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ON THE RELATIVE CATEGORY IN THE BRAKE ORBITS PROBLEM

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Dedicated to the memory of Edward Fadell and Sufian Husseini

ABSTRACT. In this paper dedicated to the memory of Edward Fadell and Sufian Husseini we show how the notion of Lusternik Schnirelmann relative category can be used to study a multiplicity problem for brake orbits in a potential well which is homeomorphic to the N–dimensional unit disk. The estimate of the relative category of the set of chords with endpoints on the $(N-1)$–unit sphere was shown to the third author by Fadell and Husseini while he was visiting the University of Wisconsin at Madison.

1. Introduction

Algebraic Topology plays a fundamental role in many areas of Mathematics. In the specific case of Calculus of Variations, Algebraic Topology provides a number of topological invariants that can be used to give lower estimates of the number of solutions to variational problems. Typically, the definition of such invariants uses appropriate variants of (relative) homology/cohomology theory or, as in the case of the celebrated Lusternik–Schnirelman category which has a lower bound in terms of the cuplength, they can be estimated using homological/cohomological techniques. Recall that the *Lusternik–Schnirelman category* of a topological space X, denoted by $\mathrm{cat}(X)$, is the minimal integer $k \in \mathbb{N}\bigcup\{+\infty\}$ such that X admits a covering formed by k closed contractible subsets. The reader will find many examples of topological invariants and their use in Nonlinear Analysis in [6] and in the references therein.

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Many of such topological invariants have a relatively straightforward definition, but possess an extremely high level of complexity as far as explicit calculations are concerned. A prototypical example of such situation is given by the *Lusternik-Schnirelman relative category*, that is the central notion of the present paper. Let us adopt the convention in [7, Definition 3.1] and give the following:

DEFINITION 1.1. Let X be a topological space and let Y be a closed subset of X. A closed subset F of X has *relative category equal to* $k \in \mathbb{N}$, and we write $\operatorname{cat}_{X,Y}(F) = k$, if k is the minimal positive integer such that there exists a family $(A_i)_{i=0}^{k}$ of open subsets of X such that $F \subset \bigcup_{i=0}^{k} A_i$, $F \cap Y \subset A_0$, and such that for all $i = 0, \ldots, k$ there exists a continuous map $h_i \colon [0,1] \times A_i \to X$ with the following properties:

(1) $h_i(0,x) = x$, $\forall x \in A_i$, $\forall i = 0, \ldots, k$;
(2) for every $i = 1, \ldots, k$:
  (a) there exists $x_i \in X \setminus Y$ such that $h_i(1, A_i) = \{x_i\}$;
  (b) $h_i\big([0,1] \times A_i\big) \subset X \setminus Y$;
(3) if $i = 0$:
  (a) $h_0(1, A_0) \subset Y$;
  (b) $h_0(\tau, A_0 \cap Y) \subset Y$, $\forall \tau \in [0,1]$.

It is fairly obvious that, for a general triple (X, Y, F) of topological spaces, an explicit calculation of $\operatorname{cat}_{X,Y}(F)$ can be extremely difficult to perform. In case X is the set of chords with extreme points on the $(N-1)$–dimensional sphere S^{N-1}, and Y is the set of the constant chords on S^{N-1}, this number plays a key role for the multiplicity of brake orbits of a natural Hamiltonian system in a potential well homeomorphic to an N-dimensional unit disk.

Edward Fadell and Sufian Husseini gave important contributions in the study of these topological invariants, and developed tools that allowed estimates of their value, which contributed also to Critical Point theory. The interested reader can find mentions of these contributions in [6] and the references therein.

In particular, Fadell and Husseini provided an estimate of the relative category of the set of chords with endpoints on the $(N-1)$–dimensional sphere modulo the constant curves. This was obtained using by an ingenious calculation, that was shown to the third author of the present paper while he was visiting the University of Wisconsin-Madison in 1991. Such a calculation is described in Proposition 2.1, and it was recently used in [13] for the proof of the Seifert conjecture about multiple brake orbits, and also in other recent multiplicity results (see, e.g., [3, 5]).

2. The estimate of the relative category

In this section we discuss the estimate on the relative category of $(\mathfrak{C}, \mathfrak{C}_0)$, where $\mathfrak{C}$ and $\mathfrak{C}_0$ are defined below.

Let $2 \le N \in \mathbb{N}$, $\mathbb{D}^N$ the unit disk in the Euclidean space $\mathbb{R}^N$, and $\mathbb{S}^{N-1} = \partial\mathbb{D}^N$. Let $\mathcal{R}$ be the reversing map $\mathcal{R}x(s) = x(1-s)$, defined in the set of curves $x\colon [0,1] \to \mathbb{D}^N$. With a slight abuse of notation we will denote by $\mathcal{R}$ also the equivalence relation induced in the set of paths in the disk. Fix $\sigma \in \,]0,1[$ and set

$$\mathfrak{C} = \{\gamma : [0,1] \to \mathbb{D}^N,\ \gamma(t) = (1-t)x_1 + tx_2,\ x_1, x_2 \in \mathbb{S}^{N-1}\}, \tag{2.1}$$

$$\mathfrak{C}_\sigma = \{\gamma \in \mathfrak{C} \,:\, |\gamma(1) - \gamma(0)| \le \sigma\}, \tag{2.2}$$

and

$$\mathfrak{C}_0 = \{\gamma \in \mathfrak{C} \,:\, \gamma(t) = x \in \mathbb{S}^{N-1}\, \forall t \in [0,1]\}. \tag{2.3}$$

Moreover for any $A \subset \mathfrak{C}$ we set

$$\tilde{A} = A/\mathcal{R}.$$

Note that $\tilde{\mathfrak{C}}$ is homeomorphic to $(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})/\mathbb{Z}_2$ where the action of $\mathbb{Z}_2$ on the product $\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}$ is given by $\mathcal{S}(A,B) = (B,A)$.

Our aim is to prove the following result.

