# Optimal solution of the liquidation problem under execution and price impact risks

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**Abstract**

We consider an investor that trades continuously and wants to liquidate an initial asset position within a prescribed time interval. As a consequence of his trading activity, during the execution of the liquidation order, the investor has no guarantees that the placed order is executed immediately, it may go unfilled, partially filled or filled in excess. The uncertainty in the execution affects the trading activity of the investor and the asset share price dynamics generating additional sources of noise: the execution risk and the price impact risk, respectively. Assuming the two sources of noise correlated and driven by the cumulative effect of the investor trading strategy, we study the problem of finding the optimal liquidation strategy adopted by the investor in order to maximize the expected revenue resulting from the liquidation. The mathematical model of the liquidation problem presented here extends the model of Almgren and Chriss (Almgren, R., Chriss, N., Optimal execution of portfolio transactions, Journal of Risk, 2000) to include execution and price impact risks. The liquidation problem is modeled as a linear quadratic stochastic optimal control problem with finite horizon and, under some assumptions about the functional form for the magnitude of execution and price impact risks, is solved explicitly. The derived solution coincides with the optimal trading strategy obtained in the absence of execution uncertainty for an asset price with a modified growth rate. This suggests that the uncertainty in the execution modifies the directional view of the investor about the future growth rate of the asset price.

**Keywords:** liquidation problem, stochastic optimal control, execution risk, price impact risk, Hamilton Jacobi Bellman equation
**AMS Subject Classifications:** 93E20, 60H10, 49L20
**JEL Codes:** C0, C61

## 1 Introduction

The liquidation problem is the problem of finding the optimal strategy adopted by an investor in order to liquidate his position on a risky asset within a prescribed time interval, called liquidation interval. The liquidation problem is widely studied in mathematical finance (see, among others, Almgren and Chriss, 2000, Almgren, 2003, Almgren, 2012, Ankirchner et al., 2016, Fatone et al., 2014, Frei and Westray, 2015, Guéant and Lehalle, 2015, Lorenz and Schied, 2012, Schied et al., 2010, Tse et al., 2013). The mathematical models of the liquidation problem studied in these papers assumes that the execution of the liquidation order influences the asset share price inducing a difference between the expected asset price and the actual price at which the trade is executed. In the financial markets usually this is the case when the liquidation order is a market order of large size.

The earliest model of the liquidation problem has been introduced by Almgren and Chriss (2000), this is a discrete time model. Continuous time versions of the Almgren and Chriss model are developed by Almgren (2003), Gatheral and Schied (2011) and Forsyth et al. (2012). In these models the asset share price is the sum of an arithmetic Brownian motion and of a term that describes the impact of the investor trading activity. The utility function is the difference between the expected revenue resulting from the execution of the liquidation order and its variance. The liquidation problem is modeled as a mean variance optimization problem that is reduced to an elementary calculus of variations problem. Several generalizations of the continuous time model introduced in Almgren (2003) have been developed. For example, Almgren (2012) studies how liquidity affects the asset share price dynamics. In Fatone et al. (2014) the presence on the market of retail investors and its consequences on the execution of the liquidation order are considered. The retail investors are modeled as an homogeneous population of small investors whose behaviour is described by a mean field game. More recently, Huang et al. (2019) use the mean field approach to model the liquidation problem in presence of multiple traders. Guéant and Lehalle (2015) assume the utility function to be a C.A.R.A. (Constant Absolute Risk Aversion) function and study the effects of limit order books on the execution of the order. In all these models the trading strategies are deterministic functions.

Trading strategies modeled as stochastic processes have been considered in Schied et al. (2010), Ankirchner et al. (2016), Cheng et al. (2017), Bulthuis et al. (2017). In Schied et al. (2010) the trajectories of the trading strategy are bounded and absolutely continuous functions of time defined in the liquidation interval. In Ankirchner et al. (2016) the effects of trends in the asset share price on the execution of the liquidation order are studied and the trading strategy is a square integrable stochastic process of time. In both papers a liquidation condition is imposed on the admissible trading strategies to require that at the end of the liquidation interval the initial asset share position is sold with probability one. The asset share price dynamic equation of Schied et al. (2010) and Ankirchner et al. (2016) is the same used by Almgren (2003) and the liquidation problem is modeled as a stochastic optimal control problem. Under some assumptions the value functions of the control problems are determined as solutions of the corresponding Hamilton Jacobi Bellman equations and of their auxiliary conditions (i.e. an initial condition in Schied et al., 2010, and a final condition in Ankirchner et al., 2016). The auxiliary condition used in Schied et al. (2010) and Ankirchner et al. (2016) is known in aeronautical engineering as fuel condition (see Bather and Chernoff, 1967, and, in the financial context, Schied et al., 2013). The fuel condition of aeronautical engineering is a final condition that guarantees that no fuel is left unused at the end of the mission planned. In the liquidation problem the same condition guarantees that at the end of the liquidation interval the investor has completed (with probability one) the sale of the asset shares initially held.

Trading strategies that are diffusion processes are considered in Cheng et al. (2017) to model the order fill uncertainty. The liquidation problem is solved in two different settings: in the first one the magnitude of order fill uncertainty is a prescribed positive constant parameter independent of the trading strategy, in the second one the magnitude of order fill uncertainty is a linear function of the optimal trading strategy. In the first setting the optimal trading strategy is found explicitly in terms of elementary functions without any constraints. In contrast, when the magnitude of uncertainty is a linear function of the trading strategy, the system of Riccati equations associated to the Hamilton Jacobi Belmann equation is solvable under some strong assumptions on the parameters of the problem and the solution, when there exists, cannot be expressed in terms of elementary functions. More recently, Bulthuis et al. (2017) have extended the model of Cheng et al. (2017) to include the uncertainty of limit order fills. The model is enriched by the addition of constraints to bound the trading strategy of limit and market orders and of a "trade director" to penalize trading strategies made simultaneously by buy side market and sell limit orders. A further extension of the model of Cheng et al. (2017) is done by Cheng et al. (2019) in the case of constant uncertainty. The new model adds to the old one a dynamic risk adjustment of the liquidation strategy. The risk adjustment is taken into account adding to the profit and loss function a quadratic term penalizing the strategies that are far from a prescribed target value.

Recently many authors investigate the noisy nature of the price impact. In Graewe et al. (2017) the authors consider linear temporary price impact and persistent price impact of limit order book (LOB) with stochastic resilience. In this context the resilience is the capacity of the LOB to revert to its normal shape after the market order placed by the large trader. The persistent impact measures the effect of the past trades on the asset share price and decays over time. In Becherer et al. (2018) the price impact is interpreted as a volume effect process and modeled as an Ornestein-Uhlenbeck stochastic process depending on the large trader holding. The assumption is motivated by the need of having a process that reverts to zero in absence of trading. The diffusion term in the stochastic volume process is interpreted as an exogenous noise induced by the trading activities of other large traders that are independent of the stochastic volume process itself. Ma et al. (2020) use an Ornestein-Uhlenbeck stochastic process to model the permanent price impact due the activity of other market participants.

The works of Cheng et. al. (2017), Bulthuis et al. (2017), on the one hand, and of Graewe et al. (2017), Becherer et al. (2018), Ma et al. (2020), on the other hand, provide new research lines about optimal execution including execution risk and stochastic price impact. Our paper is related to both these strands of literature, aimed to provide a new model of liquidation problem that takes into account the cumulative impact of trading strategy on the order execution and on the asset share price.

