Entanglement entropy for spherically symmetric regular black holes

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The Bardeen and Hayward spacetimes are here considered as standard configurations of spherically symmetric regular black holes. Assuming the thermodynamics of such objects to be analogous to standard black holes, we compute the island formula in the regime of small topological charge and vacuum energy, respectively for Bardeen and Hayward spacetimes. Late and early-time domains are separately discussed, with particular emphasis on the island formations. We single out conditions under which it is not possible to find out islands at early-times and how our findings depart from the standard Schwarzschild case. Motivated by th fact that those configurations extend Reissner-Nordström and Schwarzschild-de Sitter metrics through the inclusion of regularity behavior at $r = 0$, we show how the effects of regularity induces modifications on the overall entanglement entropy. Finally, the Page time is also computed and we thus show which asymptotic values are expected for it, for all the configurations under exam. The Page time shows slight departures than the Schwarzschild case, especially for the Hayward case, while the Bardeen regular black hole turns out to be quite indistinguishable from the Schwarzschild case.

CONTENTS

I. INTRODUCTION

Black Holes (BHs) are currently among the most intriguing objects in the universe. Understanding the physics behind them will shed light toward possible departures from Einstein's gravity as BHs are domains of strong gravity where quantum gravity effects are expected to appear. The advancement of a new era based on BH precision astronomy [1] is currently undergoing as certified by the gravitational wave detection [2] and by the impressive discovery of BH shadows [3]. Specifically, the presence of matter satisfying reasonable energy conditions inevitably leads to singularities, as shown by Penrose and Hawking singularity theorems [4, 5]. This fact arises for those theories fulfilling the *equivalence principle*, and therefore in particular for general relativity.

In addition, among all possible theoretical studies on BHs, the *information paradox* clearly represents a major issue of quantum gravity [6] and, more broadly, of general relativity and effective theories of fields. In particular, Hawking's radiation turns out to be thermal, namely the entanglement entropy outside a BH will consistently increase [7]. This occurrence is in contrast with quantum mechanics. There, entanglement entropy might eventually reach zero as it approaches the end of its evaporation, i.e., as a consequence of the pure states at the end of evaporation.

To prompt this issue, one can investigate the Page curve [8, 9], that displays the entanglement entropy time evolution, leaving *de facto* open the caveat of how the Page curve can be reproduced for the entanglement entropy of Hawking radiation. This ensures how to solve the problem of information loss quantum field theories in gravitational contexts, i.e., in curved spacetimes. Following Page's treatment, a restoration of unitarity, involving entropy decreasing after the Page time [6, 8, 9], can be found. This appears essential in conclusively solving the information paradox. Thus, a physical mechanism *fueling* the Page curve to exist could represent a key to guarantee the Page process to occur. To this end, it has been recently proposed that the Page curve arises from the effect of peculiar *islands* [10–13]. To better clarify what islands are, it is possible to note that as Hawking radia-

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tion state is employed within a region, R, outside the BH, the density matrix of R is commonly defined by taking the partial trace over the systems in $\overline{R}$. This procedure is motivated by the fact that, in so doing, one works out the complementary region of R and, so, based on the prescription of the minimal quantum extremal surface [14–16], certain regions in R, called islands, i.e., $I(\subset R)$, should be excluded from the states as they are traced out.

The subsequent strategy underlying the above prescription implies that one first extremizes the *generalized entropy* to locate the extremal points. Those, indeed, indicate the island locations and therefore the entropy minimum value becomes the fine-grained entropy of radiation [10–13]. The interest toward the concept of island increases as the same generalized entropy can be found adopting the replica method applied to the gravitational path integral [17, 18] and, moreover, the island formula can be understood by combining the AdS/BCFT correspondence and the brane world holography [19–34].

As above stated, it is widely-believed that BH singularities occur at a classical level, although they can be resolved by introducing a complete theory of quantum gravity. In this respect, to classically heal the presence of singularities, Bardeen was the first to introduce the concept of a regular BH (RBH) [35]. Such an object exhibits asymptotic flatness and a non-singular center in a static spherical symmetry and, later, has been demonstrated to arise as a genuine solution of Einstein's gravity [36]. The idea behind this RBH configuration involves a magnetic monopole source in the context of nonlinear electrodynamics [37]. Consequently, other RBH models were proposed and, remarkably, the Hayward solution was introduced, appearing as static and spherically symmetric, fulfilling the information-loss paradox [38]. The use of RBHs appears nowadays not only speculative. Indeed, they have been assumed to model neutron stars, featuring quasi-periodic oscillations. Other approaches certified that one cannot exclude topological charge effects as well as non-zero vacuum energy at $r = 0$ as it appears in the Hayward solution.