PROPOSITION 2.1. *Let $\mathfrak{C}$ and $\mathfrak{C}_0$ be defined as in* (2.1) *and* (2.3)*, respectively. Then*

$$\operatorname{cat}_{\tilde{\mathfrak{C}},\tilde{\mathfrak{C}}_0}(\tilde{\mathfrak{C}}) \ge N. \tag{2.4}$$

REMARK 2.2. In our case, property (2b) of Definition 1.1 is essential to guarantee that the relative category of $\tilde{\mathfrak{C}}$ is at least N. Indeed, if we did not require (2b) in the definition of relative category we would have $\operatorname{cat}_{\tilde{\mathfrak{C}},\tilde{\mathfrak{C}}_0}(\tilde{\mathfrak{C}}) \le \operatorname{cat}(\tilde{\mathfrak{C}}) \le 3$, where the last inequality has been proved in [15].

The proof of Proposition 2.1 will be performed using singular cohomology theory and the cup product (see e.g. [28]) with $\mathbb{Z}_2$ coefficients which, for each topological pair (X,Y), will be denoted by $H^q(X,Y)$ at any dimension $q \ge 0$. We will also exploit another cohomological invariant, called the *relative cuplength*, whose definition is the following.

DEFINITION 2.3. The number $\operatorname{cuplength}(X,Y)$ is the largest positive integer k for which there exists $\alpha_0 \in H^{q_0}(X,Y)$ ($q_0 \ge 0$) and $\alpha_i \in H^{q_i}(X)$, $i = 1, \dots, k$ such that

$$q_i \ge 1, \quad \forall\, i = 1, \dots, k,$$

and

$$\alpha_0 \cup \alpha_1 \cup \ldots \cup \alpha_k \ne 0 \text{ in } H^{q_0+q_1+\ldots+q_k}(X,Y),$$

where $\cup$ denotes the cup product.

Recall that, if $Y \ne \emptyset$, the absolute cuplenght of X is the largest positive integer k for which there exists $\alpha_i \in H^{q_i}(X)$, $i = 1, \dots, k$ such that

$$q_i \ge 1,\ \forall i = 1, \dots, k,$$

and

$$\alpha_1 \cup \ldots \alpha_k \ne 0 \text{ in } H^{q_1+\ldots+q_k}(X).$$

PROOF OF PROPOSITION 2.1. The proof is divided into four steps.

Step 1. $\mathrm{cat}_{\tilde{\mathfrak{C}},\tilde{\mathfrak{C}}_0}(\tilde{\mathfrak{C}}) \geq \mathrm{cuplength}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}}_\sigma \setminus \tilde{\mathfrak{C}}_0) + 1.$

Assume that $\mathrm{cat}_{\tilde{\mathfrak{C}},\tilde{\mathfrak{C}}_0}(\tilde{\mathfrak{C}}) = k < +\infty$. Since $\tilde{\mathfrak{C}}_0$ is not a retract of $\tilde{\mathfrak{C}}$, it is $k \geq 1$. Take $A_0, A_1, \dots, A_k$ open subsets as in definition 1.1, and let

$$\imath_r : A_r \to \tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \qquad \jmath_r : (\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \emptyset) \to (\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, A_r)$$

be inclusion maps. By property (2) of Definition 1.1, $\imath_r^* : H^q(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0) \to H^q(A_r)$ is the zero constant map for any $q \geq 1$ and any $r \geq 1$. Then, since the sequence

$$\dots \to H^{q_r}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, A_r) \xrightarrow{\jmath_r^*} H^{q_r}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0) \xrightarrow{\imath_r^*} H^{q_r}(A_r) \to \dots$$

is exact, then $\jmath_r^*$ is surjective if $q_r \geq 1$. Then for any $\alpha_r \in H^{q_r}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0)$, if $q_r \geq 1$, there exists $\beta_r \in H^{q_r}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, A_r)$ such that $\jmath_r^*(\beta_r) = \alpha_r$.

Since $\tilde{\mathfrak{C}}_0 \subset A_0$, A_0 is open and $\tilde{\mathfrak{C}}_0$ is closed, there exists $\sigma \in \,]0,1[$ such that $\tilde{\mathfrak{C}}_\sigma \subset A_0$. Moreover by property (3b) of Definition 1.1, σ can be chosen sufficiently small so that, up to consider a projection on $\tilde{\mathfrak{C}}_0$, a homotopy $\hat{h}_0$ can be built such that $\hat{h}_0(\tau, \tilde{\mathfrak{C}}_\sigma) \subset \tilde{\mathfrak{C}}_\sigma$, $\forall \tau \in [0,1]$, (while, obviously, $\hat{h}_0(1, A_0) \subset \tilde{\mathfrak{C}}_\sigma$).

Now consider the inclusion maps

$$\jmath_0 : (\tilde{\mathfrak{C}}, \tilde{\mathfrak{C}}_\sigma) \to (\tilde{\mathfrak{C}}, A_0), \quad \imath_0 : (A_0, \tilde{\mathfrak{C}}_\sigma) \to (\tilde{\mathfrak{C}}, \tilde{\mathfrak{C}}_\sigma),$$

and the (exact) sequence

$$\dots \to H^{q_0}(\tilde{\mathfrak{C}}, A_0) \xrightarrow{\jmath_0^*} H^{q_0}(\tilde{\mathfrak{C}}, \tilde{\mathfrak{C}}_\sigma) \xrightarrow{\imath_0^*} H^{q_0}(A_0, \tilde{\mathfrak{C}}_\sigma) \to \dots$$

Since $\imath_0^* : H^{q_0}(\tilde{\mathfrak{C}}, \tilde{\mathfrak{C}}_\sigma) \to H^{q_0}(A_0, \tilde{\mathfrak{C}}_\sigma)$ is the constant zero map, then $\jmath_0^*$ is surjective and for any $\alpha_0 \in H^{q_0}(\tilde{\mathfrak{C}}, \tilde{\mathfrak{C}}_\sigma)$ there exists $\beta_0 \in H^{q_0}(\tilde{\mathfrak{C}}, A_0)$ such that $\jmath_0^*(\beta_0) = \alpha_0$.