In automated financial markets the order, after being scheduled, goes into a processing system, posed in a queue and executed as quickly as possible. When placing a market order, an investor is guaranteed to execute the order as the next available price. Therefore an investor that schedules a market order gives a priority to the certainty of execution over the certainty of the execution price. However, there are no guarantees that the placed order, especially if large, is executed immediately, in fact it may go unfilled, partially filled or filled in excess. The overfill, for example, can incur when the trading system places simultaneously more orders than it needs to fill, collect their fills and cancel the excess orders afterwards. The causes of the lag between the placement and the settlement of an order can be many, from the unavailability of requested asset volume to the size of the order. Similarly, private taste shocks or beliefs (Sannikov and Skrzypacz, 2016, Kyle et al., 2017) or private information regarding the asset value and/or inventories (Du and Zhu, 2017) as well as uncertainty in order fills (Cheng et al., 2017, Bulthuis et al., 2017) can deviate the realized trading strategy of the investor from the originally scheduled trading strategy. Because of the phenomena generating this deviation are hardly predictable, we refer to all of them indifferently as execution risk. Trade urgency exacerbates the execution risk improving the probability of suboptimal trading executions and noisy market reactions (see Sannikov and Skrzypacz, 2016). Moreover, we suppose that, because more unexpected, the noise generated by earlier trades is larger than the noise due to later trades. This last assumption is in line with the empirical findings of Capponi and Cont (2019), according to which the price changes, caused by the execution of the order, depends on trade duration. Based on the previous considerations, we measure the execution risk as the cumulative effect of the scheduled trading strategy assuming that this scheduled trading strategy contributes to the execution risk not only instantaneously but also in the future with magnitude proportional to the square root of the residual trade duration. In line with Sannikov and Skrzypacz (2016), Cheng et al. (2017), Bulthuis et al. (2017) we model the effects of execution risk on the trading strategy assuming that the holding position is an Îto diffusion process whose noise term characterizes the magnitude of the execution risk. The addition of the execution risk in the dynamics of the holding position is an attempt to model the filled order instead of the scheduled order, as pointed out by Cont and Kukanov (2021). Our model generalizes the model of Cheng et al. (2017), where the magnitude of the execution risk is assumed to be constant and/or a linear function of the scheduled trading strategy.

The scheduled order reveals private information of the investor about the value of the traded asset, as a consequence, the market participants update their beliefs about the asset price generating unexpected trading activities that are quickly reflected by shocks on price volatility (see Büyüksahin and Harris, 2009). The idea that the trading activity is a driver of price volatility is not new, just think to the wide literature about the relationship between trading volume/frequency and price volatility (see, among others, Huang and Masulis, 2003, Avramov et al., 2006) or to the role of the trading activity to predict volatility supported by Jang et al. (2019). The further risk induced by the trading activity in the price volatility can be interpreted as a price impact risk. The price impact risk can also read as the difference between the price changes determined by the executed order and those due to the scheduled order. Since the price impact risk is a byproduct of the investor trading activity, similarly to execution risk, it is assumed to be driven by the cumulative effects of the scheduled trading strategies with magnitude proportional to the square root of the residual trade duration. Specifically, the price impact risk is taken into account adding to the asset share price dynamic equation a noise term driven by a Wiener process correlated to the the holding position. Both the noise terms of holding position and asset share price dynamic equations are assumed to be square root functions of the scheduled trading strategy and of the time left to reach the end of the liquidation interval. Therefore, the price impact is made by two components: the price impact cost, responsible for the changes in the price drift, and the price impact risk, responsible for the changes in the price volatility. The noisy nature of the price impact is also supported by the empirical findings of Moro et al. (2009) and by the stochastic impact models of Graewe et al. (2017), Becherer et al. (2018), Ma et al. (2020). Finally, it should be noted that assuming the price impact as a square root function of trading strategy is in line with the empirical studies of Moro et al.(2009).

Note that, because of execution risk, at the end of the liquidation interval the investor can have a residual asset position to sell in order to complete the liquidation order. In this case the residual asset position must be sold at the final time. In order to penalize trading that at the end of the liquidation interval has not completed the liquidation, we consider as utility function of the control problem the sum of the expected revenue resulting from the liquidation and of a term penalizing the trading strategies that at the end of the liquidation interval have residual amount of asset shares left unsold. The asset share price dynamic equation of the model presented in this paper is that of Almgren and Chriss (2000) except for the temporary impact term, that here is proportional to the scheduled trading strategy instead of the (actual) trading strategy, and for the presence of a further source of noise due the price impact risk. The liquidation problem consists in finding the drift of the holding position that maximizes the utility function. The liquidation problem is formulated as a linear quadratic stochastic optimal control problem that has the holding position as state variable and the scheduled trading strategy as control variable. We use the completion of squares method to derive the Hamilton Jacobi Bellman equation and the optimal feedback control. Explicit formula of the optimal scheduled trading strategy is found. The optimal scheduled trading strategy of the model considered is determined and its dependence on the model parameters is studied.

The contribution of this paper is twofold. Firstly, we introduce a new model for the execution risk that takes into account both the cumulative impact of trading strategy on the order execution and a stochastic price impact. This model has the advantage of being analytically solvable without imposing any constraints on the model parameters. Secondly, we provide a different perspective of the price impact decomposing it into two factors: the price impact cost and the price impact risk. The price impact cost refers to the additional cost incurred by the investor as a consequence of the price change due to its own trading activity. The price impact risk is the additional source of volatility induced in the price by the trading activity and reflects the reaction of the other investors to the scheduled order.

Note that in this paper we consider the liquidation problem under execution risk for a single asset. It would be interesting to extend this problem to a multiple assets problem, that is to a basket or a portfolio made of different assets. In these cases both the correlation and the co-integration of the assets must be included in the liquidation problem together with the market impact of the order flow from other market participants. One simple strategy is to consider each asset of the multidimensional problem independently and employ a liquidation strategy for individual assets, see for example, Almgren and Chriss (2000). This strategy is optimal if the assets in the portfolio do not exhibit any co-movements or dependence. In a more general framework the correlation and the co-integration of the assets and the price impact of market orders from all market participants (including the agent liquidating the basket or the portfolio) must be considered. Some attempts in this direction are given in Cartea et al. (2019), Tsoukalas et al. (2019).

This paper is structured as follows. In Section 2 we formulate the liquidation problem. In Section 3, under some hypotheses on the form of execution risk, we solve the model introduced in Section 2. In Section 4 we discuss some case studies that illustrate the behaviour of model presented in Section 2. Finally, in Section 5 some conclusions are drawn.

## 2 The model

We consider an investor that wants to liquidate within a fixed time interval, called liquidation interval, a prescribed number of shares of a risky asset traded in the financial market. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ be the set of real positive numbers and $T, Y \in \mathbb{R}$ be positive numbers. We denote by $[0, T]$ the liquidation interval and by $Y$ the initial amount of asset shares that must be sold within the time interval $[0, T]$. Let $y(t)$ be the holding position, i.e. the number of asset shares held by the investor at time $t \in [0, T]$, and $v(t, y(t)) : [0, T] \times \mathbb{R} \to \mathbb{R}$ be the scheduled trading strategy associated to the holding position $y(t)$, $t \in [0, T]$. To keep the notation simple, in the rest of paper the dependence of $v$ on $y$ is omitted and we use the shorthand notation $v(t)$ to denote $v(t, y(t))$, $t \in [0, T]$.

The scheduled trading strategy $v(t)$ is the strategy scheduled by the investor to sell the asset shares at time $t$, $t \in [0, T]$. Because of the execution risk, the (realized) holding position $y(t)$, $t \in [0, T]$, satisfies the following stochastic differential equation:

$$dy(t) = -v(t)\ dt + \phi(t, v(t))\ dW(t), \quad t \in [0, T], \tag{2.1}$$

where $\phi$ is a real function and $W(t)$, $t \in [0, T]$, is a standard Wiener process. The function $\phi$ characterizes the magnitude of the execution risk. The holding position of the investor in $[0, T]$ changes as a consequence of the desire of the investor to buy or sell (modeled in (2.1) by the term $vdt$) and of the effects of execution risk (modeled in (2.1) by the term $\phi dW$).

The presence in the holding position dynamics (2.1) of the diffusion term $\phi dW$ has many possible explanations (see, among others, Sannikov, Y., Skrzypacz, 2016, Cheng et al., 2017). For example, if the investor is a broker executing the liquidation order on the behalf of his clients, the diffusion term $\phi dW$ can model the shocks generated by the random orders of his clients. More generally, the diffusion term $\phi dW$ can model investor belief shocks (Kyle et al., 2017) or uncertainty in the order fills (Cheng et al., 2017 and Bulthuis et al., 2017). Equation (2.1) is equipped with the initial condition:

$$y(0) = Y. \tag{2.2}$$

Equation (2.1) is the state equation of the model of the liquidation problem studied in this paper, the initial condition (2.2) assigns the amount of asset shares that must be sold within the liquidation interval $[0, T]$. The function $v$ is the unknown control variable of the liquidation problem.

Let $t \in [0, T]$, we denote by $S^0(t)$ the *market* price of the asset share at time $t$, and by $S(t)$ the corresponding *execution* price (see Forsyth et al., 2012) at time $t$, that is the price realized after the sale.