As those objects exhibit horizons, it is plausible to use these configurations to investigate how the island formula works. In this paper, we therefore describe the early and late-time approximations to evaluate the island formula. To do so, we do not resort to holographic correspondence, but rather we show that the entanglement entropy due to Hawking radiation follows the Page curve once islands are involved. We compute this adopting two main spacetimes configurations, namely Bardeen and Hayward metrics. We thus investigate the effects of topological charge first and vacuum energy in computing the island formula. We evaluate the regions of the islands for small topological and vacuum energy contrubition, without limiting the analyses at large radii, but considering instead the overall domain where islands can arise. Once the aforementioned approximations of early and late-times are taken into account, we check how the Page curves are reproduced. The corresponding results deviate than the simplest cases of Schwarzschild and Reissner-Nordström as consequence of the presence of corrective terms within the RBH metrics. Physical consequences of our recipe are also discussed.

The paper is structured as follows. In Sect. II, the basic motivations behind the use of Bardeen and Hayward metrics are reported. In Sect. III, the Bardeen metric is critically analyzed, ending up with the corrections to entropy at late and early-times. The same is performed in Sect. IV, where the same is reported for the Hayward spacetime. The Page time is therefore studied in Sect. V. Finally, in Sect. VI, we report our conclusions and perspectives.

II. ISLANDS, HAWKING RADIATION AND RBH

In view of BH Hawking radiation, limited within a region denoted as R, the density matrix of R is typically determined by taking a partial trace over the complementary region $\overline{R}$. Adopting the recipe of the minimal quantum extremal surface, as outlined in Refs. [39–41], certain systems lying on $\overline{R}$ are known as islands, $I(\subset \overline{R})$. Those can be excluded from the systems that are traced out, leading to the subsequent entanglement entropy, in R, effectively determined by the systems in $R \cup I$.

Thus, the Hawking radiation entanglement entropy reads

$$S(R) = \min \left\{ \text{ext} \left[\frac{\text{Area}(\partial I)}{4G_{\text{N}}} + S_{\text{matter}}(R \cup I) \right] \right\} \ , \quad (1)$$

in which the prescription of the quantum extremal surface has been used[1].

In the above relation, $S_{\text{matter}}(R \cup I)$ denotes the entanglement entropy of the matter fields in the region $R \cup I$, whereas $\text{Area}(\partial I)$ is the area of the extremal surface that forms the boundary of the region I. Afterwards, the island rule was derived by using the *replica method* for the gravitational path integral, and so as one applies the replica trick [21, 42, 43] to gravitational theories, one gets fixed boundaries due to the replica geometries. It is therefore possible to show that the same rule, as above, can be found though the replica trick as the quantum extremal surface prescription is involved. This makes the concept of islands much more robust and so the island conjecture is expected to be applicable to all BHs [44–46].

As thermodynamics for a magnetically charged RBH appears to be equivalent than BHs for a fixed charge [47], we here conjecture as a plausible assumption that thermodynamics of RBHs is the same than BHs.

[1] This formula allows for computing entanglement entropy of Hawking radiation in R since it employs the systems in $R \cup I$, thus not tracing away systems in the *island regions*.

So that, motivated by the above point, we study the effect of islands within the context of RBHs and stress this could be possible as horizons and asymptotically flatness are respected in such regular metrics. Specifically, our islands are determined for the simplest approaches describing compact objects [48], namely involving spherically symmetric, non-rotating metric. As a byproduct, we study how the island formula is expected to be modified than Schwarzschild and Reissner-Nordström metrics through the inclusion of regularity behavior at $r = 0$.

Hereafter we shall consider

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (2)$$

the spacetimes to use, being adaptable to our RBH solutions since $f(r)$ is the unknown smooth function. By singling it out, it is possible to define the kind of RBH under exam. The original idea, developed by Bardeen in 1968, was to find $f(r)$ out to mime the Reissner-Nordström metrics, employing a charge but without singularity.

Below, we focus on Bardeen and Hayward solutions, motivating them since they provide topological charges and vacuum energy at $r = 0$.

III. ISLANDS FROM THE BARDEEN METRIC

The Bardeen metric [35, 36] represents a solution of Einstein's field equations. It appears as non-rotating BH with *topological charge*. The solution is based on finding a solution to Einstein-Maxwell equations describing a magnetically-charged BH looking similar to the traditional Reissner-Nordström BH solution, but without singularity at $r = 0$.

Looking at Eq. (2), we thus write

$$f(r) = 1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}}\,, \quad (3)$$

where q and M are the charge and mass of the magnetic monopole, respectively. The limiting case, namely the Schwarzschild BH, is clearly recovered as $q \to 0$.

This spacetime solution could be somehow reinterpreted as *quasi-Kerr solution* [49], since as it has been stated in Ref. [50], the Bardeen metric is equivalent to the Kerr one only in its rotating version[2].