Since $\tilde{\mathfrak{C}}_0$ is a strong deformation retract of $\tilde{\mathfrak{C}}_\sigma$ and since $\sigma < 1$, by excision property (recalling that $\tilde{\mathfrak{C}}_\sigma \subset A_0$) we have that for every $\hat{\alpha}_0 \in H^{q_0}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}}_\sigma \setminus \tilde{\mathfrak{C}}_0)$ there exists $\hat{\beta}_0 \in H^{q_0}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, A_0 \setminus \tilde{\mathfrak{C}}_0)$ such that

$$\jmath_0^*(\hat{\beta}_0) = \hat{\alpha}_0,$$

where $\jmath_0 : (\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}}_\sigma \setminus \tilde{\mathfrak{C}}_0) \to (\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, A_0 \setminus \tilde{\mathfrak{C}}_0)$ is the inclusion map.

Since A_i are open sets, we have

$$\begin{aligned} \hat{\beta}_0 \cup \beta_1 \cup \dots \cup \beta_k \in H^{q_0+q_1+\dots+q_k}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, (A_0 \setminus \tilde{\mathfrak{C}}_0) \cup A_1 \cup \dots \cup A_k) = \\ H^{q_o+q_1+\dots+q_k}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0) = 0. \end{aligned}$$

Moreover, by the naturality of the cup product (see [28]) we have (denoting by $\jmath$ the inclusion map)

$$\hat{\alpha}_0 \cup \alpha_1 \cup \dots \cup \alpha_k = \jmath^*(\hat{\beta}_0 \cup \beta_1 \cup \dots \cup \beta_k) = \jmath^*(0) = 0,$$

proving that $\mathrm{cuplength}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}}_\sigma \setminus \tilde{\mathfrak{C}}_0) < k$.

Step 2. $\mathrm{cuplength}(\tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}}_\sigma \setminus \tilde{\mathfrak{C}}_0) = \mathrm{cuplength}(X_\sigma, Y_\sigma)$, where

$$X_\sigma = \{[\gamma] \in \tilde{\mathfrak{C}} : |\gamma(1) - \gamma(0)| \geq \sigma\}, \quad Y_\sigma = \{[\gamma] \in \tilde{\mathfrak{C}} : |\gamma(1) - \gamma(0)| = \sigma\}.$$

This is straightforward, once one gets the existence of $H \in C^0([0,1] \times \tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0, \tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0)$ such that $H(0,x) = x \, \forall x \in \tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0$, $H(\tau, x) = x$, $\forall x \in X_\sigma$, $\forall \tau \in [0,1]$, and

$$H(1, \tilde{\mathfrak{C}} \setminus \tilde{\mathfrak{C}}_0) = X_\sigma, \quad H(1, \tilde{\mathfrak{C}}_\sigma \setminus \tilde{\mathfrak{C}}_0) = Y_\sigma.$$

Step 3. $\mathrm{cuplength}(X_\sigma, Y_\sigma) = \mathrm{cuplength}(E, \partial E)$, where E is the closed unit disk bundle over the manifold $\mathbb{P}^{N-1}$ and ∂E its boundary.
This is an immediate consequence of the fact that (X_σ, Y_σ) is homeomorphic to $(E, \partial E)$.

Step 4. $\mathrm{cuplength}(E, \partial E) \geq N - 1$.

To prove this, let us observe that

$$H^q(\mathbb{D}^{N-1}, \partial \mathbb{D}^{N-1}) = \begin{cases} 0, & \text{if } q \neq N-1, \\ \mathbb{Z}_2, & \text{if } q = N-1. \end{cases}$$

Denoting by π the canonical projection of E in $\mathbb{P}^{N-1}$, thanks to the contractibility of $\mathbb{D}^{N-1}$ we see that

$$\pi^* : H^q(E) \to H^q(\mathbb{P}^{N-1}) \text{ is an isomorphism } \forall q \geq 0. \tag{2.5}$$

Since we are considering $\mathbb{Z}_2$–coefficients there are not problems with orientation, and by [28, Corollary 5.7.18] the fiber bundle pair $((E, \partial E), \mathbb{P}^{N-1}, (\mathbb{D}^{N-1}, \partial \mathbb{D}^{N-1}), \pi)$ has a unique orientation cohomology class ζ^{N-1} with dimension $N-1$. Then, by Thom isomorphism Theorem (see [28, Theorem 5.7.10]) the homomorphism

$$\Phi : H^1(\mathbb{P}^{N-1}) \to H^{q+N-1}(E, \partial E)$$

given by $\Phi(z) = \pi^*(z) \cup \zeta^{N-1}$ is an isomorphism for any $q \geq 0$. From this fact and from (2.5) we deduce that

$$\mathrm{cuplength}(E, \partial E) \geq \mathrm{cuplength}(E).$$

Finally, using (2.5) and standard results in literature (see e.g. [28]),

$$\mathrm{cuplength}(E) = \mathrm{cuplength}(\mathbb{P}^{N-1}) = N - 1. \qquad \square$$

From a critical point theorist's perspective, it would be extremely interesting to have a proof of Proposition 2.1 that does not rely on sophisticated algebraic topological tools, like the Thom Isomorphism Theorem. For instance, one could try to adapt Rabinowitz's estimate of the category of the projective space (see [25, Section 3]). We recall that argument here for the reader's convenience.

PROPOSITION 2.4. *Let $\mathbb{P}^{N-1}$ denote the $(N-1)$–dimensional real projective space. Then* $\mathrm{cat}(\mathbb{P}^{N-1}) \geq N$.

The proof of Proposition 2.4 is carried out in different steps, each of which as its own interest. In order to pave the way, let us give the following

DEFINITION 2.5. Let E be a normed vector space and $F \subset E$ be a closed nonempty subset with $0 \notin F$. We say that F has order 1 ($ord\, F = 1$) if $F \cap (-F) = \emptyset$. We say that F has order p ($ord\, F = p$) if p is the minimum natural number such that there exist p closed subsets $F_1, \ldots, F_p$ of order 1 that verify $\cup_{i=1}^{p} F_i \supset F$. We set $ord\,(\emptyset) = 0$.

REMARK 2.6. From the above definition, it is easy to see that if A and B are two closed subsets that do not contain the zero, if $A \subset B$ then $\mathrm{ord}\ A \leq \mathrm{ord}\ B$.

