Journal of Economic Interaction and Coordination
https://doi.org/10.1007/s11403-019-00277-y

REGULAR ARTICLE

Systemic risk governance in a dynamical model of a banking system with stochastic assets and liabilities

Lorella Fatone[1] · Francesca Mariani[2]

Received: 19 October 2018 / Accepted: 6 December 2019

Abstract
We consider the problem of governing systemic risk in an assets–liabilities dynamical model of a banking system. In the model considered, each bank is represented by its assets and liabilities. The net worth of a bank is the difference between its assets and liabilities and bank is solvent when its net worth is greater than or equal to zero; otherwise, the bank has failed. The banking system dynamics is defined by an initial value problem for a system of stochastic differential equations whose independent variable is time and whose dependent variables are the assets and liabilities of the banks. The banking system model presented generalizes those discussed in Fouque and Sun (in: Fouque, Langsam (eds) Handbook of systemic risk, Cambridge University Press, Cambridge, pp 444–452, 2013) and Fatone and Mariani (J Glob Optim 75(3):851–883, 2019) and describes a homogeneous population of banks. The main features of the model are a cooperation mechanism among banks and the possibility of the (direct) intervention of the monetary authority in the banking system dynamics. By "systemic risk" or "systemic event" in a bounded time interval, we mean that in that time interval at least a given fraction of banks have failed. The probability of systemic risk in a bounded time interval is evaluated via statistical simulation. Systemic risk governance aims to maintain the probability of systemic risk in a bounded time interval between two given thresholds. The monetary authority is responsible for systemic risk governance. The governance consists in the choice of assets and liabilities of a kind of "ideal bank" as functions of time and in the choice of the rules for the cooperation mechanism among banks. These rules are obtained by solving an optimal control problem for the pseudo mean field approximation of the banking system model. Governance induces banks in the system to behave like the "ideal bank". Shocks acting on the banks' assets or liabilities are simulated. Numerical examples of systemic risk governance in the presence and absence of shocks acting on the banking system are studied.

Extended author information available on the last page of the article

Published online: 21 December 2019

1 Introduction

The notion of systemic risk refers to the risk of collapse of an entire system rather than simply the failure of its individual parts. Systemic risk and systemic risk governance are therefore important research topics that have applications in many different contexts, such as physics, biology, engineering, and finance.

We limit our attention to the modeling of systemic risk in banking systems. In this case, systemic risk refers to the collapse of banking systems due to the banks' default and it depends on the banks' interconnections. Because of the complexity of the connections between banks, systemic risk is often unpredictable and unmeasurable and the construction of a safe banking system capable of measuring and governing systemic risk plays a key role in systemic risk research. For a survey of the various aspects of systemic risk and the use of mathematical models in studying systemic risk, we refer to Fouque and Langsam (2013), Hurd (2016) and the references therein.

Models found in the literature often analyze banking activities and systemic risk under static (time-independent) or, more often, discrete-time models (see e.g., Haldane and May 2011; May and Nimalan 2010; Iori et al. 2006; Grilli et al. 2014, 2017; Tedeschi et al. 2012; Lenzu and Tedeschi 2012; Caccioli et al. 2012, 2014; Berardi and Tedeschi 2017). Among others, Berardi and Tedeschi (2017), Grilli et al. (2014), Tedeschi et al. (2012), and Lenzu and Tedeschi (2012) develop agent-based interbank networks where the banks are interconnected through credit relationships. Iori et al. (2006) construct a systemic risk contagion model based on the interbank network, and Caccioli et al. (2014) model the banking system as a bipartite network where the two groups are represented by the banks and the assets. The widespread use of discrete-time models in describing the interbank dynamics is due to the fact that these models usually allow several microeconomic variables to be included that are useful for a realistic description of a banking system. Moreover, static or discrete-time models are mathematically more tractable than continuous models.

As banking practices become more intense and seamless, it is natural to think of approximating banking activities using continuous-time models instead of discrete-time models, even if, from the mathematical point of view, they are usually more demanding. As can be seen in detail in the following, some tentative steps in this direction can be found in Fouque and Ichiba (2013), Fouque and Sun (2013), Carmona et al. (2015), Mukuddem-Petersen and Petersen (2006, 2008), Sun (2018), Biagini et al. (2019a), Fatone and Mariani (2019). In this context, it is also desirable to rigorously define the systemic risk of a banking system. In fact, especially when continuous-time models are considered, there is not a general consensus on a unique definition of systemic risk (Fouque and Langsam 2013; Hurd 2016).

The banking system model we present in this paper is situated in the framework of mean field theory. This theory, initially introduced in statistical mechanics (see e.g., Gallavotti 1999), has recently been successfully applied to the study of financial models where, in contrast to the discrete-time models mentioned above, the dynamic evolution is studied by means of a system of interacting stochastic differential equations (see among the others, Fouque and Ichiba 2013; Fouque and Sun 2013; Carmona et al. 2015; Garnier et al. 2013; Sun 2018; Biagini et al. 2019a; Fatone and Mariani 2019). In particular, under suitable assumptions, the mean field theory allows a possibly high-dimensional dynamical model (e.g., a banking system model) to be governed via a control law determined using a low-dimensional dynamical model (e.g., the mean field approximation of the banking system model). The financial dynamical models proposed in this context typically consist of an initial value problem for a system of stochastic differential equations, usually diffusion equations, whose dependent variables represent, for example, monetary reserves, wealth or other more general indicators of the health of financial institutions. These diffusion equations are usually tied together through a term in the drift that implies the network structure.

The banking system model studied in this paper generalizes those presented in Fouque and Sun (2013), Carmona et al. (2015), Fatone and Mariani (2019) and exploits some ideas taken from Haldane and May (2011), May and Nimalan (2010), Mukuddem-Petersen and Petersen (2006, 2008). In Fouque and Sun (2013), Carmona et al. (2015), the dependent variables of the stochastic differential equations that define the model are the log-monetary reserves of the banks as functions of time and there is a cooperation mechanism that regulates borrowing and lending activities among banks. Moreover, the probability of systemic risk in a bounded time interval is studied using the mean field approximation and the theory of large deviations. The model presented in Fatone and Mariani (2019) generalizes those presented in Fouque and Sun (2013), Carmona et al. (2015). In particular, in Fatone and Mariani (2019) a model with two cooperation mechanisms is studied. The first cooperation mechanism regulates the borrowing and lending activities among banks, while the second describes the borrowing and lending activities between the banks and the monetary authority. Furthermore, a technique for governing the probability of systemic risk in a bounded time interval is introduced and studied.

In Haldane and May (2011), May and Nimalan (2010), Mukuddem-Petersen and Petersen (2006, 2008), assets–liabilities models of banking systems are presented, in which each bank is modeled by its assets and liabilities. Time-independent (static) Haldane and May (2011), May and Nimalan (2010) and time-dependent (dynamic) Mukuddem-Petersen and Petersen (2006, 2008) assets–liabilities banking system models have been studied. In Mukuddem-Petersen and Petersen (2006, 2008), banks' assets and liabilities are further decomposed as the sum of more specific addenda; the time dynamics of each addendum is specified. Finally in Haldane and May (2011), May and Nimalan (2010) the analogies between systemic risk in banking systems and systemic risk in several other domains of science and engineering are explored.

In detail, this paper is concerned with measuring, monitoring and governing systemic risk in an assets–liabilities continuous-time dynamical model of a banking system. In other words, we consider a continuous-time dynamical model of a banking system where each bank holds assets and has liabilities that are stochastic processes in time. The assets and liabilities of each bank are defined implicitly as functions of time by an initial value problem for a system of stochastic differential equations. The net worth of a bank is defined as the difference between the bank's assets and liabilities and a bank is solvent when its net worth is greater than or equal to zero; otherwise, the bank has failed. A political/technical authority is responsible for managing the banking system and, in particular, for governing systemic risk. For convenience we refer to this authority as the monetary authority.

The model proposed describes a homogeneous population of banks where each bank interacts with the other banks and with the monetary authority. The homogeneity of the bank population implies that all banks in the model behave in the same way. From an economic point of view, the homogeneity of the bank population can be seen, for example, as "perfect" herding behavior occurring when the banks try to "perfectly" copy each other's behavior. For theoretical explanations of the motivations behind bank herding behavior, we refer to Acharya and Yorulmazer (2008) and the references therein. Note that the assumption of bank homogeneity makes it possible to successfully apply mean field theory. The main features of the model are a cooperation mechanism among banks that regulates interbank borrowing and lending activities and the possibility of the monetary authority's (direct) intervention in the banking system dynamics. The cooperation mechanism is based on the idea that "those who have more (assets, liabilities) give to those who have less (assets, liabilities)". The monetary authority's intervention in the banking system dynamics consists of the choice of two functions representing, respectively, the assets and liabilities of a kind of "ideal bank" as functions of time and the choice of the rules controlling the cooperation mechanism among the banks.

It is worthwhile noting that the model proposed here is a deliberate over-simplification of interbank borrowing and lending activities. Banking systems are, in reality, very complex and diverse dynamic systems and the hypothesis of a uniform bank population initially proposed in Fouque and Sun (2013) and used here is a very strong hypothesis that is far from being true in real banking systems. As well, the assumption of banks that lend to and borrow from each other in the spirit of "those who have more (assets, liabilities) give to those who have less (assets, liabilities)" is simply a game feature introduced to model lending and borrowing among banks with the final scope of better understanding the banking system and systemic risk. This assumption was initially introduced in Fouque and Sun (2013) but has been commonly accepted and used by several authors (see for example, Fouque and Ichiba 2013; Carmona et al. 2015; Fatone and Mariani 2019; more recently, Sun 2018; Biagini et al. 2019a) in order to have a banking system model that is simple enough for mathematical analysis, yet captures how banks' lending preferences affect possible multiple bank failures. Introducing incentives for lending and borrowing would make the model more realistic. Some attempts in this direction can be found in Carmona et al. (2015), Rogers and Veraart (2013), Eisert and Eufingerbi (2014) and may serve as a topic of future research. From an economic standpoint cooperation among banks can be seen, for example, as an "extreme" risk-sharing mechanism. This is the case when independent banks, having significant exposure to the same assets, begin cooperating to protect themselves from the decrease in assets values which could prejudice their interests and can lead to a crisis in the entire banking system. Finally, the hypothesis implied by the borrowing and lending mechanism adopted in these models, i.e., the potentially unlimited possibility of borrowing and lending money among banks and between banks and the monetary authority, is a very strong assumption. However, these simplifications are balanced by the fact that, in the simplified models considered, a pseudo mean field limit of the system can be easily computed and used to study systemic risk governance via an ad hoc optimal control problem. Moreover, understanding the simplified models can be seen as a preliminary step in the study of more realistic banking system models.

As emphasized in many papers dealing with discrete-time models of banking systems (see for instance, Berardi and Tedeschi 2017; Caccioli et al. 2014; Grilli et al. 2014, 2017; Iori et al. 2006; Lenzu and Tedeschi 2012; Tedeschi et al. 2012), heterogeneity influences the resilience and stability of a banking system, thereby determining the level of systemic risk. Bank heterogeneity concerns several aspects, for example, bank size, investment opportunities (Iori et al. 2006), balance-sheet distribution (Berardi and Tedeschi 2017; Iori et al. 2006), and topology of the banking network (Berardi and Tedeschi 2017; Caccioli et al. 2012; Tedeschi et al. 2012). All these aspects affect systemic risk in a banking system in different ways. The mechanism of shock propagation among financial institutions and its relation to systemic risk is also usually studied using static or discrete-time models. We have focused our attention on the models proposed in May and Nimalan (2010), Haldane and May (2011) since the authors investigate shock propagation in a static assets–liabilities model of a banking system using a mean field approach. In these papers, the effect of exogenous shocks on the individual bank is modeled as a sudden reduction in assets value.

It would be interesting to investigate how the above-mentioned properties are reflected when dealing with continuous-time dynamical models of banking systems. A first step in this direction is to analyze simple continuous-time dynamical models (like the one presented here) and then extend this study to more general, complex, diverse and realistic dynamical models. In order to do this, at least from a theoretical viewpoint, we explain the details of the banking system model studied in this paper and its potential generalizations.

We consider N banks that lend to and borrow from each other. These banks form the nodes of a network representing the banking system. From a mathematical point of view, the network is a graph of order N, i.e., a collection of N nodes with links among them. The links denote the presence of a cooperative relationship among the nodes and each link has a capacity indicating the "intensity" of the cooperation mechanism between the linked nodes. Various assumptions about the structure of the financial network, and therefore about the topology of the graph, can be made. These assumptions hold important consequences for the study of systemic risk associated with the financial network. In the simplest models, such as the one considered here and in Fouque and Sun (2013), Fatone and Mariani (2019), the bank population is homogeneous and any one of the N banks is linked in the same way to all the other banks as a lender and borrower, that is, all the links have the same capacity and the graph associated with the financial network is the clique. In particular, this means that all banks are copies of a "unique bank". More refined and realistic banking system models can be considered. For example, the banks in the model can be linked to a central bank (see e.g., Carmona et al. 2015) and they may or may not be linked among themselves. In the latter case, the graph corresponding to the financial network is a star graph. The banking system model can be generalized to consider banks of different sizes, for example, a situation characterized by the presence of big banks and small banks or big banks, medium banks and small banks. For instance, in May et al. (2008) it is shown that the topology of the USA Fedwire system, composed of some 9500 participating banks, is "highly non-random in a dissociative way", that is, there are a few big banks and each big bank is connected to many small banks. Moreover, the small banks are connected to only a few other banks, which are mainly the big banks.

More realistic models for these networks could be obtained, for example, by introducing assets and liabilities accounting, incentives for borrowing and lending, and upper bounds to the borrowing and lending activities among banks. More generally, models that are networks of networks are sometime used to study real banking systems.

The study of some of these more realistic models shows that, under some suitable assumptions, it is possible to reduce the default risk for each type of bank present in the system (i.e., big banks, medium banks, small banks), but the price of this reduction is an increase in the default of the entire system (see e.g., Haldane and May 2011). As a consequence, systemic risk must be governed. In these more complex models, the original mean field approximation cannot be used. In fact, different types of banks do not have the same "average behavior". One possibility would be to aggregate information deduced from the behavior of the mean field approximation of each type of bank present in the model according to the effective relationship between the banks. In this way, the resulting approximation could be used to govern the systemic risk associated with the original banking system. For example, the model studied in this paper could be thought of as a rough representation of the "big bank" system of a given financial network. A possible extension is a model characterized by the presence of "big banks and small banks" (and in general a model characterized by the presence of banks of different types). In this case, two mean field approximations can be considered (one for the big banks and the other for the small banks) and these two approximations can be coupled in a dynamical system that describes the qualitative interaction among the different types of banks in the model. The feasibility and effectiveness of this approach depend on the specific model under investigation, and, in particular, on the graph that expresses the interaction among the banks. For an exhaustive review on systemic risk in the context of financial networks, including the study of potential default cascades due to various contagion effects, we refer to Hurd (2016), Fouque and Langsam (2013), Biagini et al. (2019a).

Concerning the modeling of shocks, we generalize the shock propagation mechanism adopted in Haldane and May (2011), May and Nimalan (2010) to continuous-time models. In the banking system model proposed, realistic situations of banking distress due to the deterioration in the quality of banks' assets and/or liabilities can be modeled. Shocks that hit the banking system are simulated with jumps in the volatilities of the stochastic differential equations satisfied by the banks' assets and liabilities and with jumps in the correlation coefficients of the stochastic differentials of the diffusion terms that appear on the right-hand side of the assets and liabilities equations.

We use "systemic risk" or "systemic event" in a bounded time interval to refer to the fact that in that time interval at least a given fraction of the banks in the model fails. Given a banking system model, we use statistical simulation to evaluate the probability of systemic risk in a bounded time interval. The action of the cooperation mechanism among banks reduces the default probability of the individual bank at the expense of an increase in the default probability of all or almost all the banks in the banking system. The latter case is called "extreme" systemic risk.

When the number of banks in the model goes to infinity, a heuristic approximation of the banking system model called "pseudo mean field approximation" is introduced. This approximation is inspired by the mean field approximation of statistical mechanics (see e.g., Gallavotti 1999) and is based on the homogeneity of the bank population.

The pseudo mean field approximation is a stochastic dynamical system with two degrees of freedom.

We present a method to govern the probability of systemic risk in a bounded time interval. The goal of governance is to keep the probability of systemic risk in a bounded time interval between two given thresholds. Governance exploits the choice made by the monetary authority for the assets and liabilities of a kind of "ideal bank" as functions of time and the solution of a stochastic optimal control problem for the pseudo mean field approximation of the banking system model. In fact, in a homogeneous bank population when there are enough banks, all banks behave like a sort of "mean bank", the behavior of which is approximated with the behavior of the pseudo mean field approximation of the banking system model. This behavior is forced to be similar to the behavior of the "ideal bank" by solving a stochastic optimal control problem for the pseudo mean field approximation of the banking system model. With a homogeneous bank population, the governance of the pseudo mean field approximation is easily translated into the governance of the entire bank population. More specifically, it is translated into the rules for the cooperation mechanism among banks. In this way, systemic risk governance induces the individual banks to behave like the ideal bank. Shocks in the banks' assets and liabilities are simulated and numerical examples of systemic risk governance in the presence and absence of shocks are presented.

