki manifolds and quantisable Kähler manifolds (i.e. Kähler manifolds that can be obtained as quotients of some regular Sasaki manifold). For instance, we fully characterise not only the subconformal structure $(\mathcal{D},[h])$ and the almost CR structure $(\mathcal{D}, J)$ associated with a given regular shearfree manifold, but, conversely, also the regular shearfree structures with a prescribed subconformal structure or almost CR structure on the obit space. As a by-product, we establish an exact procedure for locally reconstructing all of the regular shearfree Lorentzian metrics of Kähler-Sasaki type projecting onto a prescribed quantisable Kähler manifold and we obtain the above mentioned higher dimensional version of the Robinson Theorem.

The second aim is to show that the proposed construction of regular shearfree Lorentzian manifolds can be used to describe interesting families of manifolds as, for instance, new classes of Lorentzian Einstein manifolds.

We prove that, for any choice of a real constant $\Lambda$ and of a quantisable $(n-2)$ dimensional Kähler-Einstein manifold $N$ with associated Sasaki manifold $S$, there exists a two-parameter family of Einstein metrics with Einstein constant $\Lambda$ that make $M=S \times \mathbb{R}$ a regular shearfree Lorentzian manifold of Kähler-Sasaki type. These metrics are explicitly given in terms of the Kähler metric of $N$, the contact form of the associated Sasaki manifold $S$ and two functions of the fiber coordinate $t \in \mathbb{R}$ of the trivial bundle $M=S \times \mathbb{R} \rightarrow S$. These two functions are uniquely determined by the prescribed Einstein constant $\Lambda$ of the
metric and two arbitrary real constants $B, C$ with $C>0$. We call such metrics of TaubNUT type since, for the case $N=\mathbb{C} P^{1}=S^{2}, S=S^{3}$ and $\Lambda=0$, the corresponding Ricci flat metrics are precisely the 4-dimensional Taub-NUT metrics on $S^{3} \times \mathbb{R}=\mathbb{R}^{4} \backslash\{0\}$.

We recall that any oriented 2-dimensional Riemannian manifold $N$ is Kähler and that, if such a surface is compact and with integer volume form, then it is also a quantisable Kähler manifold. Therefore, for any prescribed real number $\Lambda$, our results associate a twoparameter family of Lorentzian Einstein manifolds $(M=\mathbb{R} \times S, g)$ with Einstein constant , admitting electromagnetic plane waves, to any given compact Riemann surface $N$ with integer volume form.

Shearfree congruences of null geodesics on 4-dimensional space-times have been studied in the physics literature for a long time. In particular, their relation with the strongly pseudoconvex CR structure on the 3-dimensional orbit spaces is well known and has been discussed in various contexts and with diverse approaches. For an interesting and stimulating overview, we refer to [9,29] and to the extensive references therein. Here we tried to give a unified and, as much as possible, exhaustive approach to all of the cited relations between such four types of important geometric structures, the shearfree structures, the CR structures, the Sasaki structures and the Kähler structures. We discuss them in full generality, in arbitrary dimension, using a coordinate-free language.

As we mentioned above, the geometry of a shearfree Lorentzian manifold can be considered as the geometry of a space-time, in which there exists at least one electromagnetic plane wave. However, as just a quick look at the starry sky tells us, the Universe is pervaded by electromagnetic plane waves. This fact together with the Copernican Principle on the absence of privileged points of the Universe suggests that any Lorentzian manifold describing a realistic cosmological model should satisfy the following

Copernican Principle of visual connectedness. For any two points $x, y \in M$ with $y$ in the null cone of $x$, there is a geodesic joining $x$ and $y$ which belongs to a null shearfree congruence.
Maybe a more realistic conjecture is that such Copernican Principle is valid only locally, for sufficiently closed points $x, y$. Nonetheless we think that a local version of such principle is realistic and physically relevant. This also motivates an interesting differential geometric problem: characterise and possibly classify the Lorentzian manifolds satisfying either the above principle or one of its local variants. We believe that the results of this paper can be efficiently used to attack this problem. We plan to address it in a future work.

The paper is structured as follows. In §2, we introduce and study the notion of shearfree structure, a geometric object which underlies the metric of a shearfree Lorentzian manifold. We then show how to reconstruct all Lorentzian metrics that are compatible with a given shearfree structure. Note that the classes of Lorentzian metrics compatible with a fixed underlying shearfree structure are in natural one-to-one correspondence with the shearfree optical geometries of Robinson and Trautman ([23,24]). In §3, we introduce the notion of regular shearfree structures and we prove the one-to-one correspondence between these structures and the subconformal structures on the orbit spaces. In $\$ 4$ we study the
twisting regular shearfree structures and prove that, on a fixed Lorentzian manifold $M$, there is a bijection between this kind of shearfree structures and the strongly pseudoconvex almost CR structures on the orbit space equipped with some conformal class of positive endomorphisms. At the end of this section we introduce the regular shearfree structure of Kähler-Sasaki type, we study their relations with the quantisable Kähler manifolds and we prove the advertised generalisation of Robinson Theorem. In $\mathbb{4}$, we use the previous results to determine the Einstein Lorentzian metrics of Kähler-Sasaki type associated with a given Kähler-Einstein manifold and satisfying an appropriate ansatz. In this way we obtain the new family of Lorentzian Einstein metric mentioned above. In an appendix, we give the explicit expressions for the Christoffel symbols of a metric of Kähler-Sasaki type associated with a prescribed quantisable Kähler manifold. Such expressions are the outcomes of some tedious but very straightforward computations and are essential ingredients for the construction of the new metrics in $\$ 5$ and possibly of other types of examples.
Acknowledgments. After completing our paper, we learned that in [8,27] A. Fino, T. Leistner and A. Taghavi-Chabert, simultaneously and independently, obtained interesting results on higher dimensional shearfree congruences and Taub-NUT metrics, which partially overlap with and complement the results of this paper. We are sincerely grateful to Arman Taghavi-Chabert for bringing this to our attention. We also warmly thank Marcello Ortaggio and the anonymous referee for useful observations and suggestions.
Notation. The spaces of smooth real functions, vector fields and 1 -forms of a $n$-dimensional manifold $M$ are denoted by $\mathfrak{F}(M), X(M)$ and $\Omega^{1}(M)$, respectively. Given a tuple of vector fields $X_{1}, X_{2}, \ldots \in \mathfrak{X}(M)$, we indicate by $\left\langle X_{1}, X_{2}, \ldots\right\rangle \subset T M$ the distribution which they generate. Given a non-degenerate metric $g$, for any $X \in X(M)$ we denote $X^{b}:=g(X, \cdot)$. For any pair of 1-forms $\alpha, \beta \in \Omega^{1}(M)$, the symbol $\alpha \vee \beta$ stands for the symmetric tensor product $\alpha \vee \beta:=\frac{1}{2}(\alpha \otimes \beta+\beta \otimes \alpha)$. We write $X \in \mathcal{W}$ for any vector field $X$ tangent to a distribution $\mathcal{W} \subset T M$ and $\mathcal{L}_{X} \mathcal{W} \subset \mathcal{W}$ if $X$ preserves $\mathcal{W}$.

## 2. Shearfree Lorentzian manifolds and shearfree structures

2.1. First definitions. Let p be a null vector field on an $n$-dimensional Lorentzian manifold $(M, g)$. We associate to p the following objects:
$-\mathcal{W}:=\mathrm{p}^{\perp}$ is the codimension one distribution orthogonal to p ;

- the semipositive degenerate metric $h=g_{\mathcal{W}}:=\left.g\right|_{\mathcal{w}}$ on $\mathcal{W}$ which is induced by $g$.

A null vector field p is called shearfree if the (local) flow of p preserves the subconformal structure $(\mathcal{W},[h])$, that is

$$
\begin{equation*}
\mathcal{L}_{\mathrm{p}} X \in \mathcal{W} \quad \text { for any } X \in \mathcal{W} \quad \text { and } \quad \mathcal{L}_{\mathrm{p}} h=f h \quad \text { for some } f \in \mathcal{F}(M) . \tag{2.1}
\end{equation*}
$$

As the next lemma shows, the condition (2.1) is equivalent to

$$
\begin{equation*}
\mathcal{L}_{\mathrm{p}} g=f g+\mathrm{p}^{\mathrm{b}} \vee \eta \quad \text { for some function } f \text { and a 1-form } \eta . \tag{2.2}
\end{equation*}
$$

This demonstrates that the shearfree vector fields can be considered as generalisations of the null conformal vector fields.

Lemma 2.1. A nowhere vanishing null vector field p satisfies (2.1) if and only if it satisfies (2.2).

Proof. Assume that p satisfies (2.2). Then for any vector field $X \in \mathcal{W}=\mathrm{p}^{\perp}$

$$
g\left(\mathrm{p}, \mathcal{L}_{\mathrm{p}} X\right)=\mathrm{p}(g(\mathrm{p}, X))-\left(\mathcal{L}_{\mathrm{p}} g\right)(\mathrm{p}, X)-g([\mathrm{p}, \mathrm{p}], X)=-f g(\mathrm{p}, X)-\left(\mathrm{p}^{\mathrm{b}} \vee \eta\right)(\mathrm{p}, X)=0
$$

showing that $\mathcal{L}_{\mathrm{p}} X \in \mathcal{W}$. From this we also get

$$
\mathcal{L}_{\mathrm{p}} h=\left(\mathcal{L}_{\mathrm{p}} g\right)_{\mathcal{W}}=f g_{\mathcal{w}}+\left(\mathrm{p}^{b} \vee \eta\right)_{\mathcal{w}}=f h .
$$

Conversely, assume that p is a nowhere vanishing null vector field satisfying (2.1). Then, around any point $x_{o} \in M$, we may consider a simply connected neighbourhood $\mathcal{U}$ and a vector field $\left.\mathrm{q} \in T \mathcal{U} \backslash \mathcal{W}\right|_{\mathcal{u}}$ so that $g(\mathrm{p}, \mathrm{q})=1$. Let $\mathcal{W}^{\prime}=\operatorname{ker} \mathrm{q}_{\mathcal{W}}^{b}$ be the kernel of the $\mathrm{q}^{\mathrm{b}}$ in $\left.\mathcal{W}\right|_{\mathcal{u}}$. This determines the following direct sum decompositions of the tangent and the cotangent bundle of $\mathcal{U}$

$$
T \mathcal{U}=\langle\mathrm{p}\rangle+\mathcal{W}^{\prime}+\langle\mathrm{q}\rangle, \quad T^{*} \mathfrak{U}=\left\langle\mathrm{q}^{\mathrm{b}}\right\rangle+\mathcal{W}^{\prime *}+\left\langle\mathrm{p}^{\mathrm{b}}\right\rangle .
$$

These decompositions determine the following direct sum decomposition of the bundle $S^{2} T^{*} U$ of the symmetric square

$$
S^{2} T^{*} U=\left\langle\mathrm{p}^{\mathrm{b}}\right\rangle \vee\left\langle\mathrm{p}^{\mathrm{b}}\right\rangle+\left\langle\mathrm{p}^{\mathrm{b}}\right\rangle \vee\left\langle\mathrm{q}^{\mathrm{b}}\right\rangle+\left\langle\mathrm{q}^{\mathrm{b}}\right\rangle \vee\left\langle\mathrm{q}^{\mathrm{b}}\right\rangle+\left\langle\mathrm{p}^{b}\right\rangle \vee \mathcal{W}^{\prime *}+\left\langle\mathrm{q}^{\mathrm{b}}\right\rangle \vee \mathcal{W}^{\prime *}+\mathcal{W}^{\prime} \vee \mathcal{W}^{\prime} .
$$

Note that a symmetric $(0,2)$ tensor field vanishes identically on any pair of vector fields in $\mathcal{W}$ if and only if it takes values in $\left\langle p^{b}\right\rangle \vee\left\langle p^{b}\right\rangle+\left\langle p^{b}\right\rangle \vee\left\langle q^{b}\right\rangle+\left\langle p^{b}\right\rangle \vee \mathcal{W}^{\prime *}$. On the other hand, by assumption, $\left(\mathcal{L}_{\mathrm{p}} g-f g\right)_{\left.\mathcal{W}\right|_{\mathcal{U}}}=\mathcal{L}_{\mathrm{p}} h-f h=0$. So, by the previous observation, at the points of $\mathcal{U}$ we have that $\mathcal{L}_{\mathrm{p}} g-f g=\mathrm{p}^{b} \vee \eta$ for some (uniquely defined) 1-form $\eta$. The uniqueness of $\eta$ on $\mathcal{U}$ implies that (2.2) holds for a unique 1-form $\eta$ on $M$.

The conditions (2.1) are mostly motivated by the fact that they correspond to the two main properties of the propagating direction field $[\mathrm{p}]$ of an electromagnetic plane wave in General Relativity. Indeed, on a 4-dimensional space-time the first condition is equivalent to the property that p is geodesic, i.e. $\nabla \mathrm{p}=\lambda$ p for a function $\lambda$, and it encodes the fact that the photons travel along null geodesics. The second condition captures the fact that the null field property of the electromagnetic plane waves (i.e., in terms of the electric and magnetic fields, $\vec{E} \cdot \vec{H}=|\vec{E}|^{2}-|\vec{H}|^{2}=0$ ) is preserved along their null propagation rays ([3). It is also worth mentioning that if $\operatorname{dim} M=3$ and p is a null vector field satisfying just the first condition (i.e., it preserves the 2-dimensional distribution $\mathcal{W}=(p)^{\perp} \supset\langle\mathrm{p}\rangle$ ), then the shearfree condition $\mathcal{L}_{\mathrm{p}} h=f h, h:=g_{\mathcal{W}}$, is automatically satisfied.

Definition 2.2. A manifold $M$ with a Lorentzian metric $g$ and a shearfree vector field p is called shearfree Lorentzian manifold. The pair $(g, \mathrm{p})$ is called shearfree pair. The 1dimensional foliation of $M$, which is determined by the integral curves of the 1-dimensional distribution $\langle\mathrm{p}\rangle$, is called shearfree congruence.

### 2.2. Equivalent shearfree pairs. Standard and distinguished pairs.

## Definition 2.3.

i) Two shearfree pairs of the form $(g, \mathrm{p}),\left(g^{\prime}=\sigma g, \mathrm{p}^{\prime}=\tau \mathrm{p}\right)$ for some functions $\sigma>0$, $\tau \neq 0$, are called equivalent. The equivalence class of $(g, \mathrm{p})$ is denoted by $([g],[\mathrm{p}])$.
ii) A shearfree pair $(g, \mathrm{p})$ with autoparallel vector field p , i.e. such that $\nabla_{\mathrm{p}} \mathrm{p}=0$, is called standard.
iii) A shearfree pair is called distinguished if

$$
\begin{equation*}
\mathcal{L}_{\mathrm{p}} g=\mathrm{p}^{b} \vee \eta \quad \text { for some } \eta \in \Omega^{1}(M) \tag{2.3}
\end{equation*}
$$

Proposition $2.4([2,23])$. Let $(g, \mathrm{p})$ be a shearfree pair. Then:
(i) p is a geodesic vector field, i.e. $\nabla_{\mathrm{p}} \mathrm{p}=\lambda \mathrm{p}$ for some function $\lambda$. In other words, the corresponding shearfree congruence is geodesic.
(ii) Locally any shearfree pair $(g, \mathrm{p})$ is equivalent to a standard pair $\left(g, \mathrm{p}^{\prime}\right)=(g, \tau \mathrm{p})$. It is also locally equivalent to a standard and distinguished pair $\left(g^{\prime}, \mathrm{p}^{\prime}\right)=(\sigma g, \tau \mathrm{p})$. Such equivalent standard and distinguished pair is uniquely determined up to conformal factors $\sigma, \tau$ that are constant along the p-orbits.

Proof. (i) First of all, we observe that for any $g$-null vector field $V$ the following identity holds

$$
\begin{align*}
& g\left(\nabla_{V} V, X\right)=V(g(V, X))-g\left(V, \nabla_{V} X\right)= \\
& \quad=\left(\mathcal{L}_{V} g\right)(V, X)+g(V,[V, X])-g\left(V, \nabla_{V} X\right)=\left(\mathcal{L}_{V} g\right)(V, X)-g\left(V, \nabla_{X} V\right)= \\
& =\left(\mathcal{L}_{V} g\right)(V, X)-\frac{X(g(V, V))}{2}=\left(\mathcal{L}_{V} g\right)(V, X) \tag{2.4}
\end{align*}
$$

Since p is null and shearfree, this implies that for any vector field $X$

$$
\begin{equation*}
g\left(\nabla_{\mathrm{p}} \mathrm{p}, X\right)=\mathcal{L}_{\mathrm{p}} g(\mathrm{p}, X)=f g(\mathrm{p}, X)+\frac{1}{2} \mathrm{p}^{\mathrm{b}}(X) \eta(\mathrm{p})=\left(f+\frac{1}{2} \eta(\mathrm{p})\right) g(\mathrm{p}, X)=\lambda g(\mathrm{p}, X) \tag{2.5}
\end{equation*}
$$

From this the claim (i) follows.
(ii) Let $\left(g^{\prime}:=\sigma g, \mathrm{p}^{\prime}:=\tau \mathrm{p}\right)$ be a shearfree pair which is equivalent to ( $g, \mathrm{p}$ ) and denote by $\nabla, \nabla^{\prime}$ the Levi-Civita connections of $g$ and $g^{\prime}=\sigma g$, respectively. Let also $\mathrm{p}^{b}:=g(\mathrm{p}, \cdot)$ and $\mathrm{p}^{\prime b}:=g^{\prime}\left(\mathrm{p}^{\prime}, \cdot\right)$. By Koszul's formula, for any $X \in \mathfrak{X}(M)$

$$
\begin{aligned}
& g^{\prime}\left(\nabla_{\mathrm{p}} \mathrm{p}, X\right)=\mathrm{p}\left(g^{\prime}(X, \mathrm{p})\right)-g^{\prime}([\mathrm{p}, X], \mathrm{p})= \\
& \quad \begin{array}{l}
=(\mathrm{p}(\sigma)) \mathrm{p}^{\mathrm{b}}(X)+\sigma\left(\mathrm{p}\left(\mathrm{p}^{\mathrm{b}}(X)\right)-\mathrm{p}^{\mathrm{b}}([\mathrm{p}, X])\right)=(\mathrm{p}(\sigma)) \mathrm{p}^{\mathrm{b}}(X)+\sigma\left(\mathcal{L}_{\mathrm{p}} \mathrm{p}^{\mathrm{b}}\right)(X) \stackrel{\text { (2.2) }}{=} \\
\quad=\left(\mathrm{p}(\sigma)+\sigma f+\frac{1}{2} \sigma \eta(\mathrm{p})\right) \mathrm{p}^{\mathrm{b}}(X)
\end{array}
\end{aligned}
$$

for some function $f$ and a 1-form $\eta$. Hence $\nabla_{\mathrm{p}} \mathrm{p}=\left(\mathrm{p}(\sigma)+\sigma\left(f+\frac{1}{2} \eta(\mathrm{p})\right)\right) \mathrm{p}$ and

$$
\begin{equation*}
\nabla_{\mathrm{p}^{\prime}} \mathrm{p}^{\prime}=\tau \mathrm{p}(\tau) \mathrm{p}+\tau^{2} \nabla_{\mathrm{p}} \mathrm{p}=\tau\left(\mathrm{p}(\tau)+\tau(\mathrm{p}(\sigma)+\sigma f)+\frac{1}{2} \sigma \tau \eta(\mathrm{p})\right) \mathrm{p} \tag{2.6}
\end{equation*}
$$

On the other hand, $\mathcal{L}_{\mathrm{p}^{\prime}} g=\tau \mathcal{L}_{\mathrm{p}} g-2 \mathrm{p}^{\mathrm{b}} \vee d \tau=\tau f g+\mathrm{p}^{\mathrm{b}} \vee(\tau \eta-2 d \tau)$ and

$$
\begin{equation*}
\mathcal{L}_{\mathrm{p}^{\prime}} g^{\prime}=\mathrm{p}^{\prime}(\sigma) g+\sigma \mathcal{L}_{\mathrm{p}^{\prime}} g=\tau(\mathrm{p}(\sigma)+\sigma f) g+\mathrm{p}^{\mathrm{b}} \vee(\sigma \tau \eta-2 \sigma d \tau) . \tag{2.7}
\end{equation*}
$$