Following Ref. [36], we can shift to *Kruskal-like* coordinates[3]

The procedure is to work out the tortoise coordinate

$$r_* = \int g_{rr}dr = \int \frac{1}{f(r)}dr\,, \quad (4)$$

with the Finkelstein-like coordinates defined by the shifts:

$$u = t - r_*\,, \quad v = t + r_*\,, \quad (5)$$

allowing one to define the infinitesimal coordinates $du = dt - g_{rr}dr$, $dv = dt + g_{rr}dr$, and the subsequent spacetime, rewritten by

$$ds^2 = -f(r)dudv + r^2d\Omega^2\,, \quad (6)$$

where $d\Omega$ is the usual angular part of a given spherically-symmetric spacetime, i.e., $d\Omega^2 \equiv d\theta^2 + \sin\theta^2 d\phi^2$.

The Kruskal-like coordinates are written by defining [51]

$$U = -e^{-\kappa_+ t + \kappa_+ r_*}\,, \quad V = e^{\kappa_+ t + \kappa_+ r_*}\,, \quad (7)$$

where κ_+ in the surface gravity calculated at the outer horizon. So to compute it, we get the "+" root of $g_{tt} = 0$, which according to Ref. [52] can be written as[4] $\kappa_\pm = \frac{f'(r_\pm)}{2}$.

The metric can be finally recast under the useful form prompted by

$$ds^2 = -W^2(r)dUdV + r^2d\Omega^2\,, \quad (8)$$

with

$$W^2(r) = f(r)\frac{e^{-2\kappa_+ r_*}}{\kappa_+^2}\,, \quad (9)$$

where the *weight function* $W(r)$ represents the Jacobian to pass from one set of coordinates to another.

It is now necessary to work out horizons from the Bardeen spacetime so, given the analytic expression for the metric, the horizons results from

$$1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}} = 0\,, \quad (10)$$

showing two positive distinct solutions[5], say r_- and r_+, where as usual we conventionally require $r_- < r_+$.

Motivated by the fact that topological charges are small as numerically found, Eq. (10) can be easily solved under the prescription of small q. It is remarkable to note that we do not require large distances but small charges,

[2] The rotation is obtained through the Newman-Janis algorithm. By this property, several rotating RBHs may be found through the Newman-Janis algorithm.

[3] The procedure is the same performed in Kruskal coordinates. Since the BH is not the Schwarzschild one, we cannot claim that the coordinate change is exactly Kruskal, but rather a Kruskal replacement on a RBH solution, leading to the concept of Kruskal-like coordinates.

[4] The " + " root implies that the corresponding radius is larger than the other root(s).

[5] Any unphysical negative solution is clearly discarded into computation, consisting to non-relevant terms that do not contribute to the horizon computation.

i.e., we do not limit our treatment to large radii, but we expand around small q, implying $q/r \ll 1$, up to the third order in q/r, since $r > 0$. Thus, we have

$$f(r) \approx 1 - \frac{2M}{r} + \frac{3Mq^2}{r^3} = \frac{r^3 - 2Mr^2 + 3Mq^2}{r^3} . \quad (11)$$

From Eq. (4), we yield to

$$r_* = r - \sum_j B_j \log(r - r_j) , \quad (12)$$

having defined the auxiliary functions

$$B_j = \frac{M}{r_j}\frac{(2r_j^2 - 3q^2)}{(4M - 3r_j)} , \text{ with } \quad j = 1, 2, 3. \quad (13)$$

where the r_j's are the roots of $r_j^3 - 2Mr_j^2 + 3Mq^2 = 0$.

Thus, we have

$$\begin{aligned} r_* \approx\; & r - B_1 \log(r - r_1) \\ & - B_2 \log(r - r_2) - B_3 \log(r - r_3) , \end{aligned} \quad (14)$$

where r_1 and r_2, corresponding to $j = 1; 2$, are the aforementioned positive roots, discarding *de facto* the negative radius obtained by $f(r) = 0$, that turns out to be a unphysical solution.

We have now all the ingredients to compute the entropy contribution without and with islands, as we report in the incoming subsection.

A. Entropy without islands

The formula in Eq. (1) can be sorted out by means of the renormalizeed Newton's constant, $G_{N,r}$, under the form

$$\frac{1}{4G_{N,r}} \to \frac{1}{4G_N} + \frac{1}{\epsilon^2} , \quad (15)$$

where ϵ is the cutoff scale used in the configuration that we intend to write up. In particular, following Refs. [51, 53], we invoke

$$b_+ = (t_b, b) , \quad (16a)$$
$$b_- = (-t_b + i\frac{\pi}{\kappa_+}, b) , \quad (16b)$$
$$a_+ = (t_a, a) , \quad (16c)$$
$$a_- = (-t_a + i\frac{\pi}{\kappa_+}, a) , \quad (16d)$$

where $b_\pm$ are the boundaries of the entanglement region R and $a_\pm$ the boundaries of the island. Following Refs. [51, 53], in the configuration *without island* the entropy is simply given by

$$S_{\text{matter}} = \frac{c}{3} \log d(b_+, b_-) , \quad (17)$$

where c is the central charge, as in [53]. The latter, in the Kruskal-like coordinates, Eq. (8), can be written as

$$\begin{aligned} S_{\text{matter}} &= \frac{c}{6} \log \left[W(b_-)W(b_+)(U(b_-) - U(b_+))(V(b_+) - V(b_-))\right] \\ &= \frac{c}{6} \log \left[W^2(b)(e^{\kappa_+ t_b + \kappa_+ r_*(b)} + e^{-\kappa_+ t_b + \kappa_+ r_*(b)})(e^{\kappa_+ t_b + \kappa_+ r_*(b)} + e^{\kappa_+ t_b + \kappa_+ r_*(b)}\right] \\ &= \frac{c}{6} \log \left[4W^2(b)e^{2\kappa_+ r_*(b))} \cosh^2(\kappa_+ t_b)\right] . \end{aligned} \quad (18)$$

