# Geometry of the $p$-adic special orthogonal group $S O(3)_{p}{ }^{*}$ 

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#### Abstract

We derive explicitly the structural properties of the $p$-adic special orthogonal groups in dimension three, for all primes $p$, and, along the way, the two-dimensional case. In particular, starting from the unique definite quadratic form in three dimensions (up to linear equivalence and rescaling), we show that every element of $S O(3)_{p}$ is a rotation around an axis. An important part of the analysis is the classification of all definite forms in two dimensions, yielding a description of the rotation subgroups around any fixed axis, which all turn out to be abelian and parametrised naturally by the projective line. Furthermore, we find that for odd primes $p$, the entire group $S O(3)_{p}$ admits a representation in terms of Cardano (aka nautical) angles of rotations around the reference axes, in close analogy to the real orthogonal case. However, this works only for certain orderings of the product of rotations around the coordinate axes, depending on the prime; furthermore, there is no general Euler angle decomposition. For $p=2$, no Euler or Cardano decomposition exists.


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## 1. INTRODUCTION

Large parts of classical geometry can be interpreted, following Klein's programme [4], as statements about the orthogonal and special orthogonal groups in dimensions 2, 3, and generally $n$. It is thus no surprise that the special orthogonal groups $S O(n)_{\mathbb{R}}$ are among the most studied and most well-known groups in mathematics. The cases $n=2$ and $n=3$ stand out particularly, both for their structural properties and the possibility to visualise the action of the group on Euclidean space. In particular, $S O(2)_{\mathbb{R}}$ is the group of planar rotations, i.e., isomorphic to the commutative group of adding angles mod $2 \pi$; and $S O(3)_{\mathbb{R}}$ consists entirely of rotations around different axis, admitting essentially unique decompositions into Euler and Cardano angles.

Given the importance of $p$-adic numbers in number theory, it is natural that orthogonal groups should have been studied also over the fields $\mathbb{Q}_{p}$, which a priori are a multitude of symmetry groups, one for each nontrivial quadratic form. Just as in the real case, a fundamental property of the quadratic form is whether it is definite (i.e., only has the trivial zero) or indefinite (i.e., represents zero in nontrivial ways), which distinguishes compact symmetry groups in the latter and noncompact groups in the former case. Unlike the real case, definite quadratic forms over $\mathbb{Q}_{p}$ exist only

[^0]in dimensions two, three and four. As we shall see here, in dimensions two and three the symmetry groups, denoted by $S O(2)_{p}$ and $S O(3)_{p}$, are largely governed by structures familiar from Euclidean geometry, reinterpreted $p$-adically.

The structure of the paper is as follows. In Section 2, we review and re-derive the unique quadratic forms on dimensions three and four that do no represent 0 nontrivially ("definite" forms), up to linear and rescaling equivalence, for all primes $p$, introducing some useful notation for the rest of the paper. This material, albeit well-known, is included for the sake of a self-contained exposition. This classification allows us to define the $p$-adic special orthogonal groups $S O(3)_{p}$ and $S O(4)_{p}$ in a unique way, though we will not consider the latter afterwards in the present work.

Then, in Section 3, we begin our new contributions, by first making several basic observations about $S O(3)_{p}$, most importantly that it is compact and profinite, and that every of its elements is a rotation around some axis in $p$-adic three-space $\mathbb{Q}_{p}^{3}$. This then motivates the investigation of the rotations around a fixed given axis, which are special orthogonal transformations of the plane orthogonal to that axis, in Sections 4 and 5. They are naturally the orthogonal symmetry groups of the definite form restricted to a plane, and there are three ( $p$ odd), resp. seven ( $p=2$ ) equivalence classes of those. We denote them $S O(2)_{p}^{\kappa}$, and they all turn out to be abelian groups. Furthermore, we derive a parametrisation of each of these rotation groups by the $p$-adic projective line $P^{1}\left(\mathbb{Q}_{p}\right)$, allowing us to identify the groups $S O(2)_{p}^{\kappa}$ with certain abelian subgroups of $\operatorname{PGL}\left(2, \mathbb{Q}_{p}\right)$, illuminating in particular the composition law. Finally, in Section 6, we show that every element of $S O(3)_{p}$ has a decomposition into Cardano principal angles, for odd primes $p$ : there exist orderings of the coordinate axes such that every special orthogonal group element is a product of rotations around the axes in that order (in exactly two different ways). For $p=2$, however, no fixed ordering of the product can recover the entire group $S O(3)_{2}$, and we exhibit an example of a special orthogonal matrix that cannot be written in any of the twelve possible Euler or Cardano ways.

## 2. QUADRATIC FORMS OVER p-ADIC NUMBERS

In this section we review, and in some cases, re-derive the elementary properties we shall need about the quadratic forms over $\mathbb{Q}_{p}$, according to dimension $(n=3$ and $n=4)$ as well as type of prime $(p \equiv 1 \bmod 4, p \equiv 3 \bmod 4$ and $p=2)$. Comprehensive treatments of this material can be found in the books of Cassels [5] and Serre [6].

A quadratic form for our purposes is a homogeneous function on the $n$-dimensional $\mathbb{Q}_{p}$-vector space $V$ that can be written as

$$
Q(\boldsymbol{x})=\sum_{i j} a_{i j} x_{i} x_{j}=x^{\top} A x,
$$

where $\boldsymbol{x}=\sum_{i} x_{i} \boldsymbol{e}_{i} \in V$ is a vector, $\left(\boldsymbol{e}_{1}, \ldots, \boldsymbol{e}_{n}\right)$ is a basis of $V$, and $A$ is an $n \times n$-matrix. Throughout we will assume that $A$ is nondegenerate, i.e., has rank $n$. Equivalently, we could speak about symmetric bilinear forms $b(\boldsymbol{x}, \boldsymbol{y})$ on $V$ such that $Q(\boldsymbol{x})=b(\boldsymbol{x}, \boldsymbol{x})$, since we recover the bilinear form via $b(\boldsymbol{x}, \boldsymbol{y})=\frac{1}{2}(Q(\boldsymbol{x}+\boldsymbol{y})-Q(\boldsymbol{x})-Q(\boldsymbol{y}))$. In either case, the orthogonal group is defined as the set of linear maps on $V$ that are symmetries of the quadratic/bilinear form

$$
\begin{aligned}
O(Q) & =\{L \in \operatorname{End}(V): Q(L \boldsymbol{x})=Q(\boldsymbol{x}) \forall \boldsymbol{x} \in V\} \\
& =\{L \in \operatorname{End}(V): b(L \boldsymbol{x}, L \boldsymbol{y})=b(\boldsymbol{x}, \boldsymbol{y}) \forall \boldsymbol{x}, \boldsymbol{y} \in V\} \\
& \simeq\left\{L \in M_{n \times n}\left(\mathbb{Q}_{p}\right): L^{\top} A L=A\right\},
\end{aligned}
$$

the latter under the identification of $V$ with $\mathbb{Q}_{p}^{n}$ via the basis $\left(\boldsymbol{e}_{i}\right), \boldsymbol{x}=\sum_{i} x_{i} \boldsymbol{e}_{i} \leftrightarrow\left(x_{1}, \ldots, x_{n}\right)$, which turns the linear maps on $V$ into $n \times n$-matrices. The subset of $O(Q)$ consisting of matrices $L$ with unit determinant, $\operatorname{det} L=1$, is the special orthogonal group, denoted $S O(Q)$.

We are interested in the abstract group structure of $O(Q)$ and $S O(Q)$, which do not change when going to an equivalent form. First, $Q^{\prime}$ is similar to $Q, Q \sim Q^{\prime}$, if there exists an invertible linear map $S$ such that $Q^{\prime}(\boldsymbol{x})=Q(S \boldsymbol{x})$ for all $\boldsymbol{x} \in V$, meaning for the matrix representation $A^{\prime}$ of
$Q^{\prime}$ that $A^{\prime}=S^{\top} A S$. In that case, $O\left(Q^{\prime}\right) \simeq O(Q)$ and $S O\left(Q^{\prime}\right) \simeq S O(Q)$, the isomorphism being $O(Q) \ni L \mapsto S^{-1} L S \in O\left(Q^{\prime}\right)$. Furthermore, $Q^{\prime}$ is a scaling of $Q$ if $Q^{\prime}=t Q$ with $t \in \mathbb{Q}_{p}^{*}$; in this case, clearly $O\left(Q^{\prime}\right)=O(Q)$ and $S O\left(Q^{\prime}\right)=S O(Q)$. Hence, our first task is to classify the quadratic forms up to similarity and scaling, which we sum up into equivalence. Indeed, up to similarity, we can write every quadratic form with a diagonal matrix $A$, i.e.,

$$
Q(\boldsymbol{x})=\sum_{j} a_{j} x_{j}^{2}
$$

with $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathbb{Q}_{p}^{n}$, and $a_{j} \in \mathbb{Q}_{p}^{*}$. Multiplying $x_{j}$ by $\lambda_{j}^{-1} \neq 0$ will change $a_{j}$ to $\lambda_{j}^{2} a_{j}$, hence to classify the quadratic forms up to coordinate changes $G L_{n}\left(\mathbb{Q}_{p}\right)$, we only need to consider $a_{j} \in \mathbb{Q}_{p}^{*} /\left(\mathbb{Q}_{p}^{*}\right)^{2}$. We thus have to understand the structure of the group $K=\mathbb{Q}_{p}^{*} /\left(\mathbb{Q}_{p}^{*}\right)^{2}$. The following descriptions of $K$ are well-known [5]:

If $\mathbf{p} \neq \mathbf{2}$, then $K=\langle u, p\rangle=\{1, u, p, u p\}$, with $u$ a unit in $\mathbb{Z}_{p}$ that is not a square. Clearly $K \simeq(\mathbb{Z} / 2 \mathbb{Z})^{2}$ is the Klein group. For $p \equiv 3 \bmod 4$, we may choose $u=-1$.

If $\mathbf{p}=\mathbf{2}$, then $K=\langle-1,2,5\rangle=\{1,-1,2,-2,5,-5,10,-10\}$. In this case $K \simeq(\mathbb{Z} / 2 \mathbb{Z})^{3}$.
Given the structure of $K$, it remains to find the invariants of quadratic forms. It can be proven [5] that apart from the rank of the form, there are only two more invariants: the discriminant $d(Q)=\Pi_{j} a_{j}=\operatorname{det} A$, as well as $\varepsilon(Q)=\Pi_{j<k}\left(a_{j}, a_{k}\right)$, where $(a, b)$ is the Hilbert symbol defined as

$$
(a, b):=\left\{\begin{aligned}
1 & \text { iff } z^{2}-a x^{2}-b y^{2}=0 \text { admits nontrivial solutions } \\
-1 & \text { otherwise }
\end{aligned}\right.
$$

Now we have all the information we need in order to classify quadratic forms on $\mathbb{Q}_{p}^{n}$ for all $n$.
Theorem 1. Two quadratic forms over $\mathbb{Q}_{p}$ are similar if and only if they have same rank n, same determinant $d$ and same Hasse invariant $\varepsilon$.

Theorem 2. The quadratic form $Q$ on $\mathbb{Q}_{p}^{n}$ represents 0 nontrivially if and only if

- $n=2$ and $d \simeq-1$ in $K$;
- $n=3$ and $\varepsilon=(-1,-d)$;
- $n=4$ and either $d \nsim 1$ or $d \simeq 1$ and $\varepsilon=(-1,-1)$;
- $n \geq 5$.

We record explicitly the quadratic forms for $n=3$ up to equivalence, separate by odd and even primes $p$.

Prime p odd: There are exactly two inequivalent forms on $\mathbb{Q}_{p}^{3}$,

$$
\begin{aligned}
Q_{+}(\boldsymbol{x}) & =x_{1}^{2}-v x_{2}^{2}+p x_{3}^{2} \\
Q_{0}(\boldsymbol{x}) & =x_{1}^{2}+x_{2}^{2}+x_{3}^{2}
\end{aligned}
$$

where

$$
v= \begin{cases}-u & \text { if } p \equiv 1 \quad \bmod 4  \tag{1}\\ -1 & \text { if } p \equiv 3 \quad \bmod 4\end{cases}
$$

is a particular choice of a non-square in $\mathbb{Q}_{p}$.
Prime $\mathbf{p}=\mathbf{2 :}$ There are exactly two inequivalent forms on $\mathbb{Q}_{2}^{3}$,

$$
\begin{aligned}
Q_{+}(\boldsymbol{x}) & =x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \\
Q_{0}(\boldsymbol{x}) & =x_{1}^{2}+x_{2}^{2}-x_{3}^{2}
\end{aligned}
$$

Real Euclidean case: Also here, there are exactly two inequivalent forms on $\mathbb{R}^{3}$,

$$
\begin{aligned}
Q_{+}^{\mathbb{R}}(\boldsymbol{x}) & =x_{1}^{2}+x_{2}^{2}+x_{3}^{2}, \\
Q_{0}^{\mathbb{R}}(\boldsymbol{x}) & =x_{1}^{2}+x_{2}^{2}-x_{3}^{2} .
\end{aligned}
$$

In all cases, the real and the $p$-adic ones, one form $\left(Q_{+}\right)$is definite, in the sense that it does not represent zero nontrivially (i.e., $Q_{+}(\boldsymbol{x})=0$ iff $\boldsymbol{x}=0$ ), while the other $\left(Q_{0}\right)$ is indefinite, in the sense that it has isotropic vectors (i.e., $Q_{0}(\boldsymbol{x})=0$ for some $\boldsymbol{x} \neq 0$ ). The symmetry group $S O\left(Q_{0}\right)$ is always non-compact, as is well-known in the real case and easy to see in general in the $p$-adic case, since it has a hyperbolic component [7]. On the other hand, in the real Euclidean case, $S O\left(Q_{+}\right)$is just the real $S O(3)_{\mathbb{R}}$, which is a compact Lie group, and as we would like to preserve the compactness in the $p$-adic case, we define $S O(3)_{p}=S O\left(Q_{+}\right)$for all primes $p$. According to the above classification, this is a unique and well-defined group for every prime $p$, which will indeed turn out to be compact.

For $n=4$, we similarly focus only on the definite forms. It turns out that again, for $\mathbb{R}$ and all $\mathbb{Q}_{p}$, there is exactly one up to equivalence

$$
Q_{+}^{(4)}(\boldsymbol{x})= \begin{cases}x_{1}^{2}-v x_{2}^{2}+p x_{3}^{2}-v p x_{4}^{2} & \text { if } p \text { odd } \\ x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2} & \text { if } p=2, \\ x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2} & \text { in the real case. }\end{cases}
$$

All other inequivalent forms represent zero nontrivially (and hence their associated orthogonal groups are non-compact [7]), a fact that we will exploit later. This means, that also for $n=4$, both $\mathbb{Q}_{p}^{4}$ and $\mathbb{R}^{4}$ have essentially unique groups $S O(4)_{p}$ and $S O(4)_{\mathbb{R}}$. On the other hand, for $n \geq 5$, it is known that for all primes $p$, there are no definite quadratic forms on $\mathbb{Q}_{p}^{n}[5,6]$, unlike the real Euclidean case $\mathbb{R}^{n}$, where of course the sum of the squares of the standard coordinates is the unique definite form up to equivalence, and where we thus have an unambiguous compact Lie group $S O(n)_{\mathbb{R}}$. Because of the lack of definite quadratic forms for $n \geq 5$, we refrain from speaking of $S O(n)_{p}$ without qualification.

Even in the real Euclidean case, $n=3$ is distinguished by several geometric peculiarities. We highlight two of them, see [8, Ch. 4].
Theorem 3. Elements of $S O(3)_{\mathbb{R}}$ are rotations about axes in $\mathbb{R}^{3}$, meaning that they always have eigenvalue 1, for which the corresponding eigenspace is the rotation axis.
Theorem 4. With $R_{x}(\theta), R_{y}(\eta), R_{z}(\phi)$ denoting the rotations around the reference axes of $\mathbb{R}^{3}$ by a given angle $\theta, \eta$ and $\phi$, every $R \in S O(3)$ can be written as a product of three such rotations in any of the following forms

$$
\begin{array}{llll}
R_{x} R_{y} R_{z}, & R_{y} R_{z} R_{x}, & R_{z} R_{x} R_{y}, & R_{x} R_{z} R_{y}, \\
R_{z} R_{y} R_{x}, & R_{y} R_{x} R_{z}, \\
R_{x} R_{y} R_{x}, & R_{x} R_{z} R_{x}, & R_{y} R_{x} R_{y}, & R_{y} R_{z} R_{y}, \\
R_{z} R_{x} R_{z}, & R_{z} R_{y} R_{z} .
\end{array}
$$

These angles are called Cardano or Tait-Bryan angles, and also nautical angles, when applied to a form from the first row, and (proper) Euler angles when applied to a form from the second row.

In both cases, the choice of the angles is unique (up to isolated points) modulo $2 \pi$ radians for $\theta$ and $\phi$, and with the range of $\eta$ covering $\pi$ radians.

This theorem implies that three successive rotations relative to coordinate axes generate every rotation in $\mathbb{R}^{3}$. Hence, $S O(3)_{\mathbb{R}}$ is generated by the rotations around the three reference axes of $\mathbb{R}^{3} ;$ in the case of the Euler decomposition, actually only two reference axes. Each of these rotation groups is an infinite cyclic Lie group, so $S O(3)_{\mathbb{R}}$ is generated by two cyclic subgroups linked by a non-commutative relation.

We have left out $n=2$ until now, which may seem strange coming from the real Euclidean case, where it is the same as for all other $n$ : there is a unique definite quadratic form on $\mathbb{R}^{2}$. Also on $\mathbb{Q}_{p}^{2}$ there are definite quadratic forms, but now they are not unique up to equivalence. We will come back to them in detail below.

## 3. BASIC OBSERVATIONS ABOUT $S O(3)_{p}$

We start by deriving a few basic facts about the $p$-adic special orthogonal groups in dimension 3 from the definition. It is evidently a group under the usual matrix multiplication. In fact, it is a topological group, with the operations of multiplication and inverse being continuous with respect to the $p$-adic metric. Explicitly, the topology on $S O(3)_{p}$ is the one generated by the open balls with respect to the $p$-adic norm $\|L\|_{p}=\left\|\left(\ell_{i j}\right)_{i j}\right\|_{p}:=\max _{i, j=1,2,3}\left|\ell_{i j}\right|_{p}$, where $|\cdot|_{p}$ denotes the $p$-adic absolute value on $\mathbb{Q}_{p}$.

