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Non-linearities in Economics

by

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Introduction

I Chaos and non-linearities

The term "chaos" has been commonly used to hint non-linear dynamics characterized by an apparent absence of order and therefore incomprehensible and unpredictable behaviour. Instead, through isolated research conducted by scholars out of the scientific mainstream, it was shown that a chaotic dynamics is not only ordered [66] and therefore intelligible, but also that it is deterministic and controllable.

E. N. Lorenz, a universally recognized forerunner in the study of chaotic phenomena, first discovered in the 1960s a so-called "strange attractor" in a three-dimensional continuous-time dynamic system, when carrying out numerical experiments on convection flows [112]. He was able to publish his work in marginal journals only, as the concepts he used were unusual to say the least. Silence about this topic continued until well into late 1970s when in the United States several Physics scholars, among whom we remember M. J. Feingelbaum [32] [63], drew the attention of the scientific world to the "strange" results of this type of research.

In reality, the problem of chaotic phenomena and more generally of non-linear dynamical systems is their poor permeability to the classical instruments of investigation of dynamics so dominated by the problem of simplicity and regularity. Empirical evidence, on the contrary, shows a different world characterized by complexity and disharmony [68].

In economics, dynamics are usually non-linear and characterised by cyclical fluctuations which are called "business cycles". Burns and Mitchell (1946)[34] define business cycles as a type of fluctuation which "consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle". Imperfections may be intended as those perturbations of the equilibrium that can lead to recessions or to expansions (Fig. 0-1). The reason why we deal with chaotic models in economics is that, differently from stochastic models like the so-called "real business cycle" (RBC), the dynamics can be understood and explained in terms of the structural characteristics of the system rather than external and random shocks (i.e. endogenous instead of exogenous).

II Summary of research during the PhD course

In addition to working on the different topic of financial mathematics [143], [139], [140], [142], [141] our research during the last three years of our PhD program has been devoted to I) finding a suitable model for business cycles [137], II) looking for an indicator that could show structural changes in a signal related to chaos [144], III) applying techniques such as Recurrence Quantification Analysis (RQA) to economic time series (with a problem due to the short number of data point available for this kind of research) that may help in understanding hidden features of economic dynamics [145] and, finally, comparing time series generated by the studied models versus real ones [146], [147].

While chaos in economics may look fascinating, for good reasons chaotic economic

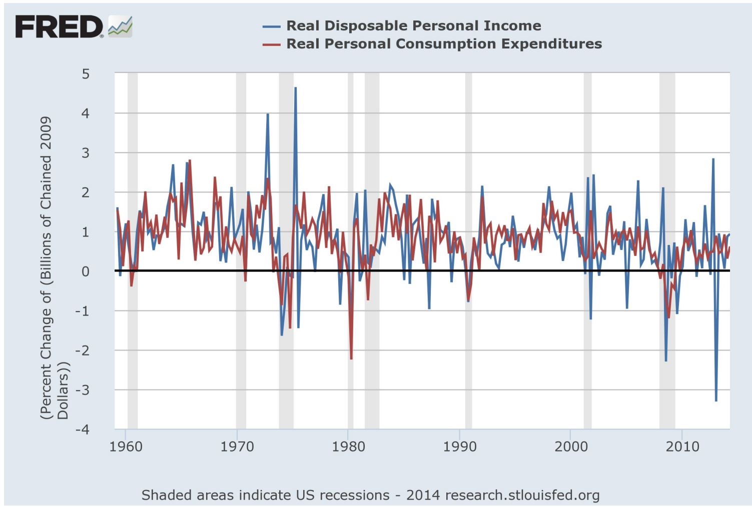


Figure 0-1: Changes in US Real Disposable Personal Income (i.e.the personal income net of income taxes) (blue - DSPIC96) and Real Personal Consumption Expenditures (red - PCECC96) 1959 (Q1) - 2014 (Q2). Source: St. Louis Fed, FRED database. Greyed areas correspond to periods of economic recessions as reckoned by FRED (Table A.1).

models to be to be considered purely intellectual constructions with little ability to explain reality. For this reason we thought that we should take up the challenge of performing a reality check. To avoid being personally involved in the matter we considered not our model but we went for a chaotic specification of the Harrod-Domar model [174] to prove that i) real data could be obtained by a suitable calibration of the model's parameters, ii) the calibrated model confirmed theoretical predictions [138].

III Scheme of this thesis

This thesis consists of three Parts derived from my research on non-linear dynamics. The First Part is formal-methodological and provides the mathematical background for the remainder. Being this a thesis on economic dynamics, its Second Part provides the related background and literature. The Third Part contains a personal specification of a business cycle model within the Kaldor-Kalecki framework. By means of numerical analysis it is shown that the proposed model is chaotic. Moreover recurrence quantification analysis and statical techniques are applied to time series derived from real macroeconomic variable and to those generated by our model's simulations. This to I) find common features if and where they do exist, II) discover some hidden features of economic dynamics and III) highlight some indicators of structural changes in the signal (i.e. in our case to look for precursors of a crash). This is followed by the above mentioned test on a given specification of the Harrod model and by the Final Remarks.