The corresponding behaviors at both late and early-times can be therefore analysed. Specifically, the behavior at late-times appears particularly interesting as the entropy could increase indefinitely eventually violating the Bekenstein bounds. Actually, for $t_b \gg 1$, Eq. (18) gives

$$S_{\text{matter}} \sim \frac{c}{3}\kappa_+ t_b , \quad (19)$$

which, as above anticipated, diverges leaving the information paradox unsolved. As possible solution, we then include islands and check whether their inclusion would modify the corresponding entropy behavior.

B. Entropy with islands

As above stated, we here include islands to check whether they modify the corresponding entropy behavior. Thus, for the sake of simplicity, we here limit our study on one island only, in agreement with previous findings, see e.g. [51, 53]. Hence, we easily get

$$
\begin{aligned}
S_{\text{matter}} &= \frac{c}{3}\log\left[\frac{d(a_+,a_-)d(b_+,b_-)d(a_+,b_+)d(a_-,b_-)}{d(a_+,b_-)d(a_-,b_+)}\right] \\
&= \frac{c}{3}\log d(b_+,b_-)d(a_+,a_-) \\
&+ \frac{c}{3}\log\left[\frac{(U(b_+)-U(a_+))(V(a_+)-V(b_+))(U(b_-)-U(a_-))(V(a_-)-V(b_-))}{(U(b_-)-U(a_+))(V(a_+)-V(b_-))(U(b_+)-U(a_-))(V(a_-)-V(b_+))}\right].
\end{aligned} \tag{20}
$$

Using

$$
S_{\text{gen}} = \frac{2\pi a^2}{G_N} + S_{\text{matter}}\,, \tag{21}
$$

in analogy to previous literature, see e.g. [51], we get

$$
\begin{aligned}
S_{\text{gen}} = \frac{2\pi a^2}{G_N} &+ \frac{c}{6}\log\left[2^4 W^2(b)W^2(a)e^{2\kappa_+(r_*(a)+r_*(b))}\cosh^2(\kappa_+ t_b)\cosh^2(\kappa_+ t_a)\right] \\
&+ \frac{c}{3}\log\left[\frac{\cosh\left(\kappa_+(r_*(a)-r_*(b))\right)-\cosh\left(\kappa_+(t_a-t_b)\right)}{\cosh\left(\kappa_+(r_*(a)-r_*(b))\right)+\cosh\left(\kappa_+(t_a+t_b)\right)}\right].
\end{aligned} \tag{22}
$$

The early-time behavior can be compared with late-times. In the former case, we assume $t_a, t_b \ll r_+$. Further, we note that if at least one island exists, its place might be close to $r=0$ since, at early-times, the entanglement entropy turns out to be small.

Bearing this in mind, one has

$$
\cosh\left(\kappa_+(r_*(a)-r_*(b))\right) \gg \cosh\left(\kappa_+(t_a-t_b)\right), \tag{23a}
$$

$$
\cosh\left(\kappa_+(r_*(a)-r_*(b))\right) \gg \cosh\left(\kappa_+(t_a+t_b)\right), \tag{23b}
$$

and rewriting Eq. (22) as

$$
\begin{aligned}
S_{\text{gen}} &\approx \frac{2\pi a^2}{G_N} + \frac{c}{6}\log\left[2^4 W^2(b)W^2(a)e^{2\kappa_+(r_*(a)+r_*(b))}\cosh^2(\kappa_+ t_b)\cosh^2(\kappa_+ t_a)\right] \\
&\approx \frac{2\pi a^2}{G_N} + \frac{c}{6}\log[W^2(a)e^{2\kappa_+ r_*(a)}] + \text{functions of } b \text{ only} \\
&= \frac{2\pi a^2}{G_N} + \frac{c}{6}\log[f(a)] + \text{functions of } b \text{ only.}
\end{aligned} \tag{24}
$$

we can now extremize S_{gen} with respect to a. This yields $\frac{4\pi a}{G_N} + \frac{c}{6}\frac{f'(a)}{f(a)} = 0$ or, more explicitly

$$
\frac{4\pi a}{G_N} + \frac{c}{6}\frac{2M\left(a^3-2aq^2\right)}{(a^2+q^2)\left((a^2+q^2)^{3/2}-2a^2M\right)} = 0\,. \tag{25}
$$

Expanding up to the third order in a, we get

$$
a \approx \sqrt{\frac{q^3}{cMG_N}\frac{6\pi q^3 - cMG_N}{2M-3q}}\,. \tag{26}
$$

The island, therefore, cannot form as the the Newton's constant contribution dominates over the third order topological term . In order to guarantee it, then, one requires

$$
q < \left(\frac{cMG_N}{6\pi}\right)^{\frac{1}{3}}, \tag{27}
$$

putting a stringent limit to the bounds over q that are allowed in the case of Bardeen RBH, with $c \geq 0$.