Theorem 5. For every prime $p$, the group $S O(3)_{p}$ is compact. As a matter of fact, it is a closed subset (with respect to the p-adic metric) of $M_{3 \times 3}\left(\mathbb{Z}_{p}\right)$, the set of matrices with p-adic integer entries, and so $S O(3)_{p} \subset S L\left(3, \mathbb{Z}_{p}\right)$.

Proof. Let $L=\left(\ell_{i j}\right)_{i j} \in S O(3)_{p}$, and write $\ell_{i j}=p^{\nu_{i j}} u_{i j}$, where $\nu_{i j}=\nu_{p}\left(\ell_{i j}\right) \in \mathbb{Z} \cup\{+\infty\}$ is the $p$ adic valuation of $\ell_{i j}$, and $u_{i j} \in \mathbb{U}_{p}$ is a unit in $\mathbb{Z}_{p}$. We need to show that $\nu_{i j} \geq 0$ for all $i, j \in\{1,2,3\}$.

When $p$ is odd, the defining condition $A=L^{\top} A L$ of $S O(3)_{p}$ implies the following three relations

$$
\begin{align*}
\ell_{11}^{2}-v \ell_{21}^{2}+p \ell_{31}^{2} & =1 \\
\ell_{12}^{2}-v \ell_{22}^{2}+p \ell_{32}^{2} & =-v \\
\ell_{13}^{2}-v \ell_{23}^{2}+p \ell_{33}^{2} & =p \tag{2}
\end{align*}
$$

The first one is equivalent to

$$
p^{2 \nu_{11}} u_{11}^{2}-v p^{2 \nu_{21}} u_{21}^{2}+p^{1+2 \nu_{31}} u_{31}^{2}=1
$$

Let us assume by contradiction that $\min \left\{\nu_{11}, \nu_{21}, \nu_{31}\right\}<0$. If $\min \left\{2 \nu_{11}, 2 \nu_{21}, 1+2 \nu_{31}\right\}=2 \nu_{11}<0$ (similarly for $\nu_{21}$ ), we multiply both sides of the equation by $p^{2\left|\nu_{11}\right|}$ to obtain

$$
u_{11}^{2}-v p^{2\left(\nu_{21}-\nu_{11}\right)} u_{21}^{2}+p^{1+2\left(\nu_{31}-\nu_{11}\right)} u_{31}^{2}=p^{2\left|\nu_{11}\right|}
$$

where every term is a $p$-adic integer. Hence, for every $k=1, \ldots, 2\left|\nu_{11}\right|$,

$$
u_{11}^{2}-v p^{2\left(\nu_{21}-\nu_{11}\right)} u_{21}^{2}+p^{1+2\left(\nu_{31}-\nu_{11}\right)} u_{31}^{2} \equiv 0 \quad \bmod p^{k}
$$

and in particular $u_{11}^{2}-v p^{2\left(\nu_{21}-\nu_{11}\right)} u_{21}^{2} \equiv 0 \bmod p$. The quadratic form $x^{2}-v y^{2}$ does not represent 0 in $\mathbb{Q}_{p}$ : if $x, y \in \mathbb{Z}_{p}$ as in our case, the only solution to $x^{2}-v y^{2} \equiv 0 \bmod p$ is $(0,0) \in(\mathbb{Z} / p \mathbb{Z})^{2}$. This gives a contradiction, since $u_{11} \in \mathbb{U}_{p}$, in particular $u_{11}^{2} \not \equiv 0 \bmod p$.

If instead $\min \left\{2 \nu_{11}, 2 \nu_{21}, 1+2 \nu_{31}\right\}=1+2 \nu_{31}<0$, we rewrite the equation in terms of $p$-adic integers as

$$
p^{2\left(\nu_{11}-\nu_{31}\right)-1} u_{11}^{2}-v p^{2\left(\nu_{21}-\nu_{31}\right)-1} u_{21}^{2}+u_{31}^{2}=p^{2\left|\nu_{31}\right|-1}
$$

Reducing it modulo $p$ we get $u_{31}^{2} \equiv 0 \bmod p$, which is in contradiction with the hypothesis that $u_{31} \in \mathbb{U}_{p}$.

The same can be done for the second and third equations of Eq. (2), since $-v, p \in \mathbb{Z}_{p}$.
When $p=2$, the defining condition $L^{\top} L=I$ implies

$$
2^{2 \nu_{1 i}} u_{1 i}^{2}+2^{2 \nu_{2 i}} u_{2 i}^{2}+2^{2 \nu_{3 i}} u_{3 i}^{2}=1, \text { for } i=1,2,3
$$

Again assuming by contradiction $\min \left\{\nu_{1 i}, \nu_{2 i}, \nu_{3 i}\right\}=\nu_{1 i}<0$ (the latter without loss of generality, by symmetry), we rewrite the last equation in terms of 2 -adic integers as

$$
u_{1 i}^{2}+2^{2\left(\nu_{2 i}-\nu_{1 i}\right)} u_{2 i}^{2}+2^{2\left(\nu_{3 i}-\nu_{1 i}\right)} u_{3 i}^{2}=2^{2\left|\nu_{1 i}\right|}
$$

As a consequence it must hold

$$
u_{1 i}^{2}+2^{2\left(\nu_{2 i}-\nu_{1 i}\right)} u_{2 i}^{2}+2^{2\left(\nu_{3 i}-\nu_{1 i}\right)} u_{3 i}^{2} \equiv 0 \quad \bmod 2^{k}, k=1, \ldots, 2\left|\nu_{1 i}\right|
$$

The quadratic form $x^{2}+y^{2}+z^{2}$ does not represent 0 in $\mathbb{Q}_{2}$ : as a matter of fact, as $x, y, z$ are in $\mathbb{Z}_{2}$, it does not so in $\mathbb{Z} / 4 \mathbb{Z}$, unless all three variables are $\equiv 0 \bmod 4$; but this is again in contradiction to $u_{1 i} \in \mathbb{U}_{p}$.

We have found $S O(3)_{p} \subset M_{3 \times 3}\left(\mathbb{Z}_{p}\right)$, and the defining condition $\operatorname{det} L=1$ implies $S O(3)_{p} \subset$ $S L\left(3, \mathbb{Z}_{p}\right)$.

Theorem 6. All elements of $S O(3)_{p}$ are rotations, i.e., they always have an eigenvalue 1, and the corresponding eigenspace is the rotation axis.

Proof. Let $L \in S O(3)_{p}$, if $\lambda$ is an eigenvalue of $L$, then also $\lambda^{-1}$ is. In fact if $x$ is an eigenvector for $\lambda$, we have $A x=L^{\top} A L x=\lambda L^{\top} A x \Rightarrow \lambda^{-1} A x=L^{\top} A x$. In other words, $\lambda^{-1}$ is an eigenvalue of $L^{\top}$ with eigenvector $A x$. On the other hand, $L$ and $L^{\top}$ share the characteristic polynomial and hence the eigenvalues.

Suppose now that none of the eigenvalues $\lambda_{i}$ of $L$, for $i=1,2,3$, are equal to 1 . Then, $\lambda_{i}=\lambda_{i}^{-1}$ for all $i$ (equivalently $\lambda_{i}^{2}=1$, thus $\lambda_{i}= \pm 1$ ), otherwise we can suppose, for example, $\lambda_{1} \neq \lambda_{1}^{-1}=\lambda_{2}$ without loss of generality, and hence $1=\operatorname{det} L=\lambda_{1} \lambda_{2} \lambda_{3}=\lambda_{3}$, which is a contradiction.

Summing up, the only case which allows the condition proved before, excluding the presence of 1 among the eigenvalues, is $\lambda_{1}=\lambda_{2}=\lambda_{3}=-1$, which is in contradiction with the condition $\operatorname{det} L=1$. Thus, $L \in S O(3)_{p}$ has always 1 as eigenvalue.

Theorem 5 means that we have well-defined projection maps

$$
\begin{aligned}
& \pi_{k}: S O(3)_{p} \longrightarrow S O(3)_{p} \quad \bmod p^{k} \subset S L\left(3, \mathbb{Z} / p^{k} \mathbb{Z}\right), \\
& L=\left(\ell_{i j}\right)_{i j} \longmapsto\left(\ell_{i j} \bmod p^{k}\right)_{i j},
\end{aligned}
$$

since all matrix entries of eligible $L$ are $p$-adic integers. The images $S O(3)_{p} \bmod p^{k}=\pi_{k}\left(S O(3)_{p}\right)$ are all finite groups, forming a projective system under the (commuting) projection maps of taking a number in $\mathbb{Z} / p^{k^{\prime}} \mathbb{Z}$ modulo $p^{k}\left(k<k^{\prime}\right)$, which by slight abuse of notation we denote by $\pi_{k}$, too. Just as the $p$-adic integers $\mathbb{Z}_{p}$ are the inverse (aka projective) limit of the rings $\mathbb{Z} / p^{k} \mathbb{Z}$ connected by the modulo $p^{k}$ projections, we conclude that $S O(3)_{p}$ is the inverse limit of the finite groups $S O(3)_{p}$ $\bmod p^{k}$ connected by the projections $\pi_{k}$. This makes $S O(3)_{p}$ a so-called profinite group.

Theorem 6 means that we can describe arbitrary element $L \in S O(3)_{p}$ by first picking an axis of rotation, $\mathbb{Q}_{p} \mathbf{n}$, with a nonzero vector $\mathbf{n} \in \mathbb{Q}_{p}^{3}$, construct the two-dimensional subspace $V=\mathbf{n}^{\perp}=$ $\{x: b(\mathbf{n}, x)=0\} \subset \mathbb{Q}_{p}^{3}$ and consider the quadratic form $Q_{+\mid V}$, which is necessarily definite. Thus, $L$ can be written as $L=L_{V}+b(\mathbf{n}, \cdot) \frac{1}{Q_{+}(\mathbf{n})} \mathbf{n}$, with a two-dimensional special orthogonal transformation $L_{V} \in S O\left(Q_{+\mid V}\right)$. In the next sections, we shall consider what forms can appear as restrictions $Q_{+\mid V}$, and which are the different appearances of $S O(2)_{p}$.

Lemma 7. For any two orthogonal bases $\mathcal{B}=\left(\boldsymbol{v}_{1}, \boldsymbol{v}_{2}, \boldsymbol{v}_{3}\right)$ and $\mathcal{C}=\left(\boldsymbol{w}_{1}, \boldsymbol{w}_{2}, \boldsymbol{w}_{3}\right)$ of $\mathbb{Q}_{p}^{3}$, there exists an orthogonal transformation $M: \mathbb{Q}_{p}^{3} \rightarrow \mathbb{Q}_{p}^{3}$ such that $M \boldsymbol{v}_{i}=\boldsymbol{w}_{i}$ for all $i=1,2,3$ if and only if $Q_{+}\left(\boldsymbol{v}_{i}\right)=Q_{+}\left(\boldsymbol{w}_{i}\right)$.

In that case, either $M$ is special or the orthogonal transformation $M^{\prime}$ sending $\mathcal{B}$ to $\mathcal{C}^{\prime}=$ $\left(\boldsymbol{w}_{1},-\boldsymbol{w}_{2}, \boldsymbol{w}_{3}\right)$ is special.

Proof. There exists a unique linear transformation $M: \mathbb{Q}_{p}^{3} \rightarrow \mathbb{Q}_{p}^{3}$ sending $\mathcal{B}$ to $\mathcal{C} . M$ is an orthogonal transformation if and only if it preserves the bilinear form on $\mathbb{Q}_{p}^{3}$ associated to $Q_{+}$. It is enough to show $b\left(\boldsymbol{v}_{i}, \boldsymbol{v}_{j}\right)=b\left(\boldsymbol{w}_{i}, \boldsymbol{w}_{j}\right)$ for all $i, j \in\{1,2,3\}$. For $i \neq j$ this holds as we assume orthogonal bases. For $i=j$ it amounts to $Q_{+}\left(\boldsymbol{v}_{i}\right)=Q_{+}\left(\boldsymbol{w}_{i}\right)$ for every $i \in\{1,2,3\}$.
$\mathcal{C}^{\prime}$ is an orthogonal basis like $\mathcal{C}$, and the orthogonal transformation $M$ exists if and only if the orthogonal $M^{\prime}$ exists. An orthogonal transformation has determinant $\pm 1$, and as $\operatorname{det} M^{\prime}=-\operatorname{det} M$, exactly one of the two will have determinant 1 .

In contrast to $\mathbb{Q}_{p}^{3}$, every vector in $\mathbb{R}^{3}$ can be normalized to 1 , as $Q_{+}(\boldsymbol{v}) \simeq 1$ modulo squares for all $\boldsymbol{v} \in \mathbb{R}^{3} \backslash \mathbf{0}$ (indeed $\mathbb{R}^{*} /\left(\mathbb{R}^{*}\right)^{2} \simeq\{ \pm 1\}$ and $Q_{+}^{\mathbb{R}}(\boldsymbol{v})>0$ for every $\left.\boldsymbol{v} \in \mathbb{R}^{3} \backslash \mathbf{0}\right)$. As a consequence, every orthogonal basis of $\mathbb{R}^{3}$ can be made orthonormal, and there always exists a proper or improper rotation mapping any orthonormal basis to any other orthonormal basis. On the other hand, in $\mathbb{Q}_{p}^{3}$ we can only consider orthogonal bases due to the existence of different kinds of vectors, depending on the value in $K=\mathbb{Q}_{p}^{*} /\left(\mathbb{Q}_{p}^{*}\right)^{2}$ that the quadratic form $Q_{+}$takes on the vectors of $\mathbb{Q}_{p}^{3} \backslash \mathbf{0}$. In this case the condition of preservation of the quadratic form becomes relevant.

Proposition 8. Given $\boldsymbol{n} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}$, for any non-zero vectors $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{n}^{\perp} \subset \mathbb{Q}_{p}^{3}$ there exists a rotation $\mathcal{R}_{\boldsymbol{n}} \in S O(3)_{p}$ such that $\mathcal{R}_{\boldsymbol{n}} \boldsymbol{v}=\boldsymbol{w}$ if and only if $Q_{+}(\boldsymbol{v})=Q_{+}(\boldsymbol{w})$.

Proof. The direct implication is trivial by definition of $S O(3)_{p}$.
Conversely, for the orthogonal pair $\boldsymbol{v}, \boldsymbol{n}$ there exists a unique vector $\boldsymbol{v}^{\prime} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}$, up to nonzero scalar multiples, which completes $\boldsymbol{v}, \boldsymbol{n}$ to an orthogonal basis $\mathcal{B}=\left(\boldsymbol{v}, \boldsymbol{v}^{\prime}, \boldsymbol{n}\right)$. Similarly, we can complete the pair $\boldsymbol{w}, \boldsymbol{n}$ to an orthogonal basis $\mathcal{C}=\left(\boldsymbol{w}, \boldsymbol{w}^{\prime}, \boldsymbol{n}\right)$.

Now we prove that $Q_{+}\left(\boldsymbol{v}^{\prime}\right)$ and $Q_{+}\left(\boldsymbol{w}^{\prime}\right)$ are in the same class in $K$. Indeed, $Q_{+}$has diagonal matrix representation on any orthogonal basis. We write the matrix representation of $Q_{+}$on the basis $\mathcal{B}$ as $M=\operatorname{diag}\left(q_{1}, q_{2}, q_{3}\right)$ for every $\boldsymbol{v}^{\prime} \in\{\boldsymbol{v}, \boldsymbol{n}\}^{\perp}$ and on $\mathcal{C}$ as $M^{\prime}=\operatorname{diag}\left(q_{1}^{\prime}, q_{2}^{\prime}, q_{3}^{\prime}\right)$ for every $\boldsymbol{w}^{\prime} \in\{\boldsymbol{w}, \boldsymbol{n}\}^{\perp}$. The hypothesis $Q_{+}(\boldsymbol{v})=Q_{+}(\boldsymbol{w})$ translates into $q_{1}^{\prime}=q_{1}$, and $q_{3}^{\prime}=q_{3}$, the latter since $\mathcal{B}$ and $\mathcal{C}$ share the third vector $n$. The determinant is an invariant of $Q_{+}$in $K$, $\operatorname{det} M \simeq \operatorname{det} M^{\prime}$ : we have $q_{1} q_{2} q_{3}=q_{1} q_{2}^{\prime} q_{3} \lambda^{2}$ for some $\lambda \in \mathbb{Q}_{p}^{*}$, which gives $q_{2}=q_{2}^{\prime} \lambda^{2}$, i.e., $Q_{+}\left(\boldsymbol{v}^{\prime}\right)=Q_{+}\left(\boldsymbol{w}^{\prime}\right) \lambda^{2}=Q_{+}\left(\lambda \boldsymbol{w}^{\prime}\right)$.

Therefore, there exists $\tilde{\boldsymbol{w}}^{\prime}=\lambda \boldsymbol{w}^{\prime}$ such that $\tilde{\mathcal{C}}=\left(\boldsymbol{w}, \tilde{\boldsymbol{w}}^{\prime}, \boldsymbol{n}\right)$ is an orthogonal basis for $\mathbb{Q}_{p}^{3}$ and $Q_{+}\left(\boldsymbol{v}^{\prime}\right)=Q_{+}\left(\tilde{\boldsymbol{w}}^{\prime}\right)$. To this basis we apply Lemma 7: there exists a special orthogonal transformation $\mathcal{R}_{n}: \mathbb{Q}_{p}^{3} \rightarrow \mathbb{Q}_{p}^{3}$ such that $\mathcal{R}_{\boldsymbol{n}} \boldsymbol{v}=\boldsymbol{w}, \mathcal{R}_{\boldsymbol{n}} \boldsymbol{v}^{\prime}=\tilde{\boldsymbol{w}}^{\prime}$ and $\mathcal{R}_{n} \boldsymbol{n}=\boldsymbol{n}$. In fact, either the orthogonal transformation $\mathcal{R}_{n}$ is already special, or we choose $\tilde{\boldsymbol{w}}^{\prime}=-\lambda \boldsymbol{w}^{\prime}$ rather than $\tilde{\boldsymbol{w}}^{\prime}=\lambda \boldsymbol{w}^{\prime}$ in the basis $\tilde{\mathcal{C}}$.