From (2.6) and (2.7), the local existence of a $\tau$ and a $\sigma$ so that $\left(g^{\prime}, \mathrm{p}^{\prime}\right)$ is standard and/or distinguished is a consequence of the existence of local solutions to the system of differential equations

$$
\begin{equation*}
\mathrm{p}(\sigma)=-\sigma f, \quad \mathrm{p}(\tau)=-\frac{1}{2} \sigma \tau \eta(\mathrm{p}) . \tag{2.8}
\end{equation*}
$$

This proves that any shearfree pair is locally equivalent to a standard and distinguished pair. We leave to the reader the checking of the last claim on uniqueness.
2.3. The shearfree structure of a shearfree Lorentzian manifold. We now introduce a geometric object that characterises the shearfree Lorentzian manifolds. For this, we observe that an equivalence class ( $[g],[\mathrm{p}]$ ) of shearfree pairs determines the codimension one distribution $\mathcal{W}=$ ker $\mathrm{p}^{b}$ and the degenerate conformal metric $[h]=\left[g_{\mathcal{W}}\right]$ on $\mathcal{W}$, that is the conformal class of the semipositive degenerate metrics induced by $g$. The kernel of such conformal metric is $\operatorname{ker}[h]=\langle\mathrm{p}\rangle$ and the conformal metric $[h]$ induces a positive definite conformal metric on the so-called screen bundle $\mathcal{W} /\langle\mathrm{p}\rangle$ (see [23]). This motivates our
Definition 2.5. Let $M$ be an $n$-dimensional manifold and $(\mathcal{W},[h])$ a pair given by a codimension one distribution $\mathcal{W} \subset T M$ and a conformal class [ $h$ ] of semipositive degenerate Riemannian metrics on $\mathcal{W}$ with one dimensional kernel $\mathcal{K}_{h}=\operatorname{ker} h \subset \mathcal{W}$.
(i) The pair $(\mathcal{W},[h])$ is called shearfree structure if it is preserved by one (hence each) vector field p in $\mathcal{K}_{h}$, that is

$$
\begin{equation*}
\mathcal{L}_{\mathrm{p}} \mathcal{W} \subset \mathcal{W} \quad \text { and } \quad \mathcal{L}_{\mathrm{p}}[h]=[h] \tag{2.9}
\end{equation*}
$$

Note that the pair $(\mathcal{W},[h])=\left(\operatorname{ker~}^{b},[g \mathcal{W}]\right)$ determined by an equivalence class $([g],[\mathrm{p}])$ of shearfree pairs is a shearfree structure.
(ii) A shearfree pair $(g, \mathrm{p})$ is called compatible with a shearfree structure $(\mathcal{W},[h])$ if ker $\mathrm{p}^{\mathrm{b}}=$ $\mathcal{W}$ and $\left.\left[g_{\mathcal{W}}\right]\right)=[h]$. In this case $g$ is called a compatible metric of the shearfree structure.
In other words, we may say that a shearfree structure is a semipositive degenerate conformal metric $[h]$ on a codimension one distribution $\mathcal{W}$ satisfying the following two conditions: (a) the kernel $\mathcal{K}_{h}=\operatorname{ker} h$ is one-dimensional and (b) $\mathcal{W}$ is invariant under any vector field $\mathrm{p} \in \mathcal{K}_{h}$.
2.4. Shearfree structures and optical geometries. The notions of shearfree structures and compatible metrics are tightly related with the optical geometries of Robinson and Trautman ( $[24])$. We recall that an optical geometry on a manifold $M$ is a triple ( $\mathcal{W}, \mathcal{K},\{g\}$ ) given by

- a codimension one distribution $\mathcal{W} \subset T M$;
- a one-dimensional subdistribution $\mathcal{K} \subset \mathcal{W}$;
- an equivalence class $\{g\}$ of Lorentzian metrics on $M$ satisfying the following conditions: - for any $g \in\{g\}$, one has $g(\mathrm{p}, \mathcal{W})=0$ for any vector field $\mathrm{p} \in \mathcal{K}$;
- any two metrics $g, g^{\prime} \in\{g\}$ are related by $g^{\prime}=\sigma g+\alpha \vee \beta$ for some positive function $\sigma: M \rightarrow \mathbb{R}$ and 1 -forms $\alpha, \beta$ with $\alpha \neq 0$ and $\alpha \mid \mathcal{w}=0$.
Now, any shearfree optical geometry ( $\mathcal{W}, \mathcal{K},\{g\}$ ) (i.e. any optical geometry admitting a shearfree vector field $\mathrm{p} \in \mathcal{K}$ ) determines an associated shearfree structure, namely the pair $\left.\left(\mathcal{W},[h]=\left[g_{\mathcal{W}}\right]\right), g \in\{g\}\right)$. In this case, the class $\{g\}$ consists of all of the compatible metrics of such shearfree structure. Conversely, any shearfree structure $(\mathcal{W},[h])$ determines the shearfree optical geometry $\left(\mathcal{W}, \mathcal{K}=\mathcal{K}_{h},\{g\}\right)$ with class $\{g\}$ given by the compatible metrics of the shearfree structure.
2.5. Reconstructions of shearfree optical geometries from shearfree structures. We now focus on the problem of reconstructing a shearfree optical geometry ( $\mathcal{W}, \mathcal{K},\{g\}$ ) starting from the associated shearfree structure ( $\mathcal{W},[h]$ ), that is of determining all shearfree metrics and shearfree pairs that are compatible with such shearfree structure.

Given a codimension one distribution $\mathcal{W} \subset T M$, a 1-form $\vartheta$ such that $\operatorname{ker} \vartheta=\mathcal{W}$ is called a defining 1 -form for $\mathcal{W}$. Note that, for any vector field $q$ which is transversal to $\mathcal{W}$, there is a unique defining 1 -form $\vartheta$ such that $\vartheta(\mathrm{q})=1$.

Definition 2.6. Let $(\mathcal{W},[h])$ be a shearfree structure on a manifold $M$ and $\mathcal{K}_{h}=\operatorname{ker} h \subset \mathcal{W}$ the corresponding one-dimensional kernel subdistribution. A rigging for $(\mathcal{W},[h])$ is a pair $\left(\mathcal{W}^{\prime}, q\right)$ given by a subdistribution $\mathcal{W}^{\prime} \subset \mathcal{W}$, which is complementary to $\mathcal{K}_{h}$ in $\mathcal{W}$, and a vector field $q$ which is transversal to $\mathcal{W}$.

Notice that any pair given by a rigging $\left(\mathcal{W}^{\prime}, \mathrm{q}\right)$ and a vector field $\mathrm{p} \in \mathcal{K}_{h}$ determines a direct sum decomposition

$$
T M=\mathcal{K}_{h}+\mathcal{W}^{\prime}+\langle\mathrm{q}\rangle
$$

and two 1 -forms $\mathrm{p}^{*}$, $\mathrm{q}^{*}$ satisfying the conditions

$$
\begin{equation*}
\mathrm{p}^{*}(\mathrm{p})=1, \quad \operatorname{ker} \mathrm{p}^{*}=\mathcal{W}^{\prime}+\langle\mathrm{q}\rangle \quad \text { and } \quad \mathrm{q}^{*}(\mathrm{q})=1, \quad \text { ker } \mathrm{q}^{*}=\mathcal{W}^{\prime}+\langle\mathrm{p}\rangle=\mathcal{W} . \tag{2.10}
\end{equation*}
$$

Note also that $\mathrm{q}^{*}$ coincides with the defining 1 -form $\vartheta=\mathrm{q}^{*}$ for $\mathcal{W}$ with $\vartheta(\mathrm{q})=1$ and it is thus uniquely determined by $\mathcal{W}$ and $q$.

With a rigging $\left(\mathcal{W}^{\prime}, q\right)$ and a degenerate metric $h \in[h]$ we associate the unique Lorentzian metric $g$ that satisfies the conditions $g(\mathrm{p}, \mathrm{q})=1, g_{\mathcal{W}}=h, g(\mathrm{p}, \mathrm{p})=g(\mathrm{q}, \mathrm{q})=0$ and $\mathcal{W}^{\prime}=\langle\mathrm{p}, \mathrm{q}\rangle^{\perp}$. This metric is

$$
\begin{equation*}
g=h+2 \mathrm{q}^{*} \vee \mathrm{p}^{*}=h+2 \vartheta \vee \mathrm{p}^{*}, \tag{2.11}
\end{equation*}
$$

where $h$ is considered as a degenerate metric on $T M$ with kernel ker $h=\mathcal{K}_{h}+\langle\mathrm{q}\rangle$.
These observations yield the following theorem, which gives a complete description of all optical geometries that are associated with a shearfree structure.

Theorem 2.7. Let $(\mathcal{W},[h])$ be a shearfree structure on $M$.
(1) For any triple $\left(h, \mathrm{p},\left(\mathcal{W}^{\prime}, \mathrm{q}\right)\right)$ formed by
(a) a degenerate metric $h \in[h]$,
(b) a nowhere vanishing vector field $\mathrm{p} \in \mathcal{K}_{h}$ and
(c) a rigging $\left(\mathcal{W}^{\prime}, \mathrm{q}\right)$,
the corresponding metric (2.11) together with the vector field p gives a shearfree pair $(g, \mathrm{p})$ which is compatible with $(\mathcal{W},[h])$.
(2) Conversely, if $(g, \mathrm{p})$ is a compatible shearfree pair for $(\mathcal{W},[h])$, then locally there is a rigging ( $\mathcal{W}^{\prime}, \mathrm{q}$ ) such that $g$ is the metric (2.11) determined by some triple $(h=$ $\left.g_{\mathcal{W}}, \mathrm{p},\left(\mathcal{W}^{\prime}, \mathrm{q}\right)\right)$. Such a rigging can be determined in two different ways:
(a) either by choosing a subdistribution $\mathcal{W}^{\prime} \subset \mathcal{W}$ which is complementary to $\mathcal{K}$ and setting q to be the vector field with $g\left(\mathrm{q}, \mathcal{W}^{\prime}+\langle\mathrm{q}\rangle\right)=0$ and $g(\mathrm{q}, \mathrm{p})=1$
(b) or by choosing a vector field q which is transversal to $\mathcal{W}$ and setting $\mathcal{W}^{\prime}=\langle\mathrm{p}, \mathrm{q}\rangle^{\perp}$.

Proof. (1) By construction, the vector field p is a nowhere vanishing null vector field of $g$ with ker $\mathrm{p}^{\mathrm{b}}=\mathcal{W}$. Moreover, $g_{\mathcal{W}}=h$ and $\mathrm{p} \in \mathcal{K}_{h}=\operatorname{ker} h$. Since ( $\mathcal{W},[h]$ ) satisfies (2.9) for any vector field in $\mathcal{K}_{h}$, it follows that p is shearfree for $g$.
(2) Let $(g, \mathrm{p})$ be a compatible pair for $(\mathcal{W}$, $[h])$, i.e. such that $\mathrm{ker}^{\mathrm{p}}=\mathcal{W}, g_{\mathcal{W}} \in[h]$ and $\mathrm{p} \in \mathcal{K}_{h}$. Then, for any vector field q such that $g(\mathrm{p}, \mathrm{q})=1$, the corresponding codimension two distribution $\mathcal{W}^{\prime}=\langle\mathrm{p}, \mathrm{q}\rangle^{\perp}$ defines a rigging $\left(\mathcal{W}^{\prime}, \mathrm{q}\right)$ for $([h], \mathcal{W})$ and the associated metric $g$ has the form (2.11). The same holds for any choice of a subdistribution $\mathcal{W}^{\prime} \subset \mathcal{W}$, which is complementary to $\langle\mathrm{p}\rangle$, and the vector field characterised by the conditions $g\left(\mathrm{q}, \mathcal{W}^{\prime}+\langle\mathrm{q}\rangle\right)=0$ and $g(\mathrm{q}, \mathrm{p})=1$.

The following corollary shows that the collection of all compatible metrics for a given shearfree structure are locally parametrised by the pairs formed by a positive function $\sigma$ and a 1 -form $\varpi$ which does not vanish on the distribution $\mathcal{K}_{h}$.

Corollary 2.8. Let $(\mathcal{W},[h])$ be a shearfree structure on $M$ and $\left(g_{o}, \mathrm{p}_{o}\right)$ be a compatible shearfree pair, hence of the form

$$
\begin{equation*}
g_{o}=h_{o}+2 \mathrm{q}_{o}^{*} \vee \mathrm{p}_{o}^{*}, \tag{2.12}
\end{equation*}
$$

where $\mathrm{q}_{o}^{*}$ and $\mathrm{p}_{o}^{*}$ are the 1 -forms determined by a triple $\left(h_{o}, \mathrm{p}_{o},\left(\mathcal{W}_{o}^{\prime}, \mathrm{q}_{o}\right)\right)$ as in Theorem 2.7, and $h_{o} \in[h]$ is considered as a degenerate metric on TM with $\operatorname{ker} h_{o}=\left\langle\mathrm{p}_{o}, \mathrm{q}_{o}\right\rangle$. Then any other compatible metric $g$ has locally the form

$$
\begin{equation*}
g=\sigma h_{o}+2 \mathrm{q}_{o}^{*} \vee \varpi=\sigma h_{o}+2 \mathrm{p}_{o}^{\mathrm{b}} \vee \varpi \quad \text { with } \mathrm{p}_{o}^{\mathrm{b}}=g_{o}\left(\mathrm{p}_{o}, \cdot\right)=\mathrm{q}_{o}^{*} \tag{2.13}
\end{equation*}
$$

for some positive function $s>0$ and a 1 -form $\varpi$ with $\varpi\left(\mathrm{p}_{o}\right) \neq 0$.
Proof. By Theorem 2.7, if $g$ is a compatible metric, then there is a triple $\left(h, \mathrm{p}_{o},\left(\mathcal{W}^{\prime}, \mathrm{q}\right)\right)$, with $h=\sigma h_{o} \in[h]$, such that $g$ has the form $g=\sigma \widetilde{h}_{o}+\mathrm{q}^{*} \vee \widetilde{\mathrm{p}}_{o}^{*}$, where:

- $\widetilde{h}_{o}$ is the extension of $h_{o}$ as a degenerate metric on $T M$ with $\operatorname{ker} \widetilde{h}_{o}=\langle\mathrm{p} o, \mathrm{q}\rangle$ and
$-\mathrm{q}^{*}$ and $\widetilde{\mathrm{p}}_{o}^{*}$ are the 1 -forms defined by

$$
\widetilde{\mathrm{p}}_{o}^{*}\left(\mathrm{p}_{o}\right)=1, \text { ker } \mathrm{p}_{o}^{*}=\mathcal{W}^{\prime}+\langle\mathrm{q}\rangle \quad \text { and } \quad \mathrm{q}^{*}(\mathrm{q})=1, \text { ker } \mathrm{q}^{*}=\mathcal{W}^{\prime}+\left\langle\mathrm{p}_{o}\right\rangle=\mathcal{W} .
$$

It follows that
(a) there exists a function $\lambda$ such that $\mathrm{q}^{*}=\lambda \mathrm{q}_{o}^{*}$;
(b) $\left\langle\mathrm{p}_{o}\right\rangle \subset \operatorname{ker}\left(\widetilde{h}_{o}-h_{o}\right)$ and hence $\widetilde{h}_{o}-h_{o}=2 \mathrm{q}_{o}^{*} \vee \varpi_{1}$ for some 1-form $\varpi_{1}$.

Therefore $g=\sigma h_{o}+2 \mathrm{q}_{o}^{*} \vee \varpi$ with $\varpi=\varpi_{1}+\lambda \widetilde{\mathrm{p}}_{o}^{*}$ and (2.13) holds.
Conversely, if $g$ has the form (2.13), then it is a Lorentzian metric for which $\mathrm{p}_{o}$ is a null vector field, $\mathrm{p}_{o}^{\perp}=\mathcal{W}$ and $\mathrm{p}_{o}$ is a shearfree vector field preserving the shearfree structure $\left(\mathcal{W},\left[h_{o}\right]\right)$. This means that $g$ is a compatible metric.

We conclude this section by giving a local explicit description of the compatible metrics for a given shearfree structure $(\mathcal{W},[h])$ in terms of some frame field aligned with the distributions $\mathcal{W}$ and $\mathcal{K}_{h} \subset \mathcal{W}$.

Let $g_{o}$ be a compatible metric for the shearfree structure ( $\left.\mathcal{W},[h]\right)$ and let $\left(h_{o}, \mathrm{p}_{o},\left(\mathcal{W}^{\prime}, \mathrm{q}_{o}\right)\right)$ be a triple, formed by a degenerate metric $h_{o} \in[h]$, a vector field $\mathrm{p}_{o} \in \mathcal{K}_{h}$ and a rigging $\left(\mathcal{W}_{o}^{\prime}, \mathrm{q}_{o}\right)$, which determines $g$ as in (2) of Theorem [2.7. Consider also a (local) frame field on $M$ of the form ( $\mathrm{p}_{o}, e_{1}, \ldots, e_{n-2}, \mathrm{q}_{o}$ ) where the vector fields $e_{i}$ are in the subdistribution $\mathcal{W}_{o}^{\prime} \subset \mathcal{W}$. We denote by ( $\mathrm{p}_{o}^{*}, e^{1}, \ldots, e^{n-2}, \mathrm{q}_{o}^{*}$ ) the dual coframe field. Any other rigging $\left(\mathcal{W}^{\prime}, q\right)$ for the shearfree structure $(\mathcal{W},[h])$ has the form

$$
\begin{equation*}
\mathrm{q}:=a \mathrm{q}_{o}+b \mathrm{p}_{o}+c^{i} e_{i}, \quad \mathcal{W}^{\prime}=\left\langle e_{i}+d_{i} \mathrm{p}_{o}\right\rangle \tag{2.14}
\end{equation*}
$$

where $a, b, c^{i}, d_{j}$ are smooth functions with $a \neq 0$ at all points. However, since by claim (2a) of Theorem 2.7, the subdistribution $\mathcal{W}^{\prime}$ can be fixed arbitrarily, with no loss of generality from now on we assume that $\mathcal{W}^{\prime}=\mathcal{W}_{o}^{\prime}$ and $d_{j} \equiv 0$.

The 1 -forms $\widetilde{\mathrm{p}}_{o}^{*}$ and $\mathrm{q}^{*}$ satisfying the conditions

$$
\begin{equation*}
\widetilde{\mathrm{p}}_{o}^{*}\left(\mathrm{p}_{o}\right)=1, \quad \operatorname{ker}_{\mathrm{p}}^{*}=\mathcal{W}_{o}^{\prime}+\langle\mathrm{q}\rangle \quad \text { and } \quad \mathrm{q}^{*}(\mathrm{q})=1, \operatorname{ker} \mathrm{q}^{*}=\mathcal{W}_{o}^{\prime}+\left\langle\mathrm{p}_{o}\right\rangle=\mathcal{W} . \tag{2.15}
\end{equation*}
$$

can be expressed in terms of the coframe field ( $\mathrm{p}_{o}^{*}, e^{i}, \mathrm{q}_{o}^{*}$ ) as

$$
\begin{equation*}
\widetilde{\mathrm{p}}_{o}^{*}=\mathrm{p}_{o}^{*}-\frac{b}{a} \mathrm{q}_{o}^{*}, \quad \mathrm{q}^{*}=\frac{1}{a} \mathrm{q}_{o}^{*} . \tag{2.16}
\end{equation*}
$$

However, for any $h=\sigma h_{o} \in[h]$, the corresponding degenerate extension to $T M$ with $\operatorname{ker} h=\left\langle\mathrm{p}_{o}, \mathrm{q}\right\rangle$ is equal to

$$
\begin{equation*}
h=\sigma h_{i j} e^{i} \vee e^{j}-2 \frac{\sigma c^{i} h_{i j}}{a} \mathrm{q}_{o}^{*} \vee e^{j}+\frac{\sigma h_{i j} c^{i} c^{j}}{a^{2}} \mathrm{q}_{o}^{*} \vee \mathrm{q}_{o}^{*} \quad h_{i j}:=h_{o}\left(e_{i}, e_{j}\right) . \tag{2.17}
\end{equation*}
$$

Since any compatible $g$ is as in (2.11) for some triples $\left(h=\sigma h_{o}, \mathrm{p}_{o},\left(\mathcal{W}^{\prime}, \mathrm{q}\right)\right)$, we conclude that in terms of the coframe field $\left(\mathrm{p}_{o}^{*}, e^{i}, \mathrm{q}_{o}^{*}\right)$ any compatible metric has the form

$$
\begin{align*}
& g=\sigma h_{i j} e^{i} \vee e^{j}+\mathrm{q}_{o}^{*} \vee\left(\frac{2}{a} \mathrm{p}_{o}^{*}-2 \frac{\sigma c^{i} h_{i j}}{a} e^{j}+\frac{\sigma h_{i j} c^{i} c^{j}}{a^{2}} \mathrm{q}_{o}^{*}-\frac{2 b}{a^{2}} \mathrm{q}_{o}^{*}\right)= \\
& \quad=\sigma\left(h_{i j} e^{i} \vee e^{j}+\mathrm{q}_{o}^{*} \vee\left(\alpha \mathrm{p}_{o}^{*}+\gamma^{i} h_{i j} e^{j}+\beta \mathrm{q}_{o}^{*}\right)\right),  \tag{2.18}\\
& \text { where } \alpha:=\frac{2}{a \sigma}, \quad \gamma^{i}:=-\frac{2 c^{i}}{a}, \quad \beta:=-\frac{2 b}{a^{2} \sigma}+\frac{h_{i j} c^{i} c^{j}}{a^{2}} . \tag{2.19}
\end{align*}
$$

This gives a parameterisation of all compatible metrics in terms of the ( $n+1$ )-tuple of functions ( $\sigma, \alpha, \beta, \gamma^{i}$ ), which are in turn determined by arbitrary functions $\sigma>0, a \neq$ $0, b, c^{i}$, as it is indicated in (2.19).