The paper is organized as follows. In Sect. 2, an assets–liabilities banking system model is defined. In Sect. 3, the definition of systemic risk in a bounded time interval is given and the implications of the presence of the cooperation mechanism and homogeneity of the bank population for the systemic risk probability are investigated. In Sect. 4, the mean field and pseudo mean field approximations of the banking system model defined in Sect. 2 are discussed. In Sect. 5, an optimal control problem for the pseudo mean field approximation of the banking system model is solved and the optimal control that is found is translated into the rules that determine the behavior of the cooperation mechanism among banks. Finally in Sect. 7, a method to govern systemic risk in a bounded time interval is presented and some numerical examples of systemic risk governance of banking systems in the presence and absence of shocks are discussed.

2 The banking system model

Let t be a real variable that denotes time and $N > 1$ be a positive integer representing the number of banks present in the banking system model at time $t = 0$. The superscript i labels the i-th bank, $i = 1, 2, \dots, N$. The activities of each bank are partitioned in the following categories: interbank loans, external assets, deposits and interbank borrowings. The assets of a bank are composed of the bank's interbank loans and external assets. The liabilities of a bank are composed of the bank's deposits and interbank borrowings. The assets a_t^i of the i-th bank at time $t \geq 0$ are the sum of the interbank loans ι_t^i at time $t \geq 0$, and the external assets e_t^i at time $t \geq 0$, of the i-th bank, $i = 1, 2, \dots, N$, that is:

$$a_t^i = \iota_t^i + e_t^i, \quad t \geq 0, \qquad i = 1, 2, \dots, N. \tag{1}$$

The liabilities l_t^i of the i-th bank at time $t \geq 0$ are the sum of the deposits d_t^i at time $t \geq 0$, and the interbank borrowings b_t^i at time $t \geq 0$, of the i-th bank, $i = 1, 2, \ldots, N$, that is:

$$l_t^i = d_t^i + b_t^i, \quad t \geq 0, \qquad i = 1, 2, \ldots, N. \tag{2}$$

The previous four categories of activities are balanced in the bank's capital. The net worth or equity of the i-th bank, c_t^i, at time $t \geq 0$, is defined as the difference between assets a_t^i at time $t \geq 0$ and liabilities l_t^i at time $t \geq 0$, of the i-th bank, $i = 1, 2, \ldots, N$, that is:

$$c_t^i = a_t^i - l_t^i, \quad t \geq 0, \qquad i = 1, 2, \ldots, N. \tag{3}$$

Equations (3) represent stylized balance sheets of the banks present in the banking system model. In this paper, we consider a continuous-time dynamical model that allows assets and liabilities to behave stochastically. This behavior is consistent with the uncertainty associated with items appearing on the balance sheets (3), namely the interbank loans and external assets (assets) and the deposits and interbank borrowings (liabilities).

We assume that a bank is solvent when its assets are greater than or equal to its liabilities, that is, we assume that the i-th bank is solvent at time $t \geq 0$, if

$$c_t^i = a_t^i - l_t^i \geq 0, \quad t \geq 0, \qquad i = 1, 2, \ldots, N. \tag{4}$$

When the net worth c_t^i, $t \geq 0$, of the i-th bank becomes negative for the first time during the time evolution, the i-th bank has failed, $i = 1, 2, \ldots, N$. The failed banks are removed from the banking system model. Note that in the models studied in this paper, the assets, liabilities and net worth of each bank are stochastic processes in time. This means, in particular, that inequality (4) must be considered for each path of the stochastic process that represents the net worth. That is, a bank can have failed on a path of its net worth and be solvent on a different path. Equations (1), (2), (3) are a simple model of bank capital; more advanced models can be found, for example, in Diamond and Rajan (2000).

In Mukuddem-Petersen and Petersen (2006, 2008), the dynamics of each addendum present on the right-hand side of (1), (2) is specified. Here instead we specify only the dynamics of the assets a_t^i, $t \geq 0$, and liabilities l_t^i, $t \geq 0$, $i = 1, 2, \ldots, N$. In fact, we assume that the banks' assets and liabilities are stochastic processes of time defined implicitly by the following system of stochastic differential equations:

$$\mathrm{d}a_t^i = a_t^i \mu_a \mathrm{d}t + a_t^i \sigma_a \mathrm{d}W_t^i, \quad t > 0, \; i = 1, 2, \ldots, N, \tag{5}$$

$$\mathrm{d}l_t^i = l_t^i \mu_l \mathrm{d}t + l_t^i \sigma_l \mathrm{d}Z_t^i, \quad t > 0, \; i = 1, 2, \ldots, N, \tag{6}$$

with initial conditions:

$$a_0^i = \tilde{a}_0^i, \quad l_0^i = \tilde{l}_0^i, \quad i = 1, 2, \ldots, N, \tag{7}$$

where $\sigma_a = \sigma_{a,t}$, $t > 0$, $\sigma_l = \sigma_{l,t}$, $t > 0$ are piecewise constant positive functions of time and μ_a, μ_l are real constants. In (7), $\tilde{a}_0^i$, $\tilde{l}_0^i$, $i = 1, 2, \ldots, N$, are random variables that, for simplicity, we assume to be concentrated at a point with probability one. Abusing the notation, we use the same symbols to denote the random variables and the points where the random variables are concentrated. We assume $\tilde{a}_0^i > 0$, $\tilde{l}_0^i > 0$, $\tilde{a}_0^i - \tilde{l}_0^i > 0$, $i = 1, 2, \ldots, N$, that is, we assume that at time $t = 0$, all banks are solvent with probability one.

The stochastic processes W_t^i, Z_t^i, $t \geq 0$, in (5), (6) are standard Wiener processes, such that $W_0^i = 0$, $Z_0^i = 0$, and $\mathrm{d}W_t^i$, $\mathrm{d}Z_t^i$, $t > 0$, are their stochastic differentials, $i = 1, 2, \ldots, N$. We assume that:

$$\begin{aligned} &\mathbb{E}(\mathrm{d}W_t^i \mathrm{d}W_t^j) = \rho_a^2\,\mathrm{d}t, \quad i \neq j, \qquad \mathbb{E}(\mathrm{d}Z_t^i \mathrm{d}Z_t^j) = \rho_l^2\,\mathrm{d}t, \quad i \neq j, \\ &\mathbb{E}(\mathrm{d}W_t^i \mathrm{d}W_t^i) = \mathbb{E}(\mathrm{d}Z_t^j \mathrm{d}Z_t^j) = \mathrm{d}t, \qquad \mathbb{E}(\mathrm{d}W_t^i \mathrm{d}Z_t^j) = 0, \\ &\qquad\qquad t > 0, \quad i, j = 1, 2, \ldots, N, \end{aligned} \tag{8}$$

where $\mathbb{E}(\cdot)$ denotes the expected value of $\cdot$, and $\rho_a = \rho_{a,t}$, $t > 0$, $\rho_l = \rho_{l,t}$, $t > 0$, are piecewise constant functions of time such that $|\rho_{a,t}| \leq 1$, $|\rho_{l,t}| \leq 1$, $t > 0$. The stochastic differentials $\mathrm{d}W_t^i$, $t > 0$, $i = 1, 2, \ldots, N$, can be represented as follows:

$$\mathrm{d}W_t^i = \rho_a d\widetilde{W}_t^0 + \sqrt{1 - \rho_a^2}\, d\widetilde{W}_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{9}$$

where $\widetilde{W}_t^j$, $t \geq 0$, $j = 0, 1, \ldots, N$, are independent standard Wiener processes such that $\widetilde{W}_0^j = 0$, $j = 0, 1, \ldots, N$, and $d\widetilde{W}_t^j$, $t > 0$, $j = 0, 1, \ldots, N$, are their stochastic differentials. The term $d\widetilde{W}_t^0$, $t > 0$, is called the common noise of the assets equations (5). Similarly the stochastic differentials $\mathrm{d}Z_t^i$, $t > 0$, $i = 1, 2, \ldots, N$, can be represented as follows:

$$dZ_t^i = \rho_l d\widetilde{Z}_t^0 + \sqrt{1 - \rho_l^2}\, d\widetilde{Z}_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{10}$$

where $\widetilde{Z}_t^j$, $t \geq 0$, $j = 0, 1, \ldots, N$, are independent standard Wiener processes such that $\widetilde{Z}_0^j = 0$, $j = 0, 1, \ldots, N$, and $d\widetilde{Z}_t^j$, $t > 0$, $j = 0, 1, \ldots, N$, are their stochastic differentials. The term $d\widetilde{Z}_t^0$, $t > 0$, is called the common noise of the liabilities equations (6). Finally we assume that $d\widetilde{W}_t^i$ and $d\widetilde{Z}_t^j$ are independent, $t > 0$, $i, j = 0, 1, \ldots, N$.

Note that in (8), the correlation coefficients ρ_a^2, ρ_l^2 between the stochastic differentials of the assets equations (5) and liabilities equations (6) are non-negative. These non-negative correlation coefficients generate the so-called "collective" behavior of the banks in the presence of a shock and are translated into the representation formulae of the stochastic differentials (9), (10). The correlation model (8) can be easily extended to more general situations.

For example, in (8), the correlation between assets and liabilities for a given bank and the correlation between assets and liabilities of different banks can be considered and modeled. In this case, the representation formulae (9), (10) must be adapted to the circumstances, using, for instance, the Cholesky factorization for the correlation matrix of the Wiener processes appearing in (8). For simplicity, we omit these generalizations here.

Note that the diffusion coefficient σ_a is the same in all assets equations (5) and that similar statements hold for the diffusion coefficient σ_l, the drift coefficients μ_a, μ_l, and the correlation coefficients ρ_a, ρ_l. Moreover, let us assume that $\tilde{a}_0^i = \tilde{a}_0$, $\tilde{l}_0^i = \tilde{l}_0$, $i = 1, 2, \dots, N$, so we have $\tilde{a}_0 > 0$, $\tilde{l}_0 > 0$, $\tilde{a}_0 - \tilde{l}_0 > 0$. With these assumptions, all banks in the model are equal, that is, the bank population described by the banking system model (3), (5), (6), (7), (8) is homogeneous. Systems composed of a homogeneous population of "individuals" are studied in statistical mechanics where the individuals are usually atoms or molecules. In extending the ideas developed in statistical mechanics to banking system models, we show that the homogeneity of the bank population implies that when N goes to infinity, all banks behave in the same way, i.e., they all behave as a kind of "mean bank". Using the language of statistical mechanics, the "mean bank" behavior is defined by the "mean field" approximation of the banking system model.

In an assets–liabilities dynamical model of a banking system [like model (3), (5), (6), (7), (8)], it is possible to study the propagation of certain types of shocks. For example, one can model shocks consisting of losses in value of the banks' external assets caused by a generalized fall in the assets market prices and/or by a generalized rise in expected defaults (see for example, Haldane and May 2011; May and Nimalan 2010). These shocks reduce the net worth of all banks at the same time, leading to an abrupt increase in the probability of systemic risk in a bounded time interval. In model (3), (5), (6), (7), (8) shocks are modeled with jumps in the volatility σ_a in the assets equations (5), while keeping σ_l constant in the liabilities equations (6) or, vice versa, with jumps of σ_l keeping σ_a constant. For simplicity, we do not consider jumps of σ_a and σ_l at the same time. In other words, the shocks acting on the banks' assets and liabilities are modeled choosing the functions $\sigma_a = \sigma_{a,t}$, $t > 0$, and $\sigma_l = \sigma_{l,t}$, $t > 0$. Moreover, in model (3), (5), (6), (7), (8), it is possible to study the banks' "collective" behavior in the presence of a shock. In fact, when a shock hits the banking system, all banks react in the same way and this "collective" behavior is modeled with a positive correlation between the stochastic differentials on the right-hand side of the assets equations (5) and/or the liabilities equations (6). That is, the banks' "collective" behavior in reaction to a shock is modeled with a jump in the functions $\rho_a = \rho_{a,t}$, $t > 0$, and/or $\rho_l = \rho_{l,t}$, $t > 0$.

Using model (3), (5), (6), (7), (8), we adapt the mechanisms used in the models presented in Fouque and Sun (2013), Fatone and Mariani (2019) to describe the cooperation among banks and we introduce the terms used to describe the intervention of the monetary authority in the banking system dynamics. To do so, we define the new variables G_t^i, H_t^i, $t \geq 0$, $i = 1, 2, \dots, N$, as follows:

$$G_t^i = \ln(a_t^i), \quad H_t^i = \ln(l_t^i), \quad t \geq 0, \quad i = 1, 2, \dots, N, \tag{11}$$

where $\ln(\cdot)$ is the logarithm of $\cdot$. First, note that the variables $G_t^i = \ln(a_t^i)$, $H_t^i = \ln(l_t^i)$, $t \geq 0$, $i = 1, 2, \dots, N$, are well-defined. In fact, at time $t = 0$, for $i = 1, 2, \dots, N$, we have $\tilde{a}_0^i = \tilde{a}_0 > 0$, $\tilde{l}_0^i = \tilde{l}_0 > 0$, $\tilde{a}_0 - \tilde{l}_0 > 0$, with probability one, and therefore Eqs. (5), (6) imply that $a_t^i > 0$, $l_t^i > 0$, with probability one, $t > 0$.

The quantities $G_t^i = \ln(a_t^i)$, $H_t^i = \ln(l_t^i)$, are, respectively, the log-assets and log-liabilities of the i-th bank at time $t \geq 0$, $i = 1, 2, \ldots, N$.

Using Itô's Lemma and Eqs. (5), (6), it is easy to see that the stochastic processes G_t^i, H_t^i, $t \geq 0$, $i = 1, 2, \ldots, N$, satisfy the following equations:

$$\mathrm{d}G_t^i = \left(\mu_a - \frac{1}{2}\sigma_a^2\right)\mathrm{d}t + \sigma_a \mathrm{d}W_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{12}$$

$$\mathrm{d}H_t^i = \left(\mu_l - \frac{1}{2}\sigma_l^2\right)\mathrm{d}t + \sigma_l \mathrm{d}Z_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{13}$$

with initial conditions:

$$G_0^i = \ln(\tilde{a}_0^i) = \ln(\tilde{a}_0), \quad H_0^i = \ln(\tilde{l}_0^i) = \ln(\tilde{l}_0), \quad i = 1, 2, \ldots, N. \tag{14}$$

Let us define the stochastic processes:

$$A_t^i = G_t^i - \left(\mu_a t - \frac{1}{2}\int_0^t \sigma_{a,\tau}^2 \mathrm{d}\tau\right), \quad t \geq 0,\ i = 1, 2, \ldots, N, \tag{15}$$

$$L_t^i = H_t^i - \left(\mu_l t - \frac{1}{2}\int_0^t \sigma_{l,\tau}^2 \mathrm{d}\tau\right), \quad t \geq 0,\ i = 1, 2, \ldots, N. \tag{16}$$

From (12), (13), (14) it is easy to see that A_t^i, L_t^i, $t \geq 0$, $i = 1, 2, \ldots, N$, satisfy the following equations:

$$\mathrm{d}A_t^i = \sigma_a \mathrm{d}W_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{17}$$

$$d\mathrm{L}_t^i = \sigma_l \mathrm{d}Z_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{18}$$

with initial conditions:

$$A_0^i = \ln(\tilde{a}_0^i) = \ln(\tilde{a}_0), \quad L_0^i = \ln(\tilde{l}_0^i) = \ln(\tilde{l}_0), \quad i = 1, 2, \ldots, N. \tag{19}$$

Let ψ_t, $t \geq 0$, be a continuous piecewise differentiable function; the notation $d\psi_t = \frac{d\psi_t}{\mathrm{d}t}\,\mathrm{d}t = (\psi_t)'\,\mathrm{d}t$, $t > 0$, denotes the "piecewise differential" of ψ_t, $t \geq 0$.

Using the ideas developed in Fouque and Sun (2013), Fatone and Mariani (2019) we modify Eqs. (17), (18) and we introduce the terms used to implement the cooperation mechanism among banks and the terms used to model the intervention of the monetary authority in the banking system dynamics. This is done by adding some drift terms to (17), (18). That is, given the continuous piecewise differentiable functions $\varphi_t > 0$, $\phi_t > 0$, $t \geq 0$, such that $\varphi_t - \phi_t > 0$, $t \geq 0$, we replace Eqs. (17), (18), respectively, with the following:

$$\mathrm{d}A_t^i = \frac{\alpha_t}{N}\sum_{j=1}^{N}\left(A_t^j - A_t^i\right)\mathrm{d}t + d\tilde{\varphi}_t + \sigma_a \mathrm{d}W_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{20}$$

$$\mathrm{d}L_t^i = \frac{\gamma_t}{N}\sum_{j=1}^{N}\left(L_t^j - L_t^i\right)\mathrm{d}t + d\tilde{\phi}_t + \sigma_l \mathrm{d}W_t^i, \quad t > 0,\ i = 1, 2, \ldots, N, \tag{21}$$

where $\tilde{\varphi}_t$, $\tilde{\phi}_t$, $t \geq 0$, are given by:

$$\tilde{\varphi}_t = \ln(\varphi_t) - \mu_a t + \frac{1}{2}\int_0^t \sigma_{a,\tau}^2 \mathrm{d}\tau, \quad t \geq 0, \tag{22}$$

$$\tilde{\phi}_t = \ln(\phi_t) - \mu_l t + \frac{1}{2}\int_0^t \sigma_{l,\tau}^2 \mathrm{d}\tau, \quad t \geq 0. \tag{23}$$

Equations (20) and (21) are completed with the initial conditions (19) and with the assumptions about the correlation coefficients (8). Hereafter, for convenience we assume $\tilde{a}_0 = \varphi_0$, $\tilde{l}_0 = \phi_0$.