We remark that there exists a transformation in $S O(3)_{p}$ rotating the line $\mathbb{Q}_{p} \boldsymbol{v}$ to the line $\mathbb{Q}_{p} \boldsymbol{w}$ if and only if $Q_{+}(\boldsymbol{v}) \simeq Q_{+}(\boldsymbol{w})$. This is trivially satisfied in $\mathbb{R}^{3}$, where any line can be rotated to any other line. However, the necessary condition of preservation of $Q_{+}$implies that there does not always exist a matrix in $S O(3)_{p}$ rotating a direction in $\mathbb{Q}_{p}^{3}$ to another one. For example, there is no element of $S O(3)_{p}$ for $p$ odd, rotating the $x$-axis of $\mathbb{Q}_{p}^{3}$ to the $z$-axis: $Q_{+}\left(\mu \boldsymbol{e}_{1}\right)=\mu^{2} \simeq 1$ and $Q_{+}\left(\lambda e_{3}\right)=\lambda^{2} p \simeq p$ belong to different classes in $K$ for every $\mu, \lambda \in \mathbb{Q}_{p}^{*}$.

The reverse implication of the previous proposition implies the following.
Proposition 9. Given a direction $\boldsymbol{n} \in \mathbb{Q}_{p}^{3}$, the action of the subgroup of $S O(3)_{p}$ of matrix rotations around $\boldsymbol{n}$ is transitive on the equivalence classes $Q_{+}^{-1}(q)=\left\{\boldsymbol{v} \in \mathbb{Q}_{p}^{3}: Q_{+}(\boldsymbol{v})=q\right\}$.

More generally, for any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}$ with $Q_{+}(\boldsymbol{v})=Q_{+}(\boldsymbol{w})$, there exists an $L \in$ $S O(3)_{p}$ with $L \boldsymbol{v}=\boldsymbol{w}$. This is achieved by choosing $\boldsymbol{n} \in\{\boldsymbol{v}, \boldsymbol{w}\}^{\perp}$.

As noted above, there are different classes of vectors in $\mathbb{Q}_{p}^{3}$, i.e., $\left\{\boldsymbol{v} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}: Q_{+}(\boldsymbol{v}) \simeq k\right\}$ for $k \in \mathbb{Q}_{p}^{*} /\left(\mathbb{Q}_{p}^{*}\right)^{2}$. This is one of the fundamental obstacles in the study of $S O(3)_{p}$, which marks a first departure from the real case.

## 4. PLANAR CASE: THE THREE (SEVEN) INCARNATIONS OF $S O(2)_{p}$

By virtue of Theorem 6, the elements of $S O(3)_{p}$ are transformations occurring in the plane $V$ orthogonal to the rotation axis $\boldsymbol{n}$ in $\mathbb{Q}_{p}^{3}$. Thus we move to classify the definite quadratic forms on $\mathbb{Q}_{p}^{2}$ and analyze the corresponding symmetry groups. We will derive a parameterization for the rotations in these groups and for $S O(3)_{p}$, and show that the restrictions of the rotations of $S O(3)_{p}$ to the plane orthogonal to their rotation axis realize all the classes of rotation groups of the $p$ adic plane. Indeed, we will show that the restrictions $Q_{+\mid V}$ of the three-dimensional definite form $Q_{+}$, where $V \subset \mathbb{Q}_{p}^{3}$ varies over all two-dimensional subspaces, exhaust all two-dimensional definite quadratic forms, up to equivalence (similarity and scaling).

Beginning with odd primes $p$, the invariants of the two-dimensional quadratic forms $Q(x, y)=$ $a x^{2}+b y^{2}, x, y \in \mathbb{Q}_{p}$, characterized by pairs $(a, b) \in K^{2}$, give the following 8 distinct equivalence classes

| $d$ | $\varepsilon$ | Representative $(a, b)$ | $(a, b)$ of equivalent forms |
| :--- | :--- | :--- | :--- |
| 1 | 1 | $(1,1)$ | $(u, u)$ <br> and $(p, p),(u p, u p)$ for $p \equiv 1 \bmod 4$ |
| 1 | -1 | $(p, p)$ for $p \equiv 3 \bmod 4$ | $(u p, u p)$ for $p \equiv 3 \bmod 4$ |
| $u$ | 1 | $(1, u)$ | $(p, u p)$ for $p \equiv 3 \bmod 4$ |
| $u$ | -1 | $(p, u p)$ for $p \equiv 1 \bmod 4$ |  |
| $p$ | 1 | $(1, p)$ |  |
| $p$ | -1 | $(u, u p)$ |  |
| $u p$ | 1 | $(1, u p)$ |  |
| $u p$ | -1 | $(u, p)$ |  |

Considering scaling, we can always make $a=1$, without loss of generality. Thus, there are four different equivalence classes of quadratic forms on $\mathbb{Q}_{p}^{2}$ up to equivalence for every odd prime $p$

$$
\begin{equation*}
x^{2}+y^{2}, \quad x^{2}+u y^{2}, \quad x^{2}+p y^{2}, \quad u x^{2}+p y^{2} \tag{3}
\end{equation*}
$$

(it will become clear in a moment why we choose the last form as written, and not as $x^{2}+u p y^{2}$ or $\left.x^{2}+p / u y^{2}\right)$.

To determine which of them represents zero, we need to check whether $d \simeq-1$ in $K$. First, $-1 \in\left(\mathbb{Q}_{p}^{*}\right)^{2}$ for $p \equiv 1 \bmod 4$, so $-1 \simeq 1 \in K$; on the other hand, $-1 \simeq u \in K$ for $p \equiv 3 \bmod 4$ because $-1 \notin\left(\mathbb{Q}_{p}^{*}\right)^{2}$. Now, $x^{2}+y^{2}$ has $d=1$ : it represents 0 for $p \equiv 1 \bmod 4$, but not for $p \equiv 3$ $\bmod 4$. $x^{2}+u y^{2}$ has $d=u$ : it represents 0 for $p \equiv 3 \bmod 4$, but not for $p \equiv 1 \bmod 4$. The last two forms of (3) do not represent 0 .

Proceeding similarly for $p=2$, we get the following 16 classes of quadratic forms on $\mathbb{Q}_{2}^{2}$

| $d$ | $\varepsilon$ | Representative $(a, b)$ | $(a, b)$ of equivalent forms |
| :---: | :---: | :--- | :--- |
| 1 | 1 | $(1,1)$ | $(2,2),(5,5),(10,10)$ |
| 1 | -1 | $(-1,-1)$ | $(-2,-2),(-5,-5),(-10,-10)$ |
| -1 | 1 | $(1,-1)$ | $(2,-2),(5,-5),(10,-10)$ |
| -1 | -1 | $\mathrm{~N} / \mathrm{A}^{\dagger}$ | $\mathrm{N} / \mathrm{A}$ |
| 2 | 1 | $(1,2)$ | $(-5,-10)$ |
| 2 | -1 | $(-1,-2)$ | $(5,10)$ |


| -2 | 1 | $(1,-2)$ | $(-1,2)$ |
| :---: | :---: | :--- | :--- |
| -2 | -1 | $(5,-10)$ | $(-5,10)$ |
| 5 | 1 | $(1,5)$ | $(-2,-10)$ |
| 5 | -1 | $(-1,-5)$ | $(2,10)$ |
| -5 | 1 | $(1,-5)$ | $(-1,5)$ |
| -5 | -1 | $(2,-10)$ | $(-2,10)$ |
| 10 | 1 | $(-2,-5)$ | $(1,-10)$ |
| 10 | -1 | $(2,5)$ | $(-1,-10)$ |
| -10 | 1 | $(1,-10)$ | $(-1,10)$ |
| -10 | -1 | $(2,-5)$ | $(-2,5)$ |

$\dagger$ No 2-adic quadratic form in dimension two can have determinant -1 together with Hasse invariant -1 .
They reduce to the following 8 classes up to scaling

$$
\begin{array}{llll}
x^{2}+y^{2}, & x^{2}-y^{2}, & x^{2}+2 y^{2}, & x^{2}-2 y^{2} \\
x^{2}+5 y^{2}, & x^{2}-5 y^{2}, & x^{2}+10 y^{2}, & x^{2}-10 y^{2}
\end{array}
$$

Only the second one represents 0 . In summary, we get the following classification.
Proposition 10. The quadratic forms on $\mathbb{Q}_{p}^{2}$ that do not represent 0 , are, up to equivalence

- the following 3 for every odd prime $p$

$$
Q_{-v}(\boldsymbol{x})=x^{2}-v y^{2}, \quad Q_{p}(\boldsymbol{x})=x^{2}+p y^{2}, \quad Q_{u p}(\boldsymbol{x})=u x^{2}+p y^{2}
$$

where $v$ is as in Eq. (1);

- the following 7 for $p=2$

$$
\begin{aligned}
Q_{1}(\boldsymbol{x}) & =x^{2}+y^{2}, & Q_{ \pm 2}(\boldsymbol{x}) & =x^{2} \pm 2 y^{2} \\
Q_{ \pm 5}(\boldsymbol{x}) & =x^{2} \pm 5 y^{2}, & Q_{ \pm 10}(\boldsymbol{x}) & =x^{2} \pm 10 y^{2}
\end{aligned}
$$

The subscript of these quadratic forms denotes their determinant.
When $p \equiv 1 \bmod 4$, the three definite forms are the restrictions of the definite three-dimensional one to the planes $x y, x z$ and $y z$. This is not true for $p \equiv 3 \bmod 4$, where $Q_{+}(\boldsymbol{x})$ reduces to $Q_{-v}(\boldsymbol{x})$ for $z=0$ and to the same $Q_{p}(\boldsymbol{x})$ for both $x=0$ and $y=0$. Similarly, the restriction of the definite three-dimensional form for $p=2$ to any reference plane $x y, x z$ and $y z$ gives only $Q_{1}(\boldsymbol{x})$ among the seven possible forms in dimension two.

In contrast to the unique special orthogonal group $S O(2)_{\mathbb{R}}$ on the real plane, there are three (compact) special orthogonal groups on $\mathbb{Q}_{p}^{2}$ for odd prime $p$, and seven for $p=2$, up to isomorphisms, according to the last proposition. We call them $S O(2)_{p}^{\kappa}$ with $\kappa$ denoting the determinant of the preserved quadratic form: these groups are

$$
\begin{equation*}
S O(2)_{p}^{\kappa}=\left\{L \in \mathcal{M}_{2 \times 2}\left(\mathbb{Q}_{p}\right): A_{\kappa}=L^{\top} A_{\kappa} L, \operatorname{det} L=1\right\} \tag{4}
\end{equation*}
$$

endowed with the common matrix product. $A_{\kappa}$ denotes the matrix representation of the quadratic form $Q_{\kappa}$ in the canonical basis

$$
A_{-v}=\left(\begin{array}{cc}
1 & 0 \\
0 & -v
\end{array}\right), \quad A_{p}=\left(\begin{array}{cc}
1 & 0 \\
0 & p
\end{array}\right), \quad A_{u p}=\left(\begin{array}{ll}
u & 0 \\
0 & p
\end{array}\right)
$$

$$
A_{1}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right), \quad A_{ \pm 2}=\left(\begin{array}{cc}
1 & 0 \\
0 & \pm 2
\end{array}\right), \quad A_{ \pm 5}=\left(\begin{array}{cc}
1 & 0 \\
0 & \pm 5
\end{array}\right), \quad A_{ \pm 10}=\left(\begin{array}{cc}
1 & 0 \\
0 & \pm 10
\end{array}\right) .
$$

We now proceed to the parametrisation of the two-dimensional rotation groups. Consider the transformation of a vector in $\mathbb{Q}_{p}^{2}$ by some $\mathcal{R}_{\kappa} \in S O(2)_{p}^{\kappa}$

$$
\binom{x^{\prime}}{y^{\prime}}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{x}{y}
$$

The orthogonality condition for $\mathcal{R}_{\kappa}$ yields the system of equations

$$
\left\{\begin{array}{l}
a^{2}+\alpha_{\kappa} c^{2}=1  \tag{5}\\
b^{2}+\alpha_{\kappa} d^{2}=\alpha_{\kappa} \\
a b+\alpha_{\kappa} c d=0
\end{array}\right.
$$

where $\alpha_{\kappa}=-v, p, \frac{p}{u}, 1, \pm 2, \pm 5, \pm 10$, respectively for the quadratic forms with $\kappa=-v, p, u p, p \neq 2$ and $\kappa=1, \pm 2, \pm 5, \pm 10, p=2$. We parametrize the solutions of this system of three equations in four $p$-adic unknowns through

$$
\begin{equation*}
\sigma= \pm \frac{c}{1+a} \in \mathbb{Q}_{p} \tag{6}
\end{equation*}
$$

for $a \in \mathbb{Z}_{p} \backslash\{-1\}$, and treat the case $a=-1$ separately. The first equation of the system (5) gives

$$
\alpha_{\kappa}=\frac{1-a^{2}}{c^{2}}
$$

when $c \neq 0$. Manipulating Eq. (6), we get $\frac{1+a}{c^{2}}=\frac{1}{(1+a) \sigma^{2}}, \sigma \neq 0$, so

$$
\begin{equation*}
\alpha_{\kappa}=\frac{1-a}{(1+a) \sigma^{2}} \Rightarrow\left(1+\alpha_{\kappa} \sigma^{2}\right) a=1-\alpha_{\kappa} \sigma^{2} \tag{7}
\end{equation*}
$$

The parameter $-\alpha_{\kappa}$ is not a square for any $\kappa$ of the considered quadratic forms (because none of them represents zero). This means that $1+\alpha_{\kappa} \sigma^{2} \neq 0$ for all $\alpha_{\kappa}$ and $\sigma \in \mathbb{Q}_{p}$. Now Eq. (7) gives

$$
a=\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}
$$

The first equation of (5) now provides

$$
c^{2}=\frac{1-a^{2}}{\alpha_{\kappa}}=\frac{4 \sigma^{2}}{\left(1+\alpha_{\kappa} \sigma^{2}\right)^{2}} \Rightarrow c= \pm_{c} \frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}}
$$

From the third equation of (5), when $a \neq 0$, we get

$$
\begin{equation*}
b^{2}=\alpha_{\kappa}\left(\frac{1}{a^{2}}-1\right) d^{2} \tag{8}
\end{equation*}
$$

which plugged into the second one, gives $d^{2}=a^{2}$, hence $d= \pm_{d} a$.
Now the second equation of (5) gives

$$
b^{2}=\alpha_{\kappa}^{2} c^{2} \Rightarrow b= \pm_{b} \frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}}
$$

We arrive at

$$
\mathcal{R}_{\kappa}(\sigma)=\left(\begin{array}{ll}
a(\sigma) & b(\sigma)  \tag{9}\\
c(\sigma) & d(\sigma)
\end{array}\right)=\left(\begin{array}{rr}
\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}} & \pm_{b} \frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}} \\
\pm_{c} \frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}} & \pm_{d} \frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}
\end{array}\right)
$$

where the signs $\pm_{b}, \pm_{c}, \pm_{d}$ are a priori unrelated. The determinant of the matrix (9) is

$$
\pm_{d}\left(\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}\right)^{2}- \pm_{b} \pm_{c} \alpha_{\kappa}\left(\frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}}\right)^{2}=\frac{ \pm_{d} 1-2 \alpha_{\kappa} \sigma^{2}\left( \pm_{d} 1+ \pm_{b} \pm_{c} 2\right) \pm_{d} \alpha_{\kappa}^{2} \sigma^{4}}{1+2 \alpha_{\kappa} \sigma^{2}+\alpha_{\kappa}^{2} \sigma^{4}}
$$

and it must be 1. This happens when $\pm_{d} 1=1$ and $\pm_{d} 1+ \pm_{b} \pm_{c} 2=-1$, i.e., $\pm_{c}=\mp_{b}$. Therefore,

$$
\mathcal{R}_{\kappa}(\sigma)=\left(\begin{array}{cc}
\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}} & \mp \frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}}  \tag{10}\\
\pm \frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}} & \frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}
\end{array}\right)
$$

with linked signs.
In the derivation, at Eq. (8), we had to assume $a \neq 0$. When $a=0$, we have $b^{2}=\alpha_{\kappa}, c^{2}=1 / \alpha_{\kappa}$, $d=0$, which provides a solution when $\alpha_{\kappa}$ is a square. In this case, $a=0$ corresponds to $\sigma= \pm c$, included in Eq. (10).

The case $c=0$ left out in the system of equations gives $\mathcal{R}_{\kappa}= \pm I$, using that the determinant is 1. Thus, $c=0$ is equivalent to $\sigma=0$ when $a \neq-1$ in Eq. (6), leading to $\mathcal{R}_{\kappa}(0)=I$. This leaves one solution of Eq. (10) unaccounted for, which is $\mathcal{R}_{\kappa}=-I$, reached from the system of equations with $a=-1$, for which $\sigma$ is not well-defined in $\mathbb{Q}_{p}$. But this sign change from the identity matrix can be included in Eq. (10) by considering $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}$.

Remark 11. Note that $\lim _{n \rightarrow \infty} p^{n}=0$ in $\mathbb{Q}_{p}$ with respect to the $p$-adic norm, since $\left|p^{n}\right|_{p}$ converges to 0 by increasing $n$. Then, $\infty$ on the $p$-adic field is formally introduced as the limit $\lim _{n \rightarrow \infty} p^{-n}$ (the inverse of 0). Indeed the matrix entries in Eq. (10) are such that

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1-\alpha_{\kappa} p^{-2 n}}{1+\alpha_{\kappa} p^{-2 n}}=\lim _{n \rightarrow \infty} \frac{p^{-2 n}\left(p^{2 n}-\alpha_{\kappa}\right)}{p^{-2 n}\left(p^{2 n}+\alpha_{\kappa}\right)}=-1 \\
& \lim _{n \rightarrow \infty} \frac{2 p^{-n}}{1+\alpha_{\kappa} p^{-2 n}}=\lim _{n \rightarrow \infty} \frac{p^{-2 n}\left(2 p^{n}\right)}{p^{-2 n}\left(p^{2 n}+\alpha_{\kappa}\right)}=0
\end{aligned}
$$

yielding $\mathcal{R}_{\kappa}(\infty)=-I$.
As a parameter $\sigma$ has a sign that is linked to its inverse (as we are going to see in a moment): it can swap between + and - variations of $b$ and $c$ even when we fix one choice of $\pm_{c}$. Hence we can keep only one sign in Eq. (10) without loss of generality. We have thus proved the following.