It is worth mentioning that Kaldor and Harrod explained in detail their models but did not write any equation. For this reason the mathematical specifications reported here are original if not stated otherwise.

Part I

Mathematical background

Chapter 1

Dynamic systems

In this Chapter we provide some basic definitions about what systems are. Namely, syestems are defined in Sec. 1.1 along with their dynamical properties such as attractors and repellers (Sec. 1.2). In Sec. 1.3 periodic points are illustrated and the Logistic map Sec. 1.4 is used as example. Cycles and limit cycles (Sec. 1.5), embedding dimension (Sec. 1.6) lead to the section on chaos Sec. 1.7. Measures of sensitive dependence on initial condition Sec. 1.8 and measures of attractors Sec. 1.9 concludes the Chapter.

1.1 Dynamic systems and their classification

The concept of dynamic system that we will define here is taken from R.E. Kalman [12] who introduced it in the 1960s when he studied the problem of linear filtering and predictions.

Roughly speaking, a system is an entity described by the so-called *state* (generally a real number or a vector of real numbers); the adjective *dynamic* is due to the fact that this state varies in time according to a suitable law.

We can formalize the concept of dynamical system by giving the following definition.

Definition 1.1 (Dynamic system). A *dynamic system* is an entity defined by the following axioms:

1. There exist an ordered set T of times, a set X of states and a function ϕ from $T \times T \times X$ to X. ϕ is named *state transition function*.

2. For all $t, \tau \in T$ and for all $x \in X$ one has that $\phi(t, \tau, x)$ represents the state at time t of a system whose initial state at time τ is x;

3. The function ϕ satisfies the following properties:

   Consistency : $\phi(\tau, \tau, x) = x$ for all $\tau \in T$, and for all $x \in X$ (as in 2),

   Composition : $\phi(t_3, t_1, x) = \phi(t_3, t_2, \phi(t_2, t_1, x))$ for all $x \in X$ and for all $t_1, t_2, t_3 \in T$ with $t_1 < t_2 < t_3$.

Definition 1.2 (Reversibility). If the state transition function ϕ is defined for any (t, τ) in $T \times T$, once assigned the initial time τ and the initial state x, the state of the system is uniquely determined for the future (i.e. for all $t > \tau$) as well as for the past, (i.e for $t < \tau$). In this case the system is said *reversible*. If the state transition function ϕ is defined only for $t \geq \tau$, then the system is said *irreversible*.

Definition 1.3 (Event, orbit and flow). For all $t \in T, x \in X$ the pair (t, x) is named *event*; moreover, for τ and x fixed, the function $t \in T \mapsto \phi(t, \tau, x) \in X$ is called *movement* of the system. The set of all movements is called *flow*. The image of the movement, i.e. the set $\{\phi(t, \tau, x) : t \in T\}$ is called *trajectory* or *orbit* of the system, i.e. the orbit passing through the state x at the time τ.

Definition 1.4 (Fixed or equilibrium point)**.** A state $x^* \in X$ is called a *fixed point* or an *equilibrium point* of the dynamics if there exist $t_1, t_2 \in T$, with $t_2 > t_1$, such that

$$\phi(t, t_1, x^*) = x^* \qquad \text{for all } t \in T \cap [t_1, t_2]; \tag{1.1}$$

x^* is said a *fixed point in an infinite time* if there exists $t_1 > T$ such that

$$\phi(t, t_1, x^*) = x^* \qquad \text{for all } t \in T \cap [t_1, +\infty[. \tag{1.2}$$

Definition 1.5 (Eventually fixed orbit)**.** An orbit is said to be *eventually fixed* if it contains a fixed point.

Definition 1.6 (Eventually fixed point)**.** A point is called *eventually fixed* if a eventually fixed orbit is generated from it.

Definition 1.7 (Stability)**.** The fixed (or equilibrium) point x^* is *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ and a $t_0 \in T$ such that for all $x \in X$ with $|x - x^*| \leq \delta$ it holds that $|\phi(t, \tau, x) - x^*| \leq \epsilon$ for any $t > t_0$.

The fixed point x^* is *asymptotically stable* if it is stable and there exists a $\delta > 0$ such that for all $x \in X$ with $|x - x^*| \leq \delta$ it holds that $\lim_{t\to\infty} |\phi(t, \tau, x) - x^*| = 0$.

The fixed point x^* is *globally asymptotically stable* if it is stable and

$$\lim_{t\to\infty} |\phi(t, \tau, x) - x^*| = 0 \qquad \text{for any } \tau \in T \text{ and } x \in X. \tag{1.3}$$

Definition 1.8 (Discrete, continuous system)**.** The system is called *discrete* if the time set T is the set of natural numbers $\{0, 1, 2, 3, \ldots\ldots\}$.
The system is called *continuous* if T is an interval of real numbers.

Definition 1.9 (Autonomous system)**.** The system is called *autonomous* if

$$\phi(t, \tau, x) = \tilde{\phi}(t - \tau, x) \tag{1.4}$$

for some suitable function $\tilde{\phi}$.

In the following we give some examples of continuous and discrete dynamic systems.

1.1.1 An example of continuous dynamic systems

Let $I = [a, b] \subset \mathbb{R}$ and let $f\colon I \times \mathbb{R} \to \mathbb{R}$.