Conversely, any Lorentzian metric of the form (2.18) for some tuple ( $\sigma, \alpha, \beta, \gamma^{i}$ ) with $\sigma>0, \alpha \neq 0$, is a compatible metric for ( $\mathcal{W},[h]$ ), since it is associated with the triple $\left(h=\sigma h_{o}, \mathrm{p}_{o},\left(\mathcal{W}_{o}^{\prime}, \mathrm{q}\right)\right)$, in which the vector field q of the rigging $\left(\mathcal{W}_{o}^{\prime}, \mathrm{q}\right)$ is defined by

$$
\begin{align*}
& \mathrm{q}=a \mathrm{q}_{o}+b \mathrm{p}_{o}+c^{i} e_{i} \\
& \text { with } a:=\frac{2}{\alpha \sigma}, \quad b:=\frac{2}{\alpha \sigma}\left(-\beta+\frac{\gamma^{\ell} \gamma^{m} h_{\ell m}}{4}\right), \quad c^{i}:=-\frac{\gamma^{i}}{\alpha \sigma} . \tag{2.20}
\end{align*}
$$

2.6. Twisting shearfree structures. Let $(\mathcal{W},[h])$ be a shearfree structure on $M$. The degree at $x \in M$ is the integer $d_{x}(\mathcal{W}):=\left.\operatorname{dim} \operatorname{Ker} d \vartheta\right|_{\mathcal{W}_{x}}$ for some (hence, for any) defining 1 -form $\vartheta$ for $\mathcal{W}$. The shearfree structure is said to be of constant degree $d$ if the degree is equal to $d$ at all points.
Lemma 2.9. For any shearfree structure $(\mathcal{W},[h])$ with defining 1 -form $\vartheta$, the one dimensional distribution $\mathcal{K}_{h}$ is in $\operatorname{ker} d \vartheta_{\mathcal{W}}$. In particular, if $\operatorname{dim} M=2 m$, the minimal possible degree is 1 and in this case $\mathcal{K}_{h}=\left.\operatorname{ker} d \vartheta\right|_{\mathcal{w}}$. If $\operatorname{dim} M=2 m+1$, the minimal possible degree is 2 .

Proof. Since $\mathrm{p} \in \mathcal{K}_{h}$ preserves $\mathcal{W}$, we have $\lambda \theta=\mathcal{L}_{\mathrm{p}} \vartheta=d \vartheta(\mathrm{p}, \cdot)+d(\vartheta(\mathrm{p}))=d \vartheta(\mathrm{p}, \cdot)$. Hence $d \vartheta(\mathrm{p}, \mathcal{W})=0$.

Definition 2.10. A shearfree structure ( $\mathcal{W},[h]$ ) is called twisting if it has constant minimal degree.

## 3. Compatible shearfree metrics of Robinson-Trautman bundles

In this section we consider our main objects of interest, the regular shearfree pairs $(g, \mathrm{p})$ and the associated shearfree structures, which we call Robinson-Trautman structures. Our aim is to characterise these shearfree structures and the corresponding optical geometries.

### 3.1. Regular shearfree pairs and Robinson-Trautman structures.

Definition 3.1. A shearfree pair ( $g, \mathrm{p}$ ) on a manifold $M$ is called regular if the vector field p is complete and generates a 1-parameter group $A=e^{t \mathrm{p}}$ of diffeomorphisms (isomorphic to $\mathbb{R}$ or $S^{1}$ ) acting freely and properly on $M$.

It is known that in this case the orbit space $S=M / A$ is a smooth manifold and the quotient map $\pi: M \rightarrow S=M / A$ is a smooth principal $A$-bundle. The corresponding shearfree structure $\left(\mathcal{W}=\operatorname{ker}^{b},[h]=\left[g_{\mathcal{W}}\right]\right)$ is $A$-invariant and the kernel distribution $\mathcal{K}_{h}$ coincides with the vertical distribution $T^{\mathrm{v}} M \subset T M$ of the bundle. This motivates the following
Definition 3.2. A shearfree structure $(\mathcal{W},[h])$ on the total space of a principal $A$-bundle $\pi: M \rightarrow S, A=\mathbb{R}$ or $S^{1}$, is called Robinson-Trautman ( $R T$ ) structure if it is $A$-invariant and the kernel distribution is $\mathcal{K}_{h}=T^{\mathrm{v}} M$. A principal $A$-bundle $\pi: M \rightarrow S$ equipped with an RT-structure ( $\mathcal{W},[h]$ ) is called Robinson-Trautman (RT) bundle.

We remark that on an $\operatorname{RT}$ bundle $(\pi: M \rightarrow S,(\mathcal{W},[h]))$ the fundamental vector field $\mathrm{p}_{o}$ which corresponds to the element $1 \in \operatorname{Lie}(A)=\mathbb{R}$, that is the velocity vector field of the 1-dimensional Lie transformation group $A=\mathbb{R}$ or $S^{1}$, gives a canonical section of $\mathcal{K}_{h}=T^{\mathrm{v}} M$.
3.2. Global standard and distinguished shearfree pairs on RT bundles. The following proposition can be considered as a global version of the claim (ii) of Proposition 2.4 on the RT bundles with structure group $\mathbb{R}$.

Proposition 3.3. On any $R T$ bundle $(\pi: M \rightarrow S,(\mathcal{W},[h]))$ with structure group $A=\mathbb{R}$ there exists a global compatible shearfree pair ( $g, \mathrm{p}$ ), which is standard and distinguished, i.e. with $\nabla_{\mathrm{p}} \mathrm{p}=0$ and $\mathcal{L}_{\mathrm{p}} g=\mathrm{p}^{\mathrm{b}} \vee \eta$ for some globally defined 1 -form $\eta$.

Proof. Let $\left\{\mathcal{U}_{\alpha}\right\}$ be a trivialising cover of the base manifold $S$. Then for each $\mathcal{U}_{\alpha}$ we have that $\left.M\right|_{\mathfrak{u}_{\alpha}} \simeq \mathcal{U}_{\alpha} \times \mathbb{R}$ and we may consider a shearfree pair $\left(g_{o}, \mathrm{p}_{o}\right)$ in which $\mathrm{p}_{o}=\partial_{t}$ for an appropriate fiber coordinate $t$. By the proof of Proposition [2.4, if the $\mathcal{U}_{\alpha}$ are sufficiently small, for each of them there is a pair $\left(\sigma_{\alpha}, \tau_{\alpha}\right)$ of strictly positive functions that are solutions to the system (2.8) over $\left.M\right|_{u_{\alpha}}=U_{\alpha} \times \mathbb{R}$. Let $\left\{\chi_{\alpha}\right\}$ be a partition of unity which is subordinated to the open cover $\left\{\mathcal{U}_{\alpha}\right\}$ of $S$ and $\left\{\tilde{\chi}_{\alpha}=\pi^{*} \chi_{\alpha}\right\}$ the corresponding family of pulled-back functions on $M$. Notice that each $\widetilde{\chi}_{\alpha}$ is constant along the integral curves of the vector field $\mathrm{p}_{\alpha}:=\sigma_{\alpha} \mathrm{p}_{o}$. This implies that the pairs ( $\left.\widetilde{\chi}_{\alpha} \cdot \sigma_{\alpha}, \widetilde{\chi}_{\alpha} \cdot \tau_{\alpha}\right)$ are solutions to (2.8). It follows that $(\sigma, \tau):=\left(\sum_{\alpha} \chi_{\alpha} \sigma_{\alpha}, \sum_{\alpha} \chi_{\alpha} \tau_{\alpha}\right)$ is a global solution to the system (2.8) and determines a global standard and distinguished pair $\left(g=\sigma g_{o}, p=\tau p_{o}\right)$. $\square$

Remark 3.4. In Proposition 3.3 the assumption $A=\mathbb{R}$ is essential. In fact, by considering appropriate quotients of such an RT bundle, one can produce examples of RT bundles with structure group $A=S^{1}$ admitting no global standard or distinguished compatible shearfree pair.

### 3.3. RT structures and subconformal structures.

Definition 3.5. A sub-Riemannian (resp. subconformal) structure on a manifold $S$ is a codimension one distribution $\mathcal{D} \subset T S$ equipped with a Riemannian metric $g^{\mathcal{D}}$ (resp. a conformal metric $\left[g^{\mathcal{D}}\right]$ ) (⿶凵1).

The next proposition gives a fundamental relation between the RT structures and the subconformal structures.

Proposition 3.6. There is a natural one-to-one correspondence between the $R T$ structures $(\mathcal{W},[h])$ on the total space of a principal $A$-bundle $\pi: M \rightarrow S, A=\mathbb{R}$ or $S^{1}$, and the subconformal structures ( $\mathcal{D}, g^{\mathcal{D}}$ ) on the base manifold $S$.

[^1]Proof. Since the codimension one distribution $\mathcal{W} \subset M$ is $A$-invariant, it projects onto a codimension one distribution $\mathcal{D}:=\pi_{*}(\mathcal{W})$ on $S$. By a similar reason, the degenerate $A$-invariant conformal metric $[h]$ on $\mathcal{W}$ projects onto a conformal metric $\left[g^{\mathcal{D}}\right]=\pi_{*}([h])$ on $\mathcal{D}$. This associates a subconformal structure $\left(\mathcal{D},\left[g^{\mathcal{D}}\right]\right)$ on $S$ with any RT structure on $M$. Conversely, given a subconformal structure ( $\mathcal{D},\left[g^{\mathcal{D}}\right]$ ) on $S$. Then:
(a) the preimage $\mathcal{W}:=\pi_{*}^{-1}(\mathcal{D})$ is an $A$-invariant codimension one distribution on $M$ which contains $T^{\mathrm{v}} M$;
(b) the pull-back $[h]:=\left(\pi^{*}\right)\left[h^{\mathcal{D}}\right]$ is an $A$-invariant degenerate conformal metric on $\mathcal{W}$ with kernel $\mathcal{K}_{h}=T^{\mathrm{v}} M$.
In particular, $(\mathcal{W},[h])$ is an RT structure on $\pi: M \rightarrow S$.
3.4. A-invariant metrics on RT bundles. Let $\pi: M \rightarrow S$ be an RT bundle with structure group $A=\mathbb{R}$ or $S^{1}$. Denote by $(\mathcal{W},[h])$ and $\left(\mathcal{D}=\pi_{*}(\mathcal{W}),\left[g^{\mathcal{D}}\right]=\pi_{*}([h])\right)$ the corresponding RT structure and subconformal structure on $M$ and $S$, respectively. Let also $\mathrm{p}_{o} \in T^{\mathrm{v}} M$ be the fundamental vector of such principal bundle.

Consider a principal connection $\mathcal{H} \subset T M$ on $M$, that is an $A$-invariant (horizontal) distribution complementary to the vertical distribution $T^{\mathrm{v}} M \subset T M$. We recall that:
(a) any vector field $Y \in \mathfrak{X}(S)$ has a unique $A$-invariant horizontal lift $Y^{\mathrm{h}}$ in $\mathcal{H} \subset T M$ projecting onto $Y$;
(b) since the kernel subdistribution $\mathcal{K}_{h} \subset \mathcal{W}$ is equal to $\mathcal{K}_{h}=T^{\mathrm{v}} M$, the intersection $\mathcal{W}^{\prime}=\mathcal{W} \cap \mathcal{H}$ is an $A$-invariant subdistribution complementary to $\mathcal{K}_{h}$.
It follows that for any vector field $Z \in \mathfrak{X}(S)$ that is transversal to $\mathcal{D}$, the pair $\left(\mathcal{W}^{\prime}=\mathcal{W} \cap\right.$ $\mathcal{H}, \mathrm{q}=Z^{\mathrm{h}}$ ) is an $A$-invariant rigging for the RT structure $(\mathcal{W},[h])$. Hence, given a subRiemannian metric $g^{\mathcal{D}} \in\left[g^{\mathcal{D}}\right]$ on the distribution $\mathcal{D} \subset T S$, the triple $\left(h=\pi^{*}\left(g^{\mathcal{D}}\right), \mathrm{p}_{o}\right.$, $\left(\mathcal{W}^{\prime}, \mathrm{q}=Z^{\mathrm{h}}\right)$ ) is $A$-invariant and determines an $A$-invariant compatible metric $g$ by (1) of Theorem [2.7. This gives the following useful result.

Theorem 3.7. Let $\pi: M \rightarrow S$ be an $R T$ bundle with fundamental vector field $\mathrm{p}_{o}, R T$ structure ( $\mathcal{W},[h]$ ) on $M$ and subconformal structure ( $\mathcal{D},\left[g^{\mathcal{D}}\right]$ ) on $S$ as above. Any pair $\left(g^{\mathcal{D}}, Z\right)$, formed by a sub-Riemannian metric $g^{\mathcal{D}} \in\left[g^{\mathcal{D}}\right]$ and a $\mathcal{D}$-transversal vector field $Z \in \mathfrak{X}(S)$, determines the $A$-invariant compatible Lorentzian metric on $M$

$$
\begin{equation*}
g:=\pi^{*}\left(g^{\mathcal{D}}\right)+\vartheta \vee \mathrm{p}_{o}^{*} \quad \text { where } \tag{3.1}
\end{equation*}
$$

(a) the tensor $\pi^{*}\left(g^{\mathcal{D}}\right)$ is considered as a degenerate metric on TM with kernel $\left\langle\mathrm{p}_{o}, Z^{\mathrm{h}}\right\rangle$,
(b) $\vartheta$ is the defining 1 -form for the distribution $\mathcal{W}=\left\langle\mathrm{p}_{o}\right\rangle+\mathcal{W}^{\prime}$ such that $\vartheta\left(Z^{\mathrm{h}}\right)=1$;
(c) $\mathrm{p}_{o}^{*}$ is the $A$-invariant defining form of $\mathcal{H}=\mathcal{W}^{\prime}+\langle\mathrm{q}\rangle$ such that $\mathrm{p}_{o}^{*}\left(\mathrm{p}_{o}\right)=1$.

The shearfree pair $\left(g, \mathrm{p}_{o}\right)$ is standard and distinguished, actually $\mathrm{p}_{o}$ is Killing.

## 4. Twisting Robinson-Trautman bundles of Kähler-Sasaki type

Let $M$ be the total space of an even dimensional twisting RT bundle ( $\pi: M \rightarrow S,(\mathcal{W},[h])$ ) with structure group $A=\mathbb{R}, S^{1}$. As we will see, the assumption that the shearfree structure is twisting is equivalent to the hypothesis that the corresponding subconformal structure $\left(\mathcal{D}=\operatorname{ker} \vartheta,\left[h^{\mathcal{D}}\right]\right)$ on $S$ is of contact type. This subconformal structure canonically determines a strongly pseudo-convex almost CR structure and, if appropriate regularity conditions are satisfied, such CR structure makes $S$ the total space of an $A^{\prime}$-bundle $\pi^{S}: S \rightarrow M$, $A^{\prime}=\mathbb{R}$ or $S^{1}$, over a quantisable Kähler manifold $\left(N, J^{N}, g^{N}\right)$.

In this section we review the main definitions and certain basic facts on subconformal structures of contact type, Sasaki manifolds and quantisable Kähler manifolds. After that we establish in detail the exact relations between the twisting RT bundles and such geometric structures.

### 4.1. Subconformal structures of contact type and CR structures.

4.1.1. Sub-Riemannian and subconformal structures of contact type.

Definition 4.1. Let $S$ be an odd dimensional manifold.
(i) A codimension one distribution $\mathcal{D} \subset T S$ on $S$ is called contact if the 2 -form $\omega^{\theta}=$ $\left.d \theta\right|_{\mathcal{D}}$, where $\theta$ is a defining form for $\mathcal{D}$, is non-degenerate. The defining form $\theta$ is called contact form.
(ii) The Reeb vector field $Z=Z^{\theta}$ of a contact form $\theta$ is the unique vector field satisfying the conditions

$$
\begin{equation*}
Z\lrcorner \theta=1, \quad Z\lrcorner d \theta=0 . \tag{4.1}
\end{equation*}
$$

A Reeb vector field $Z=Z^{\theta}$ determines a direct sum decomposition $T S=\mathcal{D}+\langle Z\rangle$ and preserves $\theta$ (indeed, $\left.\left.\mathcal{L}_{Z} \theta=d(Z\lrcorner \theta\right)+Z\right\lrcorner d \theta=0$ ) and the contact distribution $\mathcal{D}=\operatorname{ker} \theta$. The conformal class $[\theta]$ of the contact forms of $\mathcal{D}$ is globally defined.

Definition 4.2. A sub-Riemannian structure ( $\mathcal{D}, h^{\mathcal{D}}$ ) (resp. a subconformal structure $\left(\mathcal{D},\left[h^{\mathcal{D}}\right]\right)$ ) is called of contact type if the underlying distribution $\mathcal{D}$ is contact.
4.1.2. Almost, partially integrable and integrable $C R$ structures.

Definition 4.3. Let $\mathcal{D} \subset T S$ be a contact structure on a manifold $S$ and $J$ a complex structure on $\mathcal{D}$ (that is, a field $J \in \Gamma(\operatorname{End}(\mathcal{D}))$ of endomorphisms of $\mathcal{D}$ with $\left.J^{2}=-\operatorname{Id}_{\mathcal{D}}\right)$. The pair $(\mathcal{D}, J)$ is called almost CR structure. Moreover:
(i) An almost CR structure $(\mathcal{D}=\operatorname{ker} \theta, J)$ is called partially integrable if the associate 2 -form $h^{\theta}=\omega(\cdot, J \cdot), \omega=\left.d \theta\right|_{\mathcal{D}}$, is symmetric. It is called Levi form and its conformal class $\left[h^{\theta}\right]$ is globally defined.
(ii) A partially integrable structure $(\mathcal{D}, J)$ is called strongly pseudoconvex if the Levi form $h^{\theta}$ is positive (or negative) definite.
(iii) An integrable $C R$ structure $(\mathcal{D}, J)$ is a partially integrable CR structure with identically vanishing Nijenhuis tensor $N_{J} \in \mathcal{D}^{*} \otimes \mathcal{D}^{*} \otimes \mathcal{D}$, defined by

$$
N_{J}(X, Y)=[X, Y]-[J X, J Y]+J([J X, Y]+[X, J Y]), \quad X, Y \in \mathcal{D} .
$$

### 4.1.3. Correspondence between subconformal and $C R$ structures.

Theorem $4.4([2)$. Let $(S, \mathcal{D}=\operatorname{ker} \theta)$ be a contact manifold with a globally defined contact form $\theta$. There exists a one-to-one correspondence between the following two sets.

- The pairs $((\mathcal{D}, J), B)$, in which $(\mathcal{D}, J)$ is a strongly pseudoconvex almost $C R$ structure with positive Levi form $h^{\theta}>0$, and $B \in \operatorname{End}(\mathcal{D})$ is a $h^{\theta}$-symmetric and positive definite field of endomorphisms;
- The sub-Riemannian structures of contact type ( $\mathcal{D}, h$ ).