PROPOSITION 2.7. *The sphere of dimension $N-1$ has order at least $N+1$, namely*

$$\mathrm{ord}(S^{N-1}) \geq N + 1.$$

PROOF. Let us consider $F_1, \ldots, F_m$ closed sets that covers S^{N-1}. Assume that $m \leq N$ and $ord\, F_i = 1$, for each $i = 1, \ldots m-1$. To obtain the thesis, it will be sufficient to show that $ord\, F_m \geq 2$.

Consider now the maps $\phi_i \colon S^{N-1} \to \mathbb{R}$ defined by:

$$\phi_i(u) = \frac{d(u, -F_i)}{d(u, F_i) + d(u, -F_i)}, \quad i = 1, \ldots, m-1 \quad (m \leq N). \tag{2.6}$$

Since $F_i \cap (-F_i) = \emptyset$, we have that each ϕ_i is continuous. Note that

$$\phi_i(u) = 1, \quad \forall u \in F_i, \tag{2.7}$$

$$\phi_i(u) = 0, \quad \forall u \in -F_i. \tag{2.8}$$

Also, let us define $\Phi \colon S^{N-1} \to \mathbb{R}^{m-1} \subset \mathbb{R}^{N-1}$ as

$$\Phi(u) = (\phi_1(u), \ldots, \phi_{m-1}(u)). \tag{2.9}$$

From Borsuk-Ulam Theorem, there exists $\overline{u} \in S^{N-1}$ such that

$$\Phi(\overline{u}) = \Phi(-\overline{u}). \tag{2.10}$$

As a consequence, we have that

$$\overline{u} \notin F_i, \quad \forall i = 1, \ldots, m-1. \tag{2.11}$$

Indeed, recalling that $\mathrm{ord}\, F_i = 1$ for each $i = 1, \ldots, m-1$, if $\overline{u} \in F_i$, then $-\overline{u} \notin F_i$, so

$$\phi_i(\overline{u}) = 1 \neq 0 = \phi_i(-\overline{u}),$$

which is a contradiction. As a consequence, since $F_1, \ldots, F_m$ cover S^{N-1}, we have that both $\overline{u}$ and $-\overline{u}$ belong to F_m, hence $ord\, F_m \geq 2$, and we are done. $\square$

LEMMA 2.8. *Consider $K \subset E$, K closed, $0 \notin K$, K symmetric w.r.t. the origin (that is, $x \in K \Rightarrow -x \in K$). Then,*

$$\mathrm{ord}\ K \leq N + 1 \iff \exists F_1, \ldots, F_N \text{ closed sets with } F_i \cap -F_i = \emptyset, \quad \bigcup_{i=1}^{N} F_i \cup \bigcup_{i=1}^{N} (-F_i) = K. \tag{2.12}$$

PROOF. Assume $ord\,K \leq N+1$. Then there exists $F_1, \ldots, F_{N+1}$ closed sets of order 1 that covers K. We get

$$K \subset F_1 \cup (-F_1) \cup F_2 \cup (-F_2) \ldots \cup F_N \cup (-F_N) \tag{2.13}$$

Indeed, if $x \in K$, $x \notin \bigcup_{i=1}^{N} F_i$ and $x \notin \bigcup_{i=1}^{N} (-F_i)$ then $x \in F_{N+1}$ and $x \in (-F_{N+1})$, which is absurd since $ord\,F_{N+1} = 1$.

Let us now prove the converse. Since any F_i is closed and $F_i \cap -F_i = \emptyset$, for any $i = 1, \ldots, N$ there exists and open set A_i including F_i such that $\overline{A_i} \cap (-\overline{A_i}) = \emptyset$. Let us consider

$$F^*_{N+1} \colon = K \setminus (\bigcup_{i=1}^{N} A_i), \quad F^*_i = \overline{A_i}, \quad i = 1, \ldots, N, \tag{2.14}$$

that are $N+1$ closed sets that cover K. Clearly $ord\,F_i = 1$ for any $i = 1, \ldots, N$. Then to conclude the proof it suffices to prove that F^*_{N+1} has order 1 whenever $ord\,K > N$.

We first note that $F^*_{N+1} \neq \emptyset$. If not, $\cup_{i=1}^{N} A_i \supset K$, so $\cup_{i=1}^{N} \overline{A_i} \supset K$ which implies $ord\,K \leq N$.

Moreover, $F^*_{N+1} \cap (-F^*_{N+1}) = \emptyset$. Indeed if there exists $x \in F^*_{N+1}$ and $x \in -F^*_{N+1}$ we have

$$x \notin \bigcup A_i, \quad x \notin \bigcup F_i, \quad -x \notin \bigcup A_i, \quad -x \notin \bigcup F_i \tag{2.15}$$

which is absurd. □

For any $B \subset E$ denote by $\tilde{B}$ the quotient space with respect to the equivalence relation $x \sim y$ if and only if $x = y$ or $x = -y$. In $\tilde{B}$ consider the distance between equivalent classes:

$$d([u, -u], [v, -v]) = min\{|u - v|, |u + v|\}.$$

THEOREM 2.9. *Consider $S \equiv S^N = \{x \in \mathbb{R}^{N+1} \colon \|x\| = 1\}$. Let $K \subset S$ be closed and symmetric with respect to the origin. Then, $ord\,K \leq \operatorname{cat}_{\widetilde{S}} \widetilde{K} + 1$.*

PROOF. Let $\widetilde{F}_1, \ldots, \widetilde{F}_N$ be closed and contractible sets that cover $\widetilde{K}$.

We are going to show that for every $\widetilde{F}_i$, setting $F_i \coloneqq \pi^{-1}(\widetilde{F}_i)$, where $\pi(x) = [x] = \{x, -x\}$, there exists a close set F^i_0 such that

$$F_i = F^i_0 \cup (-F^i_0), \quad F^i_0 \cap (-F^i_0) = \emptyset. \tag{2.16}$$

This suffices to conclude the proof, since it implies that K can be covered by N closed sets as in Lemma 2.8, so $ord\,K \leq N+1$.