Equations (20) and (21) describe, respectively, the banks' "assets" and "liabilities". For simplicity, the variables A_t^i, L_t^i, $t \geq 0$, are called, respectively, the "assets" and "liabilities" of the i-th bank, $i = 1, 2, \ldots, N$, instead of centered log-assets and centered log-liabilities as would be more appropriate.

The functions φ_t, ϕ_t, are interpreted, respectively, as assets and liabilities of the "ideal bank" at time t, $t \geq 0$. The fact that the "ideal bank" is solvent corresponds to the assumption that $\varphi_t - \phi_t > 0$, $t \geq 0$. Recall that the functions $\tilde{\varphi}_t$, $\tilde{\phi}_t$, $t \geq 0$, of Eqs. (20) and (21) are related to φ_t, ϕ_t, $t \geq 0$, through (22) and (23). The functions $\alpha_t \geq 0$, $\gamma_t \geq 0$, $t > 0$, of (20) and (21) regulate the cooperation mechanism among banks and their choice corresponds to the rules of the cooperation mechanism among banks. This choice will later be attributed to the monetary authority and will be used to govern the systemic risk in a bounded time interval of the banking system model. The initial value problem (20), (21), (19) is completed with assumptions (8).

For $i = 1, 2, \ldots, N$, the cooperation of the i-th bank with the other banks is described by the drift terms $\frac{\alpha_t}{N}\sum_{j=1}^{N}\left(A_t^j - A_t^i\right)\mathrm{d}t$, $t > 0$, and $\frac{\gamma_t}{N}\sum_{j=1}^{N}\left(L_t^j - L_t^i\right)\mathrm{d}t$, $t > 0$, respectively, of the i-th Eq. (20) and the i-th Eq. (21). In fact the term $\frac{\alpha_t}{N}\sum_{j=1}^{N}\left(A_t^j - A_t^i\right)\mathrm{d}t$, in the i-th Eq. (20) implies that for $t > 0$ and $j = 1, 2, \ldots, N$, if at time t bank j has more "assets" than bank i (i.e., if $A_t^j > A_t^i$) assets flow from bank j to bank i, and this flow is proportional to the difference $A_t^j - A_t^i$ at the rate $\frac{\alpha_t}{N}$. The opposite happens if bank i has more "assets" than bank j (i.e., if $A_t^j < A_t^i$), $j = 1, 2, \ldots, N$. For $i = 1, 2, \ldots, N$, the term $\frac{\gamma_t}{N}\sum_{j=1}^{N}\left(L_t^j - L_t^i\right)\mathrm{d}t$, $t > 0$, in the i-th Eq. (21) is related to the "liabilities" and is analogous to the term $\frac{\alpha_t}{N}\sum_{j=1}^{N}\left(A_t^j - A_t^i\right)\mathrm{d}t$, $t > 0$, of the "assets" of the i-th Eq. (20); this term has the same effect on the liabilities as the effect that the term $\frac{\alpha_t}{N}\sum_{j=1}^{N}\left(A_t^j - A_t^i\right)\mathrm{d}t$, $t > 0$, has on the assets.

Note that the division by N in the rates $\frac{\alpha_t}{N}$, $\frac{\gamma_t}{N}$, $t > 0$, of the drift terms of Eqs. (20) and (21) is a normalization factor taken from the technical literature (see for example, Carmona et al. 2015; Garnier et al. 2013; Fatone and Mariani 2019). It plays no role in this paper.

The cooperation mechanism added in (20) and (21) is a simple implementation of the idea that "those who have more (assets, liabilities) give to those who have less (assets, liabilities)". In this sense it is a cooperation mechanism between banks. This is a very simple form of cooperation mechanism, but it is widely accepted throughout the scientific community. It was originally introduced in Fouque and Ichiba (2013), Fouque and Sun (2013), but has been generalized in various ways in a number of articles. See, for example, Garnier et al. (2013), Carmona et al. (2015), Sun (2018), Biagini et al. (2019a), Biagini et al. (2019b), Fatone and Mariani (2019).

The drift terms $d\tilde{\varphi}_t, d\tilde{\phi}_t, t > 0$, in Eqs. (20) and (21) describe the intervention of the monetary authority in the banking system dynamics. In fact the term $d\tilde{\varphi}_t, t > 0$, of Eq. (20) is responsible for the fact that the drift terms $\frac{\alpha_t}{N}\sum_{j=1}^{N}\left(A_t^j - A_t^i\right)\mathrm{d}t, t > 0$, $i = 1, 2, \ldots, N$, stabilize the trajectories of $A_t^i, t > 0, i = 1, 2, \ldots, N$, around the function $\tilde{\varphi}_t, t > 0$, and, as a consequence, stabilize the trajectories of $a_t^i, t > 0$, $i = 1, 2, \ldots, N$, around the function $\varphi_t, t > 0$. Analogously the term $d\tilde{\phi}_t, t > 0$, in Eq. (21) is responsible for the fact that the drift terms $\frac{\gamma_t}{N}\sum_{j=1}^{N}\left(L_t^j - L_t^i\right)\mathrm{d}t, t > 0$, $i = 1, 2, \ldots, N$, stabilize the trajectories of $L_t^i, t > 0, i = 1, 2, \ldots, N$, around the function $\tilde{\phi}_t, t > 0$, and, as a consequence, stabilize the trajectories of $l_t^i, t > 0$, $i = 1, 2, \ldots, N$, around the function $\phi_t, t > 0$. That is, when $\alpha_t > 0, \gamma_t > 0, t > 0$, the drift terms introduced in Eqs. (20) and (21) and expressed by $d\tilde{\varphi}_t, d\tilde{\phi}_t, t > 0$, generate a "swarming" effect in the trajectories of the assets $a_t^i, t > 0, i = 1, 2, \ldots, N$, and liabilities $l_t^i, t > 0, i = 1, 2, \ldots, N$, around $\varphi_t, \phi_t, t > 0$, respectively, that is, around the assets and liabilities of the "ideal bank". This implies that the trajectories of the net worth of the i-th bank swarm around the net worth of the "ideal bank" $\xi_t = \varphi_t - \phi_t, t > 0, i = 1, 2, \ldots, N$. This swarming effect is a key ingredient of the systemic risk governance discussed later.

Let us rewrite Eqs. (20), (21), (19) using the stochastic processes $G_t^i, H_t^i, t \geq 0$, $i = 1, 2, \ldots, N$ as dependent variables. We have:

$$\mathrm{d}G_t^i = \frac{\alpha_t}{N}\sum_{j=1}^{N}\left(G_t^j - G_t^i\right)\mathrm{d}t + d\ln(\varphi_t) + \sigma_a \mathrm{d}W_t^i, \quad t \geq 0,\ i = 1, 2, \ldots, N, \tag{24}$$

$$\mathrm{d}H_t^i = \frac{\gamma_t}{N}\sum_{j=1}^{N}\left(H_t^j - H_t^i\right)\mathrm{d}t + d\ln(\phi_t) + \sigma_l \mathrm{d}Z_t^i, \quad t \geq 0,\ i = 1, 2, \ldots, N, \tag{25}$$

with initial conditions:

$$G_0^i = \ln(\tilde{a}_0), \quad H_0^i = \ln(\tilde{l}_0), \quad i = 1, 2, \ldots, N, \tag{26}$$

where $\tilde{a}_0 = \varphi_0, \tilde{l}_0 = \phi_0$. We add assumption (8) to Eqs. (3), (11), (24), (25), (26), thereby completing the banking system model.

For simplicity, we use the same symbols to denote the variables of model (3), (5), (6), (7), (8) and those of model (3), (11), (24), (25), (26), (8). When necessary to avoid ambiguity, we specify the banking system model considered.

Note that when $\alpha_t = 0, \gamma_t = 0, t > 0$, and the functions $\varphi_t, \phi_t, t \geq 0$, are constants, there is no cooperation among banks and no intervention of the monetary authority in the banking system dynamics. In this case, model (3), (11), (24), (25), (26), (8), reduces to model (3), (5), (6), (7), (8).

3 Systemic risk in a bounded time interval

Given the banking system model (3), (5), (6), (7), (8), or (3), (11), (24), (25), (26), (8), we define two events: (i) default of a bank in a bounded time interval, (ii) systemic risk in a bounded time interval, and we introduce a probability distribution called "loss distribution" of banks that have defaulted in a bounded time interval.

Given $0 \leq \tau_1 < \tau_2 < +\infty$ and the default level $D \geq 0$, we define the event $F^i_{[\tau_1,\tau_2]}$, "default of the i-th bank in the time interval $[\tau_1, \tau_2]$", as follows:

$$F^i_{[\tau_1,\tau_2]} = \left\{ \min_{\tau_1 \leq t \leq \tau_2} c^i_t < D \right\}, \quad i = 1, 2, \ldots, N. \tag{27}$$

That is, for $i = 1, 2, \ldots, N$, the i-th bank defaults in the time interval $[\tau_1, \tau_2]$ if its net worth $c^i_t, t \geq 0$, in that time interval goes below the default level D. Recall that in this paper we have chosen $D = 0$; the inequality $\min_{\tau_1 \leq t \leq \tau_2} c^i_t < D$ is considered for each path of the stochastic process $c^i_t, \tau_1 \leq t \leq \tau_2, i = 1, 2, \ldots, N$. The failed banks are removed from the banking system model, which means that the number of banks present in the model may depend on the path of the banking system model considered and may not be constant during the time evolution.

Let int $[\cdot]$ be the integer part of the real number $\cdot$ and M be a positive integer such that $\mathrm{int}\left[\frac{N}{2}\right] \leq M \leq N$. The systemic risk (or systemic event) of type M in the time interval $[\tau_1, \tau_2]$, $\mathrm{SR}_{[\tau_1,\tau_2]}$, is the event defined as follows:

$$\mathrm{SR}^M_{[\tau_1,\tau_2]} = \{\text{at least } M \text{ banks fail in the time interval } [\tau_1, \tau_2]\}. \tag{28}$$

In this paper, we choose $M = \mathrm{int}\left[\frac{N}{2}\right] + 1$ and we write $\mathrm{SR}_{[\tau_1,\tau_2]}$ to mean $\mathrm{SR}^M_{[\tau_1,\tau_2]}$ when $M = \mathrm{int}\left[\frac{N}{2}\right] + 1$.

Let $\mathcal{P}(\cdot)$ be the probability of the event $\cdot$. Given the banking system model (3), (5), (6), (7), (8) or (3), (11), (24), (25), (26), (8) we associate to the events $F^i_{[\tau_1,\tau_2]}, i = 1, 2, \ldots, N$, and $\mathrm{SR}_{[\tau_1,\tau_2]}$ defined in (27), (28) a probability evaluated using statistical simulation. In fact the probability $\mathcal{P}(F^i_{[\tau_1,\tau_2]})$ of event $F^i_{[\tau_1,\tau_2]}, i = 1, 2, \ldots, N$, and the probability $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]})$ of event $\mathrm{SR}_{[\tau_1,\tau_2]}$ is approximated with the corresponding frequencies computed on a set of numerically simulated trajectories of the banking system model considered. Note that due to the homogeneity of the bank population, $\mathcal{P}(F^i_{[\tau_1,\tau_2]})$ does not depend on $i, i = 1, 2, \ldots, N$.

The loss distribution of the banks that have defaulted in the bounded time interval $[\tau_1, \tau_2]$ is the probability distribution of the random variable defined as the number

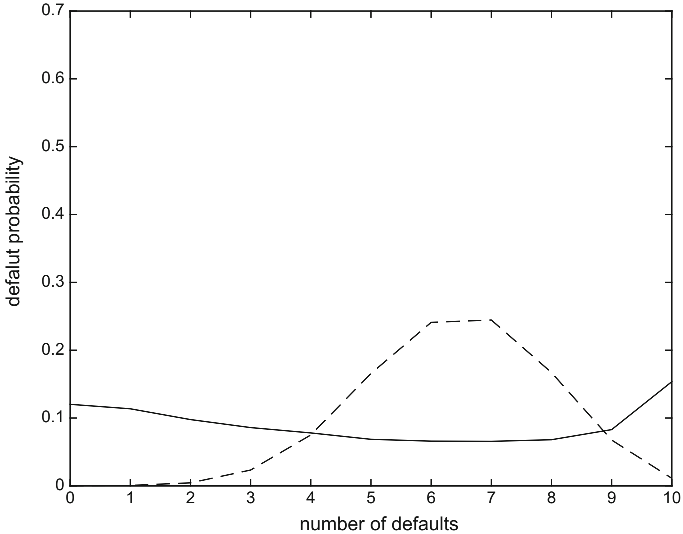

Fig. 1 Loss distribution in $[0, T]$, $T = 1$, of system (3), (11), (24), (25), (26), (8) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{1,t}$, $\sigma_l = 0.6$, $\rho_a = 0$, $\rho_l = 0$, $\varphi_t = 0.1$, $\phi_t = 0.06$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$ (solid line) and loss distribution in $[0, T]$, $T = 1$, of system (3), (5), (6), (7) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{1,t}$, $\sigma_l = 0.6$, $\rho_a = 0$, $\rho_l = 0$, $t \in [0, T]$, $T = 1$ (dashed line)

of bank defaults in the time interval $[\tau_1, \tau_2]$. Given a banking system model, the loss distribution of the banks defaulted in time interval $[\tau_1, \tau_2]$ can be approximated via statistical simulation computing the distribution of the frequencies of the appropriate events in a set of numerically simulated trajectories of the banking system model considered.

Let us study the loss distribution of the banks defaulted in the time interval $[0, T]$, $T = 1$, in banking system models (3), (5), (6), (7), (8) and (3), (11), (24), (25), (26), (8). In both models, we choose $N = 10$, and we evaluate the loss distribution of the banks defaulted in $[0, T]$, $T = 1$, using a statistical simulation starting from 10^4 numerically simulated trajectories of the models considered. We define the functions:

$$\sigma_{1,t} = 0.8, \quad t \in [0, 1], \tag{29}$$

$$\sigma_{2,t} = \begin{cases} 0.2, & t \in [0, 0.2], \\ 1, & t \in (0.2, 1], \end{cases} \tag{30}$$

$$\sigma_{3,t} = \begin{cases} 0.2, & t \in [0, 0.2], \\ 0.8, & t \in (0.2, 0.5], \\ 0.2, & t \in (0.5, 1]. \end{cases} \tag{31}$$

In Figs. 1, 2, 3, 4 and 5, the dashed line shows the loss distribution of the banks defaulted in time interval $[0, T]$, $T = 1$, of model (3), (5), (6), (7), (8), while the solid line shows the loss distribution of the banks defaulted in time interval $[0, T]$, $T = 1$, of model (3), (11), (24), (25), (26), (8). In Figs. 1, 2, 3, 4 and 5 we have: $N = 10$, $\varphi_t = 0.1$, $\phi_t = 0.06$, $\sigma_l = \sigma_{l,t} = 0.6$, $\rho_l = \rho_{l,t} = 0$, $t \in [0, T]$, $T = 1$, $\mu_a = 0.1$,

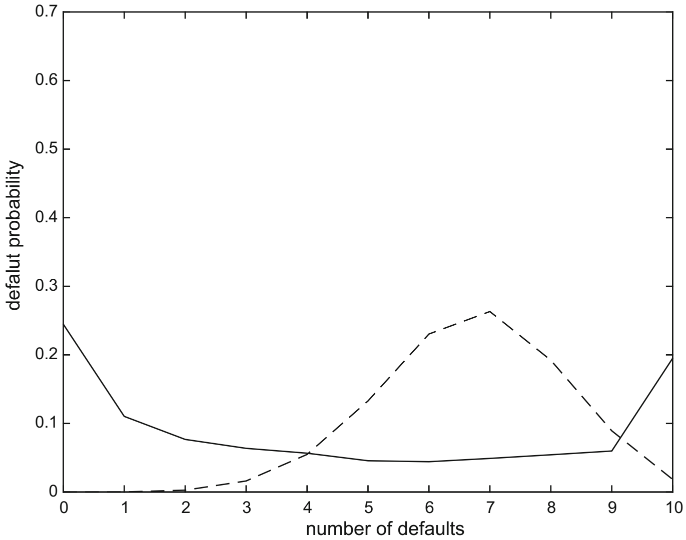

Fig. 2 Loss distribution in $[0, T]$, $T = 1$, of system (3), (11), (24), (25), (26), (8) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{2,t}$, $\sigma_l = 0.6$, $\rho_a = 0$, $\rho_l = 0$, $\varphi_t = 0.1$, $\phi_t = 0.06$, $\alpha_t = 20$, $\gamma_t = 20$, $t \in [0, T]$, $T = 1$ (solid line) and loss distribution in $[0, T]$, $T = 1$, of system (3), (5), (6), (7) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{2,t}$, $\sigma_l = 0.6$, $\rho_a = 0$, $\rho_l = 0$, $t \in [0, T]$, $T = 1$ (dashed line)