Theorem 12. A rotation of $S O(2)_{p}^{\kappa}$ takes the following matrix form in the canonical basis of $\mathbb{Q}_{p}^{2}$

$$
\mathcal{R}_{\kappa}(\sigma)=\left(\begin{array}{rr}
\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}} & -\frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}}  \tag{11}\\
\frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}} & \frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}
\end{array}\right)
$$

with $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}, \alpha_{\kappa} \in\left\{-v, p, \frac{p}{u}\right\}$ and $\alpha_{k} \in\{1, \pm 2, \pm 5, \pm 10\}$ respectively for $\kappa=-v, p$, up ( $p$ odd) and $\kappa=1, \pm 2, \pm 5, \pm 10 \quad(p=2)$.

In addition, this parametrization is one-to-one, i.e., it results in different special orthogonal transformations for different $\sigma$.

Remark 13. For every $\kappa$ of the definite rank-2 quadratic forms

$$
\begin{equation*}
\mathcal{R}_{\kappa}\left(-\frac{1}{\alpha_{\kappa} \sigma}\right)=-\mathcal{R}_{\kappa}(\sigma), \quad \sigma \in \mathbb{Q}_{p} \cup\{\infty\} \tag{12}
\end{equation*}
$$

because if we replace $\sigma \mapsto-\frac{1}{\alpha_{\kappa} \sigma}$ in the matrices (11), then the matrix elements change as follows

$$
\begin{aligned}
& \frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}} \mapsto \frac{1-\alpha_{\kappa}\left(\alpha_{\kappa} \sigma\right)^{-2}}{1+\alpha_{\kappa}\left(\alpha_{\kappa} \sigma\right)^{-2}}=\frac{\alpha_{\kappa} \sigma^{2}-1}{\alpha_{\kappa} \sigma^{2}+1}=-\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}, \\
-\frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}} \mapsto & \mapsto-\frac{2 \alpha_{\kappa}\left(-\alpha_{\kappa} \sigma\right)^{-1}}{1+\alpha_{\kappa}\left(\alpha_{\kappa} \sigma\right)^{-2}}=\frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}}, \\
& \frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}} \mapsto \frac{2\left(-\alpha_{\kappa} \sigma\right)^{-1}}{1+\alpha_{\kappa}(\alpha \sigma)^{-2}}=-\frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}} .
\end{aligned}
$$

We need $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}$ in order for this transformation to be well-defined for $\sigma=0$, too, which is guaranteed by

$$
\mathcal{R}_{\kappa}(\infty)=-\mathcal{R}_{\kappa}(0)=-I .
$$

Remark 14. As a corollary of the parameterization (11), we can see directly that the matrix entries $\frac{1-\alpha_{\kappa} \sigma^{2}}{1+\alpha_{\kappa} \sigma^{2}}, \frac{2 \sigma}{1+\alpha_{\kappa} \sigma^{2}}$ and $-\frac{2 \alpha_{\kappa} \sigma}{1+\alpha_{\kappa} \sigma^{2}}$ of the rotations are $p$-adic integers for every $\alpha_{\kappa}$, every $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}$ and every prime $p$, in accordance with Theorem 5. This can be easily checked for parameters $\sigma \in \mathbb{Z}_{p}$ :

- $1+\alpha_{\kappa} \sigma^{2} \not \equiv 0 \bmod p$, for every $\kappa, \sigma \in \mathbb{Z}_{p}, p \neq 2$, for $\kappa= \pm 2, \pm 10, \sigma \in \mathbb{Z}_{2}$, and for $\kappa=1, \pm 5, \sigma \in 2 \mathbb{Z}_{2}$. In these cases $\left(1+\alpha_{\kappa} \sigma^{2}\right)^{-1} \in \mathbb{U}$, which multiplied with $1-$ $\alpha_{\kappa} \sigma^{2}, 2 \sigma,-2 \alpha_{\kappa} \sigma \in \mathbb{Z}_{p}$ give $p$-adic integer matrix entries;
- If $\sigma \in \mathbb{Z}_{2}, \sigma \equiv 1 \bmod 2$, then $1+\sigma^{2}, 1 \pm 5 \sigma^{2} \equiv 0 \bmod 2$ : they are invertible in $\mathbb{Q}_{2}$ but not in $\mathbb{Z}_{2}$. Here the calculus modulo 2 is not enough to verify that the associated matrix entries are 2 -adic integers. But writing $\sigma=1+2 \sigma^{\prime}, \sigma^{\prime} \in \mathbb{Z}_{2}$ we get

$$
\begin{aligned}
& \frac{1-\sigma^{2}}{1+\sigma^{2}}=\frac{-4\left(\sigma^{\prime}+\sigma^{\prime 2}\right)}{2\left(1+2\left(\sigma^{\prime}+\sigma^{\prime 2}\right)\right)}=\frac{-2\left(\sigma^{\prime}+\sigma^{\prime 2}\right)}{1+2\left(\sigma^{\prime}+\sigma^{\prime 2}\right)}, \\
& \frac{2 \sigma}{1+\sigma^{2}}=\frac{2\left(1+2 \sigma^{\prime}\right)}{2\left(1+2\left(\sigma^{\prime}+\sigma^{\prime 2}\right)\right)}=\frac{1+2 \sigma^{\prime}}{1+2\left(\sigma^{\prime}+\sigma^{\prime 2}\right)} .
\end{aligned}
$$

$1+2\left(\sigma^{\prime}+{\sigma^{\prime 2}}^{2}\right) \not \equiv 0 \bmod 2$ is invertible in $\mathbb{Z}_{2}$.
We have shown that the entries of the matrices of $S O(2)_{p}^{\kappa}$ are $p$-adic integers when $\sigma \in \mathbb{Z}_{p}$, for every $\kappa$ and prime $p$. But then we can distinguish two branches for $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}$ : either $\sigma \in \mathbb{Z}_{p}$ or $\sigma^{-1} \in p \mathbb{Z}_{p}$ (including $\infty$ formally when $\sigma=0$ ). Since we want to exploit Eq. (12) for a non integer parameter, either $\sigma \in \mathbb{Z}_{p}$ or:

- $\sigma=-\frac{1}{\alpha_{\kappa} \tau}, \tau \in p \mathbb{Z}_{p}$, for $p \neq 2, \kappa=-v$ and for $p=2, \kappa=1, \pm 5$,
- $\sigma=-\frac{1}{\alpha_{\kappa} \tau}, \tau \in \mathbb{Z}_{p}$, for $p \neq 2, \kappa=p$, up and for $p=2, \kappa= \pm 2, \pm 10$.

In each of the cases of the first point

$$
\begin{aligned}
S O(2)_{p}^{\kappa} & =\left\{\mathcal{R}_{\kappa}(\sigma): \sigma \in \mathbb{Q}_{p} \cup\{\infty\}\right\} \\
& =\left\{\mathcal{R}_{\kappa}(\sigma): \sigma \in \mathbb{Z}_{p}\right\} \cup\left\{\mathcal{R}_{\kappa}\left(-\frac{1}{\alpha_{\kappa} \tau}\right): \tau \in p \mathbb{Z}_{p}\right\} \\
& =\left\{\mathcal{R}_{\kappa}(\sigma): \sigma \in \mathbb{Z}_{p}\right\} \cup\left\{-\mathcal{R}_{\kappa}(\tau): \tau \in p \mathbb{Z}_{p}\right\} .
\end{aligned}
$$

The second set includes $\mathcal{R}_{\kappa}(\infty)$ for $\tau=0$.
Similarly for all the cases in the second point

$$
S O(2)_{p}^{\kappa}=\left\{\mathcal{R}_{\kappa}(\sigma): \sigma \in \mathbb{Q}_{p} \cup\{\infty\}\right\}=\left\{\mathcal{R}_{\kappa}(\sigma): \sigma \in \mathbb{Z}_{p}\right\} \cup\left\{-\mathcal{R}_{\kappa}(\tau): \tau \in \mathbb{Z}_{p}\right\} .
$$

In this way, the parameterisation of the groups $S O(2)_{p}^{\kappa}$ is in terms of p-adic integers only. The second branch for the parameter $\sigma$ gives the same matrix entries of the first one up to a sign: these entries are p-adic integers as shown for the first branch.

This is useful when projecting $S O(3)_{p}$ modulo $p^{k}$, to be able to enumerate the elements of its subgroups of rotations around the reference axes [3].

Interlude: Euclidean geometrical interpretation. The introduction of a parameter $\sigma$ leading to $a=\frac{1-\alpha \sigma^{2}}{1+\alpha \sigma^{2}}$ in Eq. (11) is inspired by Euclidean geometry. Indeed, recalling the tangent half-angle formulae

$$
\cos \theta=\frac{1-\tan ^{2}(\theta / 2)}{1+\tan ^{2}(\theta / 2)}, \quad \sin \theta=\frac{2 \tan (\theta / 2)}{1+\tan ^{2}(\theta / 2)}
$$

and letting

$$
\begin{equation*}
\alpha_{\kappa}=1, \quad \sigma=\tan \left(\frac{\theta}{2}\right) \in \mathbb{R} \tag{13}
\end{equation*}
$$

Eq. (11) takes the form

$$
\mathcal{R}(\theta)=\left(\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right)
$$

which is the matrix for a rotation by an angle $\theta$ in the real plane. The special case $-I$ is given by $\theta=\pi$, for which $\tan (\theta / 2)$ diverges. It is as if we have been treating rotations in $\mathbb{Q}_{p}^{2}$ in terms of the tangent of the rotation angle rather than the angle itself. It is worth noticing that the usage of tangent of angles (in place of angles) in trigonometry dates back to ancient Babylon: the Babylonian table, Plimpton 322 (written about 1800 BC ), may be interpreted as a first trigonometric table, where Pythagorean triples were written as ratios between the sides of right triangles [9].

This geometrical interpretation motivates our notation $\mathcal{R}_{\kappa}(\sigma)$ for a rotation of parameter $\sigma$.
Composition laws of rotations. The composition of two rotations of $S O(2)_{p}^{\kappa}$, for any fixed $\kappa$, is given by the product of their matrix forms (11). It turns out to take the very simple form

$$
\begin{equation*}
\mathcal{R}_{\kappa}(\sigma) \mathcal{R}_{\kappa}(\tau)=\mathcal{R}_{\kappa}\left(\frac{\sigma+\tau}{1-\alpha_{\kappa} \sigma \tau}\right) \tag{14}
\end{equation*}
$$

for every $\sigma, \tau \in \mathbb{Q}_{p} \cup\{\infty\}$. For the infinitely many values of $\sigma, \tau \in \mathbb{Q}_{p}$ such that $1-\alpha_{\kappa} \sigma \tau=0$, the right-hand-side is $\mathcal{R}_{\kappa}(\infty)=-I$, according to Remark 11. If $\tau \in \mathbb{Q}_{p}$ while $\sigma$ is $\infty$ (or vice versa), the argument on the right-hand side is

$$
\lim _{n \rightarrow \infty} \frac{p^{-n}+\tau}{1-\alpha_{\kappa} p^{-n} \tau}=\lim _{n \rightarrow \infty} \frac{p^{-n}\left(1+p^{n} \tau\right)}{p^{-n}\left(p^{n}-\alpha_{\kappa} \tau\right)}=-\frac{1}{\alpha_{\kappa} \tau}
$$

so Eq. (14) becomes $-\mathcal{R}_{\kappa}(\tau)=\mathcal{R}_{\kappa}\left(-\frac{1}{\alpha_{\kappa} \tau}\right)$ in agreement with Eq. (12). If both $\sigma$ and $\tau$ are $\infty$, the above composition gives correctly the identity, since the rotation parameter on the right is

$$
\lim _{n, m \rightarrow \infty} \frac{p^{-n}+p^{-m}}{1-\alpha_{\kappa} p^{-n-m}}=\lim _{n, m \rightarrow \infty} \frac{p^{-n-m}\left(p^{m}+p^{n}\right)}{p^{-n-m}\left(p^{n+m}-\alpha_{\kappa}\right)}=0
$$

We recall that

$$
\tan (\alpha+\beta)=\frac{\tan \alpha+\tan \beta}{1-\tan \alpha \tan \beta}
$$

which is equivalent to the arguments of Eq. (14) if we use Eq. (13).
Proposition 15. $S O(2)_{p}^{\kappa}$ defined in Eq. (4) is an abelian group, for every $\kappa$ and prime $p$.

Proof. This is straightforward to prove via Eq. (14), and it is in analogy with $S O(2)_{\mathbb{R}}$, which is isomorphic to the group of addition or real angles $\theta$ modulo $2 \pi$, but reinterpreted through the halfangle tangent. Furthermore, $\mathcal{R}_{\kappa}(\sigma)^{-1}=\mathcal{R}_{\kappa}(-\sigma)$ for all $\sigma \in \mathbb{Q}_{p}$ and $\mathcal{R}_{\kappa}(\infty)^{-1}=\mathcal{R}_{\kappa}(\infty)$.

Rotations parametrized by the $p$-adic projective line. The form of the parametrization (11) of two-dimensional rotations, and their composition law (14), and in particular the necessity to add a point at infinity to the $p$-adic affine line, suggest a description in terms of projective geometry.

As usual, we identify the projective line $P^{1}\left(\mathbb{Q}_{p}\right)$ with the equivalence classes $[s: t]=\mathbb{Q}_{p}^{*}(s, t) \subset$ $\mathbb{Q}_{p}^{2} \backslash(0,0)$ of nonzero vectors under multiplication by (nonzero) scalars. Representatives are $[s: 1]$ and $\infty \equiv[1: 0]$, allowing us to think of $P^{1}\left(\mathbb{Q}_{p}\right)=\mathbb{Q}_{p} \cup\{\infty\}$ as the $p$-adic line closed with a point at infinity. The parameter in Eq. (11) is $\sigma=s / t$, which is well-defined on each equivalence class. We can write the rotations as bona fide functions of the points $[s: t]$ of $P^{1}\left(\mathbb{Q}_{p}\right)$

$$
\mathcal{R}_{\kappa}([s: t])=\left(\begin{array}{cc}
\frac{t^{2}-\alpha_{\kappa} s^{2}}{t^{2}+\alpha_{\kappa} s^{2}} & -\frac{2 \alpha_{\kappa} s t}{t^{2}+\alpha_{\kappa} s^{2}}  \tag{15}\\
\frac{2 s t}{t^{2}+\alpha_{\kappa} s^{2}} & \frac{t^{2}-\alpha_{s} s^{2}}{t^{2}+\alpha_{\kappa} s^{2}}
\end{array}\right),
$$

where $\alpha_{\kappa} \in\left\{-v, p, \frac{p}{u}\right\}$ and $\alpha_{k} \in\{1, \pm 2, \pm 5, \pm 10\}$ respectively for $\kappa=-v, p, u p$ ( $p$ odd) and $\kappa=$ $1, \pm 2, \pm 5, \pm 10(p=2)$. Furthermore, the uniqueness of the parametrization (15) as a function of $[s: t] \in P^{1}\left(\mathbb{Q}_{p}\right)$ is inherited from Theorem 12.

Looking at $\sigma$ as a ratio reveals the underlying structure, too, of the composition law (14), which in projective line coordinates becomes

$$
\begin{equation*}
\mathcal{R}_{\kappa}([s: t]) \mathcal{R}_{\kappa}([u: v])=\mathcal{R}_{\kappa}\left(\left[s v+t u: t v-\alpha_{\kappa} s u\right]\right) . \tag{16}
\end{equation*}
$$

Note that for each fixed $[s: t]$, this is a projective linear transformation of $[u: v]$ (and vice versa). Indeed, the mapping

$$
\begin{aligned}
T_{\kappa}: P^{1}\left(\mathbb{Q}_{p}\right) & \longrightarrow P G L\left(2, \mathbb{Q}_{p}\right) \\
\quad[s: t] & \longmapsto T_{\kappa}([s: t]):=\mathbb{Q}_{p}^{*}\left(\begin{array}{cc}
t & -\alpha_{\kappa} s \\
s & t
\end{array}\right)
\end{aligned}
$$

defines abelian subgroups $\mathcal{T}_{\kappa}:=T_{\kappa}\left(P^{1}\left(\mathbb{Q}_{p}\right)\right) \subset P G L\left(2, \mathbb{Q}_{p}\right)$ (the invertible $2 \times 2$-matrices over $\mathbb{Q}_{p}$ modulo scalars $\mathbb{Q}_{p}^{*}$ ), each a rational image of the projective line. $T_{\kappa}$ translates the composition law Eq. (16) into a simple matrix multiplication

$$
T_{\kappa}([s: t]) T_{\kappa}([u: v])=T_{\kappa}\left(\left[s v+t u: t v-\alpha_{\kappa} s u\right]\right)
$$

This means that we can identify $S O(2)_{p}^{\kappa}$ with the subgroup $\mathcal{T}_{\kappa} \subset P G L\left(2, \mathbb{Q}_{p}\right), T_{\kappa}$ providing the isomorphism. If, by abuse of notation, we refer to $T_{\kappa}$ as the map from $P^{1}\left(\mathbb{Q}_{p}\right)$ to its image $\mathcal{T}_{\kappa}$, the isomorphic map from $\mathcal{T}_{\kappa}$ to $S O(2)_{p}^{\kappa}$ is explicitly given by first applying $T_{\kappa}^{-1}: \mathbb{Q}_{p}^{*}\left(\begin{array}{cc}t & -\alpha_{\kappa} s \\ s & t\end{array}\right) \mapsto[s, t]$ and then parameterizing an element of $S O(2)_{p}^{\kappa}$ as in Eq. (15).