Therefore, let $\widetilde{F}$ be a closed and contractible set and $F = \pi^{-1}(\widetilde{F})$. Since F is contractible in $\widetilde{S}$ there exists a continuous map $\widetilde{\Phi} \colon [0,1] \times \widetilde{F} \to \widetilde{S}$ such that

$$\widetilde{\Phi}(0, \widetilde{u}) = \widetilde{u}, \quad \forall \widetilde{u} \in \widetilde{F}, \tag{2.17}$$

$$\widetilde{\Phi}(1,\widetilde{u}) = \widetilde{u}_0, \quad \forall \widetilde{u} \in \widetilde{F}. \tag{2.18}$$

Now the uniform continuity of $\widetilde{\Phi}$ says that

$$\exists \delta > 0\colon |t - t'| < \delta \quad \Longrightarrow \quad d(\widetilde{\Phi}(t,\widetilde{u}), \widetilde{\Phi}(t',\widetilde{u})) \le 1 \tag{2.19}$$

Now define

$$\Phi(t,u)\colon = v \in \widetilde{\Phi}(t,\widetilde{u}), \tag{2.20}$$

Since S has radius 1 we have $\|v - u\| \le 1$ and the map Φ is continuous and such that

$$\Phi(t,-u) = -\Phi(t,u), \tag{2.21}$$

$$\pi(\Phi(t,u)) = \widetilde{\Phi}(t,\pi(u)), \tag{2.22}$$

$$\Phi(0,u) = u, \quad \Phi(1,u) \in \{u_0 \; - u_0\}. \tag{2.23}$$

Then we can set

$$F_0 = \{u \in F\colon \Phi(1,u) = u_0\}, \tag{2.24}$$

$$-F_0 = \{v\colon \; -u \in F_0\} = \{v \in F\colon \Phi(1,v) = -u_0\}, \tag{2.25}$$

proving (2.16). □

REMARK 2.10. From Theorem 2.9 and Proposition 2.7 we get $\mathrm{cat}_{P_N}(P_N) \ge N+1$ where P_N is the N–dimensional projective space.

3. An application: Bos' theorem revisited

A celebrated result by Lusternik and Schnirelman [22], dated 1934, see also [16], gives the existence of at least N distinct double normals ([1]) in a convex subset of $\mathbb{R}^N$ with nonempty interior and with regular boundary. In 1963, W. Bos extended this result to the case of orthogonal geodesic chords in a convex subset of an N-dimensional Riemannian manifold (see Theorem 3.1 below).

Let us formally state this result. Let (M,g) be a Riemannian manifold with $\dim(M) = N \ge 2$ and let $\Omega \subset M$ be an open subset with smooth boundary $\partial\Omega$; $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will denote its closure. Orthogonal geodesic chords in $\overline{\Omega}$, **OGC**s for short, are noncostant geodesics $\gamma : [a,b] \to \overline{\Omega}$ that start and arrive orthogonally to $\partial\Omega$ and such that $\gamma\big(\,]a,b[\,\big) \subset \Omega$. Note that in this way the notion of principal chord in $\mathbb{R}^N$ endowed with the Euclidean metric is extended to any Riemannian manifold.

THEOREM 3.1 W. BOS, 1963, [1]. *Let (M^N,g) be a Riemannian manifold and let $\Omega \subset M$ be an open subset with $\partial\Omega$ a hypersurface of class C^2. Assume that $\overline{\Omega} = \Omega \bigcup \partial\Omega$ is* convex ([2])*, and homeomorphic to an N-dimensional disk. Then, there are at*

([1]) i.e. chords with endpoints on the boundary and speed orthogonal to it.

([2]) i.e. geodesics starting tangentially at the boundary dot not enter inside Ω for small values of the affine parameter.

least N geometrically distinct orthogonal geodesic chords in $\overline{\Omega}$ (namely having different images)

Bos's original proof employed a shortening method, due to Byrckoff, to build a flow in the space of paths with free endpoints on the boundary of the disk, along which the geodesic energy functional decreases. The topological invariant used in Bos' argument is the notion of *relative cycles* of the space of chords modulo constant chords.

We will give here an alternative proof of this result, using the notion of relative Lusternik and Schnirelmann category and the estimate given in Proposition 2.4.

SKETCH OF THE PROOF OF THEOREM 3.1 For the variational setup, we use an approach based on the notion of *geodesic with obstacle*, introduced in [23], and the pseudo-gradient theory of Palais, given in [24]. In our situation, the *obstacle* is given by the boundary of Ω. We set:

$$\mathfrak{M} = \Big\{ x \in H^1\big([0,1], \overline{\Omega}\big) : x(0), x(1) \in \partial\Omega \Big\},$$

and for $x \in \mathfrak{M}$ we introduce a space of *admissible vector fields* along x, by setting:

$$\begin{aligned}\mathcal{V}_x^- = \Big\{ \xi \in H^1\big([0,1], TM\big) : \xi(s) \in T_{x(s)}M,& \\ \xi(0) \in T_{x(0)}(\partial\Omega),\ \xi(1) \in T_{x(1)}(\partial\Omega),& \\ g\big(\xi(s), \nu_{x(s)}\big) \le 0 \text{ when } x(s) \in \partial\Omega \Big\},&\end{aligned} \tag{3.1}$$

where ν_p is the outward pointing unit normal to $\partial\Omega$ at $p \in \partial\Omega$. Roughly speaking, elements of $\mathcal{V}^-$ are vector fields V_x along curves x in $\overline{\Omega}$ with the property that $V_x(s)$ points *inwards* whenever $x(s) \in \partial\Omega$. The geodesic action functional $\mathcal{F} \colon \mathfrak{M} \to \mathbb{R}$ is defined by:

$$\mathcal{F}(x) = \frac{1}{2} \int_0^1 g(\dot{x}, \dot{x})\,\mathrm{d}s. \tag{3.2}$$

We say that $x \in \mathfrak{M}$ is a *critical curve* for the geodesic with obstacle problem if:

$$\int_0^1 g\big(\dot{x}, \tfrac{\mathrm{D}}{\mathrm{d}s}\xi\big)\,\mathrm{d}s \ge 0, \qquad \forall\, \xi \in \mathcal{V}_x^-, \tag{3.3}$$

where $\frac{\mathrm{D}}{\mathrm{d}s}$ denote covariant differentiation along x. Note that the derivative of $\mathcal{F}$ is given by:

$$\mathrm{d}\mathcal{F}(x)\xi = \int_0^1 g\big(\dot{x}, \tfrac{\mathrm{D}}{\mathrm{d}s}\xi\big)\,\mathrm{d}s. \tag{3.4}$$

Nonconstant curves $x \in \mathfrak{M}$ satisfying (3.3) are called *geodesic with obstacle in* $\overline{\Omega}$, and using (3.4) it is not hard to see that such curves are paths that arrive orthogonally to $\partial\Omega$ at the endpoints, and they satisfy the geodesic equation when they lie in Ω.