The same procedure can be carried out studying late-times. Here we assume $t_a, t_b \gg b$ which allow us to make the following approximation,see also [52],

$$
d(a_+,a_-) \approx d(b_+,b_-) \approx d(a_\pm,b_\mp) \ll d(a_\pm,b_\pm)\,, \tag{28}
$$

which leads to

$$S_{\text{matter}} \approx \frac{c}{3} \log \left[d(a_+, b_+) d(a_-, b_-)\right] . \tag{29}$$

After some algebra this can be rewritten as

$$S_{\text{matter}} = \frac{c}{3} \log \left| \frac{1}{\kappa_+} \sqrt{f(a) f(b)} e^{-2\kappa_+(a_* + b_*)} \left[e^{2\kappa_+ a_*} + e^{2\kappa_+ b_*} - 2 e^{\kappa_+(a_* + b_*)} \cosh\left(t_a - t_b\right) \right] \right| , \tag{30}$$

where $a_* = r_*(a)$ and $b_* = r_*(b)$ have been defined.

Again, maximizing with respect to t_a, we can get

$$\begin{aligned} \max_{t_a} S_{\text{matter}} = \frac{2c}{3} \log \left| \frac{e^{\kappa_+ a_*} - e^{\kappa_+ b_*}}{\kappa_+} \right| \\ + \frac{c}{6} \log \left| f(a) f(b) e^{-2\kappa_+(a_* + b_*)} \right| , \end{aligned} \tag{31}$$

where $t_a = t_b \equiv t$ and so, in this case, we need to find out the island position, by maximizing with respect to a S_{gen}.

Hence, we set $\partial_a S_{\text{gen}} = \partial_a \left(\frac{2\pi a^2}{G_N} + \max_{t_a} S_{\text{matter}} \right) = 0$, resulting in

$$\frac{4\pi a}{G_N} + \frac{c}{6} \left[\frac{f'(a) - 2\kappa_+}{f(a)} \right] + \frac{2c}{3} \frac{\kappa_+ e^{\kappa_+ a_*}}{f(a)(e^{\kappa_+ a_*} + e^{\kappa_+ b_*})} = 0 . \tag{32}$$

We now assume that the island is located very close, but outside the horizon, which means $a \approx r_+$ and $a > r_+$. This allows us to expand $f(a)$ around r_+ as

$$f(a) \approx f(r_+) + f'(r_+)(a - r_+) = 2\kappa_+ r_+ \frac{a - r_+}{r_+} , \tag{33}$$

since $f(r_+) = 0$ and $f'(a) \approx f'(r_+) = 2\kappa_+$. From Eq. (4) we get

$$a_* \approx \int^a \frac{1}{2\kappa_+(r - r_+)} dr = \frac{1}{2\kappa_+} \log \frac{a - r_+}{r_+} , \tag{34}$$

using $dr = r_+ d\left(\frac{r - r_+}{r_+}\right)$.

Consequently, the a equation, Eq. (32), acquires the form

$$\frac{4\pi r_+}{G_N} + \frac{c}{3} \frac{\sqrt{\frac{a - r_+}{r_+}}}{r_+ \frac{a - r_+}{r_+} \left(\sqrt{\frac{a - r_+}{r_+}} + e^{\kappa_+ b_*} \right)} \approx 0 , \tag{35}$$

and so, keeping the leading term in $(a - r_+)$ only, we write

$$\frac{4\pi r_+}{G_N} + \frac{c}{3} \frac{e^{-\kappa_+ b_*}}{\sqrt{r_+}\sqrt{a - r_+}} \approx 0 , \tag{36}$$

whose solution, squaring the last expression, provides the island position as

$$a \approx r_+ + \left(\frac{G_N c}{12\pi r_+} \right)^2 \frac{1}{r_+} e^{-2\kappa_+ b_*} , \tag{37}$$

that manifestly depends on b only.

We can then plug the tortoise coordinates, Eq. (14), into $e^{-2\kappa_+ b_*}$, giving

$$a \approx r_+ + \left(\frac{G_N c}{12\pi r_+} \right)^2 \frac{1}{r_+} e^{-2\kappa_+ b} (b - r_1)^{2\kappa_+ B_1} (b - r_2)^{2\kappa_+ B_2} (b - r_3)^{2\kappa_+ B_3} , \tag{38}$$

and then we end up with inserting the above value for a in the expression for the generalized entropy, Eq. (22).