We recall the following version of the Cauchy-Lipschitz Theorem from Bonsante and Da Prato[30].

Theorem 1.1 (Cauchy-Lipschitz)**.** *Assume that there exists $L > 0$ such that*

$$\big|f(t, x_1) - f(t, x_2)\big| \leq L|x_1 - x_2| \tag{1.5}$$

for any $t \in I$ and $x_1, x_2 \in \mathbb{R}$. Then for all $t_0 \in I, x_0 \in \mathbb{R}$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) & t \in I \\ x(t_0) = x_0, \end{cases} \tag{1.6}$$

has one and only one solution in $[a, b]$.

From this it follows that the ordinary differential equation

$$\dot{x} = f(t, x) \tag{1.7}$$

defines a continuous reversible dynamic system. In fact the time set is $T = I$, the state set is $X = \mathbb{R}$ and the state transition function ϕ is the function from $I \times I \times \mathbb{R}$ to $\mathbb{R}$ such that for all $t, \tau \in I, x \in \mathbb{R}$ one has that

$$\phi(t, \tau, \xi) = x(t) \tag{1.8}$$

where $x(t)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) & t \in I \\ x(\tau) = \xi \end{cases} \tag{1.9}$$

Eq. (1.9) shows that the movements are the solutions of Eq. (1.7) and, for any solution x, the corresponding orbit is the interval $\{x(t) : t \in I\}$.

The system is autonomous if f is independent from t, (i.e. in the case of a differential equation of the form $\dot{x} = f(x)$, with $f : \mathbb{R} \mapsto \mathbb{R}$ derivable function with continuous and bounded derivative), since in this case one has

$$\phi(t, \tau, x) = \phi(t - \tau, 0, x) \qquad \text{for all} \quad t, \tau, x \in \mathbb{R}. \tag{1.10}$$

An equilibrium point is a solution of the differential equation $\dot{x} = f(x)$ which is constant on an interval $J = [t_1, t_2] \subset I$. Hence the equilibrium points of the system are the solutions $x^* \in \mathbb{R}$ of the equation $f(x) = 0$.

1.1.2 Continuous dynamic system associated to a system of ordinary differential equations

The discussion contained in the previous Section 1.1.1 for a single equation, can be extended to systems of ordinary differential equations.

In fact, when $x = (x_1, x_2, \ldots, x_n)$, let $\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x})$ be a vector function from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$, and let $f_1, f_2, \ldots, f_n$ be the components of $\boldsymbol{f}$.

Assume that $f_1, f_2, \ldots, f_n$ are continuous functions in $I \times \mathbb{R}^n$, that the partial derivatives of $f_1, f_2, \ldots, f_n$ with respect to all the variables $x_1, x_2, \ldots, x_n$ exist and are continuous in $I \times \mathbb{R}^n$, and that these partial derivatives are bounded in $[a, b] \times \mathbb{R}^n$ for all $[a, b] \subset I$.

Then for all $t_0 \in I$, $\boldsymbol{x}_0 \in \mathbb{R}^n$ the vector Cauchy problem

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}(t)) & t \in I \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \end{cases} \tag{1.11}$$

has one and only one solution in the interval I.

Hence the system of ordinary differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)) \tag{1.12}$$

defines a *reversible* continuous dynamic system.

The time set is $T = I$, the state set is $X = \mathbb{R}^n$, the state transition function is the mapping ϕ from $I \times I \times \mathbb{R}^n$ to $\mathbb{R}^n$ such that for all $t, \tau \in I$, $\boldsymbol{x} \in \mathbb{R}^n$ one has that

$\phi(t,\tau,x)$ is the value in t of the unique solution of the Cauchy problem

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t,\boldsymbol{x}(t)) & t \in I \\ \boldsymbol{x}(\tau) = \boldsymbol{x} \end{cases} \tag{1.13}$$

In this case the movements are the solutions of the system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x})$ and, for any solution $\boldsymbol{x}$ of such a system of differential equations, the corresponding orbit is the curve in $\mathbb{R}^n$ of parametric equation $\boldsymbol{x} = \boldsymbol{x}(t) : t \in I$.

As before, the system is autonomous when $\boldsymbol{f}$ is independent from t, i.e. in the case of a system of differential equations of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Then an equilibrium point is a solution of the system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ which is constant on an interval $J \subset I$. Thus the equilibrium points of the system are the solutions $\boldsymbol{x}^* \in \mathbb{R}^n$ of the system of equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$.

Remark 1.1. The notion of dynamic system as outlined in Definition 1.1 is referred to the situations where the evolution of the system depends only on internal causes.

However there are situations where the evolution of the system can be modified through the action of external forces, i.e. by means of an a time-dependent input vector function $\boldsymbol{u}$. In this case Definition 1.1 can be generalized in the sense that a dynamic system is characterized by a time set T, a state set X, an input set U, a set Ω of admissible input functions from T to U, a state transition function ϕ from $T \times T \times X \times \Omega$ to X such that for all $t, \tau \in T, \boldsymbol{x} \in X, \boldsymbol{u} \in \Omega$ one has that $\phi(t,\tau,\boldsymbol{x},\boldsymbol{u})$ represents the state of the system at the time t if the state is $\boldsymbol{x}$ at the time τ and on the system acts an input function $\boldsymbol{u}$.