We assume that $S(t)$, $t \in (0,T]$, is a stochastic process modeled by the following equations:

$$S(t) = S^0(t) + \kappa(H(t) - v(t)), \qquad t \in (0,T], \tag{2.3}$$
$$S^0(t) = S^0_0 + \mu t + \gamma(y(t) - Y) + \psi B(t), \qquad t \in (0,T], \tag{2.4}$$

where $H(t) = \int_0^t \chi(s, v(s)) dZ(s)$, $t \in [0,T]$, $\chi$ is a real function such that $\chi(t,0) = 0$, $t \in [0,T]$, $\mu \in \mathbb{R}$, $\psi > 0$, $\kappa > 0$, $\gamma > 0$ and $S^0_0 > 0$ are real constants, and $B$ and $Z$ are standard Wiener processes. Note that the prices $S^0(t)$, and $S(t)$, $t \in [0,T]$, solutions, respectively, of (2.3), (2.4), are negative with positive probability. Usually this is an undesirable property since most of the time negative asset share prices are not allowed. However when $S^0_0 > 0$ and $\mu > 0$ are large enough and sufficiently small values of $T$ are considered the event "negative asset share prices" has small probability and can be tolerated, as done in Almgren (2000, 2012), Ankirchner et al. (2016), Fatone et al. (2014), Guéant and Lehalle (2015), Cheng et al. (2017).

We briefly note that when multiple correlated assets are considered, formuale (2.3), (2.4) must be adapted to the circumstances, using, for instance, the Cholesky factorization for the asset price correlation matrix of the Wiener processes appearing in (2.4). For simplicity, we omit these generalizations here but this extension is an interesting on-going research topic.

The terms $\gamma y$ in (2.4) and $-\kappa v$ in (2.3) model, respectively, the *permanent* and *temporary impacts*. The noise term $H(t)$ in (2.3) measures the additional risk at time $t$ caused by the trading activity through the cumulative effect of the trading strategy scheduled by the investor over the time interval $[0,t]$, $t \in [0,T]$.

The stochastic process $S^0(t)$, $t \in [0,T]$, describes the market price (see Cheng at al., 2017 and Forsyth et al., 2012), i.e. the asset share price that is not influenced from the temporary impact, and is defined by equation (2.4). Throughout the paper we will refer to the term $\kappa(H - v)$, responsible for the difference between the execution and the market prices, as price impact.
Moreover, we assume:

$$\mathbb{E}(dB(t), dW(t)) = \mathbb{E}(dB(t), dZ(t)) = 0, \text{ and } \mathbb{E}(dZ(t), dW(t)) = \rho dt, \tag{2.5}$$

i.e. the Wiener processes $B(t)$, $W(t)$ and $B(t)$, $Z(t)$, $t \in [0,T]$, are uncorrelated, while the Wiener processes $Z(t)$, $W(t)$, $t \in [0,T]$, are correlated with constant correlation coefficient $\rho \in [-1,1]$. In fact, unlike the market risk (driven in (2.4) by the Wiener process $B$), which is independent of the investor trading activity, the noises in (2.1) and (2.4), modeled by the Wiener processes $Z$ and $W$, are both caused by the uncertainty in the execution, so they are correlated.

Equations (2.3), (2.4) extend the asset share price dynamic equation introduced by Almgren and Chriss (2000) to the case where the trading strategy is subject to execution risk. In (2.3) the drift coefficient $v(t)$, $t \in [0,T]$, of the state equation (2.1) replaces the time derivative of the holding position used in Almgren (2003). These two terms coincide when in (2.1) we choose $\phi \equiv 0$.

Let us justify the choice made in (2.3) of the term $\kappa(H - v)$ used to model the price impact of the trading strategy on the asset share price. First of all it must be observed that when $\phi \neq 0$ the trajectories of the diffusion process (2.1) are not differentiable, therefore it is not possible to consider their time derivative as done in Almgren and Chriss (2000). Secondly, it must be noted that, in absence of the noise term $\phi dW(t)$ in (2.1), the scheduled trading strategy $v(t)$, $t \in [0,T]$, determines the holding position and, as a consequence, affects the asset share price dynamics. In contrast, when $\phi \neq 0$, in real markets, where the prices are the result of auctions, the desired strategy $v$ influences the asset share price dynamics even when, due to unexpected circumstances (modeled in (2.1) through the term $\phi dW$), the desired strategy of sale does not define completely the holding position dynamics. By choosing the scheduled trading strategy $v$, the investor selects his desirable amount of asset shares to sell. This choice influences the asset share price

dynamics and, consequently, the strategies of the other investors trading on the same asset. Therefore the investor trading activity generates an extra cost as a consequence of the price change due to the trading itself. For this reason we refer to the term $-\kappa v$ as price impact cost. Differently from the price impact cost, the term $\kappa H$ affects directly the noise of the execution price $S$ and represents the additional noise generated by the influence of the scheduled order on the market price. Such additional noise can be due, for example, to the behaviour assumed by the other investors trading on the same asset, as a consequence of the placement of the liquidation order. Because of its noisy nature, we refer to this term as price impact risk. It should be noted that the price impact cost and the price impact risk can be read as the drift and the volatility of a stochastic price impact process like those used by Graewe et al. (2017), Becherer et al. (2018) and Ma et al. (2020).
The expected revenue resulting from the liquidation at time $T$ is given by:

$$\mathbb{E}\left[-\int_0^T (S(t)-S_0^0)dy(t) + (S(T)-S_0^0)y(T)\right], \tag{2.6}$$

where $\mathbb{E}[\cdot]$ denotes the expected value of $\cdot$. In (2.6) the term

$$R = \mathbb{E}\left[(S(T)-S_0^0)y(T)\right] = \mathbb{E}\left[(S^0(T)-S_0^0)y(T)\right] + \mathbb{E}\left[\kappa(H(T)-v(T))y(T)\right] \tag{2.7}$$

represents the expected revenue resulting from the liquidation at the final time $t=T$. Since the holding position of the investor is subject to random noise (see equation (2.1)) it is possible that at the end of the liquidation interval there is a residual amount of asset shares $y(T)$ to sell or buy. This adds to the expected revenue resulting from the liquidation at the market price $S^0(T)$ (i.e. the term $\mathbb{E}\left[(S^0(T)-S_0^0)y(T)\right]$) an extra term, due to the risk of trading at the execution price $S(T)$ instead of the market price $S^0(T)$. This extra term is given by $\mathbb{E}\left[\kappa(H(T)-v(T))y(T)\right]$ in (2.7). In line with Cheng at al. (2017) we assume that:

$$\mathbb{E}\left[-\kappa v(T)y(T)\right] = \mathbb{E}\left[-\lambda y^2(T)\right], \tag{2.8}$$

where $\lambda > 0$ is a real constant.

For $t \in [0,T]$ let $\mathcal{M}_{[t,T]}$ be the set of the real-valued absolutely continuous and adapted processes in $[t,T]$. We define the set of admissible controls as the set of square integrable processes, that is:

$$\mathcal{A}_t = \left\{ g \in \mathcal{M}_{[t,T]} \ : \ \int_t^T \mathbb{E}[g^2(t)]dt < +\infty \right\}. \tag{2.9}$$

The liquidation problem is formulated as the following linear quadratic stochastic optimal control problem:

$$\max_{v\in\mathcal{A}_0} \mathbb{E}\left[-\int_0^T (S(t)-S_0^0)dy(t) + (S^0(T)-S_0^0)y(T) - \lambda y^2(T) + \kappa H(T)y(T)\right], \tag{2.10}$$

subject to the constraints (2.1), (2.2).

The penalization term $\mathbb{E}\left[-\lambda y^2(T)\right]$ in (2.10) measures the cost to sell at time $T$ the residual amount of asset shares $y(T)$ at the execution price $S(T)$ instead of the market price $S^0(T)$ (see Cheng at al., 2017). In line with Karatzas et al. (2000) for the finite-fuel control problem and Cheng et al. (2017), Bulthuis et al. (2017) for the liquidation problem, we consider a quadratic penalization term. It is worthing to note that as $\lambda \to +\infty$ the cost of selling at the end of the liquidation interval goes to infinity, i.e. the liquidation at time $T$ is not allowed and the final condition $y(T)=0$ is enforced. The condition $y(T)=0$ is the well known *finite fuel constraint* introduced by Beneš et al. (1980) and further developed by Karatzas (1985).