Remark 16. Consider the quotient topologies on $P^{1}\left(\mathbb{Q}_{p}\right)$ and $P G L\left(2, \mathbb{Q}_{p}\right)$, where $\mathbb{Q}_{p}^{2} \backslash(0,0)$ and $G L\left(2, \mathbb{Q}_{p}\right)$ are naturally endowed with their p-adic topology, and consider the topology of the $T_{\kappa}$ as subspaces of $\operatorname{PGL}\left(2, \mathbb{Q}_{p}\right)$. The map $T_{\kappa}$, from $P^{1}\left(\mathbb{Q}_{p}\right)$ to its image $\mathcal{T}_{\kappa}$, is clearly a homeomorphism, for every $\kappa$ : it is continuous, as well as its inverse $T_{\kappa}^{-1}$, as it can be directly seen from their explicit expressions above. Now, let $S O(2)_{p}^{\kappa}$ be supplied with its natural p-adic topology. The parameterization Eq. (15) is continuous in $s$ and $t$, as its components are well defined rational functions. Hence it is continuous in $[s: t]$, since the parameterization does not depend on the
representatives of the equivalence classes $[s: t]$. The inverse map, providing $[s: t]$ from such $a$ parameterized matrix $\mathcal{R}_{\kappa}([s: t])$, is given by

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right):=\left(\begin{array}{cc}
\frac{t^{2}-\alpha_{\kappa} s^{2}}{t^{2}+\alpha_{\kappa} s^{2}} & -\frac{2 \alpha_{\kappa} s t}{t^{2}+\alpha_{\kappa} s^{2}} \\
\frac{2 s t}{t^{2}+\alpha_{\kappa} s^{2}} & \frac{t^{2}-\alpha_{k} s^{2}}{t^{2}+\alpha_{\kappa} s^{2}}
\end{array}\right) \mapsto[c: 1+a]=\left[s \frac{2 t}{t^{2}+\alpha_{\kappa} s^{2}}: t \frac{2 t}{t^{2}+\alpha_{\kappa} s^{2}}\right]=[s: t]
$$

whenever $a \neq-1$, otherwise

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right) \mapsto[1: 0] .
$$

This inverse map is again continuous, thus Eq. (15) provides a homeomorphism between $P^{1}\left(\mathbb{Q}_{p}\right)$ and $S O(2)_{p}^{\kappa}$, for every $\kappa$. It immediately follows that the parameterization (11) is a homeomorphism. We have also shown that the isomorphic map between $\mathcal{T}_{\kappa}$ and $S O(2)_{p}^{\kappa}$ is a homeomorphism, for every $\kappa$.

## 5. PARAMETRIZATION OF ROTATIONS IN $S O(3)_{p}$

Returning to $p$-adic three-space $\mathbb{Q}_{p}^{3}$, we found that every element of $S O(3)_{p}$ has a fixed axis of rotation. So, now we consider a rotation around a general direction $\boldsymbol{n} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}$. Let $\boldsymbol{g}, \boldsymbol{h} \in \mathbb{Q}_{p}^{3}$ be vectors spanning $V=\boldsymbol{n}^{\perp}$. Restricting our attention to $V$ means reducing a vector $s \in \mathbb{Q}_{p}^{3}$, with coordinates $\left(s_{1}, s_{2}, s_{3}\right)$ with respect to the basis $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$, to a vector $\boldsymbol{s}_{\mid V}$ with coordinates $\left(s_{1}, s_{2}, 0\right)$. If $\boldsymbol{g}=\left(g_{1}, g_{2}, g_{3}\right), \boldsymbol{h}=\left(h_{1}, h_{2}, h_{3}\right)$ and $\boldsymbol{n}=\left(n_{1}, n_{2}, n_{3}\right)$ with respect to the canonical basis $\left(\boldsymbol{e}_{1}, e_{2}, \boldsymbol{e}_{3}\right)$ of $\mathbb{Q}_{p}^{3}$, then the canonical coordinates of $\boldsymbol{s}_{\mid V}$ are given by

$$
\left(\begin{array}{l}
w_{1} \\
w_{2} \\
w_{3}
\end{array}\right)=\left(\begin{array}{lll}
g_{1} & h_{1} & n_{1} \\
g_{2} & h_{2} & n_{2} \\
g_{3} & h_{3} & n_{3}
\end{array}\right)\left(\begin{array}{c}
s_{1} \\
s_{2} \\
0
\end{array}\right)
$$

or in short

$$
\boldsymbol{w}=M \boldsymbol{s}_{\mid V}
$$

The quadratic form preserved by $S O(3)_{p}$ is $Q_{+}(\boldsymbol{x})=\sum_{i=1}^{3} a_{i} x_{i}^{2}$ in the canonical basis where, $\left(a_{i}\right)=(1,-v, p)$ for $p \neq 2$ and $\left(a_{i}\right)=(1,1,1)$ for $p=2$. Its restriction to $V$ is

$$
\begin{aligned}
Q_{+\mid V}(\boldsymbol{w}) & =\sum_{i=1}^{3} a_{i} w_{i}^{2}=\sum_{i=1}^{3} a_{i}\left(g_{i} s_{1}+h_{i} s_{2}\right)^{2} \\
& =\left(\sum_{i=1}^{3} a_{i} g_{i}^{2}\right) s_{1}^{2}+2\left(\sum_{i=1}^{3} a_{i} g_{i} h_{i}\right) s_{1} s_{2}+\left(\sum_{i=1}^{3} a_{i} h_{i}^{2}\right) s_{2}^{2} \\
& =Q_{+}(\boldsymbol{g}) s_{1}^{2}+2 b(\boldsymbol{g}, \boldsymbol{h}) s_{1} s_{2}+Q_{+}(\boldsymbol{h}) s_{2}^{2} .
\end{aligned}
$$

If $\boldsymbol{g} \perp \boldsymbol{h}$, then

$$
Q_{+\mid V}(\boldsymbol{w})=g s_{1}^{2}+h s_{2}^{2}
$$

where $g=Q_{+}(\boldsymbol{q}), h=Q_{+}(\boldsymbol{h})$.
We consider $\boldsymbol{t} \in \mathbb{Q}_{p}^{3}$ with coordinates $\left(t_{1}, t_{2}, t_{3}\right)$ with respect to $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ such that

$$
\binom{t_{1}}{t_{2}}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{s_{1}}{s_{2}}
$$

and we want to find this matrix rotating $\boldsymbol{s}_{\mid V}$ to $\boldsymbol{t}_{\mid V}$. If $\boldsymbol{u}:=M \boldsymbol{t}_{\mid V}$, we must have

$$
Q_{+\mid V}(\boldsymbol{u})=Q_{+\mid V}(\boldsymbol{w}) .
$$

This implies

$$
\left\{\begin{array}{l}
g a^{2}+h c^{2}=g,  \tag{17}\\
g b^{2}+h d^{2}=h, \\
g a b+h c d=0
\end{array}\right.
$$

which generalizes Eq. (5) by setting

$$
\alpha:=\frac{h}{g} \in \mathbb{Q}_{p}^{*} .
$$

Now we proceed as in the previous section, and introduce

$$
\sigma= \pm \frac{c}{1+a} \in \mathbb{Q}_{p},
$$

leaving out the case $a \neq-1$ for the moment. This leads to

$$
\left(1+\alpha \sigma^{2}\right) a=1-\alpha \sigma^{2},
$$

and in order to extract $a$ we need the following results.
Proposition 17. For any orthogonal basis $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ of $\mathbb{Q}_{p}^{3}$,

$$
Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) Q_{+}(\boldsymbol{n})= \begin{cases}-v p t^{2} & \text { for } p \text { odd } \\ t^{2} & \text { for } p=2\end{cases}
$$

for some $t \in \mathbb{Q}_{p}^{*}$.
Proof. We recall that the determinant is an invariant of quadratic forms in $K=\mathbb{Q}_{p}^{*} /\left(\mathbb{Q}_{p}^{*}\right)^{2}$ under change of basis. Given bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{Q}_{p}^{3}$, there exists a unique linear transformation $L: \mathbb{Q}_{p}^{3} \rightarrow \mathbb{Q}_{p}^{3}$ mapping $\mathcal{B}$ to $\mathcal{C}$. If a quadratic form over $\mathbb{Q}_{p}$ has matrix representation $M$ with respect to $\mathcal{B}$, it has a matrix representation $M^{\prime}=L M L^{\top}$ with respect to $\mathcal{C}$. Now, $\operatorname{det} L M L^{\top}=(\operatorname{det} L)^{2}(\operatorname{det} M)$ implies $\operatorname{det} M \simeq \operatorname{det} M^{\prime}$ in $K$. Our form $Q_{+}$is represented in the canonical basis $\mathcal{B}=\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}\right)$ of $\mathbb{Q}_{p}^{3}$ by $A=\operatorname{diag}(1,-v, p)$ for $p \neq 2$ and by $A=I$ for $p=2$. We have $\operatorname{det} A=Q_{+}\left(\boldsymbol{e}_{1}\right) Q_{+}\left(\boldsymbol{e}_{2}\right) Q_{+}\left(\boldsymbol{e}_{3}\right)$, which is equal to $-v p$ when $p \neq 2$ and to 1 when $p=2$. On the other hand, $Q_{+}$has diagonal matrix form with respect to any orthogonal basis $\mathcal{C}=(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ of $\mathbb{Q}_{p}^{3}$, where its determinant is $Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) Q_{+}(\boldsymbol{n})$. Hence, $Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) Q_{+}(\boldsymbol{n}) \simeq-v p$ in $K$ for $p \neq 2$, and $Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) Q_{+}(\boldsymbol{n}) \simeq$ 1 for $p=2$.

Corollary 18. $-\alpha \in \mathbb{Q}_{p}^{*}$ is never a square.
Proof. From Proposition 17, we have

$$
Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) \simeq \begin{cases}-v p Q_{+}(\boldsymbol{n}) & \text { if } p \text { odd } \\ Q_{+}(\boldsymbol{n}) & \text { if } p=2\end{cases}
$$

which gives

$$
-\alpha=-\frac{Q_{+}(\boldsymbol{h})}{Q_{+}(\boldsymbol{g})} \simeq-Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) \simeq\left\{\begin{array}{c}
v p Q_{+}(\boldsymbol{n}), p \neq 2, \\
-Q_{+}(\boldsymbol{n}), p=2
\end{array}\right.
$$

Accordingly, $-\alpha$ is a square if and only if $Q_{+}^{(4)}(\boldsymbol{n}, s)=0$ for some $s \in \mathbb{Q}_{p}^{*}$. However $Q_{+}^{(4)}$ does not represent 0 .

This implies $1+\alpha \sigma^{2} \neq 0$ for all $\alpha \in \mathbb{Q}_{p}^{*}, \sigma \in \mathbb{Q}_{p}$, and we get

$$
a=\frac{1-\alpha \sigma^{2}}{1+\alpha \sigma^{2}}
$$

and, therefore,

$$
\mathcal{R}_{\boldsymbol{n}}(\sigma)_{\mid \boldsymbol{n}^{\perp}}=\left(\begin{array}{cc}
a(\sigma) & b(\sigma)  \tag{18}\\
c(\sigma) & a(\sigma)
\end{array}\right)=\left(\begin{array}{cc}
\frac{1-\alpha \sigma^{2}}{1+\alpha \sigma^{2}} & -\frac{2 \alpha \sigma}{1+\alpha \sigma^{2}} \\
\frac{2 \sigma}{1+\alpha \sigma^{2}} & \frac{1-\alpha \sigma^{2}}{1+\alpha \sigma^{2}}
\end{array}\right)
$$

We need to consider $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}$ as in Remark 11 in order to include the solution to Eq. (17) with $a=-1$, which is

$$
\mathcal{R}_{\boldsymbol{n}}(\infty)_{\mid \boldsymbol{n}^{\perp}}=-I
$$

We extend the result of Eq. (18) to the whole of $\mathbb{Q}_{p}^{3}$ :

$$
\left(\begin{array}{l}
t_{1} \\
t_{2} \\
t_{3}
\end{array}\right)=\left(\begin{array}{ccc}
a(\sigma) & b(\sigma) & 0 \\
c(\sigma) & a(\sigma) & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
s_{1} \\
s_{2} \\
s_{3}
\end{array}\right)
$$

choosing $t_{3}=s_{3}$, or in short

$$
\begin{equation*}
\boldsymbol{t}=\mathcal{R}_{\boldsymbol{n}}(\sigma) s \tag{19}
\end{equation*}
$$

This concludes the proof of the following theorem.
Theorem 19. A rotation of $S O(3)_{p}$ around $\boldsymbol{n}$ takes the following matrix form with respect to an orthogonal basis $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ of $\mathbb{Q}_{p}^{3}$ :

$$
\mathcal{R}_{\boldsymbol{n}}(\sigma)=\left(\begin{array}{ccc}
\frac{1-\alpha \sigma^{2}}{1+\alpha \sigma^{2}} & -\frac{2 \alpha \sigma}{1+\alpha \sigma^{2}} & 0  \tag{20}\\
\frac{2 \sigma}{1+\alpha \sigma^{2}} & \frac{1-\alpha \sigma^{2}}{1+\alpha \sigma^{2}} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

with $\sigma \in \mathbb{Q}_{p} \cup\{\infty\}$ and $\alpha=Q_{+}(\boldsymbol{h}) / Q_{+}(\boldsymbol{g})$, for every prime $p$.
Eq. (19) can be written with respect to the canonical basis $\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}\right)$ as

$$
\boldsymbol{u}=M \mathcal{R}_{\boldsymbol{n}}(\sigma) M^{-1} \boldsymbol{w}
$$

so that $M \mathcal{R}_{\boldsymbol{n}}(\sigma) M^{-1}$ is the rotation matrix on $\mathbb{Q}_{p}^{3}$ around $\boldsymbol{n}$ with respect to $\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}\right)$.
Remark 20. Rotations around the reference axes of $\mathbb{Q}_{p}^{3}$ are particular cases of Eq. (20), in which we choose the canonical basis as $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ in some order. When $p \neq 2$, if $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ equals

- $\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}\right)$, then $g=1, h=-v \Rightarrow \alpha=-v$, which determine a rotation around the third reference axis, $\mathcal{R}_{z}(\sigma)$, characterized by

$$
\left(\begin{array}{cc}
\frac{1+v \sigma^{2}}{1-v \sigma^{2}} & \frac{2 v \sigma}{1-v \sigma^{2}} \\
\frac{2 \sigma}{1-v \sigma^{2}} & \frac{1+v \sigma^{2}}{1-v \sigma^{2}}
\end{array}\right)
$$

- $\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{3}, \boldsymbol{e}_{2}\right)$, then $g=1, h=p \Rightarrow \alpha=p$, which determine a rotation around the second reference axis of the canonical basis, $\mathcal{R}_{y}(\sigma)$, characterized by

$$
\left(\begin{array}{cc}
\frac{1-p \sigma^{2}}{1+p \sigma^{2}} & -\frac{2 p \sigma}{1+p \sigma^{2}} \\
\frac{2 \sigma}{1+p \sigma^{2}} & \frac{1-p \sigma^{2}}{1+p \sigma^{2}}
\end{array}\right)
$$

- $\left(\boldsymbol{e}_{2}, \boldsymbol{e}_{3}, \boldsymbol{e}_{1}\right)$, then $g=-v, h=p \Rightarrow \alpha=-\frac{p}{v}$, which determine a rotation around the first reference axis, $\mathcal{R}_{x}(\sigma)$, characterized by

$$
\left(\begin{array}{cc}
\frac{1+\frac{p}{v} \sigma^{2}}{1-\frac{p}{v} \sigma^{2}} & \frac{2 \frac{p}{v} \sigma}{1-\frac{p}{v} \sigma^{2}} \\
\frac{2 \sigma}{1-\frac{p}{v} \sigma^{2}} & \frac{1+\frac{p}{v} \sigma^{2}}{1-\frac{p}{v} \sigma^{2}}
\end{array}\right)
$$

These matrices are associated respectively to the conservation of the rank-2 quadratic forms $Q_{-v}, Q_{p}$ and $-v y^{2}+p z^{2}$. The latter is $Q_{u p}$ if $p \equiv 1 \bmod 4$, but it is again $Q_{p}$ for $p \equiv 3 \bmod 4$, in which case $\mathcal{R}_{x}$ and $\mathcal{R}_{y}$ are characterized by the same $2 \times 2$ submatrix, only differently positioned inside the $3 \times 3$ matrix on the canonical basis.

When $p=2$ we have $Q_{+}\left(\boldsymbol{e}_{i}\right)=1, i=1,2,3$, which gives $\alpha=1$ whenever $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ equals any permutation of the canonical basis. Then, a 2-adic rotation around any reference axis of $\mathbb{Q}_{2}^{3}$ is characterized by

$$
\left(\begin{array}{cc}
\frac{1-\sigma^{2}}{1+\sigma^{2}} & -\frac{2 \sigma}{1+\sigma^{2}} \\
\frac{2 \sigma}{1+\sigma^{2}} & \frac{1-\sigma^{2}}{1+\sigma^{2}}
\end{array}\right)
$$

associated to the conservation of the rank-2 quadratic form $Q_{1}$.
Eq. (12) translates in three dimensions into

$$
\begin{equation*}
\mathcal{R}_{\boldsymbol{n}}\left(-\frac{1}{\alpha \sigma}\right)=\mathcal{R}_{\boldsymbol{n}}(\infty) \mathcal{R}_{\boldsymbol{n}}(\sigma) \tag{21}
\end{equation*}
$$

The important question we have to face now is, which of the three (seven) equivalence classes of definite quadratic forms in dimension two for odd $p(p=2)$, according to Proposition 10, arise from restricting $Q_{+}$to two-dimensional subspaces $V=\boldsymbol{n}^{\perp}$ of $\mathbb{Q}_{p}^{3}$. It will turn out that all of them are realized. Since the transformations $\mathcal{R}_{\boldsymbol{n}} \in S O(3)_{p}$ are rotations around an axis $\boldsymbol{n} \in \mathbb{Q}_{p}^{3}$ preserving the rank-3 quadratic form $Q_{+}$, this is equivalent to asking if the restrictions $\left(\mathcal{R}_{\boldsymbol{n}}\right)_{\left.\right|_{\boldsymbol{n}^{\perp}}}$, by varying the classes of $\boldsymbol{n}$, cover all the classes of special orthogonal symmetries of the $p$-adic plane preserving the definite rank-2 forms (recall that Theorem 6 identifies all elements of $S O(3)_{p}$ as such rotations around axes).