Obviously the state of the system at the time t will only depend on the initial time τ, the initial state $\boldsymbol{x}$ and the restriction of the input function $\boldsymbol{u}$ to the interval of extremes t and τ. Hence we have to assume that the state transition function ϕ

satisfies the following properties of

Consistency $\phi(\tau, \tau, \boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}(t) \quad \forall \quad (\tau, \boldsymbol{x}, \boldsymbol{u}(\cdot)) \in T \times X \times \Omega$

Composition $\boldsymbol{x}(t) = \phi(t_3, t_1, \boldsymbol{x}, \boldsymbol{u}) = \phi\left(t_3, t_2, \phi\left(t_2, t_1, \boldsymbol{x}, \boldsymbol{u}\right), \boldsymbol{u}\right)$ for each $(x, u) \in X \times \Omega$, and for each $t_1 < t_2 < t_3$

Causality If $\boldsymbol{u}, \boldsymbol{v} \in \Omega$ and $\boldsymbol{u}|_{[\tau,t]} = \boldsymbol{v}|_{[\tau,t]}$, then $\phi(t, \tau, \boldsymbol{x}, \boldsymbol{u}) = \phi(t, \tau, \boldsymbol{x}, \boldsymbol{v})$.

As we are not going to discuss how to control a dynamic system, in what follows, we shall never be concerned with the dependence of the state on an external input function u.

1.1.3 An example of a discrete time dynamic system

Definition 1.10 (Map)**.** If f is a function from X to X, then the recursive formula

$$x_{n+1} = f(x_n) \tag{1.14}$$

defines a discrete dynamic system and it is called a *map*.

In fact, if we denote by the symbol $f^{\circ n}$ the *n-th iterate* of f , i.e. for $n = 0$ the identity on X and for $n \geq 1$ the composition of f with itself n times, then the state transition function ϕ is defined by

$$\phi(t, \tau, x) = f^{\circ(t-\tau)}(x) \qquad \text{for all} \quad t, \tau \in T = \{0, 1, 2, 3, \ldots\ldots\},\ t \geq \tau, \tag{1.15}$$

since it is evident that ϕ satisfies the consistency and composition properties.

In this case a movement is a sequence $(x_n)_n$ such that $x_{n+1} = f(x_n)$ for all n, whereas an *orbit* is a set of the form $\{x_0, x_1, x_2, \ldots x_n, \ldots\}$ with $x_{n+1} = f(x_n)$ for all n.

Example 1.1. For $f(x) = x^2$, the orbit of f with initial point $x = 2$ is the set $\{2, 4, 16, 256, \ldots\}$.

Remark 1.2. Note that the dynamic system defined by Eq. (1.14) is autonomous (cfr. Eq. (1.4)) and it is reversible if and only if the function f is bijective.

Definition 1.11 (Fixed or equilibrium point for a discrete time system)**.** A *fixed point* (or a *equilibrium point*) x^* is a point of X such that $f(x^*) = x^*$. In this discrete time case, the orbit departing from x^* is the singleton $\{x^*\}$.

Example 1.2. If $f(x) = x^2$ then 0 and 1 are the only fixed points.

Example 1.3. For $f(x) = x^2$ the point $x = -1$ is not fixed for f but it is an eventually fixed point because $f(-1) = 1 \neq -1$. Instead, the point $x = 1$ is a fixed one for f.

Definition 1.12 (Periodic orbit, cycle)**.** The orbit of initial point x_0 is said *periodic* or *cycle* if there exists $m \in \mathbb{N}$ such that $f^{\circ m}(x_0) = x_0$.

Remark 1.3. Periodic orbit means that after a finite number of iterations we return to the initial point and therefore the orbit has a finite number of elements. In this case x_0 is said a *periodic* or *cyclic* point and the smallest number m such that $f^{\circ m}(x_0) = x_0$ is said the *prime period* of the orbit.

Remark 1.4. A point x_p is periodic of period n if and only if x_p is a fixed point of $f^{\circ n}$. In particular a fixed point x_p for f is the fixed for all iterates of f.

Example 1.4. If $f(x) = -x$ then $x = 0$ is the only fixed point and for all $x \neq 0$ the orbit departing from x is the set $\{x, -x\}$ and is periodic of first period $m = 2$.

Definition 1.13 (Eventually periodic orbit)**.** An orbit is said to be *eventually periodic* if it contains a periodic point. Analogously, a point is called eventually periodic if a eventually periodic orbit is generated from it.

Example 1.5. The point $x = 1$ is an eventually periodic point for the function $f(x) = x^4 - 1$, because $f(1) = 0$ and 0 is contained in the cycle $(0, -1)$.

Remark 1.5. Recursive methods to find a fixed point

Recursive expressions of the form of Eq. (1.14) are often used in numerical computations for solving equations. An example is given by the so-called 'Babylonian algorithm' to approximate the square root of a number $a > 0$ asymptotically equal to

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right). \tag{1.16}$$

A more general algorithm is the Newton method of approximating the zero of a differentiable function g

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \tag{1.17}$$

For example if $g(x) = x^2 - a$, $g'(x) = 2x$ thus the Newton algorithm in Eq. (1.17) reduces to Eq. (1.16).