Thus, recalling that we are studying the late time behavior of the solutions, we consider the following approximations

$$\cosh\left(\kappa_+(t_a + t_b)\right) \approx 2 \cosh\left(\kappa_+ t_a\right) \cosh\left(\kappa_+ t_b\right) , \tag{39a}$$

$$\cosh\left(\kappa_+(t_a + t_b)\right) \gg \cosh\left(\kappa_+(a_* - b_*)\right) \tag{39b}$$

and, using the definition for t with the additional requirement $t \gg b \gg r_+$, we can rewrite Eq. (22) as

$$S_{\rm gen} = \frac{2\pi a^2}{G_N} + \frac{c}{6}\log\left[W^2(b)W^2(a)\right] + \frac{c}{3}\log\left|\frac{1-2e^{-\kappa_+(b_*-a_*)}}{1+e^{\kappa_+(a_*-b_*-2t)}}\right| + \frac{2c}{3}\kappa_+ b_* \tag{40}$$
$$= \frac{2\pi a^2}{G_N} + \frac{c}{6}\log\left[W^2(b)W^2(a)\right] + \frac{c}{3}\log\left|-2e^{-\kappa_+(b_*-a_*)} - e^{\kappa_+(a_*-b_*-2t)}\right| + \frac{2c}{3}\kappa_+ b_*\,,$$

where we computed the logarithm expansion and used $\cosh\left(\kappa_+(a_*-b_*)\right) \approx \frac{1}{2}e^{\kappa_+(a_*-b_*)}$.

We note that the time dependence of this entropy evolves exponentially, i.e., at late-times it becomes constant. The corresponding value reads

$$S_{\rm gen} = \frac{2\pi a^2}{G_N} + \frac{c}{6}\log\left[W^2(b)W^2(a)\right] + \\ + \frac{c}{3}\log\left|-2e^{-\kappa_+(b_*-a_*)} - 1\right| + \frac{2c}{3}\kappa_+ b_*\,, \tag{41}$$

representing the asymptotic value of the entropy entering the Hawking entropy.

Finally, for $\frac{a-r_+}{r_+} = \epsilon \ll 1$ we have

$$W^2(a) = \frac{f(a)}{\kappa_+^2}e^{2\kappa_+ a_*}\,, \tag{42}$$

that can be approximated by

$$W^2(a) \approx \frac{2r_+\kappa_+}{\kappa_+^2}\epsilon \exp\left[-2\kappa_+\frac{1}{2\kappa_+}\log\epsilon\right] = \frac{2r_+}{\kappa_+}\,. \tag{43}$$

At leading order in ϵ, we have

$$S_{\rm gen} \approx \frac{2\pi r_+^2}{G_N} + \frac{c}{6}\log\left[\frac{2r_+}{\kappa_+}W^2(b)\right] + \frac{2c}{3}\kappa_+ b_*$$
$$= \frac{2\pi r_+^2}{G_N} + \frac{c}{6}\log\left[\frac{2r_+}{\kappa_+^3}f(b)e^{2\kappa_+ b_*}\right]\,, \tag{44}$$

where, using Eqs. (3) and (14), we can get

$$S_{\rm gen} \approx \frac{2\pi r_+^2}{G_N} + \frac{c}{6}\log\left[\frac{2r_+}{\kappa_+}\left(1 - \frac{2Mb^2}{(b^2+q^2)^{3/2}}\right)\right] + \\ + \frac{2c}{3}\kappa_+\left[b - \frac{M(-3q^2+2r_1^2)}{(4M-3r_1)r_1}\log(b-r_1) - \frac{M(-3q^2+2r_2^2)}{(4M-3r_2)r_2}\log(b-r_2) - \frac{M(-3q^2+2r_3^2)}{(4M-3r_3)r_3}\log(b-r_3)\right]\,. \tag{45}$$

The aforementioned expression refers to the total entropy composed by the standard Hawking part summed with the island correction. The behavior of our finding is therefor compared with the standard Schwarzschild case [53] in Tab. I, II and III.

IV. THE HAYWARD SPACETIME

More recently, the need of extending the Schwarzschild de-Sitter solution, including into a regular configuration a vacuum energy term led to introducing the Hayward solution [38]. This regular configuration resembles the physical interpretation of the Bardeen one, prompting a central flatness. As stated above, the corresponding lapse function for such a RBH implies a specific matter energy-momentum tensor that is de Sitter at the core. Again, at asymptotic distances, it recovers Minkowski. The extra-parameter responsible for flatness, hereafter Λ can be identified with a magnetic charge for a given non-linear electrodynamic theory, so that Hayward metric becomes a solution of such classes of theories as well as Bardeen one [54].

In the Hayward RBH the function $f(r)$ in Eq. (2) takes the form

$$f(r) = 1 - \frac{2Mr^2}{r^3 + 2M\Lambda^2}\,, \tag{46}$$

where, again M, is the point-like mass of the RBH, whereas Λ is intimately related to the constant term mimicking vacuum energy.