Moreover, in $\partial\Omega$, a geodesic with obstacle satisfies a certain nonlinear second order equation involving a Lagrange multiplier. Geodesics with obstacle are, roughly speaking, curves having possibly low regularity (say, C^1, or more precisely $W^{2,\infty}$), that are made up by portions that either lie on the boundary $\partial\Omega$ (the *contact set* with $\partial\Omega$), or that are geodesics segments contained in the interior Ω. While geodesics with obstacle with prescribed boundary conditions can be found in any compact Riemannian manifold with smooth boundary, arbitrary Riemannian manifolds with boundary may not contain any *true* orthogonal geodesic chord. A very elementary counterexample is depicted in Bos' paper [1], who considers a simple triangular shaped region in $\mathbb{R}^2$ with non-convex rounded corners. In this case, the lack of OGCs, i.e., segments orthogonal to the boundary at both endpoints, is immediately verified by inspection. A precise description of geodesics with obstacles can be found in [14, 23]. The point here is that when $\partial\Omega$ is convex, then geodesics with obstacle in $\overline{\Omega}$ only touch the boundary at their endpoints, and therefore they are true orthogonal geodesic chords. Thus, finding orthogonal geodesic chords is the same as finding points $x \in \mathfrak{M}$ satisfying (3.3).

The search of critical points of $\mathcal{F}$ is done using the pseudo-gradient approach of Palais, [24]. More precisely, the pseudo-gradient of $\mathcal{F}$ is a vector field in $\mathfrak{M}$ such that $\mathcal{F}$ is uniformly strictly decreasing along each one of its flow line, outside any prescribed neighborhood of the set of critical points.

The construction of this vector field following the ideas of Palais in [24] is one of the key points of the proof. It is a local argument, which is made global using partition of unity. Standard completeness arguments and the convexity assumption yield a Palais-Smale condition for the corresponding flow, which allows the proof of the classical deformation lemmas for the sublevels of $\mathcal{F}$.

This is where the topological part of the proof takes place. Recall the backwards reparameterization map $\mathcal{R}\colon \mathfrak{M} \to \mathfrak{M}$, which defines a $\mathbb{Z}_2$-action on $\mathfrak{M}$ ($\mathcal{R}^2 = \mathrm{Id}$) and observe that both the functional $\mathcal{F}$ and the notion of criticality are $\mathcal{R}$–invariant. This says that the variational problem can be cast in the quotient space $\widetilde{\mathfrak{M}} = \mathfrak{M}/\mathcal{R}$; in this space one considers the subsets $\widetilde{\mathfrak{C}} = \mathfrak{C}/\mathcal{R}$ and $\widetilde{\mathfrak{C}}_0 = \mathfrak{C}_0/\mathcal{R}$, where $\mathfrak{C}$ is the set of *chords* in $\overline{\Omega}$ and $\mathfrak{C}_0$ is the set of *trivial (i.e., constant) chords* in $\overline{\Omega}$. The notion of chord in $\overline{\Omega}$ is defined using a homeomorphism between $\overline{\Omega}$ and the standard unit disk in $\mathbb{R}^N$. By Proposition 2.1 we have

$$\mathrm{cat}_{\widetilde{\mathfrak{C}},\widetilde{\mathfrak{C}}_0}(\widetilde{\mathfrak{C}}) \geq N.$$

Let $\mathcal{D}$ denote the class of all closed $\mathcal{R}$-invariant subsets of $\mathfrak{C}$; for all $i = 1, \dots, N$, define:

$$\Gamma_i = \Big\{ D \in \mathcal{D} : \mathrm{cat}_{\widetilde{\mathfrak{C}},\widetilde{\mathfrak{C}}_0}(\widetilde{D}) \geq i \Big\}.$$

Let us denote by $\mathcal{H}$ the set of homotopies of $\mathfrak{C}$ into $\mathfrak{M}$, i.e., the set of all continuous maps $h\colon [0,1]\times\mathfrak{C}\to\mathfrak{M}$, and let us define:

$$c_i = \inf_{\substack{D\in\Gamma_i\\ h\in\mathcal{H}}} \sup\big\{\mathcal{F}\big(h(1,x)\big) : x\in\mathcal{D}\big\},\quad i=1,\dots,N.$$

Let us observe that $0\le c_i\le \sup\limits_{x\in\mathfrak{C}}\mathcal{F}(x)$, and that $c_1\le c_2\le\ldots\le c_N$. Moreover, $c_1>0$, for otherwise it would be possible to find a closed $\mathcal{R}$-invariant subset $D\subset\mathfrak{M}$ with $\mathrm{cat}_{\widetilde{\mathfrak{C}},\widetilde{\mathfrak{C}}_0}(\widetilde{D})\ge 1$ that could be *deformed* by an element $h\in\mathcal{H}$ that fixes the points of $\mathfrak{C}_0$ to a subset of $\mathfrak{M}$ that consists of curves lying in $\partial\Omega$. Moreover, using standard minimax arguments, one sees that each c_i is a critical value of $\mathcal{F}$ and, assuming the existence of only finitely many critical points of $\mathcal{F}$, the sequence (c_i) is strictly increasing. Thus, there must be at least N critical values, and it is an easy observation that they correspond to pairwise geometrically distinct orthogonal geodesic chords in $\overline{\Omega}$. □

4. The relative category in the brake orbits problem

Guided by the proof of Theorem 3.1, it has been recently obtained the proof of a conjecture due to H.Seifert on the number of brake orbits for a natural Hamiltonian system, see reference [13]. The notion of brake orbit is strictly related to that of orthogonal geodesic chord. Let us illustrate briefly a Lagrangian formulation of the brake orbits problem.