1.2 Attractors and repellers for discrete time dynamic systems

In this Section we provide some definitions concerning the behaviour of dynamic systems which, unless differently specified, are taken from R. Devaney [52], H. W. Lorenz [113] and S. Sternberg [175]. Throughout the Section $f : \mathbb{R} \mapsto \mathbb{R}$ is a twice continuously differentiable function and f^{on} denotes its n-th iterate, i.e. the composition of f with itself n times.

Definition 1.14 (Critical point)**.** We say that $x_c \in \mathbb{R}$ is a *critical* point of f if $f'(x_c) = 0$. The critical point is *non-degenerate* if $f''(x_c) \neq 0$. The critical point is *degenerate* if $f''(x_c) = 0$.

Remark 1.6. Degenerate critical points may be maxima, minima, or inflection points; non-degenerate critical points, instead, must be either maxima or minima.

Example 1.6. In $c = 0$, the function $f(x) = x^2$ has a non-degenerate critical point and $f(x) = x^n$ for $n > 2$ has a degenerate critical point. In particular c is an inflection point (i.e. a point in which the function changes concavity) for $f(x) = x^3$.

Definition 1.15 (Limit set)**.** Given $x \in X$ we call *limit set* of x the set A of points $\omega \in X$ for which there is a sequence of natural numbers $(n_i)_i \to +\infty$ such that

$$\lim_{i \to +\infty} f^{on_i}(x) = \omega. \tag{1.18}$$

Definition 1.16 (Attractor)**.** A compact (i.e. a closed and limited) set $A \subset X$ is an *attractor* of the dynamics if there is an open set U containing A such that A is the limit set of all the points in U.

Definition 1.17 (Basin of attraction)**.** The set of all x that have as limit set A is called *basin of attraction* of A.

Remark 1.7. In particular a *singleton* $\{x_a\}$ is an attractor if there exists $\delta > 0$ such that for all $x \in]x_a - \delta, x_a + \delta[$ the sequence $(f^{on}(x))_n$ has a subsequence converging to x_a.

Example 1.7. For example, if $x^* = x_a$ is a fixed point of f and $|f'(x_a)| < 1$, then there exists $\delta > 0$ such that for all $x \in]x_a - \delta, x_a + \delta[$ the sequence $(f^{on}(x))_n$ tends to x_a, and therefore a is asymptotically stable and the set $\{x_a\}$ is an attractor.

Proof. For a fixed $K \in \mathbb{R}$ such that $|f'(x_a)| < K < 1$, one has that

$$\lim_{x \to x_a} \frac{|f(x) - x_a|}{|x - x_a|} = \lim_{x \to x_a} \left| \frac{f(x) - f(a)}{x - x_a} \right| = |f'(x_a)| < K \tag{1.19}$$

and therefore there exists $\delta > 0$ such that for all $x \in]x_a - \delta, x_a + \delta[$ one has

$$|f(x) - x_a| < K|x - x_a| < \delta. \tag{1.20}$$

From this it follows by induction that for all $x \in]x_a - \delta, x_a + \delta[$ and for all $n \in \mathbf{N}$

one has that

$$f^{on}(x) \in]x_a - \delta, x_a + \delta[\qquad \text{and} \qquad |f^{0(n+1)}(x) - x_a| < K|f^{on}(x) - x_a|. \quad (1.21)$$

Hence for all $x \in]a-\delta, a+\delta[$ the distance of $f^{on}(x)$ from a decreases at a geometric rate $K < 1$ and therefore tends to 0, as desired.

□

Remark 1.8. If one has $f'(0) = 0$, then the preceding argument shows that the distance of $f^{on}(x)$ from a decreases at a geometric rate K for all $K \in]0,1[$.

This justifies the following definition:

Definition 1.18 (Superattractor)**.** A fixed point x^* such that $f'(x^*) = 0$ is called *superattractor* or *superstable*.

Remark 1.9. If $|f'(x_a)| > 1$, then, for a fixed $K \in \mathbb{R}$ such that $1 < K < |f'(x_a)|$, there exists $\delta > 0$ such that for all $x \in]x_a - \delta, x_a + \delta[$ one has $|f(x) - x_a| > K|x - x_a|$; hence the distance of $f^{on}(x)$ from a increases at a geometric rate $K > 1$ and therefore there exists $n \in \mathbb{N}$ such that $|f^{on}(x) - x_a| > \delta$.

This motivates the following definition:

Definition 1.19 (Repeller)**.** A fixed point x^* such that $|f'(x^*)| > 1$ is called unstable or a *repeller*.

Example 1.8. Let us consider a function g twice differentiable and the Newton method of Eq. (1.17). In this case

$$f(x) = x - \frac{g(x)}{g'(x)} \quad (1.22)$$

hence

$$f'(x) = 1 - \frac{g'(x)}{g'(x)} + \frac{g(x)g''(x)}{g'(x)^2} = \frac{g(x)g''(x)}{g'(x)^2}. \quad (1.23)$$

If the point x_a is a non-degenerate zero of g then x_a is a superattractive fixed point.