Such a correspondence is given by

$$
\begin{equation*}
((\mathcal{D}, J), B) \quad \longleftrightarrow \quad\left(\mathcal{D}, h:=h^{\theta} \circ B\right) \tag{4.2}
\end{equation*}
$$

Proof. It suffices to show that the correspondence $((\mathcal{D}, J), B) \mapsto\left(\mathcal{D}, h:=h^{\theta} \circ B\right)$ has an inverse. To prove this, let $h$ be a sub-Riemannian metric on $\mathcal{D}$. Denote by $K$ the field of endomorphisms of $\mathcal{D}$ defined by $K:=h^{-1} \circ \omega, \omega:=\left.d \theta\right|_{\mathcal{D}}$. Note that $K$ is $h$-skew-symmetric and, consequently, that $-K^{2}=-K \circ K>0$ is $h$-symmetric and positive. Consider the field of endomorphisms $B:=\left(-K^{2}\right)^{-\frac{1}{2}}>0$, i.e. the inverse of the (unique) positive square root of $-K^{2}$. Then the field of endomorphisms $J:=B K$ is a complex structure on $\mathcal{D}$ as the following calculation shows

$$
\begin{equation*}
J^{2}=B K B K=\left(-K^{2}\right)^{-\frac{1}{2}} K\left(-K^{2}\right)^{-\frac{1}{2}} K=\left(\left(-K^{2}\right)^{-\frac{1}{2}}\right)^{2} K^{2}=-\operatorname{Id}_{\mathcal{D}} \tag{4.3}
\end{equation*}
$$

Moreover, since $\omega=h \circ K$ and $B$ is $h$-symmetric, $h^{\theta}:=\omega(\cdot, J \cdot)$ is symmetric:

$$
\begin{align*}
h^{\theta}(X, Y)=\omega(X, J Y)=\omega\left(J^{-1} Y, X\right) & =h\left(B^{-1} Y, X\right)=h\left(Y, B^{-1} X\right) \\
& =-\omega\left(Y, K^{-1} B^{-1} X\right)=\omega(Y, J X)=h^{\theta}(Y, X) . \tag{4.4}
\end{align*}
$$

This means that $(\mathcal{D}, J)$ is a partially integrable almost CR structure. Since $B$ is positive and $h^{\theta}=h \circ B^{-1}$, the almost CR structure $(\mathcal{D}, J)$ is strongly pseudoconvex. Hence, the correspondence $(\mathcal{D}, h) \mapsto((\mathcal{D}, J), B)$ is the desired inverse map.

By the above proof, the almost CR structure ( $\mathcal{D}, J$ ) depends only on the conformal class [ $h$ ] of the sub-Riemannian metric. Hence Theorem 4.4 implies the following corollary.

Corollary 4.5. On a contact manifold ( $S, \mathcal{D}$ ) there is a canonical one-to-one correspondence between the subconformal structures $(\mathcal{D},[h])$ and the pairs $((\mathcal{D}, J),[B])$ formed by a strongly pseudoconvex almost CR structure $(\mathcal{D}, J)$ and a conformal class $[B]$ of fields of $\mathcal{D}$-endomorphisms that are positive definite with respect to the positive Levi forms of $(\mathcal{D}, J)$.

### 4.2. Sasaki and Kähler manifolds associated with Robinson-Trautman bundles.

4.2.1. Regular Sasaki manifolds. Let $(\mathcal{D}=\operatorname{ker} \theta, J)$ be a strongly pseudoconvex integrable CR structure on a contact manifold $(S, \mathcal{D}=\operatorname{ker} \theta)$ equipped with a fixed contact form $\theta$. We recall that the Reeb vector field $Z=Z^{\theta}$ is transversal to $\mathcal{D}$ and preserves $\theta$ and $\mathcal{D}$.

Definition 4.6. The CR manifold $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ is called Sasaki if the Reeb vector field $Z=Z^{\theta}$ preserves $J$, i.e., $\mathcal{L}_{Z} J=0$. Such Sasaki manifold is called regular if $Z$ generates a one-parameter group $A=\exp (\mathbb{R} Z)$ of diffeomorphisms acting freely and properly on $S$.

The Sasaki metric of a Sasaki manifold $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ is the Riemannian metric

$$
\begin{equation*}
g^{\theta}:=\theta^{2}+\frac{1}{2} h^{\theta} \tag{4.5}
\end{equation*}
$$

where $h^{\theta}$ is considered as a degenerate metric on $S$ with kernel $\langle Z\rangle$. Note that the Reeb vector field $Z=Z^{\theta}$ preserves the Sasaki metric $g^{\theta}$.

In the literature a Sasaki manifold is often defined in a different way, namely as a Riemannian manifold equipped with a unit Killing vector field $Z$ satisfying appropriate conditions. The following proposition shows that the two definitions are equivalent.

Proposition 4.7. [1] Let $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ be a Sasaki manifold with Reeb vector field $Z=Z^{\theta}$ and Sasaki metric $g=g^{\theta}$. Then $Z$ is a unit Killing vector field for $g$ and the pair $(g, Z)$ satisfies the relations

$$
\begin{equation*}
\theta=g \circ Z \quad \text { and } \quad J=\left.g^{-1} \circ d \theta\right|_{\mathcal{D}} \tag{4.6}
\end{equation*}
$$

Conversely any Riemannian manifold $(S, g, Z)$ with a unit Killing vector field $Z$ such that
(1) $\theta=g \circ Z$ is a contact form;
(2) the pair $(\mathcal{D}=\operatorname{ker} \theta, J)$, with $J:=\left(g^{-1} \circ d \theta\right)_{\mathcal{D}}=\left.\nabla^{g} Z\right|_{\mathcal{D}}$, is an integrable pseudoconvex $C R$ structure,
determines the Sasaki manifold $(S, \mathcal{D}=\operatorname{ker} \theta, J)$.
4.2.2. Correspondence between Sasaki manifolds and quantisable Kähler manifolds.

Definition 4.8. A Kähler manifold $\left(N, J^{N}, g^{N}\right)$ is called quantisable if there exists a principal $A$-bundle $\pi: S \rightarrow N$, with $A=S^{1}$ or $\mathbb{R}$, with a connection 1-form $\theta: T S \rightarrow \mathbb{R}$, whose curvature $d \theta$ is equal to the Kähler form $\omega^{N}=g^{N}\left(\cdot, J^{N} \cdot\right)$, more precisely $d \theta=\pi^{*} \omega^{N}$.

For a fixed $A=S^{1}$ or $\mathbb{R}$, the Kähler manifold $\left(N, J^{N}, g^{N}\right)$ is called $A$-quantisable if it satisfies the above condition assuming the group is $A$. By [1, Prop. 1.2], $N$ is $S^{1}$ quantisable if and only if the (Čech) cohomology class $\left[\omega^{N}\right] \in H^{2}(M, \mathbb{R})$ is integral. It is $\mathbb{R}$-quantisable if and only if $\left[\omega^{N}\right]=0$.

In the next theorem we establish a natural correspondence between the regular Sasaki manifolds and the quantisable Kähler manifolds.

Theorem 4.9. [1] Let $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ be a regular Sasaki manifold with Reeb vector field $Z=Z^{\theta}$ and denote by $A=\left\{e^{t Z}\right\} \simeq \mathbb{R}$ or $S^{1}$, the group of diffeomorphisms generated by $Z$.

Then $\pi: S \rightarrow N=S / A$ is a principal $A$-bundle and $\theta$ is a connection 1-form for such a bundle. Moreover, the $A$-invariant complex structure $J$ and the 2 -form $\omega=\left.d \theta\right|_{\mathcal{D}}$ on $\mathcal{D}$ project onto an integrable complex structure $J^{N}$ and a symplectic form $\omega^{N}$ on $N$, respectively, such that $\left(N, J^{N}, g^{N}=\omega^{N} \circ J\right)$ is a quantisable Kähler manifold.

Conversely, if $\left(N, J^{N}, g^{N}\right)$ is a quantisable Kähler manifold and $\pi: S \rightarrow N$ is a principle $A$-bundle with $A=\mathbb{R}, S^{1}$ and a connection 1 -form $\theta$ such that $d \theta=\pi^{*} \omega^{N}$, then $(S, \mathcal{D}=$ $\operatorname{ker} \theta, J)$ is a regular Sasaki manifold.

Proof. We only need to prove the second claim. For this, we observe that the equality $d \theta=\pi^{*} \omega^{N}$ implies that $\theta$ is a contact form, whose associated Reeb vector field $Z=Z^{\theta}$ coincides with the fundamental vector field of the principal bundle. Let $J$ be the field of endomorphisms of $\mathcal{D}$ defined by $J_{u}=\left(\pi_{*} \mid \mathcal{D}_{u}\right)^{-1}\left(J_{\pi(u)}^{N}\right), u \in S$. Then $(\mathcal{D}, J)$ is a $Z$-invariant strongly pseudoconvex integrable CR structure and $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ is a regular Sasaki manifold.

Given a regular Sasaki manifold $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ the associated Kähler manifold is the quantisable Kähler manifold ( $N=S / A, J^{N}, g^{N}$ ) defined in the above theorem.
4.2.3. Twisting RT bundles associated with Sasaki and Kähler manifolds. The following lemma establishes a fundamental relation between twisting RT structures and the subconformal structures of contact type.

Lemma 4.10. Let $(\pi: M \rightarrow S,(\mathcal{W},[h]))$ be an $R T$ bundle of even dimension. The $R T$ structure $(\mathcal{W},[h])$ on the bundle $\pi: M \rightarrow S$ is twisting if and only if the corresponding subconformal structure $\left(\mathcal{D},\left[h^{\mathcal{D}}\right]\right.$ ) on the base manifold $S$ is of contact type.

Proof. Let us denote by $\theta$ a defining 1-form for the distribution $\mathcal{D}=\pi_{*}(\mathcal{W}) \subset T S$ of the subconformal structure on $S$ and by $\vartheta:=\pi^{*} \theta$ the corresponding defining 1 -form for $\mathcal{W}$ on $M$. The claim follows immediately from the fact that $\left.d \theta\right|_{\mathcal{D}}$ is non-degenerate if and only if $\left.\operatorname{dim} \operatorname{ker} d \vartheta\right|_{w_{x}}=1$ for any $x \in M$, i.e. $(\mathcal{W},[h])$ is twisting.

This lemma and the previous discussion about subconformal structures of contact type motivates the following

Definition 4.11. Let $(\pi: M \rightarrow S,(\mathcal{W},[h]))$ be a twisting RT bundle of even dimension $n=2 k+2$ and $\left(\mathcal{D},\left[h^{\mathcal{D}}\right]\right)$ the corresponding subconformal structure of contact type on $S$. Let also $((\mathcal{D}, J),[B])$ be the pair given by a strongly pseudoconvex almost CR structure and a conformal class $[B]$ of positive definite endomorphisms, which corresponds to ( $\mathcal{D},\left[h^{\mathcal{D}}\right]$ ) by Corollary 4.5. The RT bundle $(\pi: M \rightarrow S,(\mathcal{W},[h]))$ is called of Kähler-Sasaki type if:
(a) there exists a global contact form $\theta$ for the contact distribution $\mathcal{D}=\operatorname{ker} \theta \subset T S$;
(b) $[B]=\left[\mathrm{Id}_{\mathcal{D}}\right]$, i.e. $\left[h^{\mathcal{D}}\right]=\left[h^{\theta}\right]$ is the conformal class of the positive Levi forms of $\mathcal{D}$;
(c) $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ is a regular Sasaki manifold, i.e. it is principle $A$-bundle over a $2 k$-dimensional quantisable Kähler manifold ( $N, J^{N}, g^{N}$ ).
The compatible Lorentzian metrics of $(M,(\mathcal{W},[h]))$ are called of Kähler-Sasaki type.
4.3. Generalised electromagnetic plane waves. Let $(M, g)$ be an orientable Lorentzian manifold with canonical volume form $\operatorname{vol}_{g}$. For any $0 \leq p \leq n$, the usual Hodge-* operator $*: \Omega^{p}(M) \rightarrow \Omega^{n-p}(M)$ is defined by

$$
\begin{equation*}
\alpha \wedge * \beta=g(\alpha, \beta) \operatorname{vol}_{g}, \quad \alpha \in \Omega^{p}(M), \quad \beta \in \Omega^{n-p}(M) . \tag{4.7}
\end{equation*}
$$

As we mentioned in the introduction, if $\operatorname{dim} M=4$, an electromagnetic plane wave on $(M, g)$ is a decomposable 2 -form $F=\vartheta \wedge \mathrm{e}^{*}$, determined by a null 1 -form $\vartheta$ and a $g$ orthogonal space-like 1-form $\mathrm{e}^{*}$, which is harmonic, that is such that $d F=0=d(* F)$.

According to Trautman ( 30$]$ ), there exist two natural possible ways to generalise the definition of electromagnetic plane wave for the Lorentzian manifolds of higher dimension $n>4$. The first way is to consider only manifolds of even dimension $n=2 k$ and assume that a "generalised plane wave" is any harmonic $k$-form, which is locally of the form $F=$ $\vartheta \wedge e^{1} \wedge \cdots e^{k-1}$ for a 1-form $\vartheta$ with $\mathrm{p}:=g^{-1} \circ \vartheta$ null and $\left\{e^{1}, \cdots, e^{k-1}\right\}$ a set of $g$-orthonormal linearly independent space-like 1 -forms. A second alternatively way (which can be used for Lorentzian manifold of arbitrary dimensions) is just to use the definition used for the 4 -manifolds. However, it seems that in higher dimensions such second definition does no longer implies that the one-dimensional distribution $\mathcal{K}=\operatorname{ker} F \cap \operatorname{ker}(* F)$ is generated by a shearfree vector field p ([15]; see also [25] ). We think that either this or some other property should be included as a part of the definition. It is an issue that we leave to a future work. Here we follow just the first way of generalising the notion of electromagnetic plane wave and we adopt the following

Definition 4.12. A generalised electromagnetic plane wave on a Lorentzian manifold $(M, g)$ of dimension $\operatorname{dim} M=2 k$ is a harmonic $k$-form $F$, which is a wedge product $F=\vartheta \wedge \alpha$ of a null 1 -form $\vartheta$ and a $(k-1)$-form $\alpha$ with the property that any null vector of the distribution $\mathcal{W}_{F}=\operatorname{ker} \vartheta$ is also in $\operatorname{ker} \alpha$.

Proposition 4.13. Let $F=\vartheta \wedge \alpha$ be a generalised electromagnetic plane wave on an oriented Lorentzian $(n=2 k)$-manifold $(M, g)$ with $n \geq 4$ and $\mathcal{W}_{F}=\operatorname{ker} \vartheta$. Then:
(1) Also the dual $k$-form $* F$ is a generalised electromagnetic plane wave, hence of the form $* F=\vartheta \wedge \beta$;
(2) Let $\mathcal{K}_{F}=\left\langle\mathrm{p}_{o}\right\rangle$ where $\mathrm{p}_{o}=g^{-1} \circ \vartheta$. Then $\mathcal{K}_{F}=\operatorname{ker} F \cap \operatorname{ker}(* F)$ and

$$
\begin{equation*}
\mathcal{L}_{\mathrm{p}_{o}} F=0 \quad \text { and } \quad \mathcal{L}_{\mathrm{p}_{o}}(* F)=0 . \tag{4.8}
\end{equation*}
$$

(3) For any vector field $\mathrm{p} \in \mathcal{K}_{F}=\left\langle\mathrm{p}_{o}\right\rangle$

$$
\begin{align*}
& \mathcal{L}_{\mathrm{p}} \vartheta=-f \vartheta \quad \text { and thus } \quad \mathcal{L}_{\mathrm{p}} \mathcal{W}_{F} \subset \mathcal{W}_{F},  \tag{4.9}\\
& \mathcal{L}_{\mathrm{p}} \alpha=f \alpha+\vartheta \wedge \gamma, \quad \mathcal{L}_{\mathrm{p}} \beta=f \beta+\vartheta \wedge \gamma^{\prime} . \tag{4.10}
\end{align*}
$$

where $f$ is a function and $\gamma, \gamma^{\prime}$ are $(k-2)$-forms with $\mathcal{K}_{F} \subset \operatorname{ker} \gamma \cap \operatorname{ker} \gamma^{\prime}$.
Proof. (1) Since $\vartheta$ is null, there exist a null vector field $\mathrm{p}_{o}$ such that $\vartheta=\mathrm{p}_{o}^{\mathrm{b}}=g\left(\mathrm{p}_{o}, \cdot\right)$ and a null vector field $\mathrm{q}_{o}$ such that $\vartheta\left(\mathrm{q}_{o}\right)=g\left(\mathrm{p}_{o}, \mathrm{q}_{o}\right)=1$. Consider a (local) frame field $\left(\mathrm{p}_{o}, e_{1}, \ldots, e_{2 k-2}, \mathrm{q}_{o}\right)$ where $\left(e_{1}, \ldots, e_{2 k-2}\right)$ is a $g$-orthonormal basis for $\mathcal{E}=\left\langle\mathrm{p}_{o}, \mathrm{q}_{o}\right\rangle^{\perp}$ and let $\left(\vartheta^{\prime}, e^{1}, \ldots, e^{k-2}, \vartheta\right)$ be its dual coframe field. Note that $\mathcal{W}_{F}=\operatorname{ker} \vartheta=\left\langle\mathrm{p}_{o}, e_{1}, \ldots, e_{2 k-2}\right\rangle$
and ker $g_{\mathcal{W}_{F}}=\left\langle\mathrm{p}_{o}\right\rangle$. Moreover, since $\left\langle\mathrm{p}_{o}\right\rangle=\operatorname{ker} g \mathcal{W}_{F} \subset \operatorname{ker} \alpha$ and $\alpha$ is determined up to terms of the form $\vartheta \wedge \gamma$, we may assume that

$$
\alpha=\sum_{i_{1}<\ldots<i_{k-1}} \alpha_{i_{1} \ldots i_{k-1}} e^{i_{1}} \wedge \ldots e^{i_{k-1}}
$$

Hence $* F=\vartheta \wedge \beta$ with $\beta={ }_{\varepsilon} \alpha$, where we denote by $*_{\varepsilon}$ the $*$-Hodge operator of the Euclidean space $\mathcal{E}$. In other words, $* F=\vartheta \wedge \beta$ for a $(k-1)$-form $\beta$ such that $(\alpha \wedge \beta)_{\mathcal{E}}=\operatorname{vol}_{\varepsilon}$ and it is therefore a generalised plane wave.
(2) Since $d F=0=d(* F)$ and $\mathrm{p}_{o} \in \operatorname{ker} F \cap \operatorname{ker}(* F)$, the Lie derivatives along $\mathrm{p}_{o}$ of $F$ and $* F$ are trivial. Furthermore, $\mathrm{p} \in \operatorname{ker} F \cap \operatorname{ker}(* F)$ if and only if p$\lrcorner \vartheta=0$ and p$\lrcorner(\alpha \wedge \beta)=0$. Since $(\alpha \wedge \beta)_{\varepsilon}=\operatorname{vol}_{\varepsilon}$, this occurs if and only if $\mathrm{p} \in\left\langle\mathrm{p}_{o}\right\rangle=\mathcal{K}_{F}$.
(3) It is sufficent to prove the claim for $\mathrm{p}=\mathrm{p}_{o}$. From (2) we have that

$$
\begin{equation*}
0=\mathcal{L}_{\mathrm{p}_{o}} F=\mathcal{L}_{\mathrm{p}_{o}} \vartheta \wedge \alpha+\vartheta \wedge \mathcal{L}_{\mathrm{p}_{o}} \alpha, \quad 0=\mathcal{L}_{\mathrm{p}_{o}}(* F)=\mathcal{L}_{\mathrm{p}_{o}} \vartheta \wedge \beta+\vartheta \wedge \mathcal{L}_{\mathrm{p}_{o}} \beta \tag{4.11}
\end{equation*}
$$

We expand $d \vartheta$ as

$$
\begin{equation*}
d \vartheta=\vartheta \wedge e^{*}+f \vartheta \wedge \vartheta^{\prime}+\sum_{i \leq j} \nu_{i j} e^{i} \wedge \mathrm{e}^{j}+e^{\prime *} \wedge \vartheta^{\prime} \tag{4.12}
\end{equation*}
$$

for some $e^{*}, e^{\prime *} \in \mathcal{E}^{*}=\left\langle e^{1}, \ldots, e^{2 k-2}\right\rangle$. Hence

$$
\mathcal{L}_{\mathrm{p}_{o}} F=-f \vartheta \wedge \alpha-e^{\prime *} \wedge \alpha+\vartheta \wedge \widetilde{\alpha}, \quad \mathcal{L}_{\mathrm{p}_{o}}(* F)=-f \vartheta \wedge \beta-e^{* *} \wedge \beta+\vartheta \wedge \widetilde{\beta}
$$

where $\widetilde{\alpha}, \widetilde{\beta}$ are the $(k-1)$-forms such that

$$
\mathcal{L}_{\mathrm{p}_{o}} \alpha=\widetilde{\alpha} \bmod \left\langle\vartheta \wedge \gamma, \gamma \in \Lambda^{k-2} \mathcal{E}^{*}\right\rangle, \quad \mathcal{L}_{\mathrm{p}_{o}} \beta=\widetilde{\beta} \bmod \left\langle\vartheta \wedge \gamma, \gamma \in \Lambda^{k-2} \mathcal{E}^{*}\right\rangle
$$