Let $\widehat{M}$ be an N-dimensional manifold with $\widehat{M}$ of class C^3 representing the configuration space of some dynamical systems and $\widehat{g}$ a Riemannian metric of class C^2. Let $V\colon\widehat{M}\to\mathbb{R}$ be a C^2–function, representing the potential energy of some conservative force acting on the system. One looks for periodic solutions $x\colon[0,T]\to\widehat{M}$ of the following Lagrangian system:

$$\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{x} = -\nabla V(x), \tag{4.1}$$

where $\frac{\mathrm{D}}{\mathrm{d}t}$ denotes the covariant derivative of the Levi–Civita connection of $\widehat{g}$ for vector fields along x, and ∇V is the gradient of V. Solutions of (4.1) satisfy the conservation law of the energy $\frac{1}{2}g(\dot{x},\dot{x})+V(x)=E$, where E is a real constant called the *energy* of the solution x. It is a classical problem to give estimate of the number of periodic solutions of (4.1) having a fixed value of the energy E. This problem has been, and still is, the main topic of a large amount of literature, also for more general autonomous Hamiltonian systems, see for instance [17, 18, 19, 21, 26] and the references therein. Among all periodic solutions of (4.1), historical importance is given to a special class called *brake orbits*; these are “pendulum-like” solutions that oscillate with constant frequency along a trajectory that joins two distinct endpoints lying in $V^{-1}(E)$.

A very famous conjecture due to H. Seifert, see [27], originally formulated under analytic regularity assumptions, asserts that, given a Lagrangian system as in (4.1), if the sublevel $V^{-1}\big(\,]-\infty,E]\,\big)$ is homeomorphic to an N–dimensional disk and E is a regular value for V, then there should exist at least N geometrically distinct brake orbits. This estimate is known to be sharp, i.e. there are examples of analytic Lagrangian systems having energy sublevels homeomorphic to an N-disk and admitting exactly N geometrically distinct brake orbits. Along the recent years, partial proofs of Seifert's conjecture have appeared in the literature, see for instance [12, 14, 18, 20, 30, 31, 32]. In particular, [18] contains a proof of the Seifert's conjecture for Euclidean metrics, when the potential is assumed even and convex. In [12], Seifert's conjecture is proved in the case $N = 2$. In [14], the conjecture is proved for perturbations of radial potentials. When the E-sublevel $V^{-1}\big(\,]-\infty, E]\,\big)$ has the topology of the annulus, the multiplicity of brake orbits is studied in [9] and [10]. Finally, in [13] the authors give a proof of the following result.

THEOREM (SEIFERT'S CONJECTURE ON BRAKE ORBITS). Let E be a regular value of the potential V, and assume that the sublevel $V^{-1}\big(\,]-\infty, E]\,\big)$ is homeomorphic to the N–dimensional disk. Then, the Lagrangian system (4.1) admits at least N geometrically distinct brake orbits of energy E.

It is now a well established fact, see [8, 11], that fixed energy brake orbits for the system (4.1) correspond to OGCs in a domain $\overline{\Omega}$ contained in the interior (and diffeomorphic to) the corresponding energy sublevel of the potential V. The metric in $\overline{\Omega}$, which is usually called the *Jacobi metric*, is conformal to g, and it makes $\partial\Omega$ strongly concave.

The central result of [13] gives a lower bound on the number of orthogonal geodesics in Riemannian strongly concave disk (3) , and satisfying a technical, nevertheless mild, additional geometric assumption. Namely, it is assumed that there exists some point in Ω through which there exists no geodesic with both endpoints on $\partial\Omega$, and which is either tangent to $\partial\Omega$ at both endpoints, or tangent to $\partial\Omega$ at one endpoint and orthogonal to $\partial\Omega$ at the other. Such assumption has a technical nature, and it is possibly inessential for the validity of the result on the number of orthogonal geodesic chords in arbitrary Riemannian disks with strongly concave boundary. The central result in [13], that entails a proof of Seifert's conjecture, can be formulated as follows. Let $\mathcal{F}$ be the energy functional defined by (3.2).

THEOREM 4.1. *Let (M^N, g) be Riemannian manifold, and let $\Omega \subset M$ be an open subset with $\partial\Omega$ a hypersurface of class C^2. Suppose that $\overline{\Omega}$ is strongly concave and homeomorphic to an N–dimensional disk. Assume also that there exists a positive M_0 such that $\mathcal{F}^{-1}\big([0, M_0^2]\big)$ contains all the "chords" in $\overline{\Omega}$ with endpoints in $\partial\Omega$, and that there exists some point in Ω through which there exists no geodesic γ such that $\mathcal{F}(\gamma) \le M_0^2$, with both endpoints on $\partial\Omega$, and which is either tangent to $\partial\Omega$ at both endpoints, or tangent to $\partial\Omega$ at one endpoint and orthogonal to $\partial\Omega$ at the other.*

---

(3) *Strong concavity* means that the geodesics that start tangentially to the boundary remain on the interior of the disk in a neighbourhood of the tangent point

Then, there are at least N geometrically distinct ([4]) *orthogonal geodesic chords in* $\overline{\Omega}$.

In order to obtain a proof of Seifert's conjecture, one proves that the Jacobi metric satisfies all the assumptions of Theorem 4.1.

Let us discuss a sketch of the proof of Theorem 4.1, which is obtained by a refinement of the proof of Theorem 3.1, using the same topological invariant given by the relative category of the space of chords modulo the constant chords. The essential difference from the convex case treated in Theorem 3.1 is that the critical points of $\mathcal{F}$ as defined in (3.3) (i.e. the geodesics with obstacle) may touch the boundary of Ω, and therefore may not be true orthogonal geodesic chords. The strong concavity property plays an essential role here, because it yields that geodesics with both endpoints on the boundary of Ω cannot remain uniformly close to $\partial\Omega$. Note that this property does not hold in Bos' counterexample [1].