Remark 1.10. As already mentioned a periodic point x_p of period n is a fixed point of $f^{\circ n}$ (n-fold composition of f).

Moreover, if p is periodic also the points

$$x_p, f(x_p), f^{\circ 2}(p), \ldots, f^{\circ n-1}(x_p) \tag{1.24}$$

are periodic and by the chain rule, the derivative of $f^{\circ n}$ in those points is the same and it is equal to

$$(f^{\circ n})'(x_p) = f^{'}(x_p)\, f'(f(x_p)) \cdots f'(f^{\circ n-1}(x_p)). \tag{1.25}$$

Definition 1.20 (Hyperbolic bifurcation point)**.** Let x_p be a periodic point of prime period n (see Remark 1.3), the point x_p is called *hyperbolic* if

$$|(f^{\circ n})'(x_p)| \neq 1 \tag{1.26}$$

The number $(f^{\circ n})'(x_p)$ is called a *hyperbolic point multiplier.*

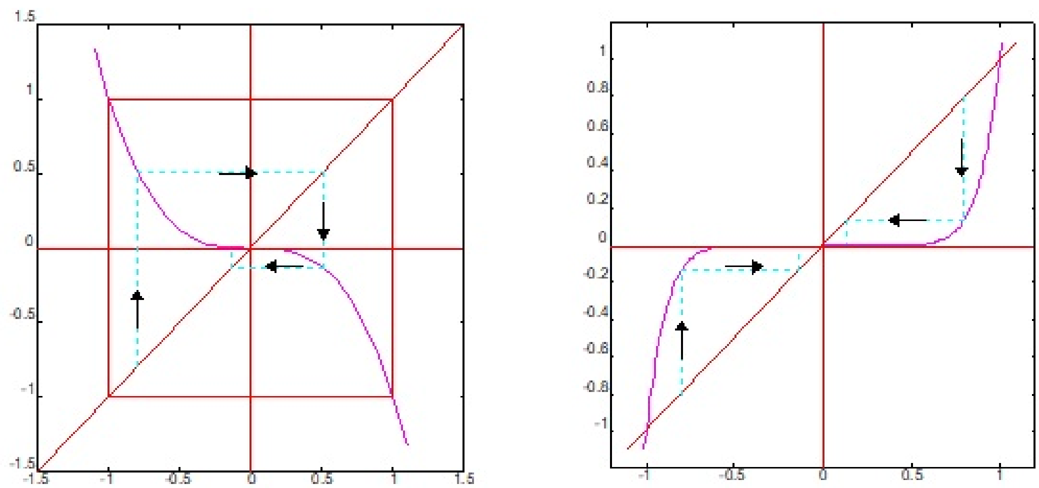
Definition 1.21 (Bifurcation point)**.** A non-hyperbolic fixed point is called a *bifurcation point.*

Definition 1.22 (Attractive periodic orbit)**.** If x_p is an attractive (respectively a repeller) fixed point for f^n then so are all the others and this is called *attractive periodic orbit.*

Definition 1.23 (Superattractive periodic orbit)**.** A periodic point is superattractive for $f^{\circ n}$ if and only if $f^{'}(s) = 0$ at least for one of the points $x_p, f(x_p), f^{\circ 2}(x_p), \ldots, f^{\circ n-1}(x_p)$.

Example 1.9. As mentioned, an attractor as well as a repeller can be a fixed or a periodic point. For example, the function $f(x) = -x^3$ has two cyclic points -1 and $+1$ of period 2 and a fixed one $x_0 = 0$ (see Figure 1-1 (a)).

It can easily be verified that x_0 is an attractor for the basin $(-1, +1)$ and that the cyclic orbit $-1, +1$ is a repeller. To show that, it is sufficient to study the function $f^{\circ 2}(x)$ for which -1 and $+1$ are fixed repeller points, and since neither is fixed for f, then they will be cyclic repellers (see Figure 1-1 (b)).



(a) Plot of $f(x) = -x^3$ (b) Plot of $f^{\circ 2}(x) = x^9$

Figure 1-1: Convergence to the attractor. Panel (a) represents $f(x) = -x^3$ that is a mirror image of $f(x) = x^3$ and Panel (b) corresponds to the graph of $f^{\circ 2}(x) = x^9$.

1.3 Existence of periodic points

In the following we recall some results that will be used in Section 1.4.

1.3.1 Schwarz derivative

Definition 1.24 (Schwarz derivative [175])**.** Let f be a one-dimensional map defined in the real field, three times derivable. The Schwarz derivative of f is

$$f^S(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{1.27}$$

The relevant property of this derivative is to preserve the sign with the composition, in the sense that if $f^S(x) > 0$ then it is also $(f^{\circ n})^S(x) > 0 \quad \forall n \in \mathbb{N}$.