Therefore the vanishing $\mathcal{L}_{\mathrm{p}_{o}} F=\mathcal{L}_{\mathrm{p}_{o}}(* F)=0$ implies that

$$
\mathcal{L}_{\mathrm{p}} \alpha=f \alpha \bmod \left\langle\vartheta \wedge \gamma, \gamma \in \Lambda^{k-2} \mathcal{E}^{*}\right\rangle, \quad \mathcal{L}_{\mathrm{p}} \beta=f \beta \quad \bmod \left\langle\vartheta \wedge \gamma, \gamma \in \Lambda^{k-2} \mathcal{E}^{*}\right\rangle
$$

and $e^{\prime *} \wedge \alpha=e^{\prime *} \wedge \beta=0$. Since $(\alpha \wedge \beta)_{\mathcal{E}}=\operatorname{vol}_{\mathcal{E}}$, this implies $e^{\prime *}=0$ and $\mathcal{L}_{\mathrm{p}_{o}} \vartheta=-f \vartheta$.
Definition 4.14. Let $(M, g)$ be a Lorentzian $2 k$-manifold. A flag structure on $M$ is a pair $(\mathcal{K}, \mathcal{W}=\operatorname{ker} \vartheta)$, determined by a null 1 -form $\vartheta$ and a one-dimensional distribution $\mathcal{K}=\operatorname{ker} g_{\mathcal{W}}$. Any generalised electromagnetic plane wave $F=\vartheta \wedge \alpha$ on $(M, g)$ determines the flag structure

$$
\left(\mathcal{K}_{F}:=\operatorname{ker} F \cap \operatorname{ker}(* F)=\operatorname{ker} g_{\mathcal{W}_{F}}, \mathcal{W}_{F}:=\operatorname{ker} \vartheta\right)
$$

which we call the flag structure of $F$.
Proposition 4.13 implies the following
Corollary 4.15. Let $(M, g)$ be a manifold of $\operatorname{dim} M=2 k$, equipped with a flag structue $\left(\mathcal{K}=\operatorname{ker} g_{\mathcal{W}}, \mathcal{W}=\operatorname{ker} \vartheta\right)$. A necessary condition for the existence of a generalised plane wave $F$ having $(\mathcal{K}, \mathcal{W})$ as its flag structure is the existence of two ( $k-1$ )-forms $\alpha, \beta$ satisfying the following conditions:
(a) $\mathcal{K} \subset \operatorname{ker} \alpha_{\mathcal{W}} \cap \operatorname{ker} \beta_{\mathcal{W}}$;
(b) $\vartheta \wedge \beta=*(\vartheta \wedge \alpha)$;
(c) any vector field $\mathrm{p} \in \mathcal{K}$ preserves $\mathcal{W}$ and there is a function $f$ such that

$$
\mathcal{L}_{\mathrm{p}} \alpha=f \alpha+\vartheta \wedge \gamma, \quad \mathcal{L}_{\mathrm{p}} \beta=f \beta+\vartheta \wedge \gamma^{\prime}
$$

for some $\gamma, \gamma^{\prime}$ such that $\operatorname{ker} g_{\mathcal{W}_{F}} \subset \operatorname{ker} \gamma \cap \operatorname{ker} \gamma^{\prime}$.
Remark 4.16. If the flag structure $(\mathcal{K}, \mathcal{W}=\operatorname{ker} \vartheta)$ is determined by a shearfree structure, then the necessary conditions of Corollary 4.15 are satisfied. If in addition $\operatorname{dim} M=4$, such necessary conditions are equivalent to say that $(\mathcal{K}, \mathcal{W})$ is the pair determined by a shearfree structure.

### 4.4. The Robinson Theorem for Lorentzian manifolds of Kähler-Sasaki type.

Theorem 4.17. Let $(M, g)$ be a $(n=2 k)$-dimensional Lorentzian manifold of KählerSasaki type with associated shearfree structure $\left(\mathcal{W},\left[h=g_{\mathcal{W}}\right]\right)$. Then locally there exists a non-trivial generalised electromagnetic plane wave $F$ with flag structure $\left(\mathcal{K}_{h}=\operatorname{ker} h, \mathcal{W}\right)$.

Proof. Let $\pi: M \rightarrow S$ and $\pi^{S}: S \rightarrow N$ be the principal bundles over the regular Sasaki manifold $(S=M / A, \mathcal{D}=\operatorname{ker} \theta, J)$ and the quantisable Kähler manifold $\left(N=S / A^{\prime}, J, g^{N}\right)$. Let also ( $z^{1}, \cdots z^{k-1}$ ) be local holomorphic coordinates on $N$ and denote by $\varphi=\varphi(z, \bar{z})$ and $\omega=i \partial \bar{\partial} \varphi$ a potential and the Kähler form of $N$. The connection 1-form $\theta=-\frac{1}{2} d u-\frac{1}{2} d^{c} \varphi=$ $-\frac{1}{2} d u-\frac{i}{2}(\bar{\partial} \varphi-\partial \varphi)$ has curvature $d \theta=i \partial \bar{\partial} \varphi=\omega$.

Since all arguments are local, we may assume that $M$ is a trivial bundle $\pi: M=A \times S \rightarrow$ $S$ with $A=\mathbb{R}$ or $S^{1}$. We denote by $t$ its fiber coordinate. Any admissible metric on $M$ has locally the form

$$
g=g^{N}+\vartheta \vee \eta
$$

where $g^{N}$ and $\vartheta$ are the pull-backs of the metric of $N$ and of the 1 -form $\theta$ of $S$, respectively, and $\eta \in \Omega^{1}(M)$ is a 1-form which can be locally written as $\eta=\alpha d t+\gamma^{i} \varphi_{i j} d z^{j}+\overline{\gamma^{i} \varphi_{i j} d z j}+\beta \vartheta$ for some real functions $\alpha, \beta$ and complex functions $\gamma^{i}$.

Let $\mathcal{F}$ and $F$ be the complex $k$-form $\mathcal{F}=\vartheta \wedge d z^{1} \wedge \cdots \wedge d z^{k-1}$ and the real $k$-form

$$
F=\operatorname{Re\mathcal {F}}=\vartheta \wedge\left(d z^{1} \wedge \cdots \wedge d z^{k-1}+\overline{d z^{1} \wedge \cdots \wedge d z^{k-1}}\right),
$$

respectively. By Lemma 4.18 below, $F$ is harmonic and thus it is a generalised electromagnetic plane wave.

Lemma 4.18. The complex form $\mathcal{F}$ is closed and coclosed.
Proof. For the closedness, just observe that $d \mathcal{F}=d\left(\vartheta \wedge d z^{1} \wedge \cdots \wedge d z^{k-1}\right)=\omega \wedge d z^{1} \cdots \wedge$ $d z^{k-1}=0$. The co-closedness follows from the fact that $* \mathcal{F}= \pm i^{k-1} \mathcal{F}$ (the sign depending on the orientation) since at each point $x \in M$ the space $\left.\left\langle d \theta, d z^{1}, \ldots, d z^{k-1}\right\rangle\right|_{x}$ is an isotropic subspace of $\left(T_{x}^{\mathbb{C}} M, g_{x}\right)$.

The proof has the following consequence.

Corollary 4.19. Assume that the Lorentzian manifold $(M, g)$ of Kähler-Sasaki type is globally trivial $M=\mathbb{R} \times S$ and that the corresponding Kähler manifold $\left(N, J^{N}, g^{N}\right)$ has a global holomorphic volume form. Then there exists on $M$ a nowhere vanishing globally defined generalised plane wave.

## 5. Einstein metrics on Robinson-Trautman bundles

Throughout this section $\pi: M=S \times \mathbb{R} \rightarrow S$ is a trivial $n$-dimensional principal bundle with structure group $A=\mathbb{R}$ over a regular Sasaki manifold $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ fibering over a quantisable Kähler manifold ( $N=S / A^{\prime}, J, g^{N}$ ) with structure group $A^{\prime}=\mathbb{R}$ or $S^{1}$. The assumption made here that the $\mathbb{R}$-bundle $\pi: M \rightarrow S$ is trivial is mostly chosen for the sake of simplicity. In fact, most parts of the following discussion remain valid under the weaker hypothesis that such a bundle is equipped with a flat connection.

The following notation is used.
$-\vartheta:=\pi^{*}(\theta)$ is the pull back of the contact form of $S$ and $\mathcal{W}=\operatorname{ker} \vartheta$ is the corresponding kernel distribution on $M$;

- $\left[h_{o}\right]$ is the conformal class of the degenerate metric $h_{o}=\left(\pi \circ \pi^{S}\right)^{*} g_{o}$ on $\mathcal{W}$;
- $\mathcal{H}_{o}=T S \subset T M$ is the standard flat connection of the trivial $\mathbb{R}$-bundle $M=S \times \mathbb{R}$;
- $\mathrm{p}_{o}=\frac{\partial}{\partial t} \in \mathfrak{X}(M)$ and $\mathrm{q}_{o}^{S} \in \mathfrak{X}(S)$ are the fundamental vector fields of the principal bundles $\pi: M \rightarrow S$ and $\pi^{S}: S \rightarrow N$, respectively;
- $\mathrm{q}_{o} \in \mathfrak{X}(M)$ is the horizontal lift of $\mathrm{q}_{o}^{S}$ on $M$ with respect to the flat connection;
- for any vector field $X$ on $N$ we denote by:
. $X^{(S)}:=X^{\mathrm{h}}$ the horizontal lift of $X$ in $\mathcal{D}=\operatorname{ker} \theta \subset T S$;
- $\widehat{X}=X^{(S)} \mathrm{h}$ the $\mathbb{R}$-invariant horizontal lift of $X^{(S)}$ in $\mathcal{H}_{o} \subset T M$.

Note that $\left(\pi: M=S \times \mathbb{R} \rightarrow S,\left(\mathcal{W},\left[h_{o}\right]\right)\right)$ is an RT bundle of Kähler-Sasaki type, the kernel distribution is $\mathcal{K}_{h}:=\operatorname{ker} h_{o}=\left\langle\mathrm{p}_{o}\right\rangle$ and the pair $\left(\mathcal{W}_{o}^{\prime}=\mathcal{W} \cap \mathcal{H}_{o}, \mathrm{q}_{o}\right)$ is a rigging for the shearfree structure ( $\mathcal{W},\left[h_{o}\right]$ ). Moreover,
(1) $\mathcal{W}_{o}^{\prime} \subset T M$ is the horizontal lift (with respect to the flat connection) of $\mathcal{D}=\operatorname{ker} \theta \subset T S$;
(2) $\mathrm{q}_{o}^{S}$ is the Reeb vector field of the contact form $\theta$;
(3) $\left.d \theta\right|_{\mathcal{D}}=\left(\pi^{S}\right)^{*}(\omega)$ where $\omega:=g_{o}(\cdot, J \cdot)$ is the Kähler form of $\left(N, J, g_{o}\right)$;
(4) for any pair of vector fields $X, Y$ on $N$, the corresponding horizontal lifts $\widehat{X}, \widehat{Y} \in \mathfrak{X}(M)$ satisfy the relations

$$
\begin{equation*}
[\widehat{X}, \widehat{Y}]=\widehat{[X, Y]}-g_{o}(X, J Y) \mathrm{q}_{o}, \quad\left[\widehat{X}, \mathrm{p}_{o}\right]=\left[\widehat{X}, \mathrm{q}_{o}\right]=\left[\mathrm{p}_{o}, \mathrm{q}_{o}\right]=0 . \tag{5.1}
\end{equation*}
$$

We conclude this section by observing that if $\left(E_{i}\right)$ is a (local) frame field for the Kähler manifold ( $N, J, g_{o}$ ), the corresponding horizontal lifts $\widehat{E}_{i}$ on $M$ form a frame field for $\mathcal{W}_{o}^{\prime}$ and the tuple of vector fields ( $\mathrm{p}_{o}, \widehat{E}_{i}, \mathrm{q}$ ) is a (local) frame field on $M$. We denote by $\left(\mathrm{p}_{o}^{*}, \widehat{E}^{i}, \mathrm{q}_{o}^{*}=\vartheta\right)$ the corresponding dual frame field.

In this section we determine a new family of Lorentzian Einstein metrics on such RT bundle belonging to the following special class of compatible metrics.
5.1. Firmly compatible metrics. By Theorem 2.7, any compatible metric on $(M=$ $S \times \mathbb{R}, \mathcal{W},[h])$ is locally determined by a triple $\left(h, \mathrm{p},\left(\mathcal{W}^{\prime}, \mathrm{q}\right)\right)$, given by a degenerate metric $h=\sigma h_{o} \in\left[h_{o}\right]$, a vector field $\mathrm{p} \in \mathcal{K}_{h}$ and a rigging $\left(\mathcal{W}^{\prime}, \mathrm{q}\right)$. With no loss of generality, we assume that $\mathrm{p}=\mathrm{p}_{o}=\frac{\partial}{\partial t}$ and the complementary subdistribution is $\mathcal{W}^{\prime}=\mathcal{W}_{o}^{\prime}$.

We will consider only globally defined compatible metrics, associated with triples ( $h, \mathrm{p}$, $\left.\left(\mathcal{W}^{\prime}, q\right)\right)$ where $q$ is a global vector field of the form

$$
\begin{equation*}
\mathrm{q}:=a \mathrm{q}_{o}+b \mathrm{p}_{o}+E, \quad a \neq 0 \tag{5.2}
\end{equation*}
$$

for some (global) functions $a, b$ and vector field $E \in \mathcal{W}_{o}^{\prime}$. The metrics that are associated with a vector field q , for which the coefficient $a$ is constant, are called strongly compatible. Our main results deal with the following even more restricted class of compatible metrics.

Definition 5.1. A compatible Lorentzian metric $g$ on $M$ is called firmly compatible if it is determined by a triple $\left(h, \mathrm{p},\left(\mathcal{W}^{\prime}, \mathrm{q}\right)\right)$ as above, in which the coefficient $a$ of q is constant and the vector field $E$ is zero.

Let $g$ be a firmly compatible metric determined by a triple $\left(h=\sigma h_{o}, \mathrm{p}_{o},\left(\mathcal{W}_{o}^{\prime}, \mathrm{q}\right)\right)$. Let also $\left(E_{i}\right)$ be a local frame field of the Kähler manifold $\left(N, J, g_{o}\right)$ and let ( $\left.\mathrm{p}_{o}, \widehat{E}_{i}, \mathrm{q}\right)$, $\left(\mathrm{p}_{o}^{*}, \widehat{E}^{i}, \mathrm{q}_{o}^{*}=\vartheta\right)$ be the corresponding frame and dual coframe fields on $M$. Then by (2.18) any firmly compatible metric on $M$ has the form

$$
\begin{align*}
& g=\sigma h_{i j} \widehat{E}^{i} \vee \widehat{E}^{j}+\vartheta \vee\left(\frac{2}{a} \mathrm{p}_{o}^{*}-\frac{2 b}{a^{2}} \vartheta\right)=\sigma\left(h_{i j} \widehat{E}^{i} \vee \widehat{E}^{j}+\vartheta \vee\left(\alpha \mathrm{p}_{o}^{*}+\beta \vartheta\right)\right) \\
& \text { where } \alpha:=\frac{2}{a \sigma}, \quad \beta:=-\frac{2 b}{a^{2} \sigma} . \tag{5.3}
\end{align*}
$$

Since $a$ is constant and any homothetic rescaling of the metric $g_{o}$ on $N$ corresponds to an (inverse) homothetic rescaling of the Reeb vector field $q_{o}$ on $S$, with no loss of generality we may assume that $a=2$ and hence, by (5.3), that

$$
\begin{equation*}
\alpha=\frac{1}{\sigma} \tag{5.4}
\end{equation*}
$$

and
$g=\sigma h_{i j} \widehat{E}^{i} \vee \widehat{E}^{j}+\vartheta \vee\left(\mathrm{p}_{o}^{*}+\widetilde{\beta} \vartheta\right)=\sigma\left(\pi^{S} \circ \pi\right)^{*}\left(g_{o}\right)+\vartheta \vee\left(\mathrm{p}_{o}^{*}+\widetilde{\beta} \vartheta\right) \quad$ where $\widetilde{\beta}:=\sigma \beta=-\frac{b}{2}$.
5.2. Einstein metrics of Taub-NUT type. In the next theorem we describe the family of Einstein metrics we advertised in the Introduction.

Theorem 5.2. Let $\pi: M=S \times \mathbb{R} \rightarrow S$ and $\pi^{S}: S \rightarrow N$ be as above and assume that the quantisable Kähler manifold $\left(N, J, g_{o}\right)$ is Einstein (2) with Einstein constant $\Lambda_{o}$.

[^2]Furthermore, for any triple of real numbers $(\Lambda, B, C)$ with $C>0$, let $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ and $\widetilde{\beta}:(0,+\infty) \rightarrow \mathbb{R}$ be the functions defined by

$$
\begin{align*}
\sigma(t) & :=\frac{1}{16 C} t^{2}+C  \tag{5.6}\\
\widetilde{\beta}(t) & :=\frac{t}{\left(t^{2}+16 C^{2}\right)^{\frac{n}{2}-1}}\left(B-\int_{1}^{t} \frac{\left(16 C^{2}+s^{2}\right)^{\frac{n}{2}-1}\left(16 C \Lambda_{0}-\Lambda\left(16 C^{2}+s^{2}\right)\right)}{4 s^{2}} d s\right) \tag{5.7}
\end{align*}
$$

Then $\widetilde{\beta}$ is a rational function admitting a unique smooth extension over $\mathbb{R}$ and the corresponding firmly compatible metric $g$ on $\left(M=S \times \mathbb{R},\left(\mathcal{W},\left[h_{o}\right]\right)\right)$

$$
\begin{equation*}
g=\sigma\left(\pi^{S} \circ \pi\right)^{*}\left(g_{o}\right)+\vartheta_{o} \vee\left(\mathrm{p}_{o}^{*}+\widetilde{\beta} \vartheta_{o}\right) \tag{5.8}
\end{equation*}
$$

is Einstein with Einstein constant $\Lambda$.
Conversely, if $g$ is a metric on $M=S \times \mathbb{R}$, which is
(a) firmly compatible, hence of the form (5.8) for some functions $\sigma, \widetilde{\beta}=\sigma \beta$ and
(b) with function $\sigma, \widetilde{\beta}$ depending just on the coordinate $t \in \mathbb{R}$,
then $g$ is Einstein with Einstein constant $\Lambda$ if and only if $\sigma$ and $\widetilde{\beta}$ are the functions defined in (5.6) and (5.7) for some choice of the constants $B$ and $C>0$.