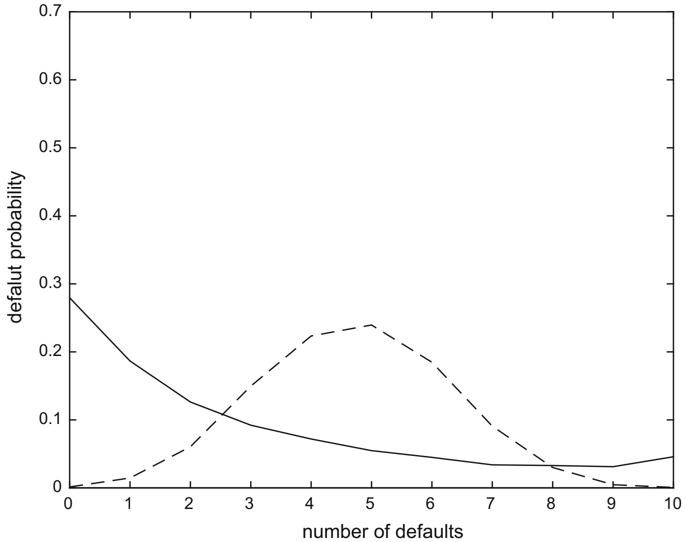

Fig. 3 Loss distribution in $[0, T]$, $T = 1$, of system (3), (11), (24), (25), (26), (8) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{3,t}$, $\sigma_l = 0.6$, $\rho_a = 0$, $\rho_l = 0$, $\varphi_t = 0.1$, $\phi_t = 0.06$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$ (solid line) and loss distribution in $[0, T]$, $T = 1$, of system (3), (5), (6), (7) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{3,t}$, $\sigma_l = 0.6$, $\rho_a = 0$, $\rho_l = 0$, $t \in [0, T]$, $T = 1$ (dashed line)

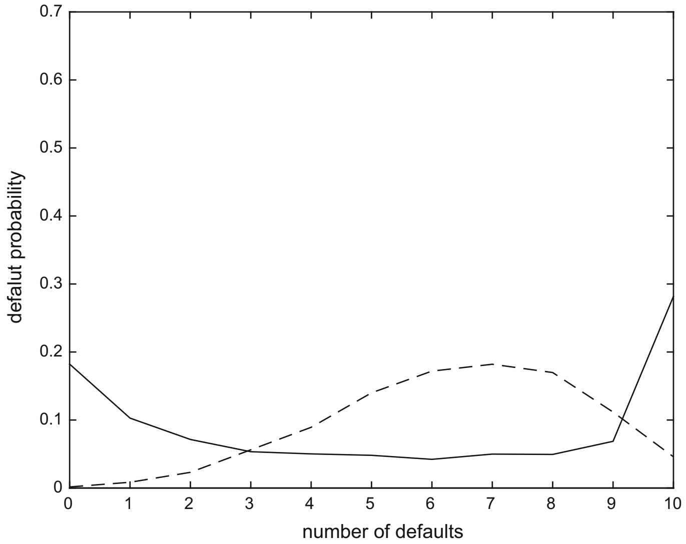

Fig. 4 Loss distribution in $[0, T]$, $T = 1$, of system (3), (11), (24), (25), (26), (8) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{1,t}$, $\sigma_l = 0.6$, $\rho_a = 0.5$, $\rho_l = 0$, $\varphi_t = 0.1$, $\phi_t = 0.06$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$ (solid line) and loss distribution in $[0, T]$, $T = 1$, of system (3), (5), (6), (7) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{1,t}$, $\sigma_l = 0.6$, $\rho_a = 0.5$, $\rho_l = 0$, $t \in [0, T]$, $T = 1$ (dashed line)

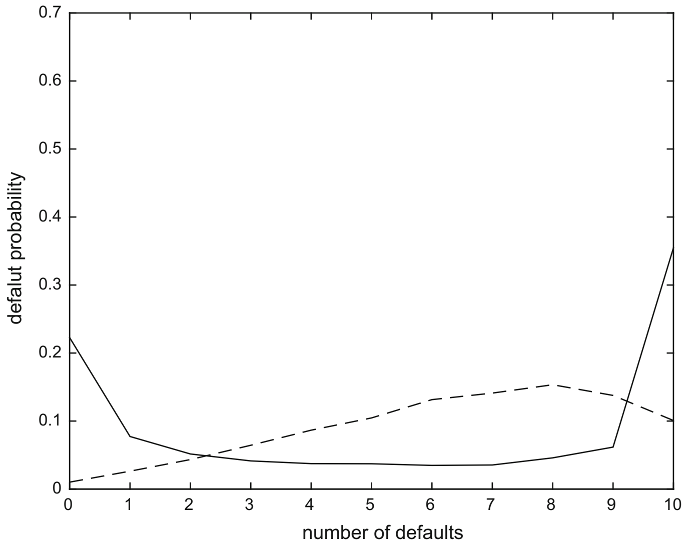

Fig. 5 Loss distribution in $[0, T]$, $T = 1$, of system (3), (11), (24), (25), (26), (8) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{1,t}$, $\sigma_l = 0.6$, $\rho_a = \sqrt{0.5}$, $\rho_l = 0$, $\varphi_t = 0.1$, $\phi_t = 0.06$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$ (solid line) and loss distribution in $[0, T]$, $T = 1$, of system (3), (5), (6), (7) when $N = 10$, $\mu_a = 0.1$, $\mu_l = 0.1$, $\sigma_a = \sigma_{1,t}$, $\sigma_l = 0.6$, $\rho_a = \sqrt{0.5}$, $\rho_l = 0$, $t \in [0, T]$, $T = 1$ (dashed line)

$\mu_l = 0.1$. Moreover in Fig. 1 we have: $\sigma_a = \sigma_{1,t}$, $\rho_a = \rho_{a,t} = 0$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$, in Fig. 2 we have: $\sigma_a = \sigma_{2,t}$, $\rho_a = \rho_{a,t} = 0$, $\alpha_t = 20$, $\gamma_t = 20$, $t \in [0, T]$, $T = 1$, in Fig. 3 we have: $\sigma_a = \sigma_{3,t}$, $\rho_a = \rho_{a,t} = 0$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$, in Fig. 4 we have: $\sigma_a = \sigma_{1,t}$, $\rho_a = \rho_{a,t} = 0.5$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$, and finally in Fig. 5 we have: $\sigma_a = \sigma_{1,t}$, $\rho_a = \rho_{a,t} = \sqrt{0.5}$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T]$, $T = 1$.

Note that the results shown in Figs. 1, 2, 3, 4 and 5 are obtained when the functions $\varphi_t = 0.1$, $\phi_t = 0.06$, $t \in [0, T]$, $T = 1$, are constants. With this choice, there is no intervention of the monetary authority in the banking system dynamics for $t \in (0, T]$ (in fact $d \ln(\varphi_t) = d \ln(\phi_t) = 0$, $t \in (0, T]$, in (24), (25)) and only the cooperation mechanism among banks is active when $t \in [0, T]$. Note that in Figs. 1, 2, 3, 4 and 5 the functions α_t, γ_t, $t \in [0, T]$, $T = 1$, are also chosen to be constant.

For model (3), (5), (6), (7), (8), the loss distribution of the banks defaulted in $[0, T]$, $T = 1$ (shown with a dashed line in Figs. 1, 2, 3, 4 and 5) is a unimodal distribution with a unique maximum corresponding to a maximizer (or to several adjacent maximizers) located in the interior of the interval $[0, N]$, $N = 10$. In contrast, when we consider model (3), (11), (24), (25), (26), (8), the loss distribution of the banks defaulted in the time interval $[0, T]$, $T = 1$ (shown with a solid line in Figs. 1, 2, 3, 4, 5) has a bump near zero defaults and a bump near N defaults and is small in between, i.e., it is a bimodal distribution with two maxima corresponding to two maximizers (or to two disjoint sets of adjacent maximizers) located at the endpoints of the interval $[0, N]$, $N = 10$. This is due to the action of the cooperation mechanism among banks in model (3), (11), (24), (25), (26), (8). Moreover, a comparison of Fig. 1 with Figs. 4 and 5 shows that the presence of a non-zero correlation (i.e., $\rho_a \neq 0$, $\rho_l = 0$) between the stochastic differentials on the right-hand side of the assets equations of the banks (Figs. 4, 5) substantially increases the probability of "extreme" systemic risk with respect to the probability of the same event in the zero correlation case (i.e., $\rho_a = \rho_l = 0$) (Fig. 1). Similar phenomena appear when volatility and correlation coefficient jumps are present in the liabilities equations.

Figures 1, 2, 3, 4 and 5 show that in a homogeneous bank population, the cooperation among banks introduced in model (3), (11), (24), (25), (26), (8) reduces the default probability of the individual bank when compared to the default probability of the individual bank in model (3), (5), (6), (7), (8) at the expense of the default probability of the entire (or almost the entire) banking system, which is greater in model (3), (11), (24), (25), (26), (8) than in model (3), (5), (6), (7), (8). Moreover, the comparison of Fig. 1 with Figs. 4 and 5 shows that this effect is enhanced by the presence of "collective" behaviors in the bank population (i.e., it is enhanced when ρ_a^2, ρ_l^2 are greater than zero). This is in agreement with the findings of Carmona et al. (2015), Fatone and Mariani (2019), Haldane and May (2011), May and Nimalan (2010), where it is shown that for a stable banking system, excessive homogeneity of the bank population is undesirable.

Note that the previous analysis can be interpreted in different ways according to the choice of the parameter M used in the definition of systemic risk. If, as is done here, we choose $M = \mathrm{int}\left[\frac{N}{2}\right] + 1$ in the definition (28) of systemic risk, we see that when α increases (i.e., when the cooperation among banks increases), the stability of the individual bank increases and the probability of systemic risk decreases. If, on the other hand, in the definition (28) M is chosen to be close enough to N (for example, $M = N - 1$ or $M = N$), we see that when α increases (i.e., when the cooperation among banks increases), the probability of systemic risk increases. The occurrence of an "extreme" systemic risk event, i.e., the event "default of $M = N - 1$ or $M = N$ banks", is often referred to as a situation in which a "cascading" and/or "contagion" effect has occurred. In fact, the intensity of the "contagion" depends on α, and increasing α means increasing the intensity of the contagion. The corresponding increase in the probability of "extreme" systemic risk can be seen as a manifestation of the induced "cascading" effect.

4 The mean field approximation and pseudo mean field approximation

For a survey of the mean field approximation in the context of statistical mechanics, see for example, Gallavotti (1999), and the references therein. We limit our attention to the use of some ideas taken from the mean field approximation of statistical mechanics in the study of the banking system models considered in the previous sections.

We begin by considering the mean field approximation of the banking system model (3), (11), (24), (25), (26), (8). When the stochastic differentials of Eqs. (24), (25) $\mathrm{d}W_t^i$, $\mathrm{d}Z_t^i$, $t \geq 0$, $i = 1, 2, \ldots, N$, are independent, that is, when in (8) we have: $\rho_a^2 = \rho_{a,t}^2 = 0$, $\rho_l^2 = \rho_{l,t}^2 = 0$, $t \geq 0$, so that in (9), (10) we have: $\mathrm{d}W_t^i = d\widetilde{W}_t^i$, $\mathrm{d}Z_t^i = d\widetilde{Z}_t^i$, $t \geq 0$, $i = 1, 2, \ldots, N$, the mean field approximation of this banking system model can be deduced by proceeding as done in Fouque and Sun (2013), Fatone and Mariani (2019). In fact, when $\rho_a^2 = \rho_{a,t}^2 = 0$, $\rho_l^2 = \rho_{l,t}^2 = 0$, $t \geq 0$, and N goes to infinity, it is easy to see that the mean field limit of (3), (11), (24), (25), (26), (8) is given by:

$$\mathcal{Y}_t = \mathcal{A}_t - \mathcal{L}_t, \quad t \geq 0, \tag{32}$$

where

$$\mathcal{A}_t = e^{\mathcal{G}_t}, \quad \mathcal{L}_t = e^{\mathcal{H}_t}, \quad t \geq 0, \tag{33}$$

and $\mathcal{G}_t$, $\mathcal{H}_t$, $t \geq 0$, satisfy the stochastic differential equations:

$$d\left(\mathcal{G}_t - \ln(\varphi_t)\right) = \alpha_t \left(\ln(\varphi_t) - \mathcal{G}_t\right) \mathrm{d}t + \sigma_a dP_t, \quad t > 0, \tag{34}$$

$$d\left(\mathcal{H}_t - \ln(\phi_t)\right) = \gamma_t \left(\ln(\phi_t) - \mathcal{H}_t\right) \mathrm{d}t + \sigma_l dQ_t, \quad t > 0, \tag{35}$$

with initial conditions:

$$\mathcal{G}_0 = \ln(\varphi_0), \quad \mathcal{H}_0 = \ln(\phi_0). \tag{36}$$

The stochastic processes P_t, Q_t, $t \geq 0$, of (34), (35) are standard Wiener processes such that $P_0 = 0$, $Q_0 = 0$, dP_t, dQ_t, $t > 0$, are their stochastic differentials and we have:

$$\mathbb{E}(dP_t dQ_t) = 0, \quad t > 0. \tag{37}$$

In the mean field approximation (32), (33), (34), (35), (36), (37), the stochastic process $\mathcal{Y}_t$, $t > 0$, represents the net worth of the "mean bank" at time $t \geq 0$. Similarly the stochastic processes $\mathcal{A}_t$, $\mathcal{L}_t$, $t > 0$, represent, respectively, the assets and liabilities of the "mean bank" at time $t \geq 0$. Due to the homogeneity of the bank population, when N goes to infinity, the assets, liabilities and net worth of the banks of model (3), (11), (24), (25), (26), (8) behave, respectively, like the assets, liabilities and net worth of the "mean bank", that is, they behave like the stochastic processes defined in (33), (32).

Let us consider the banking system model (3), (11), (24), (25), (26), (8) when the stochastic differentials of equations (24), (25) are correlated, that is, when ρ_a, ρ_l are non-zero constants. In this case as well, it is not difficult to deduce the mean field approximation of the banking system model (see for example, Carmona et al. 2015), however, for later convenience, we prefer to introduce a heuristic approximation of model (3), (11), (24), (25), (26), (8) in the limit as N goes to infinity which we call the pseudo mean field approximation that will be used in Sects. 5 and 7 to govern the probability of systemic risk in a bounded time interval. In the pseudo mean field approximation of the banking system model in question, Eqs. (34), (35) are substituted, respectively, with the equations:

$$\begin{aligned} d\left(\mathcal{G}_t - \ln(\varphi_t)\right) &= \alpha_t\left(1 - |\rho_a|\right)\left(\ln(\varphi_t) - \mathcal{G}_t\right)\mathrm{d}t \\ &\quad + g_t\,|\rho_a|\left(\ln(\phi_t) - \mathcal{H}_t\right)\mathrm{d}t + \sigma_a dP_t, \quad t > 0, \qquad (38) \\ d\left(\mathcal{H}_t - \ln(\phi_t)\right) &= \gamma_t\left(1 - |\rho_l|\right)\left(\ln(\phi_t) - \mathcal{H}_t\right)\mathrm{d}t \\ &\quad + h_t\,|\rho_l|\left(\ln(\varphi_t) - \mathcal{G}_t\right)\mathrm{d}t + \sigma_l dQ_t, \quad t > 0. \qquad (39) \end{aligned}$$

Equations (38), (39) are equipped with initial conditions (36) and assumption (37). The functions $g_t \geq 0$, $h_t \geq 0$, $t \geq 0$, are non-negative functions that will be chosen later. The pseudo mean field approximation is completed by adding Eqs. (32), (33) to (38), (39), (36), (37). In the pseudo mean field approximation (32), (33), (38), (39), (36), (37), the stochastic processes $\mathcal{Y}_t$, $\mathcal{A}_t$, $\mathcal{L}_t$, $t \geq 0$, have the same meaning as in the mean field approximation, that is, they represent, respectively, the net worth, assets and liabilities of the "pseudo mean bank" as functions of time. Equations (32), (33), (38), (39), (36), (37) define the dynamics of the "pseudo mean bank".