Proposition 1.2. *Let us consider a polynomial $Q(x)$. If all the roots of its first derivative $Q'(x)$ are real and distinct then $Q^S(x) < 0$.*

Proof. Suppose that

$$Q'(x) = \prod_{i=1}^{n}(x - a_i) \qquad \text{with } a_i \text{ real and distinct.} \tag{1.28}$$

Then the second and third derivatives are

$$Q''(x) = \sum_{j=1}^{n} \frac{\prod_{i=1}^{n}(x - a_i)}{x - a_j} = \sum_{j=1}^{n} \frac{Q'(x)}{x - a_j} \tag{1.29}$$

and

$$Q'''(x) = \sum_{\substack{j,k=1,\dots,n \\ j \neq k}} \frac{\prod_{i=1}^{n}(x - a_i)}{(x - a_j)(x - a_k)} = \sum_{\substack{j,k=1,\dots,n \\ j \neq k}} \frac{Q'(x)}{(x - a_j)(x - a_k)}. \tag{1.30}$$

Therefore

$$
\begin{aligned}
Q^S(x) &= \sum_{\substack{j,k=1,\dots,n \\ j\neq k}} \frac{1}{(x-a_j)(x-a_k)} - \frac{3}{2}\left(\sum_{j=1}^{n} \frac{1}{x-a_j}\right)^2 \\
&= -\frac{1}{2}\sum_{j=1}\left(\frac{1}{x-a_j}\right)^2 - \left(\sum_{j=1}^{n}\frac{1}{x-a_j}\right)^2 < 0.
\end{aligned}
\tag{1.31}
$$

□

Proposition 1.3. *Let us consider $h = f \circ g$. If $f^S < 0$ and $g^S < 0$ then $h^S < 0$.*

Proof. According to the chain rule the second and third derivative of h are

$$h'' = (f \circ g)'' = f''(g(x))(g'(x))^2 + f'(g(x))g''(x) \tag{1.32}$$

$$h''' = (f \circ g)''' = f'''(g(x))(g'(x))^3 + 3f''(g(x))g''(x)g'(x) + f'(g(x))g'''(x) \tag{1.33}$$

Therefore

$$h^S(x) = f^S(g(x))(g'(x))^2 + g^S(x) < 0. \tag{1.34}$$

□

In particular if f^S is negative then $(f^{\circ n})^S$ is negative for all $n > 1$. This introduces us to the following result (for a demonstration see Ref. [52]).

Theorem 1.4 (Schwarz theorem)**.** *If $f^S < 0$ and if f has n critical points then f has at most $n + 2$ attracting periodic orbits.*

1.3.2 Singer theorem

As mentioned in Def. 1.24 the Schwarz derivative preserves the sign under composition which is useful in the following theorem

Theorem 1.5 (Singer [113]). *Let f be a map from a closed interval $I \subseteq [0, b]$ onto itself; then the dynamic system $x_{n+1} = f(x_n)$ has at most one periodic orbit in the interval I if the following conditions are met:*

1. *f is a function C^3*

2. *There exists a critical point $x_c \in I$ such that:*

$$\begin{cases} f'(x) > 0 & \forall x < x_c \\ f'(x_c) = 0 & \\ f'(x) < 0 & \forall x > x_c \end{cases} \tag{1.35}$$

3. *The origin is a repeller for f, that is*

$$f(0) = 0, \quad | f'(0) |> 1 \tag{1.36}$$

4. *The Schwarz derivative is*

$$f^S(x) \leq 0 \quad \forall x \in I \setminus \{x_c\} \tag{1.37}$$

1.3.3 Sarkovsky theorem

Let us introduce the following ordering on natural numbers.

Definition 1.25 (Sarkovsky ordering [175]).

$$\begin{gathered}3 \rhd 5 \rhd 7 \rhd \quad \cdots \quad 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \quad \cdots \quad 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd \quad \cdots \\ \cdots \quad 2^n \cdot 3 \rhd 2^n \cdot 5 \rhd 2^n \cdot 7 \rhd \quad \cdots \quad \rhd 2^n \rhd \quad \cdots \quad \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1\end{gathered} \tag{1.38}$$

That is, first all odd integers except one are listed, then they are followed by 2 times that odd number, 2^2 times the odd, 2^3 times the odd, etc. This exhausts all the natural numbers with the exception of the powers of two that are listed last, in decreasing order.

Theorem 1.6 (Sarkovsky). *Let f be a continuous function with a periodic point of prime period k. If $k \rhd l$ in the Sarkovsky ordering of Def.* 1.25*, then f has also a periodic point of period l.*

As in the Sarkovsky ordering the largest number is 3 we have the following result:

Corollary 1.7. *(Period three implies all periods) If f has a periodic orbit of period three, then it has periodic orbits of all periods.*

As the set of the smallest numbers in the Sarkovsky ordering is the set of the powers of 2, it holds that:

Corollary 1.8. *If f has a periodic point of prime period k, with k not a power of two, then f has infinitely many periodic points. Conversely, if f has only finitely many periodic points, then they all necessarily have periods which are powers of two.*

1.4 Logistic map

In this Section, as an application of the concepts discussed above, we are going to investigate the discrete dynamic system generated by the so-called Logistic Map (also called *Verhulst dynamics* or *quadratic map*):

$$f_\mu(x) = \mu\, x(1-x), \quad x \in [0,1], \quad \mu \in [0,4], \tag{1.39}$$

which, despite having a simple analytical expression and graph (see Figure 1-2), may display complex behaviours depending on the value of the parameter μ.