When $\phi_0 = \chi_0 = 0$ (i.e there are no execution and price impact risks) and $\lambda \to +\infty$ (i.e. the liquidation is completed at $T$ with probability one) problem (2.10), (2.1), (2.2) reduces to the optimal execution problem solved by Almgren (2003), therefore the optimal trading strategy, solution of problem (2.10), (2.1), (2.2) when $\mu = 0$, is the Volume Weighted Average Price (VWAP) strategy that sells in each time interval an amount proportional to the predicted volume for that interval (Almgren, 2003).

# 3 The solution

The uncertainty surrounding the investor trading activity, affecting both the holding position and the asset share price, has marginal effects in time. These marginal effects depend on time and on the trading strategy of the investor and are expressed by the terms $\phi(t, v)$ and $\chi(t, v)$, $t \in [0, T]$, $v \in \mathbb{R}_+$, in (2.1) and (2.3), respectively. The market reaction to the liquidation, especially at the beginning, is large because the other market participants, who do not have a priori information about the investor intent, are taken caught unaware. As a consequence of this news, the market participants adopt different behaviours impacting both the holding position of the investor and the price dynamics. Over time, the same strategy is perceived as less unexpected. Once the trading strategy is fixed, earlier investor trades are noisier than later trades. Moreover, the greater the urgency to complete the liquidation order, due to the sizable residual asset position, roughly measured by $v(t)(T - t)$, the greater is the magnitude of execution risk (see Sannikov and Skrzypacz, 2016) and the impact on the price.

Based on the previous considerations and in line with the empirical findings of Capponi and Cont (2019), we model the marginal effects $\phi(t, v)$, and $\chi(t, v)$, $t \in [0, T]$, $v \in \mathbb{R}_+$, in (2.1) and (2.3), respectively, as follows:

$$\phi(t, v) = \phi_0 \sqrt{(T - t)v}, \; t \in [0, T], \; v \in \mathbb{R}_+, \tag{3.1}$$

$$\chi(t, v) = \chi_0 \sqrt{(T - t)v}, \; t \in [0, T], \; v \in \mathbb{R}_+, \tag{3.2}$$

where $\phi_0 > 0$, $\chi_0 > 0$.
In other words, formulae (3.1), (3.2) say that, fixed the trading strategy, the marginal effect of execution and price impact risks decreases over time and vanishes when the liquidation is completed. It is worth noting that the square root dependence on trade duration, $T - t$, is in line with the empirical findings of Capponi and Cont (2019), while the square root dependence on trading strategy, $v$, is in line with the empirical studies of Moro et al. (2009). Finally, choices (3.1), (3.2) allow for an explicit solution of problem (2.10), (2.1), (2.2) expressed in terms of elementary functions without imposing any constraints on the model parameters.

It is worth noting that in Cheng et al. (2017) the execution risk $\phi$ in (3.1) is modeled as a constant term or a linear function of the trading strategy. This means that, instead of (3.1), in Cheng et al. (2017) the authors consider

$$\phi(t, v) = m_0, \quad \text{or} \quad \phi(t, v) = m_1 v, \; t \in [0, T], \; v \in \mathbb{R}_+, \tag{3.3}$$

where $m_0$, $m_1$ are real constants. Note that when we choose the marginal effects $\phi$ and $\chi$ as in (3.1), (3.2) in (2.3) (2.4) and $\rho = 1$ in (2.5) our model reduces to a model involving only the execution risk that extends the model of Cheng et al. (2017) by considering the execution risk as a square root function of the trading strategy.

**Proposition 3.1**
Given $v \in \mathcal{A}_0$, $S$ solution of (2.3), (2.4) and $y$ solution of (2.1), (2.2), the expected revenue in (2.10) can be rewritten as follows:

$$R = \mathbb{E}\left[ -\lambda y^2(T) + \frac{\gamma}{2}(y^2(T) - Y^2) + \int_0^T \left( \mu y(t) + \left( \frac{\gamma}{2}\phi_0^2 + \kappa\rho\chi_0\phi_0 \right)(T - t)v(t) - \kappa v^2(t) \right) dt \right]. \tag{3.4}$$

*Proof.*
By (2.1), (2.2) and (2.3), (2.4) we have:

$$\begin{aligned}
-\int_0^T (S(t)-S_0^0)dy(t) &= -\int_0^T (\mu t+\gamma(y(t)-Y)+\psi B(t)-\kappa v(t)+\kappa H(t))dy(t)\\
&= -\mu\int_0^T t dy(t)-\gamma\int_0^T y(t)dy(t)+\gamma Y\int_0^T dy(t)-\psi\int_0^T B(t)dy(t)\\
&\quad+\kappa\int_0^T v(t)dy(t)-\kappa\int_0^T H(t)dy(t). \qquad (3.5)
\end{aligned}$$

Since:

$$y(t)dy(t)=\frac{1}{2}d(y^2(t))-\frac{1}{2}\phi_0^2(T-t)v(t)dt, \qquad t\in[0,T], \qquad (3.6)$$

$$B(t)dy(t)=d(B(t)y(t))-y(t)dB(t), \qquad t\in[0,T], \qquad (3.7)$$

$$H(t)dy(t)=d(H(t)y(t))-y(t)dH(t)-\rho\chi_0\phi_0(T-t)v(t)dt, \qquad t\in[0,T]. \qquad (3.8)$$

Substituting (3.6), (3.7), (3.8) into (3.5) we have:

$$\begin{aligned}
-\int_0^T (S(t)-S_0^0)dy(t)+(S^0(T)-S_0^0)y(T) &= -\kappa H(T)y(T)+\frac{\gamma}{2}(y^2(T)-Y^2)\\
&+\int_0^T\left(\mu y(t)+\left(\frac{\gamma}{2}\phi_0^2+\kappa\rho\chi_0\phi_0\right)(T-t)v(t)-\kappa v^2(t)\right)dt\\
&+\psi\int_0^T y(t)dB(t)+\kappa\phi_0\int_0^T\sqrt{(T-t)v^3(t)}dW(t)+\kappa\int_0^T y(t)dH(t). \qquad (3.9)
\end{aligned}$$

By the assumption $v\in\mathcal{A}_0$, by the Jensen inequality and by (2.1) there exists a real constant $K>0$ such that

$$\sup_{t\in[0,T]} y^2(t)\le K\left(1+\int_0^T v^2(s)ds+\sup_{t\in[0,T]}\left(\int_0^t \phi_0\sqrt{(T-s)v(s)}dW(s)\right)^2\right)<+\infty,\ t\in[0,T],$$

using the Îto isometry we have

$$\mathbb{E}\left[\left(\int_0^t \phi_0\sqrt{(T-s)v(s)}dW(s)\right)^2\right]=\int_0^t \phi_0^2(T-s)v(s)ds.$$

Finally, applying the Burkholder-Davis-Gundy inequality, there exists constants $K',K''>0$ such that

$$\mathbb{E}\left[\int_0^T y^2(t)dt\right]\le K'\mathbb{E}\left[\sup_{t\in[0,T]} y^2(t)\right]\le K''\int_0^T\mathbb{E}\left[1+\int_0^T (v^2(s)+\phi_0^2(T-s)v(s))ds\right]<+\infty.$$

Then we have:

$$\mathbb{E}\left[\int_0^T y(t)dB(t)\right]=0. \qquad (3.10)$$

By the assumption $v\in\mathcal{A}_0$ we have $\mathbb{E}\left[\int_0^T (T-t)v^3(t)dt\right]\le T\mathbb{E}\left[\int_0^T v^3(t)dt\right]<\infty$ then

$$\mathbb{E}\left[\int_0^T\sqrt{(T-t)v^3(t)}dW(t)\right]=0. \qquad (3.11)$$

Moreover, from $\mathbb{E}\left[\int_0^T (T-t)^2 v^2(t)dt\right] \le T^2\mathbb{E}\left[\int_0^T v^2(t)dt\right] < +\infty$ it follows that the stochastic process $H(t)$, $t \in [0,T]$, is a martingale and

$$\mathbb{E}\left[\int_0^T y(t)dH(t)\right] = 0. \tag{3.12}$$

Finally, substituting (3.10), (3.11), (3.12) into (3.9) we obtain (3.4). This concludes the proof.