Consider a basis change between orthogonal bases of $\mathbb{Q}_{p}^{3}$ of the kind $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n}) \mapsto(\lambda \boldsymbol{g}, \mu \boldsymbol{h}, \nu \boldsymbol{n})$, with $\lambda, \mu, \nu \in \mathbb{Q}_{p}^{*}$. The parameter $\alpha$ in the parameterization (20) is affected by this transformation as

$$
\alpha=\frac{Q_{+}(\boldsymbol{h})}{Q_{+}(\boldsymbol{g})} \mapsto \frac{Q_{+}(\mu \boldsymbol{h})}{Q_{+}(\lambda \boldsymbol{g})}=\left(\frac{\mu}{\lambda}\right)^{2} \frac{Q_{+}(\boldsymbol{h})}{Q_{+}(\boldsymbol{g})}
$$

Hence, $\alpha$ is well-defined in $K=\mathbb{Q}_{p}^{*} /\left(\mathbb{Q}_{p}^{*}\right)^{2}$; there are at most 4 classes of rotations in $S O(3)_{p}$ for $p \neq 2$ and 8 classes for $p=2$.

In Corollary 18 we found a bijective relation modulo squares between the value of the form $Q_{+}$ on a rotation axis $\boldsymbol{n}$ and $\alpha$

$$
\alpha \simeq \begin{cases}-v p Q_{+}(\boldsymbol{n}) & \text { for } p \text { odd }  \tag{22}\\ Q_{+}(\boldsymbol{n}) & \text { for } p=2\end{cases}
$$

This allows us to determine the class of a rotation realized on a plane via the class $Q_{+}(\boldsymbol{n})$ of its orthogonal axis.

As a consequence of the fact that $Q_{+}^{(4)}$ does not represent 0 , Corollary 18 showed that $\alpha \nsim-1$. Hence there are at most 3 (or 7 ) classes of rotations for odd $p$ (for $p=2$, respectively), depending on the equivalence class modulo squares of $\alpha$ :

$$
\begin{aligned}
& \alpha \in\{-v, p, u p\}, p>2 \\
& \alpha \in\{1, \pm 2, \pm 5, \pm 10\}, p=2
\end{aligned}
$$

Focusing on the language of quadratic forms, there are left at most 3 (or 7 ) equivalence classes up to scaling of restrictions of $Q_{+}$to a plane, identified by the equivalence class modulo squares of $\alpha$.

We want to see if there exist classes of $\boldsymbol{n}$ in $\mathbb{Q}_{p}^{3}$ realizing all these classes of $\alpha$, uniquely associated to the definite rank-2 forms through their determinant. We make use of Eq. (22).

When $p$ is odd, $Q_{+}(\boldsymbol{n}) \simeq-v p \alpha \Leftrightarrow n_{1}^{2}-v n_{2}^{2}+p n_{3}^{2}+v p \alpha s^{2}=0$ for some $s \in \mathbb{Q}_{p}^{*}, \boldsymbol{n}$ of canonical coordinates $\left(n_{1}, n_{2}, n_{3}\right)$. The quadratic form $n_{1}^{2}-v n_{2}^{2}+p n_{3}^{2}+v p \alpha s^{2}$ represents 0 in a nontrivial way for every $\alpha \in\{-v, p, u p\}$, for its determinant is $d \simeq-\alpha \nsimeq 1$ (Theorem 2). If $n_{1}=n_{2}=n_{3}=0$, then $v p \alpha s^{2}=0, s \in \mathbb{Q}_{p}^{*}$ is impossible. It means that $Q_{+}(\boldsymbol{n}) \simeq-v p \alpha$ admits nontrivial solutions $\boldsymbol{n}$ for every $\alpha$.

Similarly, when $p=2, Q_{+}(\boldsymbol{n}) \simeq \alpha \Leftrightarrow n_{1}^{2}+n_{2}^{2}+n_{3}^{2}-\alpha s^{2}=0$ for some $s \in \mathbb{Q}_{p}^{*}$. The form $x^{2}+$ $y^{2}+z^{2}-\alpha s^{2}$ represents 0 for every $\alpha \in\{1, \pm 2, \pm 5, \pm 10\}$, because its determinant is $d \simeq-\alpha \nsucceq 1$. In summary, we have proved the following.

Proposition 21. For every prime $p$ and every two-dimensional definite quadratic form $Q_{\kappa}$, there exists a rotation axis determined by $\boldsymbol{n} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}$ such that $Q_{+\mid \boldsymbol{n}^{\perp}} \sim Q_{\kappa}$.

In fact, when $p$ is odd,

- $Q_{+}\left(e_{3}\right)=p \Leftrightarrow \alpha \simeq-v$, as it can be seen from Remark 20 too. This is associated to the conservation of $Q_{-v}$;
- $Q_{+}\left(e_{2}\right)=-v \Leftrightarrow \alpha \simeq p$, associated to $Q_{p}$;
- $Q_{+}\left(e_{1}\right)=1 \Leftrightarrow \alpha \simeq-\frac{p}{v}$. This is associated to $Q_{u p}$ for $p \equiv 1 \bmod 4$ and $Q_{p}$ for $p \equiv 3 \bmod 4$.

On the other hand, when $p=2$, then $Q_{+}\left(\boldsymbol{e}_{i}\right)=1$ for all $i=1,2,3$, thus $\alpha=1$, associated to the conservation of $Q_{1}$.

This means that when $p \equiv 1 \bmod 4$ the canonical basis is enough to realize all the classes of rank-2 forms; however it leaves out the class associated to $\alpha \simeq u p$ for $p \equiv 3 \bmod 4$, and it realizes only one class $(\alpha \simeq 1)$ out of the seven ones for $p=2$. These remaining classes are covered by non-reference axes. Indeed, $Q_{+}$maintains some symmetry on $x, y$ for $p \equiv 3 \bmod 4$, contrary to $p \equiv 1 \bmod 4$, and the symmetry is total for $p=2$.

The axes needed to realize all the classes of definite rank-2 forms when $p \equiv 3 \bmod 4$ do not form an orthogonal basis. Moreover, in the case of $p=2$, we need seven axes, so we cannot have a basis of $\mathbb{Q}_{2}^{3}$ to realize all classes of rank- 2 forms.

Furthermore, the map $\boldsymbol{n} \mapsto Q_{+\mid \boldsymbol{n}^{\perp}}$ from vectors $\boldsymbol{n}$ in $\mathbb{Q}_{p}^{3} \backslash \mathbf{0}$ to definite quadratic forms in dimension two induces bijective correspondence between the equivalence classes of $Q_{+}(\boldsymbol{n})$ in $K$ and the rank-two definite forms $Q_{\kappa}$ up to equivalence. In fact, if $\boldsymbol{n}, \boldsymbol{n}^{\prime} \in \mathbb{Q}_{p}^{3} \backslash \mathbf{0}$ are such that $Q_{+}(\boldsymbol{n}) \nsucceq Q_{+}\left(\boldsymbol{n}^{\prime}\right)$, then $Q_{+\mid \boldsymbol{n}^{\perp}} \nsim Q_{+\mid \boldsymbol{n}^{\prime}}$. This is because, as in the proof of Proposition 17, the determinant is an invariant of quadratic forms under basis changes, and $\operatorname{det} Q_{+} \simeq$ $Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) Q_{+}(\boldsymbol{n}) \simeq Q_{+}\left(\boldsymbol{g}^{\prime}\right) Q_{+}\left(\boldsymbol{h}^{\prime}\right) Q_{+}\left(\boldsymbol{n}^{\prime}\right)$, respectively on the orthogonal bases $(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{n})$ and $\left(\boldsymbol{g}^{\prime}, \boldsymbol{h}^{\prime}, \boldsymbol{n}^{\prime}\right)$. As a consequence, if $Q_{+}(\boldsymbol{n}) \not 千 Q_{+}\left(\boldsymbol{n}^{\prime}\right)$, then $Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) \nsim Q_{+}\left(\boldsymbol{g}^{\prime}\right) Q_{+}\left(\boldsymbol{h}^{\prime}\right)$, where $Q_{+}(\boldsymbol{g}) Q_{+}(\boldsymbol{h}) \simeq \operatorname{det} Q_{+\mid \boldsymbol{n}^{\perp}}, Q_{+}\left(\boldsymbol{g}^{\prime}\right) Q_{+}\left(\boldsymbol{h}^{\prime}\right) \simeq \operatorname{det} Q_{+\mid \boldsymbol{n}^{\prime} \perp}$. Two quadratic forms $Q_{+\mid \boldsymbol{n}^{\perp}}$ and $Q_{+\mid \boldsymbol{n}^{\prime}}$ with different determinants modulo squares cannot be equivalent (Theorem 1).

It is possible to choose representatives in the three (resp. seven) classes of definite rank-2 forms for odd $p$ (reps. $p=2$ ) in such a way that if $\boldsymbol{n}, \boldsymbol{n}^{\prime} \in \mathbb{Q}_{p}^{3}$ satisfy $Q_{+}(\boldsymbol{n}) \simeq Q_{+}\left(\boldsymbol{n}^{\prime}\right)$, then $Q_{+\mid \boldsymbol{n}^{\perp}} \simeq Q_{+\mid \boldsymbol{n}^{\prime}}$ by a linear transformation (scaling is not involved). This can be shown in an abstract way, without resorting to the explicit classification of $p$-adic quadratic forms. The statement is trivial if $\boldsymbol{n}=\lambda \boldsymbol{n}^{\prime}$ for some $\lambda \in \mathbb{Q}_{p}^{*}$. Otherwise, consider a vector $\boldsymbol{v} \in \mathbb{Q}_{p}^{3}$ orthogonal to $\boldsymbol{n}, \boldsymbol{n}^{\prime}$. By Proposition 8, there exists some $\mathcal{R}_{\boldsymbol{v}}(\sigma) \in S O(3)_{p}$ such that $\mathcal{R}_{\boldsymbol{v}}(\sigma) \boldsymbol{n}=\lambda \boldsymbol{n}^{\prime}$ with $\lambda \in \mathbb{Q}_{p}^{*}$. Then, $\mathcal{R}_{\boldsymbol{v}}(\sigma)$ transforms the plane $\boldsymbol{n}^{\perp}$ to the plane $\boldsymbol{n}^{\prime \perp}$. This is a linear map, which necessarily implements the equivalence $Q_{+\mid n^{\perp}} \simeq Q_{+\mid n^{\prime} \perp}$

## 6. p-ADIC CARDANO AND EULER DECOMPOSITIONS

The previous development of $S O(3)_{p}$ showed a sufficiently close analogy to the real Euclidean $S O(3)_{\mathbb{R}}$, in that the three-dimensional space has an essentially unique definite quadratic form, and all special orthogonal transformations are actually rotations. The rotation groups around fixed axes themselves turned out to be rather more complex: there are three (for odd $p$ ) and seven (for $p=2$ ) types of rotation axes with associated different rotation groups.

Now we ask if it is possible to express any element of $S O(3)_{p}$ as a composition of rotations around the reference axes of $\mathbb{Q}_{p}^{3}$, as it happens in $\mathbb{R}^{3}$ according to Theorem 4 . In the following we will revert to calling the axes $x, y$ and $z$, to make them typographically more distinguishable.

Theorem 22 (p-adic Cardano decomposition). For every odd prime $p$, any $M \in S O(3)_{p}$ can be decomposed into

$$
M=\mathcal{R}_{z}(\zeta) \mathcal{R}_{y}(\eta) \mathcal{R}_{x}(\xi)
$$

for some parameters $\xi, \eta, \zeta \in \mathbb{Q}_{p} \cup\{\infty\}$.
Proof. Let $M \boldsymbol{e}_{1}$ be a vector with canonical coordinates $\left(m_{1}, m_{2}, m_{3}\right)$, such that $Q_{+}\left(M \boldsymbol{e}_{1}\right)=$ $Q_{+}\left(\boldsymbol{e}_{1}\right)=1$ since $M \in S O(3)_{p}$ preserves $Q_{+}$.

We need to find a composition of rotations around $\boldsymbol{e}_{2}$ and $\boldsymbol{e}_{3}$ transforming $M \boldsymbol{e}_{1}$ back to $\boldsymbol{e}_{1}$.
First, we show that there exists a $z$-rotation $\mathcal{R}_{z}(\zeta) \in S O(3)_{p}$ such that

$$
\begin{equation*}
\mathcal{R}_{z}(\zeta)^{-1} M \boldsymbol{e}_{1} \in \operatorname{span}\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{3}\right) \tag{23}
\end{equation*}
$$

This vector would be $\mathcal{R}_{z}(\zeta)^{-1} M \boldsymbol{e}_{1}=m_{1}^{\prime} \boldsymbol{e}_{1}+m_{3} \boldsymbol{e}_{3}$ for some $m_{1}^{\prime} \in \mathbb{Q}_{p}$, where the third component of $M e_{1}$ is left unchanged by a rotation around $\boldsymbol{e}_{3}$. As noted in Proposition 8, necessary and sufficient condition for the existence of such a $\mathcal{R}_{z}(\zeta)^{-1}$ is

$$
Q_{+}\left(m_{1}^{\prime} \boldsymbol{e}_{1}+m_{3} \boldsymbol{e}_{3}\right)=Q_{+}\left(M \boldsymbol{e}_{1}\right) \Leftrightarrow m_{1}^{\prime 2}+p m_{3}^{2}=1
$$

$Q_{+}\left(M \boldsymbol{e}_{1}\right)=m_{1}^{2}-v m_{2}^{2}+p m_{3}^{2}=1$ implies that $m_{1}, m_{2}, m_{3} \in \mathbb{Z}_{p}$ (cf. the compactness proof for $S O(3)_{p}$ in Theorem 5). Therefore, we resort to Hensel's Lemma to show that $f\left(m_{1}^{\prime}\right)=m_{1}^{\prime 2}-$ $1+p m_{3}^{2}$ admits roots $m_{1}^{\prime} \in \mathbb{Z}_{p} . f\left(m_{1}^{\prime}\right) \equiv m_{1}^{\prime 2}-1 \bmod p$ has zeros $m_{1}^{\prime} \equiv \pm 1 \bmod p$, in which the derivative $f^{\prime}\left(m_{1}^{\prime}\right)=2 m_{1}^{\prime} \not \equiv 0 \bmod p$. Then, Hensel's Lemma allows us to (uniquely) lift each of these solutions to a solution of the same equation $\bmod p^{k}$, converging to a $p$-adic solution $m_{1}^{\prime}$. It means that $\mathcal{R}_{z}(\zeta)^{-1}$ as in Eq. (23) exists.

Next, there exists a $y$-rotation $\mathcal{R}_{y}(\eta)^{-1} \in S O(3)_{p}$ such that

$$
\begin{equation*}
\mathcal{R}_{y}(\eta)^{-1} \mathcal{R}_{z}(\zeta)^{-1} M \boldsymbol{e}_{1}=\boldsymbol{e}_{1} \tag{24}
\end{equation*}
$$

because $Q_{+}\left(\mathcal{R}_{z}(\zeta)^{-1} M \boldsymbol{e}_{1}\right)=Q_{+}\left(\boldsymbol{e}_{1}\right)$.
Eq. (24) means that $\mathcal{R}_{y}(\eta)^{-1} \mathcal{R}_{z}(\zeta)^{-1} M \in S O(3)_{p}$ has eigenvector $\boldsymbol{e}_{1}$ with corresponding eigenvalue 1 , i.e., $\mathcal{R}_{y}(\eta)^{-1} \mathcal{R}_{z}(\zeta)^{-1} M:=\mathcal{R}_{x}(\xi)$ is a rotation around the $x$-axis.

Corollary 23. For every odd prime p, any $M \in S O(3)_{p}$ can be decomposed into

$$
\mathcal{R}_{z} \mathcal{R}_{x} \mathcal{R}_{y}, \quad \mathcal{R}_{x} \mathcal{R}_{y} \mathcal{R}_{z}, \quad \mathcal{R}_{y} \mathcal{R}_{x} \mathcal{R}_{z}
$$

respectively by certain parameters $\sigma, \tau, \omega \in \mathbb{Q}_{p} \cup\{\infty\}$.
Proof. To prove the existence of a decomposition of the kind $\mathcal{R}_{z} \mathcal{R}_{x} \mathcal{R}_{y}$, it is enough to repeat the steps of the proof of Theorem 22. In particular, given $M \boldsymbol{e}_{2}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$, there exists $\mathcal{R}_{z}(\sigma) \in S O(3)_{p}$ such that

$$
\mathcal{R}_{z}(\sigma)^{-1} M \boldsymbol{e}_{2} \in \operatorname{span}\left(\boldsymbol{e}_{2}, \boldsymbol{e}_{3}\right)
$$

In fact, there exists $m_{2}^{\prime} \in \mathbb{Z}_{p}$ such that $\mathcal{R}_{z}(\sigma)^{-1} M \boldsymbol{e}_{2}=m_{2}^{\prime} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$ and

$$
Q_{+}\left(m_{2}^{\prime} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}\right)=-v m_{2}^{\prime 2}+p m_{3}^{2}=-v=Q_{+}\left(\boldsymbol{e}_{2}\right)
$$

To show this, we apply Hensel's Lemma to $f\left(m_{2}^{\prime}\right)=v m_{2}^{2}-p m_{3}^{2}-v: f\left(m_{2}^{\prime}\right) \equiv 0 \bmod p \Leftrightarrow v m_{2}^{\prime 2} \equiv$ $v \Leftrightarrow m_{2}^{\prime} \equiv \pm 1 \bmod p$, and $f^{\prime}\left(m_{2}^{\prime}\right)=2 m_{2}^{\prime} \not \equiv 0 \bmod p$ in these solutions, which then are lifted to $p$ adic solutions.