When N goes to infinity and the functions $g_t, h_t, t \geq 0$, are chosen appropriately, the "pseudo mean bank" behavior "approximates" the behavior of the "mean bank" and as a consequence "approximates" the behavior of the banks of model (3), (11), (24), (25), (26), (8). The choice of (32), (33), (38), (39), (36), (37) and, in particular, the choice of (38), (39) as the pseudo mean field approximation is motivated by the following facts. First, when the stochastic differentials $\mathrm{d}W_t^i$, $\mathrm{d}Z_t^i$, $t > 0$, $i = 1, 2, \ldots, N$, of Eqs. (24), (25) are independent, i.e., when in (8) we have $\rho_a^2 = 0$, $\rho_l^2 = 0$, $t > 0$, the pseudo mean field approximation (32), (33), (38), (39), (36), (37) coincides with the mean field approximation (32), (33), (34), (35), (36), (37). Moreover, when the stochastic differentials $dW_t^i, t > 0, i = 1, 2, \ldots, N$, and $dZ_t^i, t > 0, i = 1, 2, \ldots, N$, in Eqs. (24), (25) are perfectly correlated, that is, when we have $|\rho_a| = 1$, $|\rho_l| = 1$, and we choose $g_t = 0$, $h_t = 0$, $t > 0$, the pseudo mean field approximation (32), (33), (38), (39), (36), (37) "coincides" with the banking system model (3), (11), (24), (25), (26), (8), with $|\rho_a| = 1$, $|\rho_l| = 1$, $t > 0$. In other words, the pseudo mean field approximation is "exact". In fact, when $|\rho_a| = 1$, $|\rho_l| = 1$, $t > 0$, and $G_0^i = \ln(\varphi_0)$, $H_0^i = \ln(\phi_0)$, $i = 1, 2, \ldots, N$, all the banks in the model satisfy the same equation and can be considered as a "unique" bank repeated N times. That is, when $|\rho_a| = 1$, $|\rho_l| = 1$, $t > 0$, the initial conditions $G_0^i = \ln(\varphi_0)$, $H_0^i = \ln(\phi_0)$, $i = 1, 2, \ldots, N$, imply that the cooperation mechanism among banks present in (24) and in (25) has no influence on the banking system dynamics. Note that the condition $|\rho_a| = 1$, $|\rho_l| = 1$, $t > 0$, implies that the Wiener processes present in the equations relative to the different banks of the model coincide, that is, $\mathrm{d}W_t^i$, $\mathrm{d}Z_t^i$, $t > 0$, in (24), (25) do not depend on i, $i = 1, 2, \ldots, N$. In this case, all the banks in the banking system model are replicated exactly by the pseudo mean field approximation (32), (33), (38), (39), (36), (37) when $|\rho_a| = 1$, $|\rho_l| = 1$, $t > 0$, and we choose $g_t = 0$, $h_t = 0$, $t > 0$. When the stochastic differentials dW_t^i, dZ_t^i, $t > 0$, $i = 1, 2, \ldots, N$, in Eqs. (24), (25) are partially correlated, that is, when in (8) we have $0 < |\rho_a| < 1$, $0 < |\rho_l| < 1$, $t > 0$, choosing the functions g_t, h_t, $t > 0$, appropriately, the pseudo mean field approximation (32), (33), (38), (39), (36), (37) "interpolates" between the extreme cases $\rho_a = 0$, $\rho_l = 0$, $t > 0$, and $|\rho_a| = 1$, $|\rho_l| = 1$, $t > 0$. Finally, in Sect. 7 in the systemic risk governance the form chosen for Eqs. (38), (39) will enable the use of the polynomial identity principle to determine the functions α_t, γ_t, $t \geq 0$, that regulate the cooperation mechanism among banks.

In Sect. 7, we explain the choice of the functions φ_t, ϕ_t, α_t, γ_t, g_t, h_t, $t \geq 0$, used to govern the systemic risk probability in a bounded time interval of model (3), (11), (24), (25), (26), (8).

Note that when $\alpha_t = 0$, $\gamma_t = 0$, $t \geq 0$, and the functions φ_t, ϕ_t, $t \geq 0$, are positive constants, there is no cooperation among banks and no intervention of the monetary authority in the banking system dynamics. In this case, we choose $g_t = 0$, $h_t = 0$, $t \geq 0$ in the pseudo mean field approximation.

5 An optimal control problem for the pseudo mean field approximation

We consider an optimal control problem for the pseudo mean field approximation (32), (33), (38), (39), (36), (37) of the banking system model (3), (11), (24), (25), (26), (8) when $0 \leq |\rho_a| < 1$, $0 \leq |\rho_l| < 1$, $t > 0$. Let n be a positive integer, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ be the n-dimensional real Euclidean space, and $\mathbb{R}^+$ be the set of the positive real numbers.

Given the positive functions φ_t, ϕ_t, $t \geq 0$, we define:

$$\mathcal{Z}_t = \mathcal{G}_t - \ln(\varphi_t), \quad t \geq 0, \tag{40}$$

and

$$\mathcal{S}_t = \mathcal{H}_t - \ln(\phi_t), \quad t \geq 0, \tag{41}$$

equations (38), (39), (36) can be rewritten as follows:

$$d\mathcal{Z}_t = \beta_a(t, \mathcal{Z}_t, \mathcal{S}_t)\mathrm{d}t + \sigma_a dP_t,\ \ t > 0, \tag{42}$$

$$d\mathcal{S}_t = \beta_l(t, \mathcal{Z}_t, \mathcal{S}_t)\mathrm{d}t + \sigma_l dQ_t,\ \ t > 0, \tag{43}$$

$$\mathcal{Z}_0 = 0, \quad \mathcal{S}_0 = 0, \tag{44}$$

where $\beta_a : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ and $\beta_l : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ are given by:

$$\beta_a = \beta_{a,t} = \beta_a(t, \mathcal{Z}, \mathcal{S}) = -\alpha_t\,(1 - |\rho_a|)\,\mathcal{Z} - |\rho_a|\,g_t\,\mathcal{S}, \quad (\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2,\ \ t > 0, \tag{45}$$

$$\beta_l = \beta_{l,t} = \beta_l(t, \mathcal{Z}, \mathcal{S}) = -\gamma_t\,(1 - |\rho_l|)\,\mathcal{S} - |\rho_l|\,h_t\,\mathcal{Z}, \quad (\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2,\ \ t > 0, \tag{46}$$

and the symbol 0 in (44) denotes the random variable concentrated at zero with probability one.

To choose the functions $\alpha_t, \gamma_t, g_t, h_t, t > 0$, of (45), (46) as done in the systemic risk governance in Sect. 7, we begin by solving the stochastic optimal control problem that follows.

Let $T_1 > 0$ be a real number, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$, $\lambda_i > 0$, $i = 1, 2, 3, 4$, and $\mathcal{B}$ be the set of the real square integrable stochastic processes defined in $[0, T_1]$. That is, a real stochastic process $\zeta = \zeta_t$, $t \in [0, T_1]$, belongs to $\mathcal{B}$ if and only if $\mathbb{E}\left(\int_0^{T_1} \zeta_t^2\,\mathrm{d}t\right) < +\infty$. We consider the following stochastic optimal control problem:

$$\min_{\beta_1, \beta_2 \in \mathcal{B}} U_{\boldsymbol{\lambda}}(\beta_1, \beta_2), \tag{47}$$

where

$$\begin{aligned} &U_{\boldsymbol{\lambda}}(\beta_1, \beta_2) \\ &\quad = \mathbb{E}\left(\int_0^{T_1} \left[|\rho_a\rho_l|(\mathcal{Z}_t - \mathcal{S}_t)^2 + \lambda_1\beta_{1,t}^2 + \lambda_2\beta_{2,t}^2 + \lambda_3(1 - |\rho_a|)\mathcal{Z}_t^2 + \lambda_4(1 - |\rho_l|)\mathcal{S}_t^2\right]\mathrm{d}t\right), \\ &\quad \beta_1,\ \beta_2 \in \mathcal{B},\ 0 \le |\rho_a| < 1,\ 0 \le |\rho_l| < 1, \end{aligned} \tag{48}$$

subject to

$$d\mathcal{Z}_t = \beta_1\,\mathrm{d}t + \sigma_a dP_t, \quad t \in [0, T_1], \tag{49}$$

$$d\mathcal{S}_t = \beta_2\,\mathrm{d}t + \sigma_l dQ_t, \quad t \in [0, T_1], \tag{50}$$

$$\mathcal{Z}_0 = 0, \quad \mathcal{S}_0 = 0. \tag{51}$$

In the control problem (47), (48), (49), (50), (51) the function $U_{\boldsymbol{\lambda}}(\beta_1, \beta_2)$ is the utility function, $\beta_1 = \beta_{1,t} = \beta_1(t, \mathcal{Z}_t, \mathcal{S}_t)$, $\beta_2 = \beta_{2,t} = \beta_2(t, \mathcal{Z}_t, \mathcal{S}_t)$, $t \in [0, T_1]$, are the control variables and $\mathcal{Z}_t, \mathcal{S}_t, t \in [0, T_1]$, are the state variables. The random variables on the right-hand side of Eq. (51) must be interpreted as already done for those of Eq. (44).

When $0 < |\rho_a| < 1$, $0 < |\rho_l| < 1$, minimizing the utility function $U_{\boldsymbol{\lambda}}(\beta_1, \beta_2)$, $\beta_1 = \beta_{1,t}$, $\beta_2 = \beta_{2,t}$, $t \in [0, T_1]$, defined in (48) means making small in the time interval $[0, T_1]$ the following quantities:

(i) the difference between the net worth of the "pseudo mean bank" and the net worth of the "ideal bank" $\mathcal{Z}_t - \mathcal{S}_t = (\mathcal{G}_t - \ln(\varphi_t)) - (\mathcal{H}_t - \ln(\phi_t))$, $t \in [0, T_1]$;
(ii) the "size" of the control variable $\beta_{1,t}$, $t \in [0, T_1]$;
(iii) the "size" of the control variable $\beta_{2,t}$, $t \in [0, T_1]$;
(iv) the "size" of $\mathcal{Z}_t$, $t \in [0, T_1]$ (and therefore the difference between $\mathcal{G}_t$, $t \in [0, T_1]$, and the function $\ln(\varphi_t)$, $t \in [0, T_1]$);
(v) the "size" of $\mathcal{S}_t$, $t \in [0, T_1]$ (and therefore the difference between $\mathcal{H}_t$, $t \in [0, T_1]$, and the function $\ln(\phi_t)$, $t \in [0, T_1]$).

These five goals correspond, respectively, to making small the addenda:

(i) $\mathbb{E}\left(\int_0^{T_1} |\rho_a \rho_l| (\mathcal{Z}_t - \mathcal{S}_t)^2 \mathrm{d}t\right)$; (ii) $\mathbb{E}\left(\int_0^{T_1} \lambda_1 \beta_{1,t}^2\, \mathrm{d}t\right)$; (iii) $\mathbb{E}\left(\int_0^{T_1} \lambda_2 \beta_{2,t}^2\, \mathrm{d}t\right)$;
(iv) $\mathbb{E}\left(\int_0^{T_1} \lambda_3 (1 - |\rho_a|) \mathcal{Z}_t^2\, \mathrm{d}t\right)$; (v) $\mathbb{E}\left(\int_0^{T_1} \lambda_4 (1 - |\rho_l|)\ \mathcal{S}_t^2\, \mathrm{d}t\right)$,

of the utility function $U_{\boldsymbol{\lambda}}$ defined in (48).

Note that when $\rho_a = 0$ and/or $\rho_l = 0$ the term $\mathbb{E}\left(\int_0^{T_1} |\rho_a \rho_l| (\mathcal{Z}_t - \mathcal{S}_t)^2 \mathrm{d}t\right)$ of $U_{\boldsymbol{\lambda}}$ is zero and, in this case, minimizing the utility function $U_{\boldsymbol{\lambda}}$ corresponds to pursuing only four of the five goals listed above, i.e., making the quantities *ii*), *iii*), *iv*), *v*) small in the time interval $[0, T_1]$.

The control problem (47), (48), (49), (50), (51) is a linear-quadratic optimal control problem (see Kolosov 1999). Following Kalman Kolosov (1999), we assume that its value function is a quadratic form in the real variables $\mathcal{Z}$, $\mathcal{S}$ with time-dependent coefficients. We have:

Proposition 1 *Under the previous assumptions when $0 \leq |\rho_a| < 1$ and $0 \leq |\rho_l| < 1$ the optimal control $\beta_1 = \beta_{1,t}$, $\beta_2 = \beta_{2,t}$, $t \in [0, T_1]$, solution of problem* (47), (48), (49), (50), (51) *is given by:*

$$\beta_1 = \beta_{1,t} = \beta_1(t, \mathcal{Z}_t, \mathcal{S}_t) = -\frac{1}{2\lambda_1}\left(2a(t)\mathcal{Z}_t + c(t)\mathcal{S}_t\right), \quad t \in [0, T_1], \tag{52}$$

$$\beta_2 = \beta_{1,t} = \beta_2(t, \mathcal{Z}_t, \mathcal{S}_t) = -\frac{1}{2\lambda_2}\left(2b(t)\mathcal{S}_t + c(t)\mathcal{Z}_t\right), \quad t \in [0, T_1], \tag{53}$$

where $\mathcal{Z}_t$, $\mathcal{S}_t$, $t \in [0, T_1]$, are the solution to the initial value problem (49), (50), (51)*. The functions $a(t)$, $b(t)$, $c(t)$, $d(t)$, $t \in [0, T_1]$, are defined by the following final value problem:*

$$\frac{\partial a}{\partial t} = \frac{a^2}{\lambda_1} + \frac{c^2}{4\lambda_2} - |\rho_a \rho_l| - \lambda_3(1 - |\rho_a|), \qquad t \in [0, T_1], \qquad a(T_1) = 0, \tag{54}$$

$$\frac{\partial b}{\partial t} = \frac{b^2}{\lambda_2} + \frac{c^2}{4\lambda_1} - |\rho_a \rho_l| - \lambda_4(1 - |\rho_l|), \qquad t \in [0, T_1], \qquad b(T_1) = 0, \tag{55}$$

$$\frac{\partial c}{\partial t} = \frac{ac}{\lambda_1} + \frac{bc}{\lambda_2} + 2|\rho_a \rho_l|, \qquad t \in [0, T_1], \qquad c(T_1) = 0, \tag{56}$$

$$\frac{\partial d}{\partial t} = -a\sigma_a^2 - b\sigma_l^2, \qquad t \in [0, T_1], \qquad d(T_1) = 0. \tag{57}$$

Note that the optimal control (52), (53) does not depend on the function $d(t)$, $t \in [0, T_1]$ that appears in (57); it depends only on the functions $a(t)$, $b(t)$, $c(t)$, $t \in [0, T_1]$. The final value problem (54), (55), (56) satisfied by $a(t)$, $b(t)$, $c(t)$, $t \in [0, T_1]$, can be solved independently from the final value problem (57) satisfied by $d(t)$, $t \in [0, T_1]$. However, the function $d(t)$, $t \in [0, T_1]$ is necessary to define the value function V (see (63)) of the control problem (47), (48), (49), (50), (51) and the formulae (52), (53) for the optimal control are deduced from the expression of the value function.

Proof Let us use the dynamic programming principle (see Kolosov 1999) to solve control problem (47), (48), (49), (50), (51). That is let

$$V(t, \mathcal{Z}, \mathcal{S}) = \min_{\beta_1, \beta_2 \in \mathcal{B}} \mathbb{E}\left(\int_t^{T_1} \left[|\rho_a \rho_l| (\mathcal{Z}_\tau - \mathcal{S}_\tau)^2 + \lambda_1 \beta_{1,\tau}^2 + \lambda_2 \beta_{2,\tau}^2 + \lambda_3 (1 - |\rho_a|) \mathcal{Z}_\tau^2 \right.\right.$$
$$\left.\left. + \lambda_4 (1 - |\rho_l|) \mathcal{S}_\tau^2 \right] \mathrm{d}\tau \,\middle|\, \mathcal{Z}_t = \mathcal{Z}, \mathcal{S}_t = \mathcal{S} \right), \quad (\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2, \; t \in [0, T_1], \tag{58}$$

be the value function of the control problem (47), (48), (49), (50), (51). The function $V(t, \mathcal{Z}, \mathcal{S})$, $(\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2$, $t \in [0, T_1]$, satisfies the following Hamilton, Jacobi, Bellman equation (see Kolosov 1999):

$$\frac{\partial}{\partial t} V(t, \mathcal{Z}, \mathcal{S}) + \frac{1}{2}\sigma_a^2 \frac{\partial^2}{\partial \mathcal{Z}^2} V(t, \mathcal{Z}, \mathcal{S}) + \frac{1}{2}\sigma_l^2 \frac{\partial^2}{\partial \mathcal{S}^2} V(t, \mathcal{Z}, \mathcal{S}) + |\rho_a \rho_l| (\mathcal{Z} - \mathcal{S})^2 +$$
$$\lambda_3 (1 - |\rho_a|) \mathcal{Z}^2 + \lambda_4 (1 - |\rho_l|) \mathcal{S}^2 + \mathcal{H}\left(\frac{\partial}{\partial \mathcal{Z}} V(t, \mathcal{Z}, \mathcal{S}), \frac{\partial}{\partial \mathcal{S}} V(t, \mathcal{Z}, \mathcal{S}) \right) = 0,$$
$$(\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2, \; t \in [0, T_1], \tag{59}$$

with final condition:

$$V(T_1, \mathcal{Z}, \mathcal{S}) = 0, \quad (\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2, \tag{60}$$

where

$$\mathcal{H}(p_1, p_2) = \min_{(\delta_1, \delta_2) \in \mathbb{R}^2} \left(\delta_1 p_1 + \lambda_1 \delta_1^2 + \delta_2 p_2 + \lambda_2 \delta_2^2 \right)$$
$$= -\frac{p_1^2}{4\lambda_1} - \frac{p_2^2}{4\lambda_2}, \quad (p_1, p_2) \in \mathbb{R}^2, \tag{61}$$

is the Hamiltonian function of the optimal control problem (47), (48), (49), (50), (51). □

Using (61) Eq. (59) becomes:

$$\frac{\partial}{\partial t}V(t,\mathcal{Z},\mathcal{S})+\frac{1}{2}\sigma_a^2\frac{\partial^2}{\partial \mathcal{Z}^2}V(t,\mathcal{Z},\mathcal{S})+\frac{1}{2}\sigma_l^2\frac{\partial^2}{\partial \mathcal{S}^2}V(t,\mathcal{Z},\mathcal{S})+|\rho_a\rho_l|\,(\mathcal{Z}-\mathcal{S})^2$$
$$+\lambda_3(1-|\rho_a|)\mathcal{Z}^2+\lambda_4(1-|\rho_l|)\mathcal{S}^2-\frac{1}{4\lambda_1}\left(\frac{\partial}{\partial \mathcal{Z}}V(t,\mathcal{Z},\mathcal{S})\right)^2$$
$$-\frac{1}{4\lambda_2}\left(\frac{\partial}{\partial \mathcal{S}}V(t,\mathcal{Z},\mathcal{S})\right)^2=0,\quad (\mathcal{Z},\mathcal{S})\in\mathbb{R}^2,\ t\in[0,T_1], \tag{62}$$

with final condition (60).