**Proposition 3.2**
The value function of stochastic optimal control problem (2.10), (2.1), (2.2) satisfies the following Hamilton Jacobi Bellmann equation:

$$\frac{\partial V(t,y)}{\partial t} + \frac{1}{4\kappa}\left(\frac{\phi_0^2}{2}(T-t)\frac{\partial^2 V(t,y)}{\partial y^2} + \left(\gamma\phi_0^2 + \kappa\rho\chi_0\phi_0\right)(T-t) - \left(\frac{\partial V(t,y)}{\partial y} + \gamma y\right)\right)^2 + \mu y = 0 \tag{3.13}$$

with final condition:

$$V(T,y) = -\lambda y^2. \tag{3.14}$$

The optimal scheduled trading strategy $v^*(t)$, $t \in [0,T]$, solution of problem (2.10), (2.1), (2.2) has the state-feedback expression:

$$\begin{aligned} v^*(t,y) = \quad & \frac{y(t)}{T-t+\alpha} - \frac{1}{4\kappa}(\mu + B)\left(T - t + \alpha - \frac{\alpha^2}{T-t+\alpha}\right) + \frac{1}{2\kappa}B(T-t) \\ & + \frac{\alpha}{2\kappa}\left(\frac{B(T-t)}{T-t+\alpha} - \frac{\kappa\phi_0^2}{T-t+\alpha}\ln\left(\frac{T-t+\alpha}{\alpha}\right)\right), \ t \in [0,T], \ y \in \mathbb{R}, \end{aligned} \tag{3.15}$$

where $\alpha = \dfrac{2\kappa}{2\lambda - \gamma} > 0$ and $B = \dfrac{\gamma}{2}\phi_0^2 + \kappa\rho\chi_0\phi_0$.
*Proof*
We use the *completion of squares* method (see Brokett, 1970). Using (3.4) the liquidation problem becomes:

$$\max_{v\in\mathcal{A}_0}\mathbb{E}\left[-\lambda y^2(T) + \frac{\gamma}{2}(y^2(T) - Y^2) + \int_0^T \left(\mu y(t) + \left(\frac{\gamma}{2}\phi_0^2 + \kappa\rho\chi_0\phi_0\right)(T-t)v(t) - \kappa v^2(t)\right)dt\right] \tag{3.16}$$

subject to constraints (2.1), (2.2). The value function associated to problem (3.16), (2.1), (2.2) is given by:

$$\begin{aligned} V(t,y) = \quad & \max_{v\in\mathcal{A}_t}\mathbb{E}_t\left[-\lambda y^2(T) + \frac{\gamma}{2}(y^2(T) - y^2(t)) + \int_t^T \left(\mu y(s) + \left(\frac{\gamma}{2}\phi_0^2 + \kappa\rho\chi_0\phi_0\right)(T-s)v(s)\right.\right. \\ & \left.\left. -\kappa v^2(s)\right)ds\right], \ t \in [0,T], \end{aligned} \tag{3.17}$$

where the maximum is taken over the class of the trading strategies solutions of (2.1), (2.2) whose scheduled trading strategy belongs to $\mathcal{A}_t$. In (3.17) $\mathbb{E}_t[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot|y(t) = y]$, $t \in [0,T]$. Applying Îto formula to $y^2(t)$, $t \in [0,T]$, and using (2.1), (2.2) we have :

$$y^2(T) = y^2(t) + \int_t^T \left(\phi_0^2(T-s)v(s) - 2y(s)v(s)\right)ds + 2\int_t^T \phi_0\sqrt{(T-s)v(s)}y(s)dW(s), \ t \in [0,T], \tag{3.18}$$

then the value function $V$ in (3.17) reduces to:

$$V(t,y) = -\kappa \min_{v\in\mathcal{A}_t} \mathbb{E}_t \quad \left[\int_t^T \left(v^2(s) - \frac{2}{\alpha}y(s)v(s) - \frac{1}{\kappa}\left(\phi_0^2\frac{\gamma}{2} + \kappa\rho\chi_0\phi_0\right)(T-s)v(s) \right.\right.$$
$$\left.\left. + \frac{1}{\alpha}\phi_0^2(T-s)v(s) - \frac{\mu}{\kappa}y(s)\right) ds + \frac{\lambda}{\kappa}y^2\right], \; t\in[0,T], \tag{3.19}$$

where $\alpha = \dfrac{2\kappa}{2\lambda-\gamma}$.
Let:

$$f_1(t) = -\frac{1}{2\kappa}(\mu+B)\left(T-t+\alpha-\frac{\alpha^2}{T-t+\alpha}\right) + \frac{\phi_0^2}{T-t+\alpha}(T-t)$$
$$+\alpha\left(\frac{B}{\kappa}\frac{T-t}{T-t+\alpha} - \frac{\phi_0^2}{T-t+\alpha}\ln\left(\frac{T-t+\alpha}{\alpha}\right)\right), \; t\in[0,T],$$
$$f_2(t) = \frac{1}{T-t+\alpha} - \frac{1}{\alpha}, \; t\in[0,T].$$

We observe that $f_1(T) = f_2(T) = 0$, $f_2'(t) = 1/(T-t+\alpha)^2$ and

$$f_1'(t) = \frac{1}{T-t+\alpha}\left(-\frac{\phi_0^2}{T-t+\alpha}(T-t) + \frac{B}{\kappa}(T-t) + f_1(t)\right) + \frac{\mu}{\kappa}, \; t\in[0,T].$$

The application of Îto formula to $f_1(t)y(t)$ and $f_2(t)y^2(t)$, $t\in[0,T]$, yields:

$$0 = f_1(t)y(t) - \int_t^T \left(f_1(s)v(s) - f_1'(s)y(s)\right) ds + \int_t^T f_1(s)\phi_0\sqrt{(T-s)v(s)}dW(s), \; t\in[0,T], \tag{3.20}$$

$$0 = f_2(t)y^2(t) - \int_t^T \left(2f_2(s)v(s)y(s) - \frac{y^2(s)}{(T-s+\alpha)^2} - f_2(s)\phi_0^2(T-s)v(s)\right) ds$$
$$+ \int_t^T 2f_2(s)y(s)\phi_0\sqrt{(T-s)v(s)}dW(s), \; t\in[0,T]. \tag{3.21}$$

Since $f_1(t)$ and $f_2(t)$ are bounded in $[0,T]$ the stochastic integrals of $\int_t^T f_1(s)\phi_0\sqrt{(T-s)v(s)}dW(s)$ and $\int_t^T 2f_2(s)y(s)\phi_0\sqrt{(T-s)v(s)}dW(s)$ has zero expectation (thought they are not necessarily martingales) and from (3.19), (3.20), (3.21) we have:

$$
\begin{aligned}
V(t,y) = -\kappa \min_{v\in\mathcal{A}_t} \mathbb{E}_t \Bigg[\int_t^T \Big(& v^2(s) - \frac{2}{\alpha} y(s)v(s) - \frac{1}{\kappa}\Big(\phi_0^2\frac{\gamma}{2} + \kappa\rho\chi_0\phi_0\Big)(T-s)v(s) + \frac{1}{\alpha}\phi_0^2(T-s)v(s) \\
&-\frac{\mu}{\kappa}y(s)\Big)\,ds + \frac{\lambda}{\kappa}y^2\Bigg] - \kappa\mathbb{E}_t\left[f_1(T)y(T) + f_2(T)y^2(T)\right] \\
= -\kappa \min_{v\in\mathcal{A}_t} \mathbb{E}_t \Bigg[\int_t^T \Big(& v^2(s) - \frac{2}{\alpha} y(s)v(s) - \frac{1}{\kappa}\Big(\phi_0^2\frac{\gamma}{2} + \kappa\rho\chi_0\phi_0\Big)(T-s)v(s)) \\
&+\frac{1}{\alpha}\phi_0^2(T-s)v(s) - \frac{\mu}{\kappa}y(s) + f_1'(s)y(s) - f_1(s)v(s) + \frac{y^2(s)}{(T-s+\alpha)^2} \\
&-\frac{2}{T-s+\alpha}y(s)v(s) + \frac{2}{\alpha}y(s)v(s) + \frac{\phi_0^2(T-s)}{T-s+\alpha}v(s) - \frac{\phi_0^2(T-s)}{\alpha}v(s)\Big)\,ds \\
&+f_1(t)y + \left(\frac{1}{T-t+\alpha} - \frac{1}{\alpha} + \frac{\lambda}{\kappa}\right)y^2\Bigg],\ t\in[0,T]. \qquad (3.22)
\end{aligned}
$$