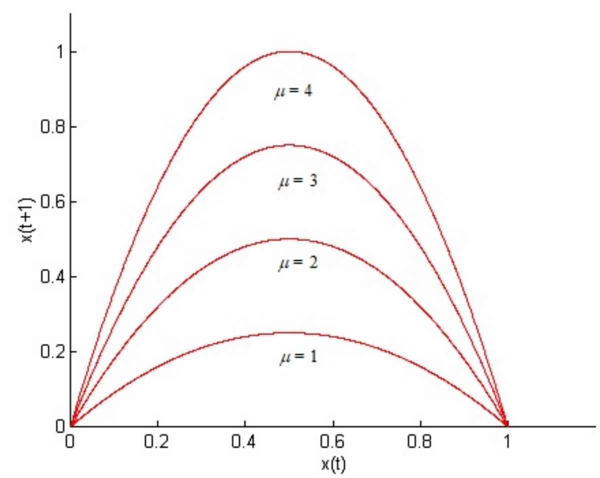


Figure 1-2: Logistic for different values of μ. The critical point is $c = 1/2$.

One of the reasons why we discuss it is due to the fact that its analysis requires the use of numerical methods similar to those used to analyse the model we propose in the third Part of this thesis. This because there are equations that cannot be resolved analytically, in fact

Theorem 1.9 (Abel–Ruffini impossibility theorem [159]). *There is no solution in radicals expression of a general polynomial equations of degree five or higher with arbitrary coefficients.*

The proof of this result was first provided incomplete by P. Ruffini and then it was completed by N.H. Abel and E. Galois who worked independently on it. In the remainder of this Section we will overcome the said problem with both graphical and numerical analysis.

1.4.1 Fixed points for the Logistic Map

First of all we note that the fixed points of the Logistic Map f_μ are $x_1^* = 0$ and $x_2^* = 1 - 1/\mu$. The first derivative $f'_\mu(x) = \mu(1-2x)$ calculated in those points is

$$f'_\mu(x_1^*) = \mu \qquad f'_\mu(x_2^*) = 2 - \mu \tag{1.40}$$

As the nature of the fixed points in the interval $[0,1]$ changes with the parameter μ, hereafter we study the different instances.

1. If $0 < \mu < 1$, there is only one fixed point $x = 0$, since the other fixed point $x_2^* = 1 - 1/\mu$ is negative.

   Note that the point $x_1^* = 0$ is attracting (but not super-attracting) since $f'_\mu(0) = \mu \in (0,1)$. Moreover for all $x \in (0,1]$ one has $0 < f(x) < x \leq 1$ and therefore the sequence $(f^{on}(x))_n$ is decreasing and converges to 0. Thus the basin of attraction of 0 is $[0,1]$

2. If $\mu = 1$, $f_1 = x(1-x)$, then 0 is the unique fixed point of f and, (with the same arguments as before), one proves that 0 is an attractor and the basin of attraction of 0 is $[0,1]$.

If $\mu > 1$, the fixed point $x_1^* = 0$ is repelling and the second fixed point $x_2^* = 1 - 1/\mu$ lies in $[0,1]$. Moreover $1 < \mu < 3$ implies $|f'_\mu(x_2^*)| = |2-\mu| < 1$; therefore the point x_2^* is an attractor and its basin depends on the parameter μ as detailed below.

3. If $1 < \mu < 2$ then the fixed point x_2^* is in $(0, 1/2)$, and the first derivative $f'_\mu(x_2^*) = 2 - \mu$ is in $(0, 1)$.

   Now we prove that the basin of attraction is $(0, 1]$.

   Proof. Note that f is strictly increasing and strictly concave in the interval $[0, 1/2]$; hence one has that

   – $x \in [0, x_2^*[\quad \Longrightarrow \quad 0 \leq f_\mu(x) < f_\mu(x_2^*) = x_2^* \quad \text{and} \quad f_\mu(x) > x,$

   – $x \in]x_2^*, 1/2] \quad \Longrightarrow$
   $x_2^* = f_\mu(x_2^*) < f_\mu(x) \leq f_\mu(1/2) = \mu/4 \leq 1/2 \quad \text{and} \quad 0 < f_\mu(x) < x.$

   From this it follows by induction that for all $n \in \mathbb{N}$ one has that

$$0 \leq f_\mu^{on}(x) < x_2^* \quad \text{and} \quad f_\mu^{o(n+1)}(x) > f_\mu^{on}(x), \qquad \text{for all } x \in [0, x_2^*[,$$
$$x_2^* < f_\mu^{on}(x) \leq 1/2 \quad \text{and} \quad f_\mu^{o(n+1)}(x) < f_\mu^{on}(x), \qquad \text{for all } x \in]x_2^*, 1/2].$$

   Hence the sequence $(f_\mu^{on}(x))_n$ is increasing and converging to x_2^* for all $x \in [0, x_2^*[$ and is decreasing and converging to x_2^* for all $x \in]x_2^*, 1/2]$.

   Finally for all $x \in]1/2, 1]$