Following Kalman Kolosov (1999), we assume that the value function solution of problem (62), (60) is of the form:

$$V(t,\mathcal{Z},\mathcal{S})=a(t)\mathcal{Z}^2+b(t)\mathcal{S}^2+c(t)\mathcal{Z}\mathcal{S}+d(t),\qquad (\mathcal{Z},\mathcal{S})\in\mathbb{R}^2,\ t\in[0,T_1], \tag{63}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $t\in[0,T_1]$, are functions to be determined. Substituting (63) in (62), (60) and using the polynomial identity principle, it is easy to see that the final value problem for the Hamilton, Jacobi, Bellman equation (62), (60) reduces to the final value problem (54), (55), (56), (57).

Problem (54), (55), (56), (57) is a final value problem for a system of Riccati ordinary differential equations. In general systems of this kind have only local solutions. This means that, in general, a solution of (54), (55), (56), (57) in the time interval $[0,T_1]$ may not exist. When this is the case, assumption (63) about the form of the value function is not good enough to solve problem (47), (48), (49), (50), (51) and we do not study the control problem (47), (48), (49), (50), (51) any further. Hereafter, we assume that the final value problem (54), (55), (56), (57) has a solution defined in $[0,T_1]$.

From knowledge of the value function V defined in (63) solution of (62), (60) the optimal control $\beta_1=\beta_{1,t}$, $\beta_2=\beta_{2,t}$, $t\in[0,T_1]$, solution of (47), (48), (49), (50), (51) is determined using the formulae:

$$\beta_1=\beta_{1,t}=\beta_1(t,\mathcal{Z}_t,\mathcal{S}_t)=-\frac{1}{2\lambda_1}\frac{\partial}{\partial \mathcal{Z}}V(t,\mathcal{Z},\mathcal{S})\Big|_{\mathcal{Z}=\mathcal{Z}_t,\mathcal{S}=\mathcal{S}_t}$$
$$=-\frac{1}{2\lambda_1}\left(2a(t)\mathcal{Z}_t+c(t)\mathcal{S}_t\right),\qquad t\in[0,T_1], \tag{64}$$

$$\beta_2=\beta_{2,t}=\beta_2(t,\mathcal{Z}_t,\mathcal{S}_t)=-\frac{1}{2\lambda_2}\frac{\partial}{\partial \mathcal{S}}V(t,\mathcal{Z},\mathcal{S})\Big|_{\mathcal{Z}=\mathcal{Z}_t,\mathcal{S}=\mathcal{S}_t}$$
$$=-\frac{1}{2\lambda_2}\left(2b(t)\mathcal{S}_t+c(t)\mathcal{Z}_t\right),\qquad t\in[0,T_1], \tag{65}$$

where $\mathcal{Z}_t$, $\mathcal{S}_t$, $t\in[0,T_1]$, are the solution of (49), (50), (51) when β_1=$\beta_{1,t}$, β_2=$\beta_{2,t}$, $t\in[0,T_1]$, are given by (64), (65).

Problem (47), (48), (49), (50), (51) is the optimal control problem used to govern the pseudo mean field approximation (32), (33), (38), (39), (36), (37) of banking system model (3), (11), (24), (25), (26), (8). In fact, when $0 \le |\rho_a| < 1$, $0 \le |\rho_l| < 1$, given the optimal control β_1, β_2 defined in (64), (65), we determine the functions β_a, β_l of (45), (46) imposing the identities $\beta_a(\mathcal{Z}, \mathcal{S}) = \beta_1(\mathcal{Z}, \mathcal{S})$, $\beta_l(\mathcal{Z}, \mathcal{S}) = \beta_2(\mathcal{Z}, \mathcal{S})$, $(\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2$, and using the polynomial identity principle in the variables $(\mathcal{Z}, \mathcal{S}) \in \mathbb{R}^2$. We have:

$$\alpha_t = \frac{a(t)}{\lambda_1(1 - |\rho_a|)}, \quad \gamma_t = \frac{b(t)}{\lambda_2(1 - |\rho_l|)}, \quad t \in [0, T_1], \quad 0 \le |\rho_a|, |\rho_l| < 1, \tag{66}$$

and

$$g_t = \frac{c(t)}{2\lambda_1 |\rho_a|}, \quad h_t = \frac{c(t)}{2\lambda_2 |\rho_l|}, \quad t \in [0, T_1], \quad 0 < |\rho_a|, |\rho_l| < 1, \tag{67}$$

or

$$h_t = 0 \quad \text{and/or} \quad g_t = 0, \quad t \in [0, T_1], \quad \rho_a = 0 \quad \text{and/or} \quad \rho_l = 0. \tag{68}$$

We point out that when $\rho_a = 0$ and/or $\rho_l = 0$, the function $c(t) = 0$, $t \in [0, T_1]$ is a solution of (56). Moreover, note that the use of the polynomial identity principle in the deduction of (66), (67), (68) is possible thanks to the form of Eqs. (38), (39) of the pseudo mean field approximation.

Recall that the function α_t, $t \in [0, T_1]$, defined in (66) is a function that, when substituted in (24), induces the trajectories of the logarithm of the assets to swarm around $\ln(\varphi_t)$, $t \in [0, T_1]$; it therefore induces the trajectories of the assets to swarm around φ_t, $t \in [0, T_1]$. Similarly the function γ_t, $t \in [0, T_1]$, defined in (66) is a function that, when substituted in (25), induces the trajectories of the logarithms of the liabilities to swarm around $\ln(\phi_t)$, $t \in [0, T_1]$; it therefore induces the trajectories of the liabilities to swarm around ϕ_t, $t \in [0, T_1]$.

Remember that in (24), (25) the constraints $\alpha_t \ge 0$, $\gamma_t \ge 0$, $t \in [0, T_1]$, must be satisfied. When they are not satisfied by the choices of α_t, γ_t, $t \in [0, T_1]$, made in (66), they are enforced. In the numerical experiments discussed in Sect. 7, when the functions α_t and/or γ_t, $t \in [0, T_1]$, determined using (66), are negative, we choose $\alpha_t = 0$ and/or $\gamma_t = 0$, $t \in [0, T_1]$.

Note that formulae (66), (67), (68) provide a choice of the functions α_t, γ_t, g_t, h_t when $t \in [0, T_1]$; to choose these functions when $t > 0$, the previous formulae must be adapted to account for the repeated solution of control problems similar to the one considered here.

6 Systemic risk governance

Let $T_2 > 0$ be a real number. We consider the problem of governing the probability of systemic risk in the time interval $[0, T_2]$ in model (3), (11), (24), (25), (26), (8) in the absence or presence of shocks acting on the banking system. Given τ_1, τ_2 such that $0 \le \tau_1 < \tau_2 \le T_2$, and the interval $[\tau_1, \tau_2] \subseteq [0, T_2]$, let us consider the governance of systemic risk in the time interval $[\tau_1, \tau_2]$. The goal of governance is to keep the probability of systemic risk in the time interval $[\tau_1, \tau_2]$, $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]})$, between two given thresholds. Systemic risk governance pursues its goal by trying to keep the assets, liabilities and net worth of the banks in the model "close", respectively, to the assets, liabilities and net worth of the "ideal bank", that is close, respectively, to the functions $\varphi_t > 0$, $\phi_t > 0$ and $\xi_t = \varphi_t - \phi_t > 0$, $t \in [\tau_1, \tau_2]$. Given the choice of functions φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$, governance is based on the solution to the optimal control problem (47), (48), (49), (50), (51) and on its relationship with the banking system model (3), (11), (24), (25), (26), (8) when the functions α_t, γ_t, $t \in [\tau_1, \tau_2]$, are chosen by adapting formula (66) deduced for the time interval $[0, T_1]$ to the time interval $[\tau_1, \tau_2]$. In fact the choice of the functions α_t, γ_t, $t \in [\tau_1, \tau_2]$, obtained adapting formula (66) to the time interval $[\tau_1, \tau_2]$, creates a "swarming" effect of the banks' assets and liabilities around, respectively, the functions φ_t, ϕ_t, $t \in [\tau_1, \tau_2]$. As a consequence, it creates a "swarming" effect of the banks' net worth around the function ξ_t, $t \in [\tau_1, \tau_2]$.

We assume that the decisions about systemic risk governance in the time interval $[\tau_1, \tau_2]$ are taken at time $t = \tau_1$. In detail, to pursue the goal of keeping the probability of systemic risk in the time interval $[\tau_1, \tau_2]$, $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]})$, between two given thresholds, the first step at time $t = \tau_1$ is to choose the functions φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$ appropriately. In fact, it is easy to see that by increasing $\xi_t > 0$, $t \in [\tau_1, \tau_2]$, the systemic risk probability in $[\tau_1, \tau_2]$ decreases and that by decreasing $\xi_t > 0$, $t \in [\tau_1, \tau_2]$, the systemic risk probability in $[\tau_1, \tau_2]$ increases. Moreover, since $\xi_t = \varphi_t - \phi_t > 0$, $t \in [\tau_1, \tau_2]$, ξ_t, $t \in [\tau_1, \tau_2]$, can be increased by increasing φ_t and leaving ϕ_t, $t \in [\tau_1, \tau_2]$ unchanged, by decreasing ϕ_t leaving φ_t, $t \in [\tau_1, \tau_2]$ unchanged, or by changing φ_t and ϕ_t, $t \in [\tau_1, \tau_2]$ simultaneously. Similarly, ξ_t, $t \in [\tau_1, \tau_2]$, can be decreased either by decreasing φ_t leaving ϕ_t, $t \in [\tau_1, \tau_2]$ unchanged, by increasing ϕ_t leaving φ_t, $t \in [\tau_1, \tau_2]$ unchanged, or by changing φ_t and ϕ_t, $t \in [\tau_1, \tau_2]$ simultaneously.

Given the thresholds S_1, S_2, such that $0 \leq S_1 < S_2 < 1$, and φ_{τ_1}, ϕ_{τ_1}, $\xi_{\tau_1} = \varphi_{\tau_1} - \phi_{\tau_1}$ we want to choose the functions φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$, such that the probability of systemic risk in the time interval $[\tau_1, \tau_2]$ satisfies the following inequalities:

$$S_1 \leq \mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]}) \leq S_2. \tag{69}$$

Note that the lower bound $S_1 > 0$ to the probability of systemic risk is introduced to avoid that the monetary authority governing the systemic risk (i.e., for example reacting to an adverse shock hitting the banking system) forces the systemic risk probability to an unnecessary small value, thereby penalizing the operation of the banking system with the prescription of an unnecessary high cooperation level. This last purpose is pursued by choosing $S_1 > 0$ appropriately. If this is not a goal of the monetary authority in the systemic risk governance, the choice $S_1 = 0$ is allowed.

We define some simple rules that are used to choose the functions φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$ in order to satisfy (69). At time $t = \tau_1$, we start by making the "simplest" possible choice of φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$, that is, we choose $\varphi_t = \varphi_{\tau_1}$, $\phi_t = \phi_{\tau_1}$, $\xi_t = \xi_{\tau_1}$, $t \in [\tau_1, \tau_2]$. Corresponding to this choice, the functions α_t, γ_t,

$t \in [\tau_1, \tau_2]$, are determined by adapting formula (66) to the time interval $[\tau_1, \tau_2]$ and the probability of systemic risk in the time interval $[\tau_1, \tau_2]$, $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]})$, is evaluated via statistical simulation. Note that $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]})$ depends not only on the functions $\varphi_t, \phi_t, \alpha_t, \gamma_t$, $t \in [\tau_1, \tau_2]$, but also on the random variables $a^i_{\tau_1}, l^i_{\tau_1}, c^i_{\tau_1} = a^i_{\tau_1} - l^i_{\tau_1}$, $i = 1, 2, \ldots, N$. Based on the value of $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]})$, the following actions are taken:

Strategy 1 if $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]}) > S_2$ the monetary authority changes the functions φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$, to "swarm" the trajectories of the net worth of the banking system model (3), (11), (24), (25), (26), (8) "upward", that is, the monetary authority increases $\xi_t > 0$, $t \in [\tau_1, \tau_2]$. This is done in one of the following ways:

Strategy 1a increasing $\varphi_t > 0$ leaving $\phi_t > 0$, $t \in [\tau_1, \tau_2]$ unchanged;
Strategy 1b decreasing $\phi_t > 0$ leaving $\varphi_t > 0$, $t \in [\tau_1, \tau_2]$ unchanged;
Strategy 1c changing both $\varphi_t > 0$ and $\phi_t > 0$, $t \in [\tau_1, \tau_2]$.

Strategy 2 if $\mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]}) < S_1$ the monetary authority changes the functions φ_t, ϕ_t, $\xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$, to "swarm" the trajectories of the net worth of the banking system model (3), (11), (24), (25), (26), (8) "downward", that is the monetary authority decreases $\xi_t > 0$, $t \in [\tau_1, \tau_2]$. This is done in one of the following ways:

Strategy 2a decreasing $\varphi_t > 0$ leaving $\phi_t > 0$, $t \in [\tau_1, \tau_2]$ unchanged;
Strategy 2b increasing $\phi_t > 0$ leaving $\varphi_t > 0$, $t \in [\tau_1, \tau_2]$ unchanged;
Strategy 2c changing both $\varphi_t > 0$ and $\phi_t > 0$, $t \in [\tau_1, \tau_2]$.

Strategy 3 if $S_1 \le \mathcal{P}(\mathrm{SR}_{[\tau_1,\tau_2]}) \le S_2$, the monetary authority leaves the functions $\varphi_t, \phi_t, \xi_t = \varphi_t - \phi_t$, $t \in [\tau_1, \tau_2]$, unchanged.

Note that at time $t = \tau_1$, the monetary authority makes its decisions about systemic risk governance in the time interval $[\tau_1, \tau_2]$ assuming that the volatilities σ_a, σ_l and the correlation coefficients ρ^2_a, ρ^2_l remain constant in the time interval at the value that they have at time $t = \tau_1$. That is, the monetary authority does not expect volatility and/or correlation shocks to hit the banking system in the time interval $[\tau_1, \tau_2]$. It simply reacts to them after they have occurred.

The choice of acting on the assets φ_t, $t \in [\tau_1, \tau_2]$, or liabilities ϕ_t, $t \in [\tau_1, \tau_2]$, of the "ideal bank" depends on the kind of shock that must be confronted. For example, in the presence of a volatility shock on the assets side that occurred before $t = \tau_1$ (the systemic risk governance decision time), that is, reacting to a jump in the function σ_a that occurred before $t = \tau_1$, it is natural at time $t = \tau_1$ to increase/decrease ξ_t, $t \in [\tau_1, \tau_2]$, simply by increasing/decreasing $\varphi_t > 0$, $t \in [\tau_1, \tau_2]$, leaving $\phi_t > 0$, $t \in [\tau_1, \tau_2]$ unchanged. In other words, in this situation, it is natural to limit the monetary authority's action to *Strategies 1a*, *2a*, and *3*. Similarly, in the presence of a volatility shock on the liabilities side that occurred before $t = \tau_1$, that is, reacting to the presence of a jump in the function σ_l that occurred before $t = \tau_1$, it is natural at time $t = \tau_1$ to increase/decrease ξ_t, $t \in [\tau_1, \tau_2]$, simply by decreasing/increasing $\phi_t > 0$, $t \in [\tau_1, \tau_2]$, leaving $\varphi_t > 0$, $t \in [\tau_1, \tau_2]$ unchanged. That is, in this situation, it is natural to limit the monetary authority's action to *Strategies 1b*, *2b*, and *3*.

When possible, the strategy of increasing the "ideal bank's" assets is more desirable for the well-being of the economy than decreasing its liabilities. In fact, increasing the assets induces similar behavior in the assets of the banks in the banking system and this keeps the wheels of the economy turning, while decreasing the liabilities induces similar behavior in the banks' liabilities, thereby slowing down the economy. At the extremes, when possible, the monetary authority should prefer *Strategies 1a*, *2b*, and *3* to *Strategies 1b*, *1c*, *2a*, *2c*.