Proof. Consider a (local) frame field $\left(\mathrm{p}_{o}, \widehat{E}_{i}, \mathrm{q}\right)$ and the dual coframe field $\left(\mathrm{p}_{o}^{*}, \widehat{E}^{i}, \mathrm{q}_{o}^{*}=\vartheta\right)$ on $M$ as described in $\$ 5.1$. We set

$$
g_{i j}:=g_{o}\left(E_{i}, E_{j}\right), \quad \omega_{i j}:=g_{o}\left(E_{i}, J E_{j}\right), \quad J_{i}^{j}=g^{j k} \omega_{k i}, \quad\left[E_{i}, E_{j}\right]=c_{i j}^{k} E_{k}, \quad \widetilde{\beta}:=\beta \sigma
$$

According to this notation, we have $J\left(E_{j}\right)=J_{j}^{\ell} E_{\ell}$. As we mentioned above, in terms of the above coframe field, any firmly compatible metric with $a \equiv 2$ has the form (5.5). A tedious (but straightforward) computation based on Koszul's formula (see §Appendix A for details) shows that the Christoffel symbols of the Levi-Civita connection in this frame field, i.e. the functions $\Gamma_{A B}^{C}$ defined by

$$
\begin{aligned}
& \nabla_{E_{i}} E_{j}=\boldsymbol{\Gamma}_{i j}^{k} E_{k}+\boldsymbol{\Gamma}_{i j}^{\mathrm{p}_{o}} \mathrm{p}_{o}+\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}} \mathrm{q}_{o}, \quad \quad \nabla_{E_{i}} \mathrm{p}_{o}=\boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{k} E_{k}+\boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{p}_{o} \mathrm{p}_{o}}+\boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{q}_{o}} \mathrm{q}_{o}, \\
& \nabla_{E_{i}} \mathrm{q}_{o}=\boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{k} E_{k}+\boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{\mathrm{p}_{o}} \mathrm{p}_{o}+\boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{\mathrm{q}_{o}} \mathrm{q}_{o}, \quad \quad \nabla_{\mathrm{p}_{o}} E_{j}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} j}{ }^{k} E_{k}+\boldsymbol{\Gamma}_{\mathrm{p}_{o} j}{ }_{\mathrm{p}} \mathrm{p}_{o} \mathrm{p}_{o}+\boldsymbol{\Gamma}_{\mathrm{p}_{o} j}{ }^{\mathrm{q}_{o}} \mathrm{q}_{o},
\end{aligned}
$$

$$
\begin{aligned}
& \nabla_{\mathrm{q}_{o}} E_{j}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} j}{ }^{k} E_{k}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} j}{ }^{\mathrm{p}_{o}} \mathrm{p}_{o}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} j}{ }^{\mathrm{q}_{o}} \mathrm{q}_{o}, \quad \nabla_{\mathrm{q}_{o}} \mathrm{p}_{o}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}{ }^{k} E_{k}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{p}_{o}}{ }^{\mathrm{p}_{o}} \mathrm{p}_{o}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}{ }^{\mathrm{q}_{o}} \mathrm{q}_{o}, \\
& \nabla_{\mathrm{q}_{o}} \mathrm{q}_{o}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}{ }^{k} E_{k}+\boldsymbol{\Gamma}_{\mathrm{q}_{o}}{ }_{\mathrm{q}_{o}} \mathrm{p}_{o} \mathrm{p}_{o}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}{ }^{\mathrm{q}_{o}} \mathrm{q}_{o},
\end{aligned}
$$

are equal to

$$
\begin{align*}
& \boldsymbol{\Gamma}_{i j}^{m}=g^{m k} g_{o}\left(\nabla_{E_{i}}^{o} E_{j}, E_{k}\right)+\frac{1}{2 \sigma} \widehat{E}_{i}(\sigma) \delta_{j}^{m}+\frac{1}{2 \sigma} \widehat{E}_{j}(\sigma) \delta_{i}^{m}-\frac{1}{2 \sigma} g_{i j} g^{m k} \widehat{E}_{k}(\sigma), \\
& \boldsymbol{\Gamma}_{i j}^{\mathrm{p}_{o}}=g_{i j}\left(-\mathrm{q}_{o}(\sigma)+2 \widetilde{\beta} \mathrm{p}_{o}(\sigma)\right), \quad \boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}}=-\frac{\omega_{i j}}{2}-g_{i j} \mathrm{p}_{o}(\sigma),  \tag{5.9}\\
& \boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} i}^{m}=-\frac{J_{i}^{m}}{4 \sigma}+\mathrm{p}_{o}(\sigma) \frac{\delta_{i}^{m}}{2 \sigma}, \quad \boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} i}{ }^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{q}_{o}}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} i}{ }^{\mathrm{q}_{o}}=0,  \tag{5.10}\\
& \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} i}^{m}=-\widetilde{\beta} \frac{J_{i}^{m}}{2 \sigma}+\frac{\mathrm{q}_{o}(\sigma) \delta_{i}^{m}}{2 \sigma}, \quad \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} i}^{\mathrm{p}_{o}}=\widehat{E}_{i}(\widetilde{\beta}),  \tag{5.11}\\
& \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{p}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{o}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{P}_{o}}^{\mathrm{q}_{o}}=0, \quad \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{\mathrm{q}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o}}{ }^{\mathrm{q}_{o}}=0,  \tag{5.12}\\
& \boldsymbol{\Gamma}_{\mathrm{P}_{o} \mathrm{q}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}^{m}=0, \quad \quad \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{q}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{p}_{o}}^{\mathrm{p}_{o}}=\mathrm{p}_{o}(\widetilde{\beta}), \quad \quad \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{q}_{o}}^{\mathrm{q}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}^{\mathrm{q}_{o}}=0,  \tag{5.13}\\
& \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{m}=-\frac{g^{m k}}{2 \sigma} \widehat{E}_{k}(\widetilde{\beta}), \quad \quad \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{\mathrm{q}_{o}}=\mathrm{q}_{o}(\widetilde{\beta})+2 \widetilde{\beta} \mathrm{p}_{o}(\widetilde{\beta}), \quad \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{\mathrm{q}_{o}}=-\mathrm{p}_{o}(\widetilde{\beta}) . \tag{5.14}
\end{align*}
$$

Using these expressions, we may directly compute the components $\mathrm{R}_{A B C}{ }^{D}$ of the Riemann curvature tensor $\left.{ }^{(3)}\right)$ in the frame $\left(X_{A}\right)=\left(\mathrm{p}_{o}, \widehat{E}_{1}, \ldots, \widehat{E}_{n-2}, \mathrm{q}_{o}\right)$. Note that if $X_{A}, X_{B}$ are commuting vector fields of the frame, then the corresponding components $\mathrm{R}_{A B C}{ }^{D}$ are given by the formula

$$
\mathrm{R}_{A B C}{ }^{D}=X_{A}\left(\boldsymbol{\Gamma}_{B C}{ }_{C}^{D}\right)-X_{B}\left(\boldsymbol{\Gamma}_{A C}{ }_{C}^{D}\right)-\boldsymbol{\Gamma}_{A C}{ }_{C}^{F} \boldsymbol{\Gamma}_{B F}^{D}+\boldsymbol{\Gamma}_{B C}{ }_{C}^{F} \boldsymbol{\Gamma}_{A F}^{D} .
$$

By (5.1), the only non-commuting pairs in the frame are those with $X_{A}=\widehat{E}_{i}$ and $X_{B}=\widehat{E}_{j}$. The corresponding components $\mathrm{R}_{i j C}{ }^{D}$ are given by

$$
\mathrm{R}_{i j C}{ }^{D}=\widehat{E}_{i}\left(\boldsymbol{\Gamma}_{j C}^{D}\right)-\widehat{E}_{j}\left(\boldsymbol{\Gamma}_{i C}^{D}\right)-\boldsymbol{\Gamma}_{i C}^{F} \boldsymbol{\Gamma}_{j F}^{D}+\boldsymbol{\Gamma}_{j C}^{F} \boldsymbol{\Gamma}_{i F}^{D}-c_{i j}^{k} \boldsymbol{\Gamma}_{k C}^{D}+\omega_{i j} \boldsymbol{\Gamma}_{\mathrm{q}_{o}}{ }_{C}^{D} .
$$

Using these two expressions, we can determine all components $\operatorname{Ric}_{A B}=\mathrm{R}_{D A B}{ }^{D}$ of the Ricci tensor and write down the Einstein equations $\operatorname{Ric}_{A B}=\Lambda g_{A B}$. We list these equations below. In those expressions, the terms that are struck out are those which are immediately seen to be 0 on the base of the above expressions for the Christoffel symbols $\boldsymbol{\Gamma}_{A B}{ }^{C}$. We also use the shorthand notation

$$
\begin{equation*}
\widehat{\operatorname{Ric}}_{i j}:=\widehat{E}_{m}\left(\boldsymbol{\Gamma}_{i j}^{m}\right)-\widehat{E}_{i}\left(\boldsymbol{\Gamma}_{m j}^{m}\right)-\boldsymbol{\Gamma}_{m j}^{\ell} \boldsymbol{\Gamma}_{i \ell}^{m}+\boldsymbol{\Gamma}_{i j}^{\ell} \boldsymbol{\Gamma}_{m \ell}^{m}-c_{m i}^{r} \boldsymbol{\Gamma}_{r j}^{m} . \tag{5.15}
\end{equation*}
$$

Under the ansatz (5.23), $\widehat{\operatorname{Ric}}_{i j}$ is equal to the pull-back on $M$ of the Ricci tensor $R_{i j}$ of the base manifold ( $N, g_{o}$ ) (see the observations after (5.23) below). Using this notation, the

[^3]Einstein equations for a metric of the form (5.5) take the form

$$
\begin{aligned}
& \operatorname{Ric}_{i j}=\mathrm{R}_{m i j}{ }^{m}+\mathrm{R}_{\mathrm{p}_{o} i j}{ }^{\mathrm{p}}+\mathrm{R}_{\mathrm{q}_{o} i j}{ }^{\mathrm{q}_{o}}= \\
& =\widehat{E}_{m}\left(\boldsymbol{\Gamma}_{i j}^{m}\right)-\widehat{E}_{i}\left(\boldsymbol{\Gamma}_{m j}{ }_{j}^{m}\right)-\boldsymbol{\Gamma}_{m j}{ }_{j} \boldsymbol{\Gamma}_{i \ell}^{m}-\boldsymbol{\Gamma}_{m j}{ }^{\mathrm{p}_{o}} \boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{m}-\boldsymbol{\Gamma}_{m j}{ }^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{m}+ \\
& +\boldsymbol{\Gamma}_{i j}^{\ell} \boldsymbol{\Gamma}_{m \ell}{ }^{m}+\boldsymbol{\Gamma}_{i j}{ }^{\mathrm{p}}{ }_{o} \boldsymbol{\Gamma}_{m \mathrm{p}_{o}}^{m}+\boldsymbol{\Gamma}_{i j}{ }^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{m \mathrm{q}_{o}}{ }^{m}-c_{m i}^{r} \boldsymbol{\Gamma}_{r j}^{m}+\omega_{m i} \boldsymbol{\Gamma}_{\mathrm{q}_{o} j}{ }^{m}+
\end{aligned}
$$

$$
\begin{align*}
& +\boldsymbol{\Gamma}_{i j}^{\ell} \boldsymbol{\Gamma}_{\mathrm{q}_{o} 冘}^{\mathrm{q}_{o}}+\boldsymbol{\Gamma}_{i j}^{\mathrm{p}_{o}} \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}^{\mathrm{q}_{o}}+\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{\mathrm{q}_{o}}= \\
& =\widehat{\operatorname{Ric}}_{i j}-\boldsymbol{\Gamma}_{m j}{ }^{\mathrm{p}_{o}} \boldsymbol{\Gamma}_{i \mathrm{P}_{o}}^{m}-\boldsymbol{\Gamma}_{m j}{ }^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{m}+\boldsymbol{\Gamma}_{i j}^{\mathrm{p}_{o}} \boldsymbol{\Gamma}_{m \mathrm{P}_{o}}^{m}+\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{m \mathrm{q}_{o}}^{m}+\omega_{m i} \boldsymbol{\Gamma}_{\mathrm{q}_{o} j}^{m}+ \\
& +\mathrm{p}_{o}\left(\boldsymbol{\Gamma}_{i j}^{\mathrm{p}_{o}}\right)-\boldsymbol{\Gamma}_{\mathrm{p}_{o},}{ }^{\ell} \boldsymbol{\Gamma}_{i \ell}^{\mathrm{p}_{o}}+\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{q}_{o}}{ }^{\mathrm{p}} \mathrm{\Gamma}_{o} \mathrm{q}_{o}\left(\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}}\right)-\boldsymbol{\Gamma}_{\mathrm{q}_{o} j}{ }^{\ell} \boldsymbol{\Gamma}_{i \ell}^{\mathrm{q}_{o}}+ \\
& +\overline{\boldsymbol{\Gamma}}_{i j}^{\mathbf{R}_{o}} \boldsymbol{F}_{\mathrm{q}_{o} \mathrm{P}_{o}}^{\mathrm{q}_{o}}+\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}} \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{\mathrm{q}_{o}}=\sigma \Lambda g_{i j}  \tag{5.16}\\
& \mathrm{Ric}_{i \mathrm{p}_{o}}=\mathrm{R}_{m i \mathrm{p}_{o}}{ }^{m}+\mathrm{R}_{\mathrm{p}_{o} i \mathrm{p}_{o}}{ }^{\mathrm{p}_{o}}+\mathrm{R}_{\mathrm{q}_{o} i \mathrm{p}_{o}}{ }^{\mathrm{q}_{o}}=
\end{align*}
$$

$$
\begin{align*}
& +\Gamma_{i p_{o}}^{l} \Gamma_{q_{o} k}^{q_{o}}+\bar{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{P}_{o} \Gamma_{q_{o} \mathrm{P}_{a}}^{q_{o}}}+\bar{\Gamma}_{\mathrm{T}_{o}}^{q_{o}} \Gamma_{q_{o} q_{o}}^{q_{o}}=0  \tag{5.17}\\
& \operatorname{Ric}_{i \mathrm{q}_{o}}=\mathrm{R}_{m i \mathrm{q}_{o}}{ }^{m}+\mathrm{R}_{\mathrm{p}_{o} i \mathrm{q}_{o}}{ }^{\mathrm{p}_{o}}+\mathrm{R}_{\mathrm{q}_{o} i \mathrm{q}_{o}}{ }^{\mathrm{q}_{o}}=
\end{align*}
$$

$$
\begin{align*}
& +\underline{\Gamma}_{i q_{0}}^{l} \Gamma_{q_{0} k}^{q_{0}}+\bar{\Gamma}_{i q_{0}}^{\mathrm{P}_{0}} \Gamma_{q_{0} \mathrm{P}_{a}}^{q_{0}}+\bar{\Gamma}_{i \dot{q}_{0}}^{q_{0}} \Gamma_{q_{0} q_{0}}^{q_{0}}=0 \tag{5.18}
\end{align*}
$$

$$
\begin{aligned}
& \operatorname{Ric}_{\mathrm{p}_{o} \mathrm{q}_{o}}=\mathrm{R}_{m \mathrm{p}_{o} \mathrm{q}_{o}}{ }^{m}+\mathrm{R}_{\mathrm{q}_{o} \mathrm{p}_{o} \mathrm{q}_{o}}{ }^{\mathrm{q}_{o}}=
\end{aligned}
$$

$$
\begin{align*}
& \operatorname{Ric}_{\mathrm{p}_{o} \mathrm{p}_{o}}=\mathrm{R}_{m \mathrm{p}_{o} \mathrm{p}_{o}}{ }^{m}+\mathrm{R}_{\mathrm{q}_{o} \mathrm{P}_{o} \mathrm{P}_{o}}{ }^{\mathrm{q}_{o}}= \tag{5.20}
\end{align*}
$$

$$
\begin{align*}
& \operatorname{Ric}_{\mathrm{q}_{o} \mathrm{q}_{o}}=\mathrm{R}_{m \mathrm{q}_{o} \mathrm{q}_{o}}{ }^{m}+\mathrm{R}_{\mathrm{p}_{o} \mathrm{q}_{o} \mathrm{q}_{o}}{ }^{\mathrm{p}_{o}}= \tag{5.21}
\end{align*}
$$

$$
\begin{aligned}
& +\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{\ell} \boldsymbol{\Gamma}_{m \ell}{ }^{m}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}^{\mathrm{p}_{o}} \boldsymbol{\Gamma}_{m \mathrm{p}_{o}}{ }^{m}+\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}{ }_{\mathrm{q}}^{o} \boldsymbol{\Gamma}_{m \mathrm{q}_{o}}{ }^{m}+
\end{aligned}
$$

We decompose this system into three subsystems, namely:
(a) the set of equations (5.20) - (5.22), concerning the Ricci curvatures $\operatorname{Ric}_{\mathrm{p}_{o} \mathrm{p}_{o}}, \operatorname{Ric}_{\mathrm{p}_{o} \mathrm{q}_{o}}=$
$\operatorname{Ric}_{\mathrm{q}_{o} \mathrm{p}_{o}}$ and $\operatorname{Ric}_{\mathrm{q}_{o} \mathrm{q}_{o}}$;
(b) the equations (5.17) and (5.19), concerning the "mixed" curvatures $\operatorname{Ric}_{\mathrm{p}_{o} i}$ and $\mathrm{Ric}_{\mathrm{q}_{o} j}$;
(c) the equations (5.16).

These three subsystems are in general tightly coupled. However under the assumption

$$
\begin{equation*}
\widehat{E}_{i}(\sigma)=\widehat{E}_{i}(\widetilde{\beta})=0, \quad \text { for each } 1 \leq i \leq n-2 . \tag{5.23}
\end{equation*}
$$

the system becomes much more treatable and can be solved. Note that these equations together with (5.1) imply

$$
\begin{equation*}
\mathrm{q}_{o}(\sigma)=\mathrm{q}_{o}(\widetilde{\beta})=0 \tag{5.24}
\end{equation*}
$$

and hence that the functions $\sigma, \widetilde{\beta}: M=S \times \mathbb{R} \rightarrow \mathbb{R}$ depend only on the coordinate of the fiber $\mathbb{R}$. If we now assume (5.23) (and its consequence (5.24)), we have that:
(1) The Christoffel symbols $\boldsymbol{\Gamma}_{i j}^{m}$ coincide with the functions $\Gamma_{i j}^{k}=g^{m k} g_{o}\left(\nabla_{E_{i}}^{o} E_{j}, E_{k}\right)$ and they are equal to the (lifts to $M$ of the) Christoffel symbols of the Levi-Civita connection $\nabla^{o}$ of the Kähler manifold $\left(N, g_{o}, J\right)$ with respect to the frame field $\left(E_{i}\right)$. In particular, the functions $\widehat{\mathrm{Ric}_{i j}}$ defined in (5.15) coincide with the (lifts to $M$ of the) components of the Ricci curvature $\operatorname{Ric}^{N}$ of $\left(N, J, g_{o}\right)$.
(2) The equations (5.17) and (5.19) reduce to

$$
\begin{align*}
& \frac{1}{4 \sigma}\left(-\widehat{E}_{m}\left(J_{i}^{m}\right)+\widehat{E}_{i}\left(J_{m}^{m}\right)+\Gamma_{i m}^{r} J_{r}^{m}-\Gamma_{m r}^{m} J_{i}^{r}+c_{m i}^{\ell} J_{\ell}^{m}\right)-\frac{\mathrm{p}_{o}(\sigma)}{2 \sigma}\left(\Gamma_{i m}^{m}-\Gamma_{m i}^{m}+c_{m i}^{m}\right)= \\
& \quad=\frac{1}{4 \sigma}\left(-\left(\nabla_{E_{m}} J\right)_{i}^{m}+\left(\nabla_{E_{i}} J\right)_{m}^{m}\right)+\left(\frac{J_{\ell}^{m}}{4 \sigma}-\frac{\mathrm{p}_{o}(\sigma) \delta_{\ell}^{m}}{2 \sigma}\right)\left(\Gamma_{i m}^{\ell}-\Gamma_{m i}^{\ell}+c_{m i}^{\ell}\right)=0  \tag{5.25}\\
& \frac{\widetilde{\beta}}{2 \sigma}\left(-\widehat{E}_{m}\left(J_{i}^{m}\right)+\widehat{E}_{i}\left(J_{m}^{m}\right)+\Gamma_{i m}^{r} J_{r}^{m}-\Gamma_{m r}^{m} J_{i}^{r}+c_{m i}^{\ell} J_{\ell}^{m}\right)= \\
& \quad=\frac{\widetilde{\beta}}{2 \sigma}\left(-\left(\nabla_{E_{m}}^{o} J\right)_{i}^{m}+\left(\nabla_{E_{i}}^{o} J\right)_{m}^{m}\right)+\frac{J_{\ell}^{m}}{4 \sigma}\left(\Gamma_{i m}^{\ell}-\Gamma_{m i}^{\ell}+c_{m i}^{\ell}\right)=0 \tag{5.26}
\end{align*}
$$

We observe that, being the Levi-Civita connection $\nabla^{\circ}$ of the Kähler manifold ( $N, g_{o}, J$ ) with trivial torsion (thus, $\Gamma_{i m}^{\ell}-\Gamma_{m i}^{\ell}+c_{m i}^{\ell} \equiv 0$ ) and such that $\nabla^{o} J=0$, it follows that (5.25) and (5.26) are identically satisfied. This means that the query for Einstein metrics now reduces just to looking for solutions to the equations (5.16) and (5.20) - (5.22).