In analogy to the Bardeen case, we perform below the same computation to get the Hawking entropy corrections without and with islands.

A. Islands from the Hayward spacetime

As the derivation in Sec. III is valid for every spherically symmetric and static metric, we can immediately write

$$a \approx r_+ + \left(\frac{G_N c}{12\pi r_+}\right)^2 \frac{1}{r_+} e^{-2\kappa_+ b_*}\,, \tag{47}$$

and

$$S_{\rm gen} \approx \frac{2\pi r_+^2}{G_N} + \frac{c}{6}\log\left[\frac{2r_+}{\kappa_+^3} f(b) e^{2\kappa_+ b_*}\right]. \tag{48}$$

In this case though, the tortoise coordinate can be exactly computed to give

$$r_* = r - 2M\sum_{j=1}^{3} H_j \log(r - r_j)\,, \tag{49}$$

having defined

$$H_j = \frac{r_j}{4M - 3r_j},\ \text{with}\ j = 1, 2, 3\,. \tag{50}$$

in this case the r_j's denote the positive roots of the polynomial $2M\Lambda^2 - 2Mr_j^2 + r_j^3$. In order to ensure the presence of at least one event horizon we need to impose $\Lambda \leq \frac{4}{3\sqrt{3}}M$, because otherwise the discriminant of the polynomial is negative resulting in one negative root and two complex roots.

We can use Eq. (49) to obtain

$$a \approx r_+ + \left(\frac{G_N c}{12\pi r_+}\right)^2 \frac{1}{r_+}\,(b - r_1)^{H_1}\,(b - r_2)^{H_2}\,(b - r_3)^{H_3}\,, \tag{51}$$

and

$$\begin{aligned} S_{\rm gen} \approx{}& \frac{2\pi r_+^2}{G_N} + \frac{c}{6}\log\left[\frac{2r_+}{\kappa_+^3}\left(1 - \frac{2Mb^2}{b^3 + 2M\Lambda^2}\right) e^{2\kappa_+ b_*}\right] \\ &+ \frac{2c\kappa_+}{6}\left[b - 2M\left(\frac{r_1\log(b - r_1)}{4M - 3r_1} + \frac{r_2\log(b - r_2)}{4M - 3r_2} + \frac{r_2\log(b - r_3)}{4M - 3r_3}\right)\right]. \end{aligned} \tag{52}$$

Formally both solutions, Eq. (45) and Eq. (52), appear similar. Moreover, the result above found appears compatible with previous findings, see e.g. [55]. However, the physical mechanisms behind the aforementioned results is extremely different, leading to corrections that look like different as well. The above computed entropy clearly deviates from the Bardeen case. Analogy and differences are prompted in Tab. I, II and III where we define

$$\delta S \equiv \frac{\Delta S}{S_{\rm Schwarz.}} = \frac{S_{B;H} - S_{\rm Schwarz.}}{S_{\rm Schwarz.}}\,. \tag{53}$$

	Schwarzschild	**Bardeen**	**Hayward**
		$q = 0.005$	$\Lambda = 0.05$
$S_{\rm gen}$	26.91746	27.19917	27.16725
δS	0	0.01046577	0.009279884
		$q = 0.01$	$\Lambda = 0.1$
$S_{\rm gen}$	26.91746	27.19774	27.06968
δS	0	0.01041237	0.005654933
		$q = 0.015$	$\Lambda = 0.15$
$S_{\rm gen}$	26.91746	27.19534	26.90579
δS	0	0.01032337	0.0004338177

TABLE I. Values generated from Eq. (45) and Eq. (52) choosing $M = G = c = 1$ and $b = 10$

	Schwarzschild	**Bardeen**	**Hayward**
		$q = 0.005$	$\Lambda = 0.05$
$S_{\rm gen}$	30.57981	30.86149	30.82505
δS	0	0.00921124	0.008019731
		$q = 0.01$	$\Lambda = 0.1$
$S_{\rm gen}$	30.57981	30.85995	30.71373
δS	0	0.009160889	0.004379252
		$q = 0.015$	$\Lambda = 0.15$
$S_{\rm gen}$	30.57981	30.85738	30.52656
δS	0	0.009076965	0.001741265

TABLE II. Values generated from Eq. (45) and Eq. (52) choosing $M = G = c = 1$ and $b = 50$

	Schwarzschild	**Bardeen**	**Hayward**
		$q = 0.005$	$\Lambda = 0.05$
$S_{\rm gen}$	34.86887	35.15051	35.10875
δS	0	0.008077054	0.006879337
		$q = 0.01$	$\Lambda = 0.1$
$S_{\rm gen}$	34.86887	35.14885	34.9812
δS	0	0.008029439	0.003221389
		$q = 0.015$	$\Lambda = 0.15$
$S_{\rm gen}$	34.86887	35.14608	34.76658
δS	0	0.007950076	0.002933656

TABLE III. Values generated from Eq. (45) and Eq. (52) choosing $M = G = c = 1$ and $b = 100$

V. THE PAGE TIME

We previously studied the entropy behavior of our system both at early and late-times. We saw that initially it increases linearly in time (see Eq. (19)) but, when enough radiation is emitted, an island is formed and the entropy remains constant as in Eq. (45). The time at which this transition happens is called *Page time*. More generally the Page time is defined as the time when the entropy becomes constant and it is usually indicated by t_{Page}, and estimated imposing that the *entropy in the configuration without island is equal to that with island*.