The existence of the decomposition with respect to the $x, y$ and $z$ axes follows straightforwardly from Theorem 22: if any $M^{-1} \in S O(3)_{p}$ can be written as $M^{-1}=\mathcal{R}_{z}(\zeta) \mathcal{R}_{y}(\eta) \mathcal{R}_{x}(\xi)$ for certain parameters $\xi, \eta, \zeta$, then any $M \in S O(3)_{p}$ can be decomposed into $M=\mathcal{R}_{x}(\xi)^{-1} \mathcal{R}_{y}(\eta)^{-1} \mathcal{R}_{z}(\zeta)^{-1}:=$ $\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right) \mathcal{R}_{z}\left(\zeta^{\prime}\right)$, where $\xi^{\prime}=-\xi$ if $\xi \in \mathbb{Q}_{p}$ or $\xi=\xi^{\prime}$ are infinite, and similarly for $\eta^{\prime}, \zeta^{\prime}$.

One similarly proves the existence of the nautical decomposition $\mathcal{R}_{y} \mathcal{R}_{x} \mathcal{R}_{z}$ for $S O(3)_{p}$ from the one as $\mathcal{R}_{z} \mathcal{R}_{x} \mathcal{R}_{y}$.

Theorem 24 ( $p$-adic Cardano decomposition, $p \equiv 1 \bmod 4$ ). For every odd prime $p \equiv 1 \bmod 4$, any $M \in S O(3)_{p}$ can be decomposed into

$$
M=\mathcal{R}_{x}(\zeta) \mathcal{R}_{z}(\eta) \mathcal{R}_{y}(\xi)
$$

for some parameters $\xi, \eta, \zeta \in \mathbb{Q}_{p} \cup\{\infty\}$.
By applying this to $M^{-1}$, we get a decomposition $M=\mathcal{R}_{y}(\zeta) \mathcal{R}_{z}(\eta) \mathcal{R}_{x}(\xi)$.
Proof. The strategy is similar to the previous Theorem 22. We look at $M \boldsymbol{e}_{2}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$, which has $m_{1}^{2}-v m_{2}^{2}+p m_{3}^{2}=Q_{+}\left(M \boldsymbol{e}_{2}\right)=Q_{+}\left(\boldsymbol{e}_{2}\right)=-v$. We first look for a $\zeta$ with $\mathcal{R}_{x}(\zeta)^{-1} M \boldsymbol{e}_{2}=$ $m_{1} \boldsymbol{e}_{1}+m_{2}^{\prime} \boldsymbol{e}_{2}$, which is guaranteed if $-v m_{2}^{\prime 2}=-v m_{2}^{2}+p m_{3}^{2}$, because in this way $Q_{+}\left(m_{1} \boldsymbol{e}_{1}+\right.$ $\left.m_{2}^{\prime} \boldsymbol{e}_{2}\right)=Q_{+}\left(M \boldsymbol{e}_{2}\right)=Q_{+}\left(\boldsymbol{e}_{2}\right)$, and we can invoke Proposition 8 . We prove that we can find $m_{2}^{\prime}$ with $m_{2}^{\prime 2}=m_{2}^{2}-\frac{p}{v} m_{3}^{2}$. Indeed, note first that $m_{2} \not \equiv 0 \bmod p$, since otherwise $-v=Q_{+}\left(M \boldsymbol{e}_{2}\right) \equiv m_{1}^{2}$ $\bmod p$, which is impossible since $-v$ is a non-square in $\mathbb{Z}_{p}$ and hence modulo $p$ (it is here that we use $p \equiv 1 \bmod 4$, since then -1 is a square in $\mathbb{Q}_{p}$, thus $v$ and $-v$ are both non-squares). Thus, we have solutions $m_{2}^{\prime} \equiv \pm m_{2} \bmod p$, which are both nonzero. Since the derivative $2 m_{2}^{\prime}$ is then nonzero, as well, we can invoke Hensel's Lemma to obtain a solution of the equation in $\mathbb{Z}_{p}$.

Now we proceed as before: since now $\mathcal{R}_{x}(\zeta)^{-1} M \boldsymbol{e}_{2} \perp \boldsymbol{e}_{3}$, we can find $\eta$ such that $\mathcal{R}_{z}(\eta)^{-1} \mathcal{R}_{x}(\zeta)^{-1} M \boldsymbol{e}_{2}=\boldsymbol{e}_{2}$ (once more invoking Proposition 8). But this means that $\mathcal{R}_{z}(\eta)^{-1} \mathcal{R}_{x}(\zeta)^{-1} M$ is a rotation around the $y$-axis, i.e., $\mathcal{R}_{z}(\eta)^{-1} \mathcal{R}_{x}(\zeta)^{-1} M=\mathcal{R}_{y}(\xi)$ for some $\xi$, and we are done.

Remark 25. Suppose a general Cardano or Euler type decomposition

$$
S O(3)_{p} \ni M=\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma) \mathcal{R}_{\boldsymbol{n}_{2}}(\tau) \mathcal{R}_{\boldsymbol{n}_{3}}(\omega)
$$

where $\boldsymbol{n}_{1} \perp \boldsymbol{n}_{2}$ and $\boldsymbol{n}_{2} \perp \boldsymbol{n}_{3}$ (in the Cardano decomposition, $\boldsymbol{n}_{1} \perp \boldsymbol{n}_{3}$, while in the Euler one $\boldsymbol{n}_{1}=\boldsymbol{n}_{3}$ ). We have

$$
\begin{aligned}
M=\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma) \mathcal{R}_{\boldsymbol{n}_{2}}(\tau) \mathcal{R}_{\boldsymbol{n}_{3}}(\omega) & \Leftrightarrow \mathcal{R}_{\boldsymbol{n}_{2}}(\tau)^{-1} \mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M=\mathcal{R}_{\boldsymbol{n}_{3}}(\omega) \\
& \Leftrightarrow \mathcal{R}_{\boldsymbol{n}_{2}}(\tau)^{-1} \mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3}=\boldsymbol{n}_{3}
\end{aligned}
$$

Now, $\mathcal{R}_{\boldsymbol{n}_{2}}(\tau)^{-1}$ preserves the component of the vector $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3}$ along $\boldsymbol{n}_{2}$, which must be 0 to get $\boldsymbol{n}_{3}$ as a result, therefore $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3} \perp \boldsymbol{n}_{2}$.

This shows that $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3} \perp \boldsymbol{n}_{2}$ is a necessary condition for the existence of a decomposition $M=\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma) \mathcal{R}_{\boldsymbol{n}_{2}}(\tau) \mathcal{R}_{\boldsymbol{n}_{3}}(\omega)$. Since orthogonal transformations preserve $Q_{+}$, we also have $Q_{+}\left(\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3}\right)=Q_{+}\left(\boldsymbol{n}_{3}\right)$.

Conversely, these conditions are also sufficient. To be precise, assume that there exists a vector $\boldsymbol{v} \perp \boldsymbol{n}_{2}$ with $\boldsymbol{n}_{1}^{\top} \boldsymbol{v}=\boldsymbol{n}_{1}^{\top} M \boldsymbol{n}_{3}$ and $Q_{+}(\boldsymbol{v})=Q_{+}\left(M \boldsymbol{n}_{3}\right)=Q_{+}\left(\boldsymbol{n}_{3}\right)$. Then, by Proposition 8 there exists $\sigma$ such that $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3}=\boldsymbol{v} \perp \boldsymbol{n}_{2}$. Note that after the choice of $\sigma$, it follows that $\tau$ and $\omega$ are determined uniquely.

Remark 26. The Cardano decompositions for $S O(3)_{p}$ of the kind $\mathcal{R}_{x} \mathcal{R}_{z} \mathcal{R}_{y}$ and $\mathcal{R}_{y} \mathcal{R}_{z} \mathcal{R}_{x}$ do not exist in general for odd primes $p \equiv 3 \bmod 4$. In fact, one can construct counterexamples of matrices $M \in S O(3)_{p}$ for which there does not exist $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma) \in S O(3)_{p}$ such that $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3} \perp$ $\boldsymbol{n}_{2}$ according to Remark 25, by exploiting $Q_{+}\left(\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3}\right)=Q_{+}\left(\boldsymbol{n}_{3}\right)$.

As an example, let us show the existence of special orthogonal transformations that cannot be written in the Cardano product form $\mathcal{R}_{x} \mathcal{R}_{z} \mathcal{R}_{y}$ (the form $\mathcal{R}_{y} \mathcal{R}_{z} \mathcal{R}_{x}$ follows a closely similar reasoning). According to the recipe, we consider $M \boldsymbol{e}_{2}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$, which has $m_{1}^{2}+m_{2}^{2}+$ $p m_{3}^{2}=Q_{+}\left(M e_{2}\right)=Q_{+}\left(e_{2}\right)=1$. To get a corresponding Cardano decomposition, we would need $\sigma$ with $\mathcal{R}_{x}(\sigma)^{-1} M \boldsymbol{e}_{2}=m_{1} \boldsymbol{e}_{1}+m_{2}^{\prime} \boldsymbol{e}_{2}$ such that $m_{2}^{\prime 2}=m_{2}^{2}+p m_{3}^{2}$, but this is impossible if $m_{2}=0$, since $p$ is not a square in $\mathbb{Q}_{p}$. It remains to show that this case can occur, namely there exists an $M \in S O(3)_{p}$ with $M \boldsymbol{e}_{2}=m_{1} \boldsymbol{e}_{1}+m_{3} \boldsymbol{e}_{3}$. By Proposition 9, it is sufficient to find solutions of $m_{1}^{2}+p m_{3}^{2}=1$, which indeed has two roots for $m_{1}$, as we can see by first solving it modulo $p$, resulting in $m_{1} \equiv \pm 1 \bmod p$, and then using Hensel's Lemma.

Remark 27. For odd primes p, none of the six possible Euler decompositions for $S O(3)_{p}$ exist in general. We indicate how to construct counterexamples, following the same strategy as in the previous remark, based on Remark 25.
(i $\mathcal{E}$ ii) Nonexistence of $X Y X$ and YXY: Let us focus on the decomposition form $\mathcal{R}_{x} \mathcal{R}_{y} \mathcal{R}_{x}$ (the form $\mathcal{R}_{y} \mathcal{R}_{x} \mathcal{R}_{y}$ follows a closely similar reasoning). According to the recipe, we consider $M \boldsymbol{e}_{1}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$, which has $m_{1}^{2}-v m_{2}^{2}+p m_{3}^{2}=Q_{+}\left(M \boldsymbol{e}_{1}\right)=Q_{+}\left(\boldsymbol{e}_{1}\right)=1$. To get a corresponding Euler decomposition, we would need $\xi$ with $\mathcal{R}_{x}(\xi)^{-1} M \boldsymbol{e}_{1}=m_{1} \boldsymbol{e}_{1}+m_{3}^{\prime} \boldsymbol{e}_{3}$ such that $p m_{3}^{\prime 2}=-v m_{2}^{2}+p m_{3}^{2}$, but this is impossible if $m_{2} \not \equiv 0 \bmod p$. It remains to show that this case can occur, namely that there exists an $M \in S O(3)_{p}$ with $m_{2} \not \equiv 0 \bmod p$. By Hensel's Lemma, this boils down to finding solutions to $m_{1}^{2}-v m_{2}^{2} \equiv 1 \bmod p$ in integers modulo $p$, such that $m_{2} \not \equiv 0 \bmod p$, which always exist. (Indeed, for $p \equiv 3 \bmod 4$, when $-v=1$, we can choose $m_{2}=1$.)
(iii) Nonexistence of XZX: According to the recipe, we consider $M \boldsymbol{e}_{1}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$, which has $m_{1}^{2}-v m_{2}^{2}+p m_{3}^{2}=Q_{+}\left(M \boldsymbol{e}_{1}\right)=Q_{+}\left(\boldsymbol{e}_{1}\right)=1$. To get a corresponding Euler decomposition, we would need $\xi$ with $\mathcal{R}_{x}(\xi)^{-1} M \boldsymbol{e}_{1}=m_{1} \boldsymbol{e}_{1}+m_{2}^{\prime} \boldsymbol{e}_{2}$, such that $-v m_{2}^{\prime 2}=-v m_{2}^{2}+p m_{3}^{2}$, but this is impossible if $m_{2}=0$. This case can occur: there exists an $M \in S O(3)_{p}$ with $m_{2}=0$, because $m_{1}^{2}+p m_{3}^{2}=1$ always admits solutions according to Hensel's Lemma.
(iv) Nonexistence of YZY: Similarly, consider $M e_{2}=m_{1} e_{1}+m_{2} e_{2}+m_{3} e_{3}$, which has $m_{1}^{2}-$ $v m_{2}^{2}+p m_{3}^{2}=Q_{+}\left(M e_{2}\right)=Q_{+}\left(\boldsymbol{e}_{2}\right)=-v$. To get a corresponding Euler decomposition, we would need $\eta$ with $\mathcal{R}_{y}(\eta)^{-1} M \boldsymbol{e}_{2}=m_{1}^{\prime} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}$, such that $m_{1}^{\prime 2}=m_{1}^{2}+p m_{3}^{2}$, but this is impossible if $m_{1}=0$. However, there exists an $M \in S O(3)_{p}$ with $m_{1}=0$, because $-v m_{2}^{2}+p m_{3}^{2}=-v$ always admits solutions, as before using Hensel's Lemma.
( $v \mathcal{G}$ vi) Nonexistence of ZXZ and ZYZ: Let us focus on $\mathcal{R}_{z} \mathcal{R}_{y} \mathcal{R}_{z}$ (the form $\mathcal{R}_{z} \mathcal{R}_{x} \mathcal{R}_{z}$ follows a closely similar reasoning), and consider $M \boldsymbol{e}_{3}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$, which has $m_{1}^{2}-v m_{2}^{2}+p m_{3}^{2}=$ $Q_{+}\left(M e_{3}\right)=Q_{+}\left(e_{3}\right)=p$. To get a corresponding Euler decomposition, we would need $\zeta$ with $\mathcal{R}_{z}(\zeta)^{-1} M \boldsymbol{e}_{3}=m_{1}^{\prime} \boldsymbol{e}_{1}+m_{3} \boldsymbol{e}_{3}$, such that $m_{1}^{\prime 2}=m_{1}^{2}-v m_{2}^{2}$. This ends in a contradiction if the right hand side happens to be a non-square in $\mathbb{Q}_{p}$. Indeed, we will show that $m_{1}^{2}-v m_{2}^{2}=v p^{2}$ can occur. To see this, consider first the condition $m_{1}^{2}-v m_{2}^{2}+p m_{3}^{2}=p$, from which we get $m_{1}^{2}-v m_{2}^{2} \equiv 0$ $\bmod p$, and since $v$ is a non-square modulo $p$, this means that $m_{1} \equiv m_{2} \equiv 0 \bmod p$, i.e., we can write $m_{1}=p x, m_{2}=p y$ with $x, y \in \mathbb{Z}_{p}$, and our condition after cancellation of $p$ factors becomes $p x^{2}-v p y^{2}+m_{3}^{2}=1$. First solving modulo $p$ and then using Hensel's Lemma shows that this always has solutions for $m_{3}$, regardless of the integers $x$ and $y: m_{3}= \pm \sqrt{1-p x^{2}+v p y^{2}} \in \mathbb{Q}_{p}$. This means we can choose $m_{1}$ and $m_{2}$ freely as multiples of $p$, and always satisfy the quadratic form constraint. On the other hand, we claim that there are solutions of $m_{1}^{2}-v m_{2}^{2}=v p^{2}$, which upon substituting $x$ and $y$ becomes $x^{2}-v y^{2}=v$. Indeed, the quadratic form on the left hand side does not represent zero nontrivially, but $x^{2}-v y^{2}-v z^{2}$ does. So, we can take any solution of $x^{2}-v y^{2}-v z^{2}=0$, which necessarily must have $z \neq 0$. By homogeneity, we can assume $z=1$, which gives the desired solution.

Remark 28. When $p=2$, none of the Cardano and Euler decompositions exist for all of $S O(3)_{2}$. The counterexample is a single matrix

$$
M=\left(\begin{array}{ccc}
-2 & -2 & \sqrt{-7} \\
\frac{1}{2}(\sqrt{-7}+1) & \frac{1}{2}(\sqrt{-7}-1) & 2 \\
\frac{1}{2}(\sqrt{-7}-1) & \frac{1}{2}(\sqrt{-7}+1) & 2
\end{array}\right) .
$$

Note that -7 is a square in $\mathbb{Q}_{2}$, and indeed $\sqrt{-7}=1+2^{2}+2^{4}+2^{5}+2^{7}+\ldots$, so that $\frac{1}{2}(\sqrt{-7}+1)=$ $1+2+2^{3}+2^{4}+2^{6}+\ldots$ is odd, and $\frac{1}{2}(\sqrt{-7}-1)=2+2^{3}+2^{4}+2^{6}+\ldots$ is even in $\mathbb{Z}_{2}$.

As a matter of fact, $M$ does not have a decomposition of any of the six Cardano and six Euler forms shown in Theorem 4. Let us show that it cannot be written as a product $\mathcal{R}_{x} \mathcal{R}_{y} \mathcal{R}_{z}$. Namely, by Remark 25 it is enough to show that there does not exist a $\xi$ such that $\mathcal{R}_{x}(\xi)^{-1} M \boldsymbol{e}_{3} \perp \boldsymbol{e}_{2}$. With $M \boldsymbol{e}_{3}=m_{1} \boldsymbol{e}_{1}+m_{2} \boldsymbol{e}_{2}+m_{3} \boldsymbol{e}_{3}$ such that $m_{1}^{2}+m_{2}^{2}+m_{3}^{2}=1$, we would need a vector $m_{1} \boldsymbol{e}_{1}+m_{3}^{\prime} \boldsymbol{e}_{3}$ with quadratic form evaluating to 1, i.e., $m_{3}^{\prime 2}=m_{2}^{2}+m_{3}^{2}=8$, which however is not a square in $\mathbb{Q}_{2}$; contradiction.