The choice between Strategies *1a*, *1b*, *1c*, or *2a*, *2b*, *2c* is based on the comparison of these strategies from the systemic risk point of view. A possible criterion for this comparison is to evaluate the corresponding loss distributions of the banks that have defaulted in the time interval $[\tau_1, \tau_2]$. The strategy associated with the loss distribution with the "smallest tail" must be considered as the best strategy. For simplicity, we do not pursue this goal here.

We now discuss some numerical experiments of systemic risk governance. We present the results obtained considering the governance of systemic risk in the next year during a period of 2 years in model (3), (11), (24), (25), (26), (8) in the absence or presence of shocks acting on the banking system. Governance decisions are taken at the beginning of each quarter during the 2-year period studied. For simplicity, we consider only volatility shocks on either the assets side or the liabilities side. The occurrence of these shocks is simulated with jumps in the volatilities σ_a, σ_l, of the stochastic differential equations of the assets (24) and liabilities (25), respectively. Note that together with jumps in the volatility coefficients, we sometimes consider jumps in the correlation coefficients ρ_a, ρ_l, of the stochastic differentials on the right-hand side of Eqs. (24), (25). Moreover, when there are no shocks acting on the banking system or when the monetary authority faces a volatility shock on the assets side, we consider as possible only the actions described in *Strategies 1a*, *2a*, *3* and in *Strategies 1a*, *2b*, *3*. Similarly, when the monetary authority faces a volatility shock on the liabilities side, we consider as possible only the actions described in *Strategies 1a*, *2b*, *3* and in *Strategies 1b*, *2b*, *3*.

In the experiments, we study a banking system model with $N = 10$ banks with a time horizon T_2 of 3 years, i.e., the time unit is equal to 1 year and $T_2 = 3$. We assume that governance decisions are made quarterly, i.e., the time step of governance decisions is $\Delta\tau = 1/4$. In the time interval $[0, T_2]$, we consider the time intervals $[\tau_1^j, \tau_2^j] \subset [0, T_2]$, $T_2 = 3$, where $\tau_1^j = j \cdot \Delta\tau$ and $\tau_2^j = \tau_1^j + 1$, $j = 0, 1, \dots, 8$, and governance decisions are made at the times $t = \tau_1^j$, $j = 0, 1, \dots, 8$. That is, at time $t = \tau_1^j$ the decision is made relative to systemic risk in the time interval $[\tau_1^j, \tau_2^j]$.

In the time intervals $[\tau_1^j, \tau_2^j]$, $j = 0, 1, \dots, 8$, the model (3), (11), (24), (25), (26), (8) reduces to the following (sub)-models:

$$c_t^i = a_t^i - l_t^i, \quad t \in [\tau_1^j, \tau_2^j],\ i = 1, 2, \dots, N,\ \ j = 0, 1, \dots, 8, \tag{70}$$

where the stochastic processes:

$$G_t^i = \ln(a_t^i), \quad H_t^i = \ln(l_t^i), \quad t \in [\tau_1^j, \tau_2^j],\ i = 1, 2, \dots, N,\ \ j = 0, 1, \dots, 8, \tag{71}$$

satisfy the following system of stochastic differential equations:

$$
\begin{aligned}
\mathrm{d}G_t^i &= \frac{\alpha_t}{N}\sum_{k=1}^{N}\left(G_t^k - G_t^i\right)\mathrm{d}t + d\ln(\varphi_t) + \sigma_a \mathrm{d}W_t^i, \\
&\qquad t \in (\tau_1^j, \tau_2^j],\ i = 1, 2, \ldots, N,\ \ j = 0, 1, \ldots, 8, \qquad (72)\\
\mathrm{d}H_t^i &= \frac{\gamma_t}{N}\sum_{k=1}^{N}\left(H_t^k - H_t^i\right)\mathrm{d}t + d\ln(\phi_t) + \sigma_l \mathrm{d}Z_t^i, \\
&\qquad t \in (\tau_1^j, \tau_2^j],\ i = 1, 2, \ldots, N,\ \ j = 0, 1, \ldots, 8, \qquad (73)
\end{aligned}
$$

with initial conditions:

$$
G^i_{\tau_1^0} = \ln(\tilde{a}_0), \quad H^i_{\tau_1^0} = \ln(\tilde{l}_0), \quad i = 1, 2, \ldots, N, \tag{74}
$$

$$
G^i_{\tau_1^j} = G^i_{\tau_2^{j-1}}, \quad H^i_{\tau_1^j} = H^i_{\tau_2^{j-1}}, \quad i = 1, 2, \ldots, N,\ \ j = 1, 2, \ldots, 8, \tag{75}
$$

and the assumption:

$$
\begin{aligned}
&\mathbb{E}(\mathrm{d}W_t^i \mathrm{d}W_t^k) = \rho_a^2\,\mathrm{d}t,\ \ i \neq k, \qquad \mathbb{E}(\mathrm{d}Z_t^i \mathrm{d}Z_t^k) = \rho_l^2\,\mathrm{d}t,\ \ i \neq k,\\
&\mathbb{E}(\mathrm{d}W_t^i \mathrm{d}W_t^i) = \mathbb{E}(\mathrm{d}Z_t^k \mathrm{d}Z_t^k) = \mathrm{d}t, \qquad \mathbb{E}(\mathrm{d}W_t^i \mathrm{d}Z_t^k) = 0,\\
&\qquad t \in [\tau_1^j, \tau_2^j], \quad i, k = 1, 2, \ldots, N,\ \ j = 0, 1, \ldots, 8.
\end{aligned} \tag{76}
$$

For $j = 0, 1, \ldots, 8$, the functions $\alpha_t, \gamma_t, t \in [\tau_1^j, \tau_2^j]$, in (72), (73) are obtained by adapting formula (66), that is relative to the time interval $[0, T_1]$, to the time interval $[\tau_1^j, \tau_2^j]$. In each time interval $[\tau_1^j, \tau_2^j]$, $j = 0, 1, \ldots, 8$, the probability of systemic risk of the corresponding sub-model (70), (71), (72), (73), (74), (75), (76) is evaluated via a statistical simulation starting from 10^4 numerically generated trajectories of the corresponding sub-model. These trajectories are obtained by finite differences using the explicit Euler method with time step $\Delta t = 10^{-4}$ to numerically solve the stochastic differential equations (72), (73) with auxiliary conditions (74), (75), (76).

In order to keep the probability of systemic risk in each time interval $[\tau_1^j, \tau_2^j]$, $j = 0, 1, \ldots, 8$, between the thresholds S_1 and S_2, we provide the monetary authority with a pre-defined set of functions that can be used to push the trajectories of the assets and liabilities of the j-th sub-model (70), (71), (72), (73), (74), (75), (76) "upward" or "downward", or leave them "unchanged", $j = 0, 1, \ldots, 8$. That is for the assets, we define the functions:

$$
A_{j,n_a}\!: \qquad \varphi_t = \varphi_{t,j,n_a} = \begin{cases} \frac{n_a}{8}\,(t - \tau_1^j) + \varphi_{\tau_1^j}, & t \in [\tau_1^j, \tau_1^j + \Delta\tau],\\ \frac{n_a}{8}\,\Delta\tau + \varphi_{\tau_1^j}, & t \in (\tau_1^j + \Delta\tau, \tau_2^j], \end{cases}
$$
$$
j = 0, 1, \ldots, 8, \ \ n_a = -8, -7, \ldots, 0, \ldots, 7, 8. \tag{77}
$$

Similarly, for the liabilities, we define the functions:

$$L_{j,n_l}: \qquad \phi_t = \phi_{t,j,n_l} = \begin{cases} \frac{n_l}{8}\,(t-\tau_1^j) + \phi_{\tau_1^j}, & t \in [\tau_1^j, \tau_1^j + \Delta\tau], \\ \frac{n_l}{8}\,\Delta\tau + \phi_{\tau_1^j}, & t \in (\tau_1^j + \Delta\tau, \tau_2^j], \end{cases}$$
$$j = 0, 1, \ldots, 8, \quad n_l = -8, -7, \ldots, 0, \ldots, 7, 8. \tag{78}$$

Finally, based on (77), (78) for the net worth, we define the functions:

$$P_{j,n_a,n_l}: \ \xi_t = \xi_{t,j,n_a,n_l} = \varphi_{t,j,n_a} - \phi_{t,j,n_l}, \quad t \in [\tau_1^j, \tau_2^j],$$
$$j = 0, 1, \ldots, 8, \quad n_a\, n_l = -8, -7, \ldots, 0, \ldots, 7, 8. \tag{79}$$

Note that for $j = 0, 1, \ldots, 8$ in (77), when $0 < n_a \le 8$ (or $-8 \le n_a < 0$), the function φ_{t,j,n_a}, is a non-decreasing (or non-increasing) piecewise linear function of t, while when $n_a = 0$, the function φ_{t,j,n_a} is a constant. Consequently, for $j = 0, 1, \ldots, 8$, the choice of functions φ_{t,j,n_a} with $0 < n_a \le 8$ $(-8 \le n_a < 0)$ in (77) pushes the trajectories of the assets of the j-th sub-model (70), (71), (72), (73), (74), (75), (76) "upward" ("downward"), while the choice $n_a = 0$ leaves the trajectories of the assets of the j-th sub-model (70), (71), (72), (73), (74), (75), (76) "unchanged". Similar statements adapted to the circumstances hold for the choices $0 < n_l \le 8$, $-8 \le n_l < 0$, $n_l = 0$ of the functions ϕ_{t,j,n_l} in (78) and for the trajectories of the liabilities in the j-th sub-model (70), (71), (72), (73), (74), (75), (76). Note that the implementation of *Strategy 1*, *2*, or *3* with the choices made in (77), (78) is only illustrative. Many other choices of the functions representing the assets and liabilities of the "ideal bank" are possible and lead to results analogous to the ones discussed here.

To measure the quality and the cost of the systemic risk governance implemented in the experiments, we define four performance indices. Let

$$\eta_j = \mathcal{P}(\mathrm{SR}_{[\tau_1^j, \tau_2^j]}), \quad j = 0, 1, \ldots, 8, \tag{80}$$

and let $\boldsymbol{\eta} = (\eta_0, \eta_1, \ldots, \eta_8) \in \mathbb{R}^9$ be the vector of the systemic risk governance procedure implemented in the experiments. The systemic risk norm $\mathcal{N}_{\mathrm{SR}}$ is defined as follows:

$$\mathcal{N}_{\mathrm{SR}} = \|\boldsymbol{\eta}\|_2\,, \tag{81}$$

where $\|\boldsymbol{\eta}\|_2$ denotes the Euclidean norm of the vector $\boldsymbol{\eta}$. The index $\mathcal{N}_{\mathrm{SR}}$ is used to measure the quality of the systemic risk governance. Note that small values of the index $\mathcal{N}_{\mathrm{SR}}$ correspond to high-quality systemic risk governance and that in the numerical experiments discussed here, when the governance goal (69) is achieved in every 1-year time interval contained in $[0, T_2]$, we have $3S_1 \le \mathcal{N}_{\mathrm{SR}} \le 3S_2$.

The indices $\mathcal{C}_{\mathrm{SR}}^c$, $\mathcal{C}_{\mathrm{SR}}^\alpha$, $\mathcal{C}_{\mathrm{SR}}^\gamma$ are used to measure the cost of the systemic risk governance. The first index $\mathcal{C}_{\mathrm{SR}}^c$ measures the "cost associated with the choice of the assets and liabilities" of the "ideal bank" defined in (77) and (78), while the indices $\mathcal{C}_{\mathrm{SR}}^\alpha$,

$\mathcal{C}_{\text{SR}}^{\gamma}$ measure "the cost associated with the choice of the functions α_t, γ_t, $t \in [0, T_2]$, that regulate the cooperation mechanism among banks". More specifically, in each 1-year period considered in the governance procedure, we define the cost associated with the choice of the assets and liabilities of the "ideal bank" as the absolute value of the angular coefficient of the linear part of the piecewise linear functions in (77), (78). In this way, in the period $[\tau_1^j, \tau_2^j]$, the cost of choosing A_{j,n_a}, defined in (77), is $\frac{|n_a|}{8}$, $n_a = -8, -7, \ldots, 0, \ldots, 7, 8$, and similarly, the cost of choosing L_{j,n_l}, defined in (78), is $\frac{|n_l|}{8}$, $n_l = -8, -7, \ldots, 0, \ldots, 7, 8$. Finally, the cost of choosing P_{j,n_a,n_l} given in (79), is defined as $\frac{|n_a|}{8} + \frac{|n_l|}{8}$, $n_a = -8, -7, \ldots, 0, \ldots, 7, 8$, $n_l = -8, -7, \ldots, 0, \ldots, 7, 8$. The total cost measured by the index $\mathcal{C}_{\text{SR}}^{c}$ of the systemic risk governance procedure defined above is given by the sum over j of the cost of the trajectories P_{j,n_a,n_l}, $j = 0, 1, \ldots, 8$, used in the procedure. The indices $\mathcal{C}_{\text{SR}}^{\alpha}$, $\mathcal{C}_{\text{SR}}^{\gamma}$ are given, respectively, by the sum of the means of α_t, γ_t in the time intervals $[\tau_1^j, \tau_2^j]$, $j = 0, 1, \ldots, 8$, used in the systemic risk governance procedure. In other words, recalling Eqs. (72), (73) and defining:

$$\bar{\alpha}_j = \frac{1}{\Delta\tau} \int_{\tau_1^j}^{\tau_2^j} \alpha_t \, \mathrm{d}t, \quad t \in [\tau_1^j, \tau_2^j], \quad j = 0, 1, \ldots, 8, \tag{82}$$

$$\bar{\gamma}_j = \frac{1}{\Delta\tau} \int_{\tau_1^j}^{\tau_2^j} \gamma_t \, \mathrm{d}t, \quad t \in [\tau_1^j, \tau_2^j], \quad j = 0, 1, \ldots, 8, \tag{83}$$

we have:

$$\mathcal{C}_{\text{SR}}^{\alpha} = \sum_{j=0}^{8} \bar{\alpha}_j, \qquad \mathcal{C}_{\text{SR}}^{\gamma} = \sum_{j=0}^{8} \bar{\gamma}_j. \tag{84}$$

In the numerical experiments, the indices $\mathcal{N}_{\text{SR}}$, $\mathcal{C}_{\text{SR}}^{c}$, $\mathcal{C}_{\text{SR}}^{\alpha}$, $\mathcal{C}_{\text{SR}}^{\gamma}$ change significantly depending on the circumstances (i.e., the presence or absence of volatility and correlation shocks) faced during the 2-year period of the systemic risk governance procedure. Moreover, covering the entire history of governance, that is, covering a 2-year governance period composed of nine quarterly decisions, the indices defined above measure only a "overall" quality and cost of the systemic risk governance procedure.

Table 1 shows the numerical results obtained in the systemic risk governance of model (70), (71), (72), (73), (74), (75), (76). In the experiments presented, the monetary authority pursues the goal of keeping the probability of systemic risk in the next year between the thresholds $S_1 = 0.01$ and $S_2 = 0.05$ by implementing the actions associated with *Strategies 1*, *2*, *3* through the choice of the functions A_{j,n_a}, L_{j,n_l}, P_{j,n_a,n_l}, $j = 0, 1, \ldots, 8$, $n_a = -8, -7, \ldots, 0, \ldots, 7, 8$, $n_l = -8, -7, \ldots, 0, \ldots, 7, 8$, defined, respectively, in (77), (78), (79). In detail, for $j = 0, 1, \ldots, 8$, the monetary authority runs through the possible choices of the functions listed in (77), (78), (79) in their natural order according to *Strategies 1*, *2*, *3* starting from the choice $A_{j,0}$, $L_{j,0}$, $P_{j,0,0} = A_{j,0} - L_{j,0}$. Via statistical simulation, it then evaluates the probability of systemic risk in the next year associated with each choice of the previous functions

Table 1 Numerical experiments with $\mu_a = \mu_l = 0.1$, $\lambda_i = 0.1$, $i = 1, 2, 3, 4$, $\varphi_0 = 0.6$, $\phi_0 = 0.2$, $S_1 = 0.01$, $S_2 = 0.05$

Experiment	σ_a	σ_l	ρ_a	ρ_l	Strategies	$\mathcal{N}_{\mathrm{SR}}$ ($\mathcal{N}_{\mathrm{SR}}$ no gov)	$\mathcal{C}^c_{\mathrm{SR}}$	$\mathcal{C}^\alpha_{\mathrm{SR}}$	$\mathcal{C}^\gamma_{\mathrm{SR}}$
1	0.3	0.3	0	0	*1a, 2a, 3*	0.08 (0.06)	0.28	4.17	4.17
					1a, 2b, 3	0.08 (0.06)	0.28	4.17	4.17
2	0.3	0.3	$\rho_{1,t}$	0	*1a, 2a, 3*	0.07 (0.13)	0.33	5.59	4.59
					1a, 2b, 3	0.06 (0.13)	0.33	5.59	4.59
3	$\sigma_{4,t}$	0.3	0	0	*1a, 2a, 3*	0.17 (0.71)	1.26	12.91	12.91
					1a, 2b, 3	0.19 (0.71)	1.21	11.77	11.77
4	$\sigma_{4,t}$	0.3	$\rho_{1,t}$	0	*1a, 2a, 3*	0.26 (0.77)	1.53	21.99	15.72
					1a, 2b, 3	0.25 (0.77)	1.34	19.33	13.07
5	0.3	$\sigma_{4,t}$	0	0	*1a, 2b, 3*	0.20 (0.72)	1.21	11.77	11.77
					1b, 2b, 3	0.18 (0.72)	1.11	10.41	10.41
6	0.3	$\sigma_{4,t}$	0	$\rho_{1,t}$	*1a, 2b, 3*	0.27 (0.77)	1.36	13.45	19.71
					1b, 2b, 3	0.50 (0.77)	2.01	24.45	30.69

considered. The first choice that gives a probability of systemic risk in the next year that satisfies (69) is chosen as the systemic risk governance decision. The choice of functions α_t, γ_t corresponding to the previous choices of functions φ_t, ϕ_t, is made by adapting (66) to the circumstances. If none of the functions listed in (77), (78), (79) gives a probability of systemic risk in the next year that satisfies (69), the governance procedure is not able to reach its goal in the time interval considered, in which case the governance procedure makes the best choice available in (77), (78), (79) and tries to reach its goal in the following time interval.