Since $J$ is a complex structure, we have $J_{m}^{m}=\operatorname{tr}(J)=0$ and $J_{\ell}^{m} J_{m}^{\ell}=\operatorname{tr}\left(J^{2}\right)=-(n-2)$. Using this together with the identity $\widehat{\operatorname{Ric}}_{i j}=\operatorname{Ric}_{i j}^{N}$ and the expressions (5.9) - (5.14), we see that (5.16) is equivalent to

$$
\begin{equation*}
\operatorname{Ric}_{i j}^{N}=-\left(2 \mathrm{p}_{o}(\widetilde{\beta}) \mathrm{p}_{o}(\sigma)+\left(\frac{n-4}{2 \sigma}\left(\mathrm{p}_{o}(\sigma)\right)^{2}-\frac{1}{4 \sigma}+\mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)\right) 2 \widetilde{\beta}-\sigma \Lambda\right) g_{i j} \tag{5.27}
\end{equation*}
$$

and that the equations (5.20) - (5.22) are equivalent to

$$
\begin{align*}
& 2 \mathrm{p}_{o}\left(\mathrm{p}_{o}(\widetilde{\beta})\right)+\frac{(n-2)}{\sigma} \mathrm{p}_{o}(\widetilde{\beta}) \mathrm{p}_{o}(\sigma)+\frac{(n-2)}{4 \sigma^{2}} \widetilde{\beta}=\Lambda,  \tag{5.28}\\
& \frac{n-2}{4 \sigma^{2}}\left(-2 \sigma \mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)+\left(\mathrm{p}_{o}(\sigma)\right)^{2}+\frac{1}{4}\right)=0,  \tag{5.29}\\
& 2 \widetilde{\beta}\left(2 \mathrm{p}_{o}\left(\mathrm{p}_{o}(\widetilde{\beta})\right)+\frac{(n-2)}{\sigma} \mathrm{p}_{o}(\widetilde{\beta}) \mathrm{p}_{o}(\sigma)+\frac{(n-2)}{4 \sigma^{2}} \widetilde{\beta}\right)=2 \Lambda \widetilde{\beta} . \tag{5.30}
\end{align*}
$$

We observe that, since the left hand side in (5.27) is independent of the fiber coordinates of the bundle $\pi^{S} \circ \pi: M \rightarrow N$, such equation might be satisfied only if $\left(N, g_{o}, J\right)$ is Kähler-Einstein, i.e. Ric $=\Lambda^{o} g$ for some Einstein constant $\Lambda^{o}$, and $\sigma$ and $\widetilde{\beta}$ satisfy the equation

$$
\begin{equation*}
2 \mathrm{p}_{o}(\widetilde{\beta}) \mathrm{p}_{o}(\sigma)+\left(\frac{n-4}{2 \sigma}\left(\mathrm{p}_{o}(\sigma)\right)^{2}-\frac{1}{4 \sigma}+\mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)\right) 2 \widetilde{\beta}-\sigma \Lambda+\Lambda^{o}=0 \tag{5.31}
\end{equation*}
$$

Moreover, since (5.30) is manifestly implied by (5.28) and we look for solutions in which $\sigma>0$ at all points, it now suffices to find solutions $\sigma>0$ and $\widetilde{\beta}$ of the system of the just three ordinary differential equations (5.31), (5.28) and (5.29).

Now we show that the equation (5.28) follows from the others. Indeed, using the fact that $\sigma>0$ and differentiating the whole term inside parentheses in (5.29) along $\mathrm{p}_{o}$, we immediately get that $\sigma$ satisfies

$$
\begin{equation*}
\mathrm{p}_{o}\left(\mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)\right)=0 \tag{5.32}
\end{equation*}
$$

Using this property and replacing $\mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)$ by the expression $\mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)=\frac{1}{2 \sigma}\left(\left(\mathrm{p}_{o}(\sigma)\right)^{2}+\frac{1}{4}\right)$ which follows from (5.29), one can check that the derivative along $\mathrm{p}_{o}$ of the equation (5.31) is the multiple by $\mathrm{p}_{o}(\sigma)$ of the equation (5.28). This proves that the latter is implied by (5.31) in case $\mathrm{p}_{o}(\sigma) \neq 0$.

This means that we are now reduced to find solutions $\sigma>0$ and $\widetilde{\beta}$ to the o.d.e. system

$$
\begin{align*}
& 2 \mathrm{p}_{o}(\widetilde{\beta}) \mathrm{p}_{o}(\sigma)+\left(\frac{n-4}{2 \sigma}\left(\mathrm{p}_{o}(\sigma)\right)^{2}-\frac{1}{4 \sigma}+\mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)\right) 2 \widetilde{\beta}-\sigma \Lambda+\Lambda^{o}=0  \tag{5.33}\\
& -2 \sigma \mathrm{p}_{o}\left(\mathrm{p}_{o}(\sigma)\right)+\left(\mathrm{p}_{o}(\sigma)\right)^{2}+\frac{1}{4}=0 \tag{5.34}
\end{align*}
$$

We now observe that (5.32) shows that each solution $\sigma=\sigma(t)$ to (5.34) is a quadratic polynomial $\sigma(t)=C_{2} t^{2}+C_{1} t+C_{0}$. Actually, it is equivalent to say that such positive quadratic polynomial has discriminant equal to $-\frac{1}{4}$. So, by an appropriate coordinate change $t \mapsto t+c$, we may always assume that it has the simpler form

$$
\begin{equation*}
\sigma(t)=\frac{1}{16 C} t^{2}+C \quad \text { for some } C>0 \tag{5.35}
\end{equation*}
$$

From this we get that (5.33) is equivalent to the first order linear differential equation

$$
\begin{equation*}
\frac{t}{4 C} \frac{d \widetilde{\beta}}{d t}+\left(\frac{n-4}{\frac{t^{2}}{16 C}+C}\left(\frac{t}{8 C}\right)^{2}-\frac{1}{2\left(\frac{t^{2}}{16 C}+C\right)}+\frac{1}{4 C}\right) \widetilde{\beta}-\left(\frac{t^{2}}{16 C}+C\right) \Lambda+\Lambda^{o}=0 \tag{5.36}
\end{equation*}
$$

The general solutions on $(0,+\infty)$ of this equation have the form

$$
\begin{array}{r}
\widetilde{\beta}(t):=e^{-\int_{1}^{t} a(s) d s}\left(B_{0}-\int_{1}^{t} b(s) e^{\int_{1}^{s} a(\tilde{s}) d \check{s}} d s\right) \quad \text { with } \\
a(t):=\frac{8 C}{t}\left(\frac{n-4}{2\left(\frac{t^{2}}{16 C}+C\right)}\left(\frac{t}{8 C}\right)^{2}-\frac{1}{4\left(\frac{t^{2}}{16 C}+C\right)}+\frac{1}{8 C}\right)=\frac{(n-3) t^{2}-16 C^{2}}{t^{3}+16 C^{2} t} \text { and } \\
b(t):=-\frac{4 C}{t}\left(\frac{t^{2}}{16 C}+C\right) \Lambda+\frac{4 C}{t} \Lambda_{o} . \tag{5.37}
\end{array}
$$

We now observe that $e^{\int_{1}^{t} a(s) d s}=B_{1} \frac{\left(t^{2}+16 C^{2}\right)^{\frac{n}{2}-1}}{t}$ for some $B_{1} \in \mathbb{R}$ and that $b(s) e^{\int_{1}^{s} a(\check{s}) d \check{s}}=$ $B_{1} \frac{\left(16 C^{2}+s^{2}\right)^{\frac{n}{2}-1}\left(16 C \Lambda_{0}-\Lambda\left(16 C^{2}+s^{2}\right)\right)}{4 s^{2}}$. Hence, since $n$ is even, it is a rational function of $s^{2}$. Plugging these expressions into (5.37), we see that $\widetilde{\beta}(t)$ is a rational function that is well defined at $t=0$ and hence on the whole real axis. Setting $B=B_{0} / B_{1}$, we get (5.7).

Remark 5.3. As a corollary of the proof, we have that if the quantisable Kähler manifold $\left(N, J, g^{o}\right)$ is not Einstein, there is no firmly compatible Einstein metric on the trivial $\mathbb{R}$ bundle $M=S \times \mathbb{R}$.

As we will shortly see, the classical Taub-NUT metrics can be considered as firmly compatible metrics (5.7) on the 4-dimensional RT bundle $\pi: M=S^{3} \times \mathbb{R} \rightarrow \mathbb{R}$, where $S^{3}$ is considered as a Sasaki manifold over the round sphere $S^{2}=\mathbb{C} P^{1}$. Due to this, we call the metrics of Theorem 5.2 of Taub-NUT type.
5.3. 4-dimensional Taub-NUT metrics and higher dimensional analogues. Let $(S, \mathcal{D}=\operatorname{ker} \theta, J)$ and $\left(N, J, g_{o}\right)$ be a regular Sasaki manifold and its associated KählerEinstein manifold, respectively, as in Theorem 5.2. Consider a (local) trivialisation $\left.S\right|_{u}=$ $\mathcal{U} \times A$ on some open set $U \subset N$ and denote by $u$ a coordinate for the fiber $A=\mathbb{R}, S^{1}$ of the Sasaki manifold. In this way, the contact form $\theta$ takes the form $\theta=d u+\eta$ for some appropriate 1 -form $\eta$ in $\Omega^{1}(\mathcal{U})$. Then, using the coordinates $(t, u)$ for the fiber $\mathbb{R} \times A$ of the (trivialised) bundle $\pi^{S} \circ \pi:\left.M\right|_{\mathcal{U}}=\left.S\right|_{\mathcal{U}} \times \mathbb{R} \rightarrow \mathcal{U} \subset N$, the metrics of Theorem 5.2 read as

$$
\begin{align*}
& g=\sigma(t) g_{o}+\widetilde{\beta}(t)\left\{(d u+\eta) \vee\left(\frac{1}{\widetilde{\beta}(t)} d t+d u+\eta\right)\right\}= \\
& =\sigma(t) g_{o}+\widetilde{\beta}(t)\left(\left(d u+\eta+\frac{1}{2 \widetilde{\beta}(t)} d t\right)-\frac{1}{2 \widetilde{\beta}(t)} d t\right) \vee\left(\left(d u+\eta+\frac{1}{2 \widetilde{\beta}(t)} d t\right)+\frac{1}{2 \widetilde{\beta}(t)} d t\right) \\
& =\sigma(t) g_{o}+\widetilde{\beta}(t)\left(d u+\eta+\frac{1}{2 \widetilde{\beta}(t)} d t\right)^{2}-\frac{1}{4 \widetilde{\beta}(t)} d t^{2}= \\
& \quad=\sigma(t) g_{o}+\widetilde{\beta}(t)\left(\frac{1}{\Lambda_{o}} d v+\eta\right)^{2}-\frac{1}{4 \widetilde{\beta}(t)} d t^{2} \tag{5.38}
\end{align*}
$$

where, in the last step, we replaced the fiber coordinates $(t, u)$ by

$$
\begin{equation*}
(t, u) \longmapsto\left(t, v:=\Lambda_{o}\left(u+\int_{0}^{t} \frac{1}{2 \widetilde{\beta}(s)} d s\right)\right) \tag{5.39}
\end{equation*}
$$

If we now take $N=\mathbb{C} P^{1}=S^{2}$ as Kähler manifold and consider its standard spherical coordinates $(\phi, \psi)$, the round metric $g_{o}$ of constant curvature $\kappa=\Lambda_{o}$ on $N=S^{2}$ and the associated contact form $\theta$ on the associated Sasaki manifold $S^{3}$ (which fibers on $S^{2}$ by means of the Hopf map) have the coordinate expressions

$$
\begin{equation*}
g_{o}=\frac{1}{\Lambda_{o}}\left(d \psi^{2}+\sin \psi^{2} d \phi^{2}\right), \quad \theta=d u+\eta=d u+\frac{1}{\Lambda_{o}} \cos \psi d \phi . \tag{5.40}
\end{equation*}
$$

On the other hand the functions (5.7) corresponding to Ricci flat metrics $(\Lambda=0)$ are

$$
\begin{equation*}
\widetilde{\beta}(t)=\Lambda_{o}\left(\check{B} \frac{t}{t^{2}+16 C^{2}}-4 C \frac{t^{2}-16 C^{2}}{t^{2}+16 C^{2}}\right) \quad \text { with } \quad \check{B}:=\frac{B}{2 \Lambda_{o}}\left(1+16 C^{2}\right)+4 C\left(1-16 C^{2}\right) . \tag{5.41}
\end{equation*}
$$

Plugging (5.40) and (5.41) into (5.38), we obtain the following coordinate expression for the Ricci flat metrics on the 4-manifold $M=S^{3} \times \mathbb{R}=\mathbb{R}^{4} \backslash\{0\}$ :

$$
\begin{align*}
g=\frac{1}{\Lambda_{o}} & \left(\frac{1}{16 C} t^{2}+C\right)\left(d \psi^{2}+\sin \psi^{2} d \phi^{2}\right)+ \\
& +\frac{1}{\Lambda_{o}} \frac{\check{B} t-4 C t^{2}+64 C^{3}}{t^{2}+16 C^{2}}(d v+\cos \psi d \phi)^{2}-\frac{1}{\Lambda_{o}} \frac{t^{2}+16 C^{2}}{\left.4\left(\check{B} t-4 C t^{2}+64 C^{3}\right)\right)} d t^{2} \tag{5.42}
\end{align*}
$$

If we now set

$$
\begin{align*}
\ell:=\sqrt{C} . \quad m:=\frac{\check{B}}{32 \ell^{3}} & =\frac{1}{32 C^{\frac{3}{2}}}\left(\frac{B}{2 \Lambda_{o}}\left(1+16 C^{2}\right)+4 C\left(1-16 C^{2}\right)\right) \\
& \quad \text { and apply the coordinate change } t \longmapsto \check{t}=\frac{t}{4 \ell}=\frac{t}{4 \sqrt{C}} \tag{5.43}
\end{align*}
$$

the metrics (5.42) take the very familiar coordinate expression of the (rescaled) Taub-NUT metrics

$$
\begin{align*}
g:=\frac{1}{\Lambda_{o}}\left\{( \check { t } ^ { 2 } + \ell ^ { 2 } ) \left(d \psi^{2}\right.\right. & \left.+\sin ^{2} \psi d \phi^{2}\right)+ \\
& \left.+\frac{2 m \check{t}+\ell^{2}-\check{t}^{2}}{\check{t}^{2}+\ell^{2}} 4 \ell^{2}(d v+\cos \psi d \phi)^{2}-\frac{\check{t}^{2}+\ell^{2}}{2 m \check{t}+\ell^{2}-\check{t}^{2}} d \check{t}^{2}\right\} \tag{5.44}
\end{align*}
$$

Many other 4-dimensional Einstein metrics can be determined in exactly the same way: it suffices to impose a different value $\Lambda \neq 0$ for the desired Einstein constant and/or to replace the Kähler manifold $N=S^{2}$ by some other compact Riemann surface, as e.g. $T^{2}=S^{1} \times S^{1}$ or a compact quotient of the unit disk $\Delta \subset \mathbb{C}$, equipped with some metric of constant curvature.

In order to generate explicit examples of higher dimensional Einstein metrics of TaubNUT type, one might follow the same procedure as above, starting, for instance, from some higher dimensional homogeneous flag manifolds $N=G^{\mathbb{C}} / P$ of a complex semisimple Lie group, equipped with its unique (up to a homothety) invariant Kähler-Einstein metric of positive Einstein constant $\Lambda_{o}>0$. For instance, one might consider the 4-dimensional manifolds $N=\mathbb{C} P^{2}$ or $N=\mathbb{C} P^{1} \times \mathbb{C} P^{1}$ equipped with the Fubini-Study metric or the cartesian product of two round metrics, respectively. In all these cases, for any choice of a constant $\Lambda \in \mathbb{R}$ and of a Sasaki manifold $(S, \mathcal{D}=\theta, J)$ projecting onto $N$ (about the regular Sasaki manifolds that are associated with the compact flag manifolds, see for instance [1] and references therein), Theorem 5.2 and the above discussion yield explicit coordinate expressions for Lorentzian Einstein metrics with any prescribed Einstein constant $\Lambda$ over $M=S \times \mathbb{R}$.

## Appendix A. The Christoffel symbols of the Levi-Civita connection of a Lorentzian metric of Kähler-Sasaki type

In this appendix, we give the explicit expressions of the Christoffel symbols of the LeviCivita connection of a compatible metric on the trivial RT bundle $\pi: M=S \times \mathbb{R} \rightarrow S$ as described in \$5.1. We compute them in terms of the frame field ( $\mathrm{p}_{o}, \widehat{E}_{i}, \mathrm{q}_{o}$ ) and its dual coframe field ( $\mathrm{p}_{o}^{*}, \widehat{E}^{i}, \mathrm{q}_{o}^{*}$ ), associated with a local frame field $\left(E_{i}\right)$. More precisely, we list here the real functions $\boldsymbol{\Gamma}_{A B}^{C}$ which determine the covariant derivatives $\nabla_{X_{A}} X_{B}=\boldsymbol{\Gamma}_{A B}{ }_{B} X_{C}$ for any choice of two vector fields $X_{A}, X_{B}$ of the tuple ( $\mathrm{p}_{o}, \widehat{E}_{i}, \mathrm{q}_{o}$ ) for a metric $g$ of the form (see (2.18))

$$
\begin{equation*}
g=\sigma\left(g_{i j} \widehat{E}^{i} \vee \widehat{E}^{j}+\mathrm{q}_{o}^{*} \vee\left(\alpha \mathrm{p}_{o}^{*}+\gamma^{i} g_{i j} \widehat{E}^{j}+\beta \mathrm{q}_{o}^{*}\right)\right), \quad g_{i j}=g_{o}\left(E_{i}, E_{j}\right) . \tag{A.1}
\end{equation*}
$$

First we compute the Christoffel symbols under the simplifying assumption $\sigma \equiv 1$. Then one can directly check that the elements of the dual coframe field ( $\mathrm{p}_{o}^{*}, \widehat{E}^{1}, \ldots, \widehat{E}^{n-2}, \mathrm{q}_{o}^{*}$ ) satisfy the identities

$$
\begin{array}{r}
\widehat{E}^{i}=g\left(g^{i k} \widehat{E}_{k}-\frac{\gamma^{i}}{\alpha} \mathrm{p}_{o}, \cdot\right), \quad \mathrm{p}_{o}^{*}=g\left(\frac{2}{\alpha} \mathrm{q}_{o}+\frac{1}{\alpha^{2}}\left(\gamma^{m} \gamma^{k} g_{m k}-4 \beta\right) \mathrm{p}_{o}-\frac{\gamma^{m}}{\alpha} \widehat{E}_{m}, \cdot\right), \\
\mathrm{q}_{o}^{*}=g\left(\frac{2}{\alpha} \mathrm{p}_{o}, \cdot\right) . \tag{A.2}
\end{array}
$$

Using these and the expansion $X=\widehat{E}^{i}(X) \widehat{E}_{i}+\mathrm{p}_{o}^{*}(X) \mathrm{p}_{o}+\mathrm{q}_{o}^{*}(X) \mathrm{q}_{o}$ of any vector field $X \in \mathfrak{X}(M)$ in terms of our frame field, we may compute the covariant derivatives $\nabla_{X_{A}} X_{B}$ for any pair $X_{A}, X_{B}$ of vector fields of the frame $\left\{\widehat{E}_{i}, \mathrm{p}_{o}, \mathrm{q}_{o}\right\}$ using Koszul's formula

$$
\begin{align*}
g\left(\nabla_{X} Y, Z\right)=\frac{1}{2}(X(g(Y, Z))+Y( & g(X, Z))-Z(g(X, Y))- \\
& -g([X, Z], Y)-g([Y, Z], X)+g([X, Y], Z)) . \tag{A.3}
\end{align*}
$$

Note also that for any pair of commuting vector fields $X_{A}, X_{B}$ one has $\nabla_{X_{A}} X_{B}=\nabla_{X_{B}} X_{A}$ because $\nabla$ has trivial torsion. From this information and by some (long and tedious, but very straightforward) computations we get the following list of covariant derivatives in the case $\sigma \equiv 1$. In the next formulas we use the notation $g_{i j}:=g_{o}\left(E_{i}, E_{j}\right), \omega_{i j}:=g_{o}\left(E_{i}, J E_{j}\right)$, $\left[E_{i}, E_{j}\right]=c_{i j}^{k} E_{k}$ and we set

$$
\begin{aligned}
& S_{i j \mid k}:=\frac{1}{2}\left(\frac{\gamma^{\ell}}{2} g_{o}\left(E_{i}, J E_{k}\right) g_{o}\left(E_{\ell}, E_{j}\right)+\frac{\gamma^{\ell}}{2} g_{o}\left(E_{j}, J E_{k}\right) g_{o}\left(E_{\ell}, E_{i}\right)+\right. \\
&\left.-\frac{\gamma^{\ell}}{2} g_{o}\left(E_{i}, J E_{j}\right) g_{o}\left(E_{\ell}, E_{k}\right)\right) .
\end{aligned}
$$