Now, adding and subtracting to (3.22) the term: $\frac{1}{4\kappa}\int_t^T \left(\left(-\frac{\kappa\phi_0^2}{T-s+\alpha} + B\right)(T-s) + \kappa f_1(s)\right)^2 ds,\ t \in [0,T]$, we obtain:

$$
\begin{aligned}
V(t,y) = -\kappa \min_{v\in\mathcal{A}_t} \mathbb{E}\Bigg[\int_t^T \Big(& v(s) - \frac{y(s)}{T-s+\alpha} + \frac{1}{4\kappa}(\mu+B)\Big(T-s+\alpha - \frac{\alpha^2}{T-s+\alpha}\Big) \\
&-\frac{1}{2\kappa}B(T-s) - \frac{\alpha}{2\kappa}\Big(\frac{B(T-s)}{T-s+\alpha} - \frac{\kappa\phi_0^2}{T-s+\alpha}\ln\Big(\frac{T-s+\alpha}{\alpha}\Big)\Big)\Big)^2 ds\Bigg] \\
&+ c(t) - \kappa f_1(t)y - \left(\frac{\kappa}{T-t+\alpha} + \frac{\gamma}{2}\right)y^2,\ t\in[0,T],\ y\in\mathbb{R}, \qquad (3.23)
\end{aligned}
$$

where $c'(t) = -\frac{1}{4\kappa}\left(\left(-\frac{\kappa\phi_0^2}{T-t+\alpha} + B\right)(T-t) + \kappa f_1(t)\right)^2$, $t\in[0,T]$, and $c(T)=0$.
By straightforward computations it is easy to verify that the maximum in (3.23) is attained at $v = v^*$ where $v^*$ is given by (3.15) and

$$
V(t,y) = a(t)y^2 + b(t)y + c(t),\ t\in[0,T],\ y\in\mathbb{R}, \qquad (3.24)
$$

where:

$$
\begin{aligned}
a(t) &= -\frac{\gamma}{2} - \frac{\kappa}{T-t+\alpha},\ t\in[0,T], &(3.25)\\
b(t) &= -\kappa f_1(t),\ t\in[0,T]. &(3.26)
\end{aligned}
$$

Note that the functions $a(t)$, $b(t)$, $t\in[0,T]$, are solutions of the following system of Riccati equations:

$$
\begin{aligned}
a'(t) &= -\frac{1}{\kappa}\left(a(t)+\frac{\gamma}{2}\right)^2, & t\in[0,T], \quad (3.27)\\
b'(t) &= \frac{1}{\kappa}\left(a(t)+\frac{\gamma}{2}\right)\left(\phi_0^2(T-t)\left(a(t)+\frac{\gamma}{2}\right) + B(T-t) - b(t)\right) - \mu, & t\in[0,T] \quad (3.28)
\end{aligned}
$$

with final conditions: $a(T) = -\lambda$, $b(T) = 0$.

Finally, by straightforward computations, it is easy to verify that the value function $V$ satisfies the Hamilton Jacobi Bellmann equation (3.13) with final condition (3.14). This concludes the proof.

**Corollary 3.1**
In the limit as $\lambda \to +\infty$ the optimal scheduled trading strategy reduces to:

$$v^*(t,y) = \frac{y(t)}{T-t} - \frac{1}{4\kappa}(\mu - B)(T-t),\ t \in [0,T],\ y \in \mathbb{R}. \tag{3.29}$$

*Proof.* It easily follows taking the limit of (3.15) as $\lambda \to +\infty$.

Recall that when $\phi_0 = 0$ the optimal scheduled trading strategy $v^*$ in (3.29) is the optimal trading strategy of Almgren (2003) under constant directional view about the asset price evolution (see Ankirchner et al. 2016). When we also have zero drift ($\mu = 0$) the optimal trading strategy is the VWAP strategy that sells in each time interval an amount of asset shares proportional to the predicted volume for that interval (see Almgren, 2003). On the other hand, it is worth to note that, when $\phi_0 \neq 0$, the optimal scheduled trading strategy in (3.29) is the optimal trading strategy of Almgren (2003) for a modified asset price $S$ with drift given by $\tilde{\mu} = \mu - B = \mu - \frac{\gamma}{2}\phi_0^2 - \kappa\rho\chi_0\phi_0$. To be specific, under execution risk ($\phi_0 \neq 0$) the investor modifies his directional view about the future asset price growth rate passing from $\mu$ to $\mu - B$. It should also be noted that, when $\rho \geq 0$ (i.e. when there is a non negative correlation between execution and price impact risk) the asset drift $\tilde{\mu}$ in presence of execution risk is smaller than the asset drift $\mu$ in absence of execution risk. In contrast, when $\rho < 0$ (i.e. when there is a negative correlation between execution and price impact risk), we have $\tilde{\mu} > \mu$. After all, it is legitimate to believe that execution and price impact risk are positive correlated. In fact, when execution risk affects the trading strategy determining a decrease on the amount of asset shares sold with respect to the scheduled amount, we expect that the price impact risk causes a simultaneous increase on the asset share price. We can conclude that, assuming a positive correlation between execution and price impact risk, the presence of execution risk changes the directional view of the investor regarding the future price movement causing an asset share return expectation lower than in absence of execution risk.

**Proposition 3.3**
Let $y^*(t)$, $t \in [0,T]$, be the optimal holding position of problem (3.16), (2.1), (2.2) as $\lambda \to +\infty$ we have $\lim_{t \to T^-} y^*(t) = 0$ a.s..
*Proof.*
Here we follow Delyon and Hu (2006). From Corollary 3.1 substituting $v^*$, given by formula (3.29), into (2.1) we obtain that the optimal holding position $y^*(t)$, $t \in [0,T]$, associated to problem (3.16), (2.1), (2.2) with $\phi(t,v) = \phi_0\sqrt{(T-t)v}$ and $\chi(t,v) = \chi_0\sqrt{(T-t)v}$, $t \in [0,T]$, $v \in \mathbb{R}_+$, is solution of the following problem:

$$dy^*(t) = -\left(\frac{y^*(t)}{T-t} - \frac{1}{4\kappa}(\mu - B)(T-t)\right)dt + \phi_0\sqrt{y^*(t) - \frac{1}{4\kappa}(\mu - B)(T-t)^2}dW(t),\ t \in [0,T], \tag{3.30}$$

$$y^*(0) = Y. \tag{3.31}$$

Let $\tilde{y}(t) = y^*(t) - \frac{1}{4\kappa}(\mu - B)(T-t)^2$, $t \in [0,T]$, applying Îto's formula to $y^*(t)$, $t \in [0,T]$, it is easy to verify by straightforward computations that $\tilde{y}$ is solution of:

$$d\tilde{y}(t) = -\left(\frac{\tilde{y}(t)}{T-t} - \frac{1}{2\kappa}(\mu - B)(T-t)\right)dt + \phi_0\sqrt{\tilde{y}(t)}dW(t),\ t \in [0,T], \tag{3.32}$$

$$\tilde{y}(0) = \tilde{y}_0, \tag{3.33}$$

where $\tilde{y}_0 = Y - \frac{1}{4\kappa}(\mu - B)T^2$.
Applying Îto's formula to $\frac{\tilde{y}(t)}{T-t}$, $t \in [0,T]$, we deduce:

$$\frac{\tilde{y}(t)}{T-t} = \frac{\tilde{y}_0}{T} + \frac{1}{2\kappa}(\mu - B)t + \phi_0 \int_0^t \frac{\sqrt{\tilde{y}(s)}}{T-s} dW(s),\ t \in [0,T]. \tag{3.34}$$

Since the stochastic process $\left\{ \frac{\sqrt{\tilde{y}(t)}}{T-t} \right\}_{t\in[0,T]}$ is locally bounded a.s., then $M(t) = \int_0^t \frac{\sqrt{\tilde{y}(s)}}{T-s} dW(s)$, $t \in [0,T]$, is a martingale with quadratic variation:

$$\langle M \rangle(t) = \int_0^t \frac{\tilde{y}(s)}{(T-s)^2} ds,\ t \in [0,T]. \tag{3.35}$$

Note that $\langle M \rangle(t) \to +\infty$ as $t \to T^-$ and there exists a constant $K > 0$ such that $\langle M \rangle(t) \leq \frac{K}{T-t}$, $t \in [0,T]$.
Applying Dambis-Dubins-Schwarz theorem (see Klebaner, 2012), we have that there exists a standard one-dimensional Brownian motion $\hat{B}$ such that:

$$M(t) = \hat{B}(\langle M \rangle(t)),\ t \in [0,T]. \tag{3.36}$$

Substituting (3.36) into (3.34) we have:

$$\tilde{y}(t) = (T-t)\left( \frac{\tilde{y}_0}{T} + \frac{1}{2\kappa}(\mu - B)t + \phi_0 \hat{B}(\langle M \rangle(t)) \right),\ t \in [0,T]. \tag{3.37}$$

Finally, since the limit of $t\hat{B}(1/t)$ as $t \to 0$ goes to zero by the Law of Large Numbers for Brownian motions, we have that:

$$\lim_{t \to T^-} (T-t)\hat{B}(\langle M \rangle(t)) = 0 \text{ a.s.}, \tag{3.38}$$

and

$$\lim_{t \to T^-} y^*(t) = \lim_{t \to T^-} \left( \tilde{y}(t) + \frac{1}{4\kappa}(\mu - B)(T-t)^2 \right) = 0 \text{ a.s.}. \tag{3.39}$$

This concludes the proof.