Note that when the correlation coefficients ρ_a^2 and/or ρ_l^2 increase, the "swarming" effect induced by the cooperation mechanism among banks in (24) and/or (25) decreases. Recall that in the extreme case of $\rho_a^2 = 1$, $\rho_l^2 = 1$, the cooperation mechanism has no effect anymore. Therefore, when $\rho_a^2 = 1$, $\rho_l^2 = 1$, governing the systemic risk probability is only possible by increasing the net worth of the "ideal bank".

Let $\sigma_{4,t}$, $t \in [0, T_2]$, $T_2 = 3$, be defined as follows:

$$\sigma_{4,t} = \begin{cases} 0.3, & t \in [0, 0.2], \\ 0.8, & t \in (0.2, 0.5], \\ 0.3, & t \in (0.5, 3]. \end{cases} \tag{85}$$

In the experiments in Table 1, positive shocks acting on the assets or on the liabilities of the banks are modeled by considering the choices $\sigma_a = \sigma_{4,t}$, $t \in [0, T_2]$, $T_2 = 3$, or $\sigma_l = \sigma_{4,t}$, $t \in [0, T_2]$, $T_2 = 3$. Moreover, let $\rho_{1,t}$, $t \in [0, T_2]$, $T_2 = 3$, be defined as follows:

$$\rho = \rho_{1,t} = \begin{cases} 0, & t \in [0, 0.2], \\ 0.5, & t \in (0.2, 0.5], \\ 0, & t \in (0.5, 3]. \end{cases} \tag{86}$$

The "collective" behavior of the banks in the presence of shocks is modeled assuming a positive correlation in the noise terms of the assets or of the liabilities equations of (70), (71), (72), (73), (74), (75), (76). In particular, in some experiments we consider the choices $\rho_a = \rho_{1,t}, t \in [0, T_2], T_2 = 3$, or $\rho_l = \rho_{1,t}, t \in [0, T_2], T_2 = 3$.

Note that the functions $\sigma_{4,t}, t \in [0, T_2], T_2 = 3$, defined in (85), and $\rho_{1,t}, t \in [0, T_2]$, $T_2 = 3$, defined in (86), "jump together" at time $t = 0.2$ and $t = 0.5$.

The remaining parameters of the model used in the experiments shown in Table 1 are: $\mu_a = 0.1$, $\mu_l = 0.1$, $\tilde{a}_0 = \varphi_0 = 0.6$, $\tilde{l}_0 = \phi_0 = 0.4$, $\lambda_i = 0.1$, $i = 1, 2, 3, 4$. Note that the previous choices guarantee $\xi_t = \varphi_t - \phi_t > 0, t \in [0, T_2], T_2 = 3$.

Table 1 shows the values of the indices $\mathcal{N}_{\mathrm{SR}}, \mathcal{C}^c_{\mathrm{SR}}, \mathcal{C}^\alpha_{\mathrm{SR}}, \mathcal{C}^\gamma_{\mathrm{SR}}$ obtained in the experiments. In the seventh column of Table 1, next to the value of $\mathcal{N}_{\mathrm{SR}}$, the value of $\mathcal{N}_{\mathrm{SR}}$ obtained in absence of governance is shown within brackets. In the absence of governance, we choose $\mu_a = 0.1$, $\mu_l = 0.1$, $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T_2]$, $T_2 = 3$, $\tilde{a}_0 = \varphi_0 = 0.6$, $\tilde{l}_0 = \phi_0 = 0.4$, and we evaluate the probability of systemic risk in the next year at the beginning of each quarter. These choices of parameter values guarantee that the probability of systemic risk in the next year at time $t = 0$ lies between the thresholds $S_1 = 0.01$ and $S_2 = 0.05$. Note that with the previous choices in the absence of governance, the values of the indices $\mathcal{C}^c_{\mathrm{SR}}, \mathcal{C}^\alpha_{\mathrm{SR}}, \mathcal{C}^\gamma_{\mathrm{SR}}$ are always equal to 0, 90, 90, respectively. When necessary, the values of the last three columns of Table 1 obtained in the presence of governance may be compared with values of the indices $\mathcal{C}^c_{\mathrm{SR}} = 0$, $\mathcal{C}^\alpha_{\mathrm{SR}} = 90$, $\mathcal{C}^\gamma_{\mathrm{SR}} = 90$ that correspond to the absence of governance.

An overview of Table 1 shows that from the systemic risk governance perspective, the performance of Strategies *1a, 2a, 3* versus Strategies *1a, 2b, 3* and of Strategies *1a, 2b, 3* versus Strategies *1b, 2b, 3* is approximately the same.

Experiments 1 and 2 of Table 1 show that when the volatilities σ_a, σ_l are constant and there are no correlation shocks in the time interval $[0, T_2], T_2 = 3$, the presence or absence of governance does not make a significant difference provided that, in the absence of governance at time $t = 0$, a good choice is made for the assets and liabilities of the "ideal bank" in the 1-year period beginning at time $t = 0$ and for the constant values of $\alpha_t, \gamma_t, t \in [0, T_2], T_2 = 3$. That is, when the volatilities σ_a, σ_l are constants and there are no correlation shocks in the time interval $[0, T_2], T_2 = 3$, the systemic risk governance is substantially reduced to the choice for the assets and liabilities of the "ideal bank" at time $t = 0$ and for the constant values of the functions α_t, γ_t, $t \in [0, T_2]$, $T_2 = 3$. When this choice is made correctly, continuing with constant assets and liabilities functions of the "ideal bank" or with small variations in each successive time interval $[\tau_1^j, \tau_2^j]$, $j = 1, 2, \ldots, 8$, is enough to keep the probability of systemic risk in the next year between the given thresholds. The functions α_t, γ_t, $t \in [0, T_2]$, $T_2 = 3$, are chosen according to the rules established in Sect. 5 in the study of the control problem for the pseudo mean field approximation of the banking system. In this case, the possibly positive value of $\mathcal{C}^c_{\mathrm{SR}}$ in the presence of governance (compared with $\mathcal{C}^c_{\mathrm{SR}} = 0$ in the absence of governance) is certainly compensated by the smaller values of the indices $\mathcal{C}^\alpha_{\mathrm{SR}}, \mathcal{C}^\gamma_{\mathrm{SR}}$ with respect to the values corresponding to the absence of governance (i.e., $\mathcal{C}^\alpha_{\mathrm{SR}}$=90, $\mathcal{C}^\gamma_{\mathrm{SR}} = 90$). Moreover, note that in these cases we have $3S_1 \leq \mathcal{N}_{\mathrm{SR}} \leq 3S_2$.

Note that in Experiment 1, the choice of the functions $\alpha_t = 10, \gamma_t = 10, t \in [0, T_2]$, $T_2 = 3$, made in the absence of governance together with the choices of the other parameters of the problem guarantees that the probability of systemic risk in the next year at time $t = 0$ will be between the thresholds $S_1 = 0.01$ and $S_2 = 0.05$ and it gives values of $\mathcal{N}_{\mathrm{SR}}$ in the presence and absence of governance of the same order of magnitude. For an idea of the consequences of changing the values $\alpha_t = 10$, $\gamma_t = 10$, $t \in [0, T_2]$, $T_2 = 3$, we mention that if we fix $\alpha_t = 1$, $\gamma_t = 1$, $t \in [0, T_2]$, $T_2 = 3$ in Experiment 1, leaving all the remaining parameters unchanged, in the absence of governance, we have $\mathcal{C}^c_{\mathrm{SR}} = 0$, $\mathcal{C}^{\alpha}_{\mathrm{SR}} = 9$, $\mathcal{C}^{\gamma}_{\mathrm{SR}} = 9$, but we have $\mathcal{N}_{\mathrm{SR}} = 0.30$.

As expected, the comparison between Experiments 1 and 3 of Table 1 shows that the governance of systemic risk in the presence of volatility shocks is more demanding than governance in the absence of shocks. This can be seen by comparing the cost indices $\mathcal{C}^c_{\mathrm{SR}}$, $\mathcal{C}^{\alpha}_{\mathrm{SR}}$, $\mathcal{C}^{\gamma}_{\mathrm{SR}}$ and the quality indices $\mathcal{N}_{\mathrm{SR}}$ of Experiments 1 and 3. Furthermore, the situation worsens when both a volatility and a correlation shock act on the liabilities equations of the banking system. This last fact can be seen by comparing the performance indices of Experiments 1, 3 and 4 of Table 1. In particular, the comparison of the performance indices of Experiments 3 and 4 shows that the values of the cost indices $\mathcal{C}^c_{\mathrm{SR}}$, $\mathcal{C}^{\alpha}_{\mathrm{SR}}$, $\mathcal{C}^{\gamma}_{\mathrm{SR}}$ increase significantly going from Experiment 3 to Experiment 4 despite the fact that the index $\mathcal{N}_{\mathrm{SR}}$ signals that the quality of the governance is decreasing. In fact, in Experiments 3 and 4, we have $\mathcal{N}_{\mathrm{SR}} > 3S_2$. This means that during the 2-year period studied, it was not always possible to satisfy (69). Note that the index $\mathcal{N}_{\mathrm{SR}}$ in Experiment 4 is greater than in Experiment 3, which shows that the "collective" behavior of the banks induced by the non-zero correlation ρ_a makes governance more difficult.

Similar observations can be made when a volatility shock acts on the banks' liabilities (compare, for example, Experiments 1 and 5 of Table 1) and when positive correlation is present in the noise terms of the liabilities equations ρ_l (compare, for example, Experiments 5 and 6).

Moreover, note that in Experiments 3, 4, 5, 6, the value of the index $\mathcal{N}_{\mathrm{SR}}$ is always greater than $3S_2 = 0.15$. This is due to the fact that when governance faces the shock for the first time, it is unable to reach its goal of keeping the probability of systemic risk in the next year within the assigned thresholds.

We conclude by noting that in the experiments presented in Table 1, the systemic risk governance procedure proposed is able to reach its goal, i.e., it is able to keep the probability of systemic risk in the next year between the assigned thresholds at a reasonable cost.

7 Conclusions

A new quantitative approach to measure, monitor and govern systemic risk in an assets–liabilities continuous-time dynamical model of banking system was investigated. The strategy is situated within the theory of mean field equations. As emphasized in the Introduction, as a mathematical abstraction of a banking network, the proposed model is a deliberate and extreme oversimplification of a real banking system model. However, based on earlier work, its essential assumptions have produced interesting and potentially important insights.

The model proposed describes a homogeneous population of banks where each bank is represented by its assets and liabilities, which are time-dependent interacting stochastic processes. The net worth of a bank is defined as the difference between assets and liabilities of the bank and a bank is solvent when its net worth is greater or equal to zero; otherwise, the bank has failed. The main features of the model are a cooperation mechanism among banks that regulates interbank borrowing and lending activities and the possibility of the (direct) intervention of the monetary authority in the banking system dynamics. We use systemic risk or systemic event in a bounded time interval to refer to the fact that in that time interval at least a given fraction of the banks fails. The monetary authority is responsible for the systemic risk governance, which aims to keep the probability of systemic risk in a bounded time interval between two given thresholds. The rules of governance are obtained by solving an optimal control problem for the pseudo mean field approximation of the banking system model. The numerical examples of systemic risk governance in the presence and absence of shocks acting on the banking system show the effectiveness of the proposed approach in governing systemic risk.

The simple banking system model presented in this paper may be extended to more general models by relaxing one or more of the hypotheses used here, and the governance of systemic risk in these more refined and realistic banking system models can be investigated. These extensions are on-going research topics.

References

Acharya V, Yorulmazer T (2008) Information contagion and bank herding. J Money Credit Bank 40:215–231

Berardi S, Tedeschi G (2017) From banks strategies to financial (in)stability. Int Rev Econ Finance 47:255–272

Biagini F, Fouque JP, Frittelli M, Meyer-Brandis T (2019a) A unified approach to systemic risk measures via acceptance sets. Math Finance 29(1):329–367

Biagini F, Mazzon A, Meyer-Brandis T (2019b) Financial asset bubbles in banking networks. SIAM J Financ Math 10(2):430–465

Caccioli F, Catanach TA, Farmer JD (2012) Heterogeneity, correlations and financial contagion. Adv Complex Syst 15(Supp02):1250058

Caccioli F, Shrestha M, Moore C, Farmer JD (2014) Stability analysis of financial contagion due to overlapping portfolios. J Bank Finance 46:233–245

Carmona R, Fouque JP, Sun LH (2015) Mean field games and systemic risk. Commun Math Sci 13(4):911–933

Diamond DW, Rajan RG (2000) A theory of bank capital. J Finance 55(6):2431–2465

Eisert T, Eufingerbi C (2014) Interbank network and bank bailouts: insurance mechanism for non-insured creditors? SSRN Electron J. https://doi.org/10.2139/ssrn.1976383

Fatone L, Mariani F (2019) Systemic risk governance in a dynamical model of a banking system. J Glob Optim 75(3):851–883

Fouque JP, Ichiba T (2013) Stability in a model of interbank lending. SIAM J Financ Math 4(1):784–803

Fouque JP, Langsam J (eds) (2013) Handbook of systemic risk. Cambridge University Press, Cambridge

Fouque JP, Sun LH (2013) Systemic risk illustrated. In: Fouque JP, Langsam J (eds) Handbook of systemic risk. Cambridge University Press, Cambridge, pp 444–452

Gallavotti G (1999) Statistical mechanics: a short treatise. Springer, New York

Garnier J, Papanicolaou G, Yang TW (2013) Large deviations for a mean field model of systemic risk. SIAM J Financ Math 4:151–184

Grilli R, Tedeschi G, Gallegati M (2014) Bank interlinkages and macroeconomic stability. Int Rev Econ Finance 34:72–88

Grilli R, Iori G, Stamboglis N, Tedeschi G (2017) A networked economy: a survey on the effect of interaction in credit markets. In: Introduction to agent-based economics, chapter 10. Elsevier, pp 229–252
Haldane AG, May RM (2011) Systemic risk in banking ecosystems. Nature 469:351–355
Hurd TR (2016) Contagion! Systemic risk in financial networks. Springer briefs in quantitative finance. Springer, Berlin
Iori G, Jafarey S, Padilla FG (2006) Systemic risk on the interbank market. J Econ Behav Org 61(4):525–542
Kolosov GE (1999) Optimal design of control systems: stochastic and deterministic problems. CRC Press, New York
Lenzu S, Tedeschi G (2012) Systemic risk on different interbank network topologies. Phys A Stat Mech Appl 391(18):4331–4341
May RM, Nimalan A (2010) Systemic risk: the dynamics of model banking systems. J R Soc Interface 7:823–838
May RM, Levin SA, Sugihara G (2008) Complex systems: ecology for bankers. Nature 451:893–895
Mukuddem-Petersen J, Petersen MA (2006) Bank management via stochastic optimal control. Automatica 42:1395–1406
Mukuddem-Petersen J, Petersen MA (2008) Optimizing asset and capital adequacy management in banking. J Optim Theory Appl 137(1):205–230
Rogers LCG, Veraart LAM (2013) Faliure and rescue in an interbank network. Manag Sci 59(4):882–898
Sun LH (2018) Systemic risk and interbank lending. J Optim Theory Appl 179(2):400–424
Tedeschi G, Mazloumian A, Gallegati M, Helbing D (2012) Bankruptcy cascades in interbank markets. PLoS ONE 7(12):e52749

Affiliations

Lorella Fatone[1] · Francesca Mariani[2]

✉ Francesca Mariani
f.mariani@univpm.it

Lorella Fatone
lorella.fatone@unicam.it

1 Dipartimento di Matematica, Università di Camerino, Via Madonna delle Carceri 9, 62032 Camerino, MC, Italy

2 Dipartimento di Scienze Economiche e Sociali, Università Politecnica delle Marche, Piazza Martelli 8, 60121 Ancona, AN, Italy