Likewise, it cannot be decomposed in the form $\mathcal{R}_{y} \mathcal{R}_{x} \mathcal{R}_{z}$, because then we would need $\mathcal{R}_{y}(\eta)^{-1} M \boldsymbol{e}_{3} \perp \boldsymbol{e}_{1}$, which amounts to a vector $m_{2} \boldsymbol{e}_{2}+m_{3}^{\prime} \boldsymbol{e}_{3}$ with quadratic form evaluating to 1 , i.e., $m_{3}^{\prime 2}=m_{1}^{2}+m_{3}^{2}=-3$, which however is not a square in $\mathbb{Q}_{2}$, as it is $\equiv 5 \bmod 8$; contradiction.

Any other decomposition results in a contradiction of the same type, because, as can be checked, every column of $M$ has the property that, while the squares of its elements sum to 1 , the sum of any two squares is not a square in $\mathbb{Q}_{2}$. To prove this, it is enough to consider all occurring cases and check that the sum in question is not a square modulo 8 or 16 .

Although these results are in contrast to the Euclidean case, $S O(3)_{p}$ is still generated by its subgroups $G_{x}, G_{y}, G_{z}$ of rotations around the reference axes, at least for all odd primes $p$. Now we will find the multiplicity of the above Cardano representations of $S O(3)_{p}$ for odd $p$.

Proposition 29. If $M \in S O(3)_{p}$ and $M=\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta)$, then there exists at least another distinct Cardano representation of $M$ along the same axes

$$
M=\mathcal{R}_{x}(\infty) \mathcal{R}_{x}(\xi) \mathcal{R}_{y}\left(\frac{1}{\alpha \eta}\right) \mathcal{R}_{z}(\infty) \mathcal{R}_{z}(\zeta) .
$$

Proof. A systematic ambiguity of order 2 in the products $\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta)$ is based on the relation

$$
\mathcal{R}_{x}(\infty) \mathcal{R}_{y}(\infty) \mathcal{R}_{z}(\infty)=I
$$

By using it with Eq. (21) we get

$$
\begin{aligned}
M & =\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta) \\
& =\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{x}(\infty) \mathcal{R}_{y}(\infty) \mathcal{R}_{z}(\infty) \mathcal{R}_{z}(\zeta) \\
& =\mathcal{R}_{x}(\infty) \mathcal{R}_{x}(\xi)\left(\mathcal{R}_{y}(\eta)+\mathcal{R}_{x}(\infty)\left[\mathcal{R}_{y}(\eta), \mathcal{R}_{x}(\infty)\right]\right) \mathcal{R}_{y}(\infty) \mathcal{R}_{z}(\infty) \mathcal{R}_{z}(\zeta) \\
& =\mathcal{R}_{x}(\infty) \mathcal{R}_{x}(\xi)\left(\begin{array}{ccc}
-e(\eta) & 0 & f(\eta) \\
0 & 1 & 0 \\
g(\eta) & 0 & -e(\eta)
\end{array}\right) \mathcal{R}_{z}(\infty) \mathcal{R}_{z}(\zeta),
\end{aligned}
$$

where the product of $\mathcal{R}_{x}(\infty)$ with the commutator of matrices $\left[\mathcal{R}_{y}(\eta), \mathcal{R}_{x}(\infty)\right]$ provides the transformation $\eta \mapsto-\eta$ (an infinite parameter remains the same) on the parameter of the $y$-rotation (sign change of the off-diagonal elements), which together with $\mathcal{R}_{y}(\infty)$ globally gives $\eta \mapsto \frac{1}{\alpha \eta}$ (sign change of the diagonal entries).

We give two elementary results in order to prove that the Cardano representation of $S O(3)_{p}$ along the axes $x, y$ and $z$ is exactly twofold for odd $p$.

Proposition 30. For odd $p$, and all $\mathcal{R}_{x}(\xi)$, $\mathcal{R}_{x}\left(\xi^{\prime}\right), \mathcal{R}_{y}(\eta), \mathcal{R}_{y}\left(\eta^{\prime}\right) \in S O(3)_{p}$,

$$
\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta)=\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right) \Leftrightarrow \mathcal{R}_{x}(\xi)=\mathcal{R}_{x}\left(\xi^{\prime}\right), \mathcal{R}_{y}(\eta)=\mathcal{R}_{y}\left(\eta^{\prime}\right)
$$

Proof. This is immediate to prove by equating two matrices of the kind

$$
\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & a(\xi) & b(\xi) \\
0 & c(\xi) & a(\xi)
\end{array}\right)\left(\begin{array}{ccc}
e(\eta) & 0 & f(\eta) \\
0 & 1 & 0 \\
g(\eta) & 0 & e(\eta)
\end{array}\right)=\left(\begin{array}{ccc}
e(\eta) & 0 & f(\eta) \\
b(\xi) g(\eta) & a(\xi) & b(\xi) e(\eta) \\
a(\xi) g(\eta) & c(\xi) & a(\xi) e(\eta)
\end{array}\right)
$$

thanks to the fact that $e(\eta), a(\xi) \neq 0$ for every parameter.
Corollary 31. Let $\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{\boldsymbol{n}}(\sigma)=\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right) \mathcal{R}_{\boldsymbol{n}}\left(\sigma^{\prime}\right), p>2$. If $\left(\mathcal{R}_{x}(\xi), \mathcal{R}_{y}(\eta), \mathcal{R}_{\boldsymbol{n}}(\sigma)\right) \neq$ $\left(\mathcal{R}_{x}\left(\xi^{\prime}\right), \mathcal{R}_{y}\left(\eta^{\prime}\right), \mathcal{R}_{\boldsymbol{n}}\left(\sigma^{\prime}\right)\right)$ then $\mathcal{R}_{\boldsymbol{n}}(\sigma) \neq \mathcal{R}_{\boldsymbol{n}}\left(\sigma^{\prime}\right)$.

Proof. We prove the contrapositive: if $\mathcal{R}_{\boldsymbol{n}}(\sigma)=\mathcal{R}_{n}\left(\sigma^{\prime}\right)$, then

$$
\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{\boldsymbol{n}}(\sigma)=\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right) \mathcal{R}_{\boldsymbol{n}}\left(\sigma^{\prime}\right) \Rightarrow \mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta)=\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right)
$$

By Proposition 30 , this is equivalent to $\left(\mathcal{R}_{x}(\xi), \mathcal{R}_{y}(\eta), \mathcal{R}_{\boldsymbol{n}}(\sigma)\right)=\left(\mathcal{R}_{x}\left(\xi^{\prime}\right), \mathcal{R}_{y}\left(\eta^{\prime}\right), \mathcal{R}_{\boldsymbol{n}}\left(\sigma^{\prime}\right)\right)$.
Theorem 32. Every $M \in S O(3)_{p}$, for odd prime p, has exactly two distinct Cardano decompositions with respect to the $x, y$ and $z$ axes

$$
M=\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta)
$$

for some $\xi, \eta, \zeta \in \mathbb{Q}_{p} \cup\{\infty\}$, and

$$
M=\mathcal{R}_{x}(\infty) \mathcal{R}_{x}(\xi) \mathcal{R}_{y}\left(\frac{1}{\alpha \eta}\right) \mathcal{R}_{z}(\infty) \mathcal{R}_{z}(\zeta)
$$

Proof. We look for nontrivial solutions of $\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta)=\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right) \mathcal{R}_{z}\left(\zeta^{\prime}\right)$. Using the parameterization (20), we have

$$
\begin{aligned}
\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta) & =\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & a(\xi) & b(\xi) \\
0 & c(\xi) & a(\xi)
\end{array}\right)\left(\begin{array}{ccc}
e(\eta) & 0 & f(\eta) \\
0 & 1 & 0 \\
g(\eta) & 0 & e(\eta)
\end{array}\right)\left(\begin{array}{ccc}
l(\zeta) & m(\zeta) & 0 \\
n(\zeta) & l(\zeta) & 0 \\
0 & 0 & 1
\end{array}\right) \\
& =\left(\begin{array}{ccc}
e(\eta) l(\zeta) & e(\eta) m(\zeta) & f(\eta) \\
a(\xi) n(\zeta)+b(\xi) g(\eta) l(\zeta) & a(\xi) l(\zeta)+b(\xi) g(\eta) m(\zeta) & b(\xi) e(\eta) \\
c(\xi) n(\zeta)+a(\xi) g(\eta) l(\zeta) & c(\xi) l(\zeta)+a(\xi) g(\eta) m(\zeta) & a(\xi) e(\eta)
\end{array}\right) .
\end{aligned}
$$

We equate two matrices of this kind, whose entries we call $m_{i j}$ and $m_{i j}^{\prime}$ respectively ( $i, j=1,2,3$ ). Thus,

$$
m_{13}=m_{13}^{\prime} \Leftrightarrow f(\eta)=f\left(\eta^{\prime}\right),
$$

equivalent to $-\frac{2 p \eta}{1+p \eta^{2}}=-\frac{2 p \eta^{\prime}}{1+p \eta^{\prime 2}} \Leftrightarrow 2 p\left(\eta-\eta^{\prime}\right)\left(1-p \eta \eta^{\prime}\right)=0 \Leftrightarrow \eta=\eta^{\prime}$ or $\eta^{\prime}=1 /(p \eta)$. Furthermore, $f(\eta)=f\left(\eta^{\prime}\right) \Leftrightarrow g(\eta)=g\left(\eta^{\prime}\right)$, since $g(\eta)=\frac{2 \eta}{1+p \eta^{2}}$.

If $\eta=\eta^{\prime} \in \mathbb{Q}_{p} \cup\{\infty\}$, then $\mathcal{R}_{y}(\eta)=\mathcal{R}_{y}\left(\eta^{\prime}\right)$, otherwise $\eta^{\prime}=1 /(p \eta) \in \mathbb{Q}_{p} \cup\{\infty\}$ and

$$
\mathcal{R}_{y}(\eta)=\left(\begin{array}{ccc}
-e\left(\eta^{\prime}\right) & 0 & f\left(\eta^{\prime}\right) \\
0 & 1 & 0 \\
g\left(\eta^{\prime}\right) & 0 & -e\left(\eta^{\prime}\right)
\end{array}\right)
$$

As a consequence, from $m_{11}=m_{11}^{\prime}, m_{12}=m_{12}^{\prime}, m_{23}=m_{23}^{\prime}, m_{33}=m_{33}^{\prime}$ we deduce $l(\zeta)= \pm l\left(\zeta^{\prime}\right)$, $m(\zeta)= \pm m\left(\zeta^{\prime}\right), a(\xi)= \pm a\left(\xi^{\prime}\right), b(\xi)= \pm b\left(\xi^{\prime}\right)$, where the + sign is always related to $\eta=\eta^{\prime}$ and the sign to $\eta \eta^{\prime}=1 / p$. This is because $e(\eta) \neq 0$ for every $\eta \in \mathbb{Q}_{p} \cup\{\infty\}$, as $e(\eta)=\frac{1-p \eta^{2}}{1+p \eta^{2}}=0 \Leftrightarrow \eta^{2}=1 / p$, but $1 / p$ is not a square; $e(\infty)=-1$.

Then, $m_{22}=m_{22}^{\prime}$ is always satisfied, whereas

$$
\begin{aligned}
m_{21}=m_{21}^{\prime} & \Leftrightarrow a(\xi) n(\zeta)+b(\xi) g(\eta) l(\zeta)=a\left(\xi^{\prime}\right) n\left(\zeta^{\prime}+b\left(\xi^{\prime}\right) g\left(\eta^{\prime}\right) l\left(\zeta^{\prime}\right)\right. \\
& \Leftrightarrow a(\xi) n(\zeta)= \pm a(\xi) n\left(\zeta^{\prime}\right) \Leftrightarrow n(\zeta)= \pm n\left(\zeta^{\prime}\right) \\
m_{31}=m_{32} & \Leftrightarrow c(\xi) n(\zeta)+a(\xi) g(\eta) l(\zeta)=c\left(\xi^{\prime}\right) n\left(\zeta^{\prime}\right)+a\left(\xi^{\prime}\right) g\left(\eta^{\prime}\right) l\left(\zeta^{\prime}\right) \\
& \Leftrightarrow c(\xi) n(\zeta)= \pm c\left(\xi^{\prime}\right) n(\zeta) \Leftrightarrow c(\xi)= \pm c\left(\xi^{\prime}\right)
\end{aligned}
$$

Note that $a(\xi) \neq 0$ for every $\xi \in \mathbb{Q}_{p} \cup\{\infty\}$, because $a(\xi)=\frac{1+\frac{p}{v} \xi^{2}}{1-\frac{p}{v} \xi^{2}}=0 \Leftrightarrow \xi^{2}=-v / p$ but $-v / p$ is not a square; $a(\infty)=-1$.

In the last step it might be $n(\zeta)=\frac{2 \zeta}{1-v \zeta^{2}}=0 \Leftrightarrow \zeta=0$, for which the condition $m(\zeta)= \pm m\left(\zeta^{\prime}\right)=$ 0 gives $\mathcal{R}_{z}(\zeta)=\mathcal{R}_{z}\left(\zeta^{\prime}\right)=I$. This implies $\mathcal{R}_{x}(\xi)=\mathcal{R}_{x}\left(\xi^{\prime}\right), \mathcal{R}_{y}(\eta)=\mathcal{R}_{y}\left(\eta^{\prime}\right)$ by Corollary 31, in particular $c(\xi)=c\left(\xi^{\prime}\right)$, in agreement with what we deduced above. Then, $m_{32}=m_{32}^{\prime}$ is satisfied.

We have found that $\mathcal{R}_{x}(\xi) \mathcal{R}_{y}(\eta) \mathcal{R}_{z}(\zeta)=\mathcal{R}_{x}\left(\xi^{\prime}\right) \mathcal{R}_{y}\left(\eta^{\prime}\right) \mathcal{R}_{z}\left(\zeta^{\prime}\right)$ if and only if either

$$
\left(\mathcal{R}_{x}(\xi), \mathcal{R}_{y}(\eta), \mathcal{R}_{z}(\zeta)\right)=\left(\mathcal{R}_{x}\left(\xi^{\prime}\right), \mathcal{R}_{y}\left(\eta^{\prime}\right), \mathcal{R}_{z}\left(\zeta^{\prime}\right)\right)
$$

or
which concludes the proof.
Remark 33. The above results of the duplicity of the Cardano decomposition can be understood qualitatively, and for any $p$, from Remark 25.

Indeed, assume that $S O(3)_{p} \ni M=\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma) \mathcal{R}_{\boldsymbol{n}_{2}}(\tau) \mathcal{R}_{\boldsymbol{n}_{3}}(\omega)$, where $\boldsymbol{n}_{1} \perp \boldsymbol{n}_{2} \perp \boldsymbol{n}_{3}$. We have seen that this is possible if and only if $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3} \perp \boldsymbol{n}_{2}$, and chosen $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)$, the other two rotations are uniquely determined. There are at most two solutions, since there is only one free parameter in conditions for $\boldsymbol{v}=\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)^{-1} M \boldsymbol{n}_{3}$, and they boil down to a single quadratic equation. On the other hand, with $\mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)$, another solution of this condition is always $\mathcal{R}_{\boldsymbol{n}_{1}}(\infty) \mathcal{R}_{\boldsymbol{n}_{1}}(\sigma)$.

## 7. DISCUSSION

We have seen that not only is there a well-motivated $p$-adic special orthogonal group $S O(3)_{p}$, but that it shares many geometric features with its real counterpart $S O(3)_{\mathbb{R}}$, although there are also crucial differences, owing to the specific number theory of $\mathbb{Z}_{p}$ for different primes $p$, in particular distinguishing odd $p$ from $p=2$, and within odd primes between $p \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$.

These differences manifested themselves most blatantly in the discussion of Cardano and Euler angle decompositions of general special orthogonal transformations in terms of rotations around the reference axes: they are not available in general for $p=2$, but certain (all) Cardano decompositions hold for odd primes $p \equiv 3 \bmod 4(p \equiv 1 \bmod 4)$. We leave open the possibility of modified principal angle decompositions beyond the use of the reference axes.

Note that we have treated the groups essentially as algebraic groups, and in future development, it may pay off to follow the analytic approach of $p$-adic Lie groups and Lie algebras [10], which allows for a local description in terms of infinitesimal generators.

An open question that we have left, might be answerable in this way, which is that after the cyclicity of the two-dimensional rotation groups $S O(2)_{p}^{\kappa}$. Since the groups are definitely profinite, the correct question seems to be whether the groups $S O(2)_{p}^{\kappa}$ are procyclic. To prove this, we consider their projections modulo $p^{k}, \pi_{k}\left(S O(2)_{p}^{\kappa}\right) \subset S L\left(2, \mathbb{Z} / p^{k} \mathbb{Z}\right)$, and should show that for every $p$ and $\kappa$, and sufficiently large $k$ these are cyclic, cf. [3].

Finally, we want to discuss our motivation for considering $p$-adic spatial rotations in the first place: it grows out of Volovich's $p$-adic quantum theory, in which Euclidean space is replaced by $p$-adic space to define the underlying phase space [11], see also [12-14]. The idea is to realise quantum systems as unitary representations of the symmetry group of the $p$-adic space, according to Noether's theorem [15], and this has been realised for the displacement operations in position and momentum, resulting in a $p$-adic Heisenberg-Weyl algebra of position and momentum operators. What has not been done yet is to develop a quantum theory of $p$-adic angular momentum. By the same philosophy, this is identical to the classification of all the projective unitary irreducible representations of $S O(3)_{p}$. We get infinitely many such representations from reducing the group modulo $p^{k}$, noting that $S O(3)_{p} \bmod p^{k}$ are finite groups, for which the irreps can be found by standard tools [16]. Indeed, in [2, 3], this has been done for reduction modulo $p$ and $p^{2}$, for certain odd primes. An open question is whether all irreps of $S O(3)_{p}$ arise in this way. We suspect that this problem cannot be solved using standard $p$-adic techniques based on Hensel's Lemma, but may require non-standard $p$-adic analysis and Gretel's Lemma [17].

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[^0]:    * The present work grew out of a BSc thesis [1] and two MSc theses [2, 3] at the University of Camerino, co-supervised at the Autonomous University of Barcelona.
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