The process $\tilde{y}$, solution of the stochastic differential equation (3.32), is the diffusion process of Deylon and Hu (2206) constructed by adding to the process $\hat{y}$, solution of $d\hat{y}(t) = \phi_0 \sqrt{\hat{y}(t)}$, $t \in [0,T]$, the extra drift term $-\hat{y}(t)/(T-t) + 1/2\kappa(\mu - B)(T-t)$. As $t \to T^-$ this last term becomes increasingly strong forcing the process $\tilde{y}$ to hit 0 at $t = T$ a.s. (see Deylon and Hu, 2006, Whitaker et al., 2016). When $B = \mu$ a popular discretization of the stochastic differential equation (3.32) is the Modified Diffusion Bridge introduced by Durham and Gallant (2002). Note that the process $\tilde{y}$, solution of (3.32), is absolutely continuous with respect to the conditioned process $\hat{y}|0$, that is the process $\hat{y}$ conditioned on hitting 0 a.s. at $t = T$.

The processes $y^*(t)$, $\tilde{y}(t)$, $t \in [0,T]$, solutions of (3.30), (3.31) and of (3.32), (3.33), are Extend Cox Ingersoll Ross (ECIR) square root processes (Hull and White, 1990) with reversion rate $-1/(T-t)$ and time dependent equilibrium levels given, respectively, by $\frac{1}{4\kappa}(\mu - B)(T-t)^2$ and $\frac{1}{2\kappa}(\mu - B)(T-t)^2$.

By straightforward computations we obtain that the expected value of $y^*(t)$, $t \in [0,T]$, is given by $\mathbb{E}(y^*(t)) = \left(\frac{Y}{T} + \frac{1}{4\kappa}(\mu - B)t\right)(T-t)$, $t \in [0,T]$. When $\mu > B$, i.e. when the asset growth rate $\mu$ is large enough or the execution risk parameter $\phi_0$ is small enough, the expected value of the optimal holding position is a concave function of time. This means that the investor, on average, liquidates the initial asset position more quickly over time. This is the behaviour of an investor believing that the asset price will rise in the future, and, as a consequence, postpones selling in time to take advantage of the asset price increase. Otherwise, when $\mu < B$, i.e. when the asset growth rate $\mu$ is small enough or the execution risk parameter $\phi_0$ is large enough, the expected value of the optimal strategy is a convex function of time. This means that the investor, on average, liquidates the initial asset position more slowly over time. This is the behaviour of an investor believing that the asset price is likely to decrease in the future, and, as a consequence, sells more quickly at the beginning of the liquidation to avoid drawbacks due to the asset price decrease.

Differently to Cheng et al. (2017), where the execution risk affects the optimal trading strategy only in its diffusion term, in our model the execution risk also affects the drift of the optimal trading strategy changing the directional view of the investor about the price movements. It is interesting to observe that when we choose $\gamma = -2\kappa\rho\chi_0$ we have $B = 0$ and, in this case, the drift of the optimal holding position $y^*$, solution of (3.30), (3.31), does not depend on $\phi_0$. This happens only if we choose $\rho < 0$, i.e. if we assume that asset share price and holding position are negatively correlated.

In conclusion, both the optimal trading strategy found by Cheng et al. (2017) in the case of constant or linear execution risk (see equations (3.3)) and the optimal trading strategy found in this paper when the execution risk is modeled as in (3.1) are obtained solving suitable stochastic optimal control problems using the standard completion of square method. In Cheng et al. (2017), when the execution risk is constant, the optimal trading strategy is found explicitly in terms of elementary functions without any constraints. In contrast, when the magnitude of uncertainty is a linear function of the trading strategy, the solution, when there exists, cannot be expressed in terms of elementary functions. The liquidation problem studied in this paper generalizes the model of Cheng et al. (2017) since it takes into account both execution and price impact risks. This problem is modeled as a linear quadratic stochastic optimal control problem with finite horizon and, under some ad hoc assumptions on the functional form of the magnitude of execution and price impact risks, is solved explicitly. More precisely the problem has an explicit solution expressed by elementary functions and this solution is obtained without imposing any constraints on the model parameters. In addition to what done in Cheng et al. (2017), we prove that when the penalization term goes to infinity the optimal solution satisfies the liquidation condition almost surely. Unlike the optimal trading strategy found by Cheng et al. (2017) in the case of constant execution risk, the optimal trading strategy derived here depends on the magnitude of the execution risk and of the price impact risk. Moreover, in the limit as $\lambda \to +\infty$, the optimal trading strategy at time $t$ (see equation (3.29)) differs from the one found in Cheng et al. (2017) for the term $\frac{1}{4\kappa}(\mu - B)(T-t)$, $t \in [0,T]$. This term takes into account both the execution risk and the price impact risk and is responsible for the modification of the directional view about the asset price growth rate of the investor, which changes from $\mu$, in absence of the execution risk, to $\mu - B$, in presence of the execution risk.

## 4 Case studies

In this section we analyze the behaviour of the optimal trading strategy obtained in Proposition 3.2 in two case studies that differ for the orders considered. Moreover, we compare the optimal trading strategy obtained in Proposition 3.2 with the adaptive VWAP strategy (also called constant uncertainty trading strategy) of Cheng et al. (2017). The adaptive VWAP strategy is the solution of problem (2.10), (2.1), (2.2) when $\chi_0 = 0$ and $\phi = m_0$, where $m_0$ is a real constant (see equations (3.3)). When $\phi_0 = \chi_0 = 0$ the optimal trading strategy obtained in Proposition 3.2 and the adaptive VWAP strategy of Cheng et al. (2017) coincide with the deterministic VWAP strategy of Almgren and Chriss (2000). For shortness, in the rest of Section we call the optimal holding position and trading strategy obtained in Proposition 3.2 square root uncertainty holding position and trading strategy, respectively.

We simulate, with the explicit Euler method, the optimal trading strategy, solution of (2.10), (2.1), (2.2), and the adaptive VWAP strategy of Cheng et al. (2017) using as simulation parameters those used in Almgren and Chriss (2000) and Cheng et al. (2017). To guarantee a fair comparison between the two models, across all simulations we generate the trajectories using the same Brownian motions. Specifically, assuming the trading year made by 252 trading days, we consider as time unit a trading day and we choose: the initial asset share position to liquidate $Y = 10^6$, the liquidation interval of one day, i.e. $T = 1$, the initial asset share price $S_0^0 = 50\$/share$, an annual volatility of 30%, i.e. $\sigma = 0.3/\sqrt{252} \cdot 50(\$/(share)(1/\sqrt{day})$, and zero annual return, i.e. $\mu = 0(\$/share)(1/day)$. Moreover, we choose: the permanent impact parameter $\gamma = 2.5 \times 10^{-7}\$/share^2$, the temporary impact parameter $\kappa = 2.5 \times 10^{-6}(\$/share^2)day$, the correlation parameter $\rho = 0$ and $\lambda = 1000\kappa$. Let $p_0 > 0$, in line with Cheng et al. (2017), we choose $m_0 = p_0 Y$, $\phi_0 = p_0\sqrt{Y/T}share^{1/2}$ and $\chi_0 = \sigma$. With these choices the executed orders have on average $p_0$ deviation from the placed orders per day. In fact at each time $t \in [0, T]$ the constant $\phi_0$ multiplies $\sqrt{v(t)(T-t)}$, where $v(t)$ is roughly of order $Y/T$. Given $Y$ and $T$ the difficulty of liquidation increases when the execution risk parameter $p_0$ increases. The aim of this section is to analyze the behaviour of the optimal trading strategy obtained in Proposition 3.2 when "easy" and "difficult" orders are considered. We consider "difficult" a liquidation order with small values of $p_0$ and we consider "easy" a liquidation order with large values of $p_0$. Specifically in the numerical experiments we choose $p_0 = 10\%$ for the "easy" order and $p_0 = 30\%$ for the "difficult" order.