As in the case of the proof of Theorem 3.1, one defines a class of infinitesimal admissible variations, denoted by $\mathcal{V}^-$, as in (3.1), and also a subclass of $\mathcal{V}^-$ denoted by $\mathcal{V}^+$. Elements of $\mathcal{V}^+$ satisfy the additional requirement of changing their direction near $\partial\Omega$: $V_x \in \mathcal{V}^+$ points inward when x touches $\partial\Omega$, and outward when x is at a certain (prescribed) small distance from $\partial\Omega$.

As in Theorem 3.1, a path in Ω will be called $\mathcal{V}^-$-critical when it is fixed by the flow of every local vector field in the class $\mathcal{V}^-$. However, in the nonconvex case, there are $\mathcal{V}^-$-critical curves that are *not* OGCs, but rather curves that belong to the more general class of the *geodesics with obstacle*.

The set of $\mathcal{V}^-$-critical paths that are not OGCs is denoted by Z^-. Remarkably, the strong convavity assumption implies in particular that geodesics with obstacle that are orthogonal to $\partial\Omega$ at the endpoints and that have bounded length, the contact set with $\partial\Omega$ consists of a uniformly bounded number of disjoint intervals and isolated points.

Using the two classes $\mathcal{V}^-$ and $\mathcal{V}^+$ described above, one constructs a global flow on $\mathfrak{M}$, which plays the role of the flow of a *pseudo-gradient* vector field (see [24]) for the geodesic action functional. The pseudo-gradient field is constructed locally in two distinct regions of the space of admissible paths whose mutual distance is strictly positive, and then made global using convex linear combinations. Reference [24] provides the basic tools for the globalization of local flows, using partitions of unity.

One of the main technical parts of the proof of Theorem 4.1 consists in the construction of a special set Λ_* of paths that satisfies the following properties:

([4]) two orthogonal geodesic chords $\gamma_1, \gamma_2 \colon [0,1] \to \overline{\Omega}$ are *geometrically distinct* if $\gamma_1\big([0,1]\big) \neq \gamma_2\big([0,1]\big)$.

- Λ_* is invariant by the pseudo-grandient flow, and the geodesic action functional is strictly decreasing along the inward pointing flow lines that are near the *entrance of* Λ_* ([5]);
- Λ^* contains in its interior all the geodesics with obstacle that are orthogonal to $\partial\Omega$ at both endpoints;
- Λ^* does *not* contain true OGCs;
- Λ_* is *topologically trivial*, meaning that it can be continuously retracted to a set of curves lying entirely on $\partial\Omega$, through a retraction that fixes the constant curves. Roughly speaking, this implies that points in Z^- are not counted in the minimax argument, and therefore the topological invariant, i.e., relative Lusternik–Schnirelman category employed in the minimax argument gives a lower bound for true OGCs.

The global flow is defined using homotopies associated to the infinitesimal variations in $\mathcal{V}^+$ in Λ_*, to the more general variations in $\mathcal{V}^-$ far from Λ_*, and with convex combinations near the entrance set of Λ_*. The technical geometric assumption of Theorem 4.1 mentioned above is used to construct the continuous retraction of Λ_* onto a set of curves lying in $\partial\Omega$.

Once the pseudo-gradient flow and the set Λ_* have been defined, the proof of Theorem 4.1 follows standard general ideas from minimax theory. By the above construction, the fixed points of our flow that lie outside of Λ_* are OGCs. The minimax procedure detects a number of points fixed by the flow (outside Λ_*) which is greater than or equal to the Lusternik-Schnirelman relative category of a set which again has the topology of the quotient space $(\mathbb{S}^{N-1} \times \mathbb{S}^{N-1})/\mathcal{R}$, where $\mathcal{R}(A, B) = (B, A)$ (category relative to the diagonal of $\mathbb{S}^{N-1} \times \mathbb{S}^{N-1}$). Such number is greater or equal to N, as proved in Proposition 2.1. Since the set Λ_* can be continuously retracted to a set consisting of curves lying in the boundary of Ω, geodesics with obstacle do not contribute to the count of those fixed points detected by minimax. This implies that the strongly concave Riemannian N-disk under consideration possesses at least N distinct OGCs, proving our desired result.

This multiplicity result can be proved even for the more general Hamiltonian systems of *classical type*. A Hamiltonian system is said of classical type if it is even and strictly convex with respect to the generalized momenta, i.e. if it can be written as

$$H(q, p) = K(q, p) + V(q),$$

where $K(q, \cdot)\colon \mathbb{R}^N \to \mathbb{R}$ is even and strictly convex for each q. This kind of Hamiltonian were introduced in [29], where A. Weinstein proved the existence of a brake orbit

([5]) Given a semi-group $(\phi_t)_{t\geq 0}$ of homeomorphisms of a topological space $\mathcal{X}$, and given a subset $\mathcal{Y} \subset \mathcal{X}$ which is ϕ_t-invariant for all t, the *entrance* of $\mathcal{Y}$ is the set of the $x \in Y$ such that there exists $\delta_x > 0$ such that $\phi_t(x) \notin Y$ for any $t \in]-\delta_x, 0[$. In our concrete setting, the entrance set of Λ_* is denoted by Γ_* and it is defined in [13], cf. formula (5.9).

for Hamiltonian systems of classical type. While the brake orbits of a natural Hamiltonian system can be identified with the OGCs of a disk endowed with a Riemannian metric, for Hamiltonian systems of classical type the disk has to be endowed with a *Finsler* metric, as it has been proved in [4]. As a consequence, the multiplicity of OGCs in Finsler manifold with boundary, studied in [2], plays a central role in this extension.

The proof of existence of N geometrically distinct brake orbits for a $2N$–dimensional Hamiltonian system of classical type is given in [5] and it follows the same scheme of the proof of Theorem 4.1, but some technicalities arise due to the lack of regularity of the Finsler metric on the zero section of the tangent bundle.

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