We thus perform this computation here, starting from Eqs. (19) and (45) and imposing

$$\frac{c}{3}\kappa_+ t_{\text{Page}} \approx \frac{2\pi r_+^2}{G_N}\,, \tag{54}$$

obtaining a Page time of the form

$$t_{\text{Page}} \approx \frac{6\pi r_+^2}{c G_N \kappa_+}\,, \tag{55}$$

where we only kept the Bekenstein-Hawking term, i.e. $\frac{2\pi r_+^2}{G_N}$, for the sake of simplicity.

Recalling how to set the BH temperature up by

$$T_{\text{BH}} = \frac{\kappa_+}{2\pi}\,, \tag{56}$$

and assuming the regular configuration provides the same temperature, as motivated by recent studies, see e.g. [56], we thus take $T_{\text{BH}} = T_{\text{RBH}}$, with the latter the corresponding effective temperature for RBHs.

In Tab. IV, we report some numerical results for the Page time in our spacetime configurations. A particular attention has been devoted to the relative variation with respect to the Schwarzschild spacetime, i.e.,

$$\delta t \equiv \frac{\Delta t}{t_S} = \frac{t_{B;H} - t_S}{t_S}\,. \tag{57}$$

As emphasized above, the presented numerical results show that there is no clear evidence in favor of discrepancies occurring for RBHs than BHs. This means that the thermodynamics, as well as islands, are comparable to standard solutions, see e.g. [57]. Significant departures are found in the context of Hayward solution, whereas the Bardeen spacetime does not show significant changes compared with Schwarzschild. This implies that the inclusion of regular behaviors does not seem to influence the Page time and, consequently, the net island formation.

	Schwarzschild	**Bardeen**	**Hayward**
		$q = 0.005$	$\Lambda = 0.05$
t_{page}	150.7964	150.7964	150.891
δt	0	1.171946×10^{-10}	0.0006273523
		$q = 0.01$	$\Lambda = 0.1$
t_{page}	150.7964	150.7964	151.1792
δt	0	1.875313×10^{-9}	0.002538055
		$q = 0.015$	$\Lambda = 0.15$
t_{page}	150.7964	150.7964	151.6743
δt	0	9.49575×10^{-9}	0.005821291

TABLE IV. Table of indicative values generated from Eq. (55) with the above choices of free parameters and $M = G = c = 1$. The numerics involved into computations reflect the need of small changes and unspecified Λ terms. The most significant departures from the Schwarzschild case are found for Hayward spacetime, while in the Bardeen configuration, the outcomes appear to be negligibly small.

VI. FINAL OUTLOOKS AND PERSPECTIVES

We studied how the presence of non-singular regions for RBH configurations influences the island formula. To do so, we worked out the Bardeen and Hayward spacetimes and evaluated the corresponding thermodynamics of such objects, computing the island formula for both those metrics.

We distinguished two main cases, i.e., late and early-times, where we approximated the corresponding solutions. We emphasized in which regions the islands can exist, underlying how the effects of topological charge and vacuum energy influence the overall result. We pointed out our findings in the regime of arbitrary radii with small topological charge and unspecified vacuum energy magnitudes.

As those configurations are matchable with compact objects, we compared our outcomes with respect to the Reissner-Nordström and Schwarzschild-de Sitter BHs. We thus compared the effects of regularity behavior at $r = 0$ of our solutions with predictions made from the free terms, q and Λ. We also computed the Page time, emphasizing its asymptotic values in time and compared with previous findings. Our results indicated that the effects due to regular configurations that we investigated appeared to be more evident by tuning the free parameters of our RBHs, especially for the Hayward case. The Bardeen spacetime, on the contrary, did not depart significantly from the Schwarzschild BH. Hence, if from the one hand it appeared evident that the presence of non-singular behavior modified the entropy contribution to islands, on the other hand the predicted changes induced by regular solutions did not significantly modify the outcomes got in the case of the Schwarzschild BH.

In view of our results, it appears evident to investigate other RBHs for the island formula, including the effects of rotation and/or working out non linear electrodynamic contributions to the lapse function, with the aim of comparing them with those obtained for singular spacetimes. Moreover, the task of using regular solution with compact object is still debated, so the need of working out rotating solutions and/or those providing quadrupole terms would appear interesting for future works. Further, as future development we will focus on metrics, exhibiting horizons, that however may also show unphysical island regions.

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