In Figure 1 we plot the sample trajectory of the optimal square root uncertainty holding position (solid line) and of the optimal constant uncertainty optimal holding position (dotted line) obtained with $p_0 = 10\%$ (left panel) and $p_0 = 30\%$ (right panel). Looking at Figure 1 we observe that at the beginning of the liquidation interval the optimal holding positions are very close each other and close to the VWAP holding position that corresponds to a linear reduction of holdings over the liquidation interval. As time approaches to the liquidation horizon, the optimal square root uncertainty holding position moves away from the optimal constant uncertainty holding position and, except for the time interval $[0.7, 0.8]$, is under the optimal constant uncertainty holding position. This behaviour depends on the choices made of $\lambda$, $\mu$ and $\rho$. In fact, as explained in Section 3, the choice $\lambda = 1000\kappa$ implies that $\alpha \simeq 0$ and the optimal square root uncertainty holding position approaches to the holding position solution of (3.30), (3.31) whose expected value for $\mu = 0$ and $B > 0$ is a convex function of time. As $p_0$ increases the parameter $B$ increases and the convexity of the optimal square root holding position increases. Otherwise, when $\lambda = 1000\kappa$ the constant uncertainty holding position of Cheng et al. (2017) approaches to the adaptive VWAP holding position whose expected value for $\mu = 0$ is a linear function of time.

In Figure 2 we plot the sample trajectories of the optimal square root uncertainty trading strategy (solid line) and of the optimal constant uncertainty trading strategy (dotted line) obtained with $p_0 = 10\%$ (left panel) and $p_0 = 30\%$ (right panel). Looking at the sample trajectories of the optimal trading strategies shown in Figure 2 we observe that the optimal constant uncertainty strategy is larger and more unstable than the optimal square root uncertainty strategy and this effect is more evident towards the end of the liquidation interval where the optimal constant uncertainty strategy spikes up significantly to achieve the full liquidation. This fact is expected since in the square root uncertainty case it is possible to avoid the uncertainty choosing the trading strategy equal to zero, meanwhile in the constant uncertainty case this is not possible (see Bulthuis et al., 2017).

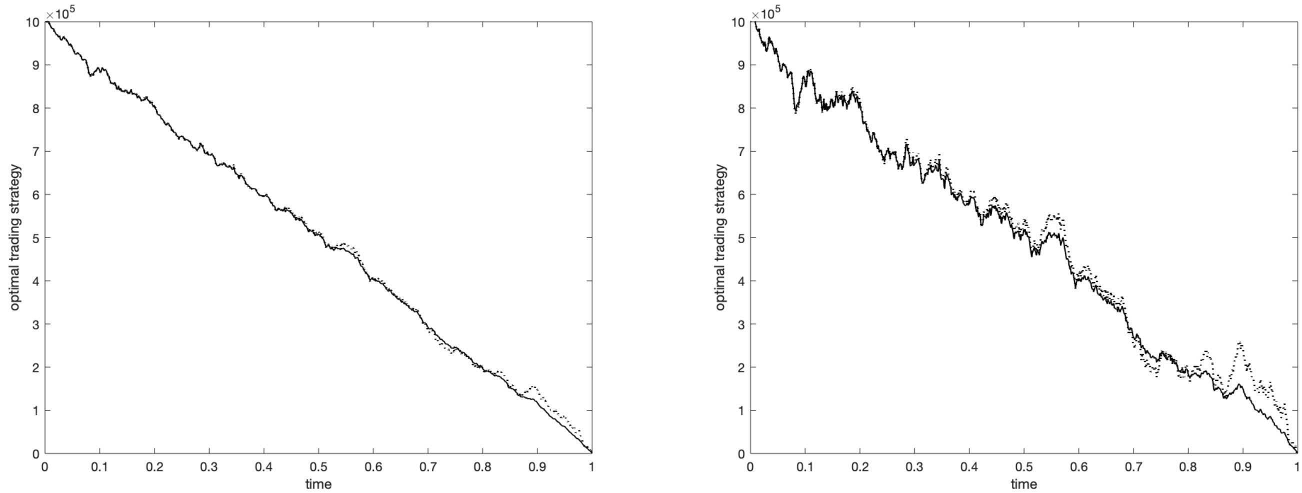


Figure 1: Sample trajectories of the optimal square root uncertainty optimal holding position (solid line) and of the optimal constant uncertainty optimal holding position (dotted line) obtained with $p_0 = 10\%$ (left panel) and $p_0 = 30\%$ (right panel).

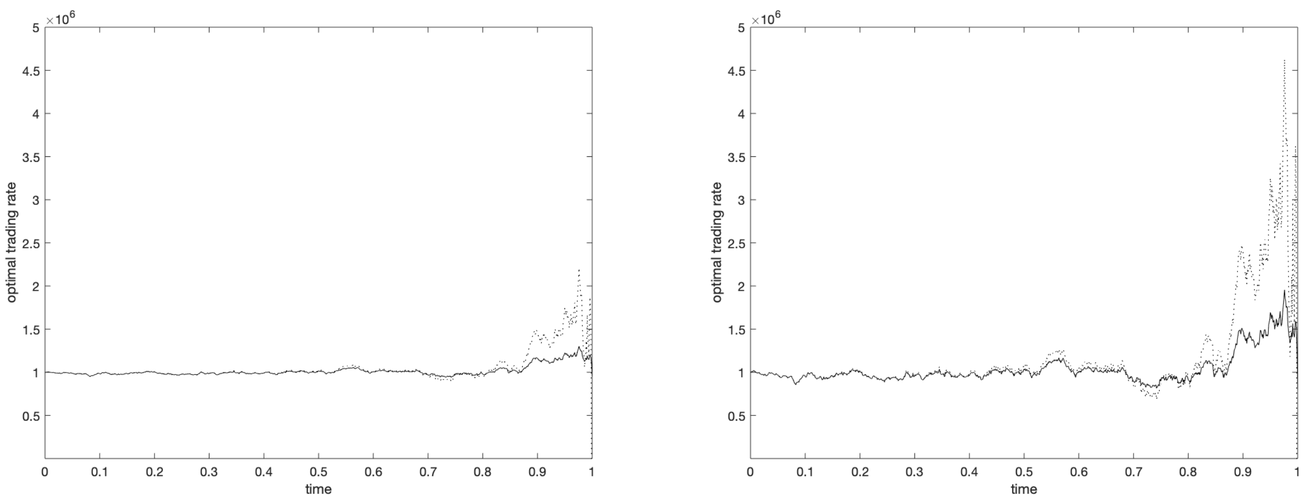


Figure 2: Sample trajectories of the optimal square root uncertainty optimal trading strategy (solid line) and of the optimal constant uncertainty optimal trading strategy (dotted line) obtained with $p_0 = 10\%$ (left panel) and $p_0 = 30\%$ (right panel).

## 5 Conclusions

We have presented a new model of liquidation problem that takes into account execution and price impact risks. Under the assumption that uncertainty in execution affects both holding position and asset share price dynamics and that the magnitude of execution and price impact risks is proportional to the square root of the residual asset position, we have modeled the liquidation problem as a linear quadratic stochastic optimal control problem and we have solved it. When the liquidation condition is enforced, i.e. when the liquidation is completed at the final time of the liquidation interval, the optimal holding position is an ECIR square root process and belongs to the class of processes proposed by Delyon and Hu (2006). The model has the advantage of having explicit solution expressed by elementary functions obtained without imposing any constraints on the model parameters. The optimal trading strategy found in presence of execution risk coincides with the optimal trading strategy in absence of execution risk for a modified asset price suggesting that, under execution risk, the investor modifies his directional view about the asset price growth rate. This finding supports the idea of the existence of a shadow price in friction markets of Kallsen and Muhle-Karbe (2010) and of Mariani et al. (2019). The proposed liquidation model may be extended and adapted, for example, to take into account multiple assets and more sophisticated models of stochastic price impact. These topics will be certainly the goals of future works.

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