Here is the list:

$$
\begin{align*}
& \nabla_{\widehat{E}_{i}} \widehat{E}_{j}=\left(g^{m k} g_{o}\left(\nabla_{E_{i}}^{o} E_{j}, E_{k}\right)+g^{m k} S_{i j \mid k}+\frac{\gamma^{m} \omega_{i j}}{4}\right) \widehat{E}_{m}+\left(\frac{1}{2 \alpha} \widehat{E}_{i}\left(\gamma^{k} g_{j k}\right)+\frac{1}{2 \alpha} \widehat{E}_{j}\left(\gamma^{k} g_{i k}\right)-\right. \\
& \left.-\frac{1}{4 \alpha} \gamma^{m} \gamma^{k} g_{m k} \omega_{i j}-\frac{\gamma^{m}}{\alpha} g_{o}\left(\nabla_{E_{i}}^{o} E_{j}, E_{m}\right)-\frac{\gamma^{m}}{\alpha} S_{i j \mid m}\right) \mathrm{p}_{o}-\frac{\omega_{i j}}{2} \mathrm{q}_{o} . \\
& \nabla_{\widehat{E}_{i}} \mathrm{p}_{o}=\frac{\alpha g^{m k} \omega_{i k}}{4} \widehat{E}_{m}+\left(\frac{1}{2 \alpha} \widehat{E}_{i}(\alpha)+\frac{1}{2 \alpha} \mathrm{p}_{o}\left(\gamma^{k}\right) g_{i k}-\frac{\gamma^{m} \omega_{i m}}{4}\right) \mathrm{p}_{o} \\
& \nabla_{\widehat{E}_{i}} \mathrm{q}_{o}=\left(\frac{g^{m k}}{4} \widehat{E}_{i}\left(\gamma^{t} g_{t k}\right)-\frac{g^{m k}}{4} \widehat{E}_{k}\left(\gamma^{t} g_{t i}\right)+\frac{g^{m k}}{2} \beta \omega_{i k}-\frac{\gamma^{\ell}}{4} c_{i r}^{t} g_{t t} g^{m r}-\right. \\
& \left.-\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{i}(\alpha)+\frac{\gamma^{m}}{4 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}\right) \widehat{E}_{m}+ \\
& +\left(\frac{1}{\alpha} \widehat{E}_{i}(\beta)+\frac{1}{4 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \widehat{E}_{i}(\alpha)-\frac{1}{4 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\frac{\beta}{\alpha^{2}} \widehat{E}_{i}(\alpha)+\frac{1}{\alpha^{2}} \beta \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\right. \\
& \left.-\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{i}\left(\gamma^{t} g_{t m}\right)+\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{m}\left(\gamma^{t} g_{i t}\right)-\frac{\gamma^{m}}{2 \alpha} \beta \omega_{i m}\right) \mathrm{p}_{o}+\left(\frac{1}{2 \alpha} \widehat{E}_{i}(\alpha)-\frac{1}{2 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}\right) \mathrm{q}_{o} \\
& \nabla_{\mathrm{p}_{o}} \widehat{E}_{i}=\frac{\alpha g^{m k}}{4} \omega_{i k} \widehat{E}_{m}+\left(\frac{1}{2 \alpha} \widehat{E}_{i}(\alpha)+\frac{1}{2 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\frac{\gamma^{m}}{4} \omega_{i m}\right) \mathrm{p}_{o} \\
& \nabla_{\mathrm{p}_{o}} \mathrm{p}_{o}=\left(\mathrm{p}_{o}(\log \alpha)\right) \mathrm{p}_{o} \\
& \nabla_{\mathrm{p}_{o}} \mathrm{q}_{o}=\left(\frac{1}{4} \mathrm{p}_{o}\left(\gamma^{m}\right)-\frac{g^{m k}}{4} \widehat{E}_{k}(\alpha)\right) \widehat{E}_{m}+\left(\frac{1}{\alpha} \mathrm{p}_{o}(\beta)-\frac{\gamma^{m}}{4 \alpha} \mathrm{p}_{o}\left(\gamma^{i}\right) g_{i m}+\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{m}(\alpha)\right) \mathrm{p}_{o} \\
& \nabla_{\mathrm{q}_{o}} \widehat{E}_{i}=\left(\frac{g^{m k}}{4} \widehat{E}_{i}\left(\gamma^{t} g_{t k}\right)-\frac{g^{m k}}{4} \widehat{E}_{k}\left(\gamma^{t} g_{t i}\right)+\frac{g^{m k}}{2} \beta \omega_{i k}-\frac{\gamma^{\ell}}{4} c_{i r}^{t} g_{t \ell} g^{m r}-\right. \\
& \left.-\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{i}(\alpha)+\frac{\gamma^{m}}{4 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}\right) \widehat{E}_{m}+ \\
& +\left(\frac{1}{\alpha} \widehat{E}_{i}(\beta)+\frac{1}{4 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \widehat{E}_{i}(\alpha)-\frac{1}{4 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\frac{\beta}{\alpha^{2}} \widehat{E}_{i}(\alpha)+\frac{1}{\alpha^{2}} \beta \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\right. \\
& \left.-\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{i}\left(\gamma^{t} g_{t m}\right)+\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{m}\left(\gamma^{t} g_{i t}\right)-\frac{\gamma^{m}}{2 \alpha} \beta \omega_{i m}\right) \mathrm{p}_{o}+\left(\frac{1}{2 \alpha} \widehat{E}_{i}(\alpha)-\frac{1}{2 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}\right) \mathrm{q}_{o} \tag{A.10}
\end{align*}
$$

$$
\begin{gather*}
\nabla_{\mathrm{q}_{o}} \mathrm{p}_{o}=\left(\frac{g^{m k}}{4} \mathrm{p}_{o}\left(\gamma^{i}\right) g_{i k}-\frac{g^{m k}}{4} \widehat{E}_{k}(\alpha)\right) \widehat{E}_{m}+\left(\frac{1}{\alpha} \mathrm{p}_{o}(\beta)-\frac{\gamma^{m}}{4 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{t m}+\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{m}(\alpha)\right) \mathrm{p}_{o}  \tag{A.11}\\
\nabla_{\mathrm{q}_{o}} \mathrm{q}_{o} \\
=\left(\frac{g^{m k}}{2} \mathrm{q}_{o}\left(\gamma^{i}\right) g_{i k}-\frac{g^{m k}}{2} \widehat{E}_{k}(\beta)-\frac{\gamma^{m}}{2 \alpha} \mathrm{q}_{o}(\alpha)+\frac{\gamma^{m}}{2 \alpha} \mathrm{p}_{o}(\beta)\right) \widehat{E}_{m}+ \\
+\left(\frac{1}{\alpha} \mathrm{q}_{o}(\beta)+\frac{1}{2 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{q}_{o}(\alpha)-\frac{1}{2 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{p}_{o}(\beta)-\frac{2}{\alpha^{2}} \beta \mathrm{q}_{o}(\alpha)+\frac{2}{\alpha^{2}} \beta \mathrm{p}_{o}(\beta)-\right.  \tag{A.12}\\
\left.\quad-\frac{\gamma^{m}}{2 \alpha} \mathrm{q}_{o}\left(\gamma^{i}\right) g_{i m}+\frac{\gamma^{m}}{2 \alpha} \widehat{E}_{m}(\beta)\right) \mathrm{p}_{o}+\left(\frac{1}{\alpha} \mathrm{q}_{o}(\alpha)-\frac{1}{\alpha} \mathrm{p}_{o}(\beta)\right) \mathrm{q}_{o}
\end{gather*}
$$

We now denote by $\widetilde{g}$ a compatible metric as above, determined by the conformal factor $\sigma=1$ and by $g=\sigma \tilde{g}$ another compatible metric, which is determined by an arbitrary conformal factor $\sigma>0$. The Levi-Civita connection $D$ of $g$ is related with the Levi-Civita connection $\nabla$ of $\widetilde{g}$ by the formula (see e.g. [4, Th. 1.159])

$$
\begin{equation*}
D_{X} Y=\nabla_{X} Y+X(\varphi) Y+Y(\varphi) X-g(X, Y) \operatorname{grad}(\varphi), \quad \varphi:=\frac{1}{2} \log \sigma \tag{A.13}
\end{equation*}
$$

On the other hand, by (A.2),

$$
\begin{equation*}
\operatorname{grad} \varphi=(\operatorname{grad} \varphi)^{\widehat{E}_{i}} \widehat{E}_{i}+(\operatorname{grad} \varphi)^{\mathrm{p}_{o}} \mathrm{p}_{o}+(\operatorname{grad} \varphi)^{\mathrm{q}_{o}} \mathrm{q}_{o} \tag{A.14}
\end{equation*}
$$

where

$$
\begin{align*}
(\operatorname{grad} \varphi)^{\widehat{E}_{i}} & :=g^{i k} \widehat{E}_{k}(\varphi)-\frac{\gamma^{i}}{\alpha} \mathrm{p}_{o}(\varphi)  \tag{A.15}\\
(\operatorname{grad} \varphi)^{\mathrm{p}_{o}} & :=\frac{2}{\alpha} \mathrm{q}_{o}(\varphi)+\frac{1}{\alpha^{2}}\left(\gamma^{m} \gamma^{k} g_{m k}-4 \beta\right) \mathrm{p}_{o}(\varphi)-\frac{\gamma^{m}}{\alpha} \widehat{E}_{m}(\varphi)  \tag{A.16}\\
(\operatorname{grad} \varphi)^{\mathrm{q}_{o}} & :=\frac{2}{\alpha} \mathrm{p}_{o}(\varphi) \tag{A.17}
\end{align*}
$$

Combining (А.4) - (A.12) with (A.13), (A.15) - (A.17), we see that the Christoffel symbols $\Gamma_{A B}^{C}$ of an arbitrary compatible metric $g$ are

$$
\begin{align*}
\boldsymbol{\Gamma}_{i j}^{m}= & g^{m k} g_{o}\left(\nabla_{E_{i}}^{o} E_{j}, E_{k}\right)+g^{m k} S_{i j \mid k}+\frac{\gamma^{m} \omega_{i j}}{4}+\frac{1}{2 \sigma} \widehat{E}_{i}(\sigma) \delta_{j}^{m}+\frac{1}{2 \sigma} \widehat{E}_{j}(\sigma) \delta_{i}^{m} \\
& \quad-\frac{g_{i j}}{2 \sigma}\left(g^{m k} \widehat{E}_{k}(\sigma)-\frac{\gamma^{m}}{\alpha} \mathrm{p}_{o}(\sigma)\right)  \tag{A.18}\\
\boldsymbol{\Gamma}_{i j}^{\mathrm{p}_{o}}= & \frac{1}{2 \alpha} \widehat{E}_{i}\left(\gamma^{k} g_{j k}\right)+\frac{1}{2 \alpha} \widehat{E}_{j}\left(\gamma^{k} g_{i k}\right)-\frac{1}{4 \alpha} \gamma^{m} \gamma^{k} g_{m k} \omega_{i j}-\frac{\gamma^{m}}{\alpha} g_{o}\left(\nabla_{E_{i}}^{o} E_{j}, E_{m}\right)-\frac{\gamma^{m}}{\alpha} S_{i j \mid m} \\
& \quad-\frac{g_{i j}}{2 \sigma}\left(\frac{2}{\alpha} \mathrm{q}_{o}(\sigma)+\frac{1}{\alpha^{2}}\left(\gamma^{m} \gamma^{k} g_{m k}-4 \beta\right) \mathrm{p}_{o}(\sigma)-\frac{\gamma^{m}}{\alpha} \widehat{E}_{m}(\sigma)\right)  \tag{A.19}\\
\boldsymbol{\Gamma}_{i j}^{\mathrm{q}_{o}}= & -\frac{\omega_{i j}}{2}-\frac{g_{i j}}{\alpha \sigma} \mathrm{p}_{o}(\sigma) \tag{A.20}
\end{align*}
$$

$$
\begin{align*}
& \boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} i}^{m}=\frac{\alpha g^{m k} \omega_{i k}}{4}+\frac{1}{2 \sigma} \mathrm{p}_{o}(\sigma) \delta_{i}^{m}  \tag{A.21}\\
& \boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} i}^{\mathrm{p}_{o}}=\frac{1}{2 \alpha} \widehat{E}_{i}(\alpha)+\frac{1}{2 \alpha} \mathrm{p}_{o}\left(\gamma^{k}\right) g_{i k}-\frac{\gamma^{m} \omega_{i m}}{4}+\frac{1}{2 \sigma} \widehat{E}_{i}(\sigma)  \tag{A.22}\\
& \boldsymbol{\Gamma}_{i \mathrm{p}_{o}}^{\mathrm{q}_{o}}=\boldsymbol{\Gamma}_{\mathrm{p}_{o} i}{ }^{\mathrm{q}_{o}}=0  \tag{A.23}\\
& \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} i}{ }^{m}=\frac{g^{m k}}{4} \widehat{E}_{i}\left(\gamma^{t} g_{t k}\right)-\frac{g^{m k}}{4} \widehat{E}_{k}\left(\gamma^{t} g_{t i}\right)+\frac{g^{m k}}{2} \beta \omega_{i k}-\frac{\gamma^{\ell}}{4} c_{i r}^{t} g_{t \ell} g^{m r}- \\
& -\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{i}(\alpha)+\frac{\gamma^{m}}{4 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{t i}+\frac{1}{2 \sigma} \mathrm{q}_{o}(\sigma) \delta_{i}^{m}-\frac{\gamma^{t}}{4 \sigma} g_{t i}\left(g^{m k} \widehat{E}_{k}(\sigma)-\frac{\gamma^{m}}{\alpha} \mathrm{p}_{o}(\sigma)\right)  \tag{A.24}\\
& \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} i}{ }^{\mathrm{p}_{o}}=\frac{1}{\alpha} \widehat{E}_{i}(\beta)+\frac{1}{4 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \widehat{E}_{i}(\alpha)-\frac{1}{4 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\frac{1}{\alpha^{2}} \beta \widehat{E}_{i}(\alpha)+ \\
& +\frac{1}{\alpha^{2}} \beta \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}-\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{i}\left(\gamma^{t} g_{t m}\right)+\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{m}\left(\gamma^{t} g_{i t}\right)-\frac{\gamma^{m}}{2 \alpha} \beta \omega_{i m}- \\
& -\frac{\gamma^{t}}{4 \sigma} g_{t i}\left(\frac{2}{\alpha} \mathrm{q}_{o}(\sigma)+\frac{1}{\alpha^{2}}\left(\gamma^{m} \gamma^{k} g_{m k}-4 \beta\right) \mathrm{p}_{o}(\sigma)-\frac{\gamma^{m}}{\alpha} \widehat{E}_{m}(\sigma)\right)  \tag{A.25}\\
& \boldsymbol{\Gamma}_{i \mathrm{q}_{o}}^{\mathrm{q}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} i}{ }^{\mathrm{q}_{o}}=\frac{1}{2 \alpha} \widehat{E}_{i}(\alpha)-\frac{1}{2 \alpha} \mathrm{p}_{o}\left(\gamma^{t}\right) g_{i t}+\frac{1}{2 \sigma} \widehat{E}_{i}(\sigma)-\frac{\gamma^{t} g_{t i}}{2 \alpha \sigma} \mathrm{p}_{o}(\sigma)  \tag{A.26}\\
& \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{P}_{o}}{ }^{m}=0  \tag{A.27}\\
& \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{p}_{o}}^{\mathrm{p}_{o}}=\mathrm{p}_{o}(\log (\alpha \sigma))  \tag{A.28}\\
& \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{p}_{o}}^{\mathrm{q}_{o}}=0  \tag{A.29}\\
& \boldsymbol{\Gamma}_{\mathrm{P}_{o} \mathrm{q}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}^{m}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{P}_{o}}^{m}=\frac{1}{4} \mathrm{p}_{o}\left(\gamma^{m}\right)-\frac{g^{m k}}{4} \widehat{E}_{k}(\alpha)-\frac{\alpha}{4 \sigma}\left(g^{m k} \widehat{E}_{k}(\sigma)-\frac{\gamma^{m}}{\alpha} \mathrm{p}_{o}(\sigma)\right)  \tag{A.30}\\
& \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{q}_{o}}^{\mathrm{p}_{o}}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{p}_{o}}{ }^{\mathrm{p}_{o}}=\frac{1}{\alpha} \mathrm{p}_{o}(\beta)-\frac{\gamma^{m}}{4 \alpha} \mathrm{p}_{o}\left(\gamma^{i}\right) g_{i m}+\frac{\gamma^{m}}{4 \alpha} \widehat{E}_{m}(\alpha)+\frac{1}{2 \sigma} \mathrm{q}_{o}(\sigma)- \\
& -\frac{1}{2 \sigma}\left(\mathrm{q}_{o}(\sigma)+\frac{1}{2 \alpha}\left(\gamma^{m} \gamma^{k} g_{m k}-4 \beta\right) \mathrm{p}_{o}(\sigma)-\frac{\gamma^{m}}{2} \widehat{E}_{m}(\sigma)\right)  \tag{A.31}\\
& \boldsymbol{\Gamma}_{\mathrm{p}_{o} \mathrm{q}_{o}} \mathrm{q}_{o}=\boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{p}_{o}}{ }^{\mathrm{q}_{o}}=0  \tag{A.32}\\
& \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}{ }^{m}=\frac{g^{m k}}{2} \mathrm{q}_{o}\left(\gamma^{i}\right) g_{i k}-\frac{g^{m k}}{2} \widehat{E}_{k}(\beta)-\frac{\gamma^{m}}{2 \alpha} \mathrm{q}_{o}(\alpha)+\frac{\gamma^{m}}{2 \alpha} \mathrm{p}_{o}(\beta)- \\
& -\frac{\beta}{2 \sigma}\left(g^{m k} \widehat{E}_{k}(\sigma)-\frac{\gamma^{m}}{\alpha} \mathrm{p}_{o}(\sigma)\right)  \tag{A.33}\\
& \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{q}_{o}}{ }^{\mathrm{p}_{o}}=\frac{1}{\alpha} \mathrm{q}_{o}(\beta)+\frac{1}{2 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{q}_{o}(\alpha)-\frac{1}{2 \alpha^{2}} \gamma^{m} \gamma^{k} g_{m k} \mathrm{p}_{o}(\beta)-\frac{2}{\alpha^{2}} \beta \mathrm{q}_{o}(\alpha)+ \\
& +\frac{2 \beta}{\alpha^{2}} \mathrm{p}_{o}(\beta)-\frac{\gamma^{m}}{2 \alpha} \mathrm{q}_{o}\left(\gamma^{i}\right) g_{i m}+\frac{\gamma^{m}}{2 \alpha} \widehat{E}_{m}(\beta)- \\
& -\frac{\beta}{\sigma}\left(\frac{1}{\alpha} \mathrm{q}_{o}(\sigma)+\frac{1}{2 \alpha^{2}}\left(\gamma^{m} \gamma^{k} g_{m k}-4 \beta\right) \mathrm{p}_{o}(\sigma)-\frac{\gamma^{m}}{2 \alpha} \widehat{E}_{m}(\sigma)\right)  \tag{A.34}\\
& \boldsymbol{\Gamma}_{\mathrm{q}_{o} \mathrm{o}_{o}} \mathrm{q}_{o}=\frac{1}{\alpha} \mathrm{q}_{o}(\alpha)-\frac{1}{\alpha} \mathrm{p}_{o}(\beta)+\frac{1}{\sigma} \mathrm{q}_{o}(\sigma)-\frac{\beta}{\alpha \sigma} \mathrm{p}_{o}(\sigma) \tag{A.35}
\end{align*}
$$

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[^0]:    2020 Mathematics Subject Classification. 83C20, 83C50, 53C25, 32V05, 32V30, 53C17.
    Acknowledgments. D.A. and G.S. were partially supported by the Australian Research Council, Discovery Grant DP130103485. D.A. was supported also by the grant no. 18-00496S of the Czech Science Foundation.

[^1]:    ${ }^{1}$ We do not assume that $\mathcal{D}$ is bracket generating, i.e. a contact distribution. The structures with this additional hypothesis are discussed in 4.1.1 below.

[^2]:    ${ }^{2}$ We recall that this occurs if and only if $S$ is Sasaki-Einstein (see e.g. [5, Ch.11]).

[^3]:    ${ }^{3}$ Following [10], we define $R$ by the formula $R_{X Y} Z:=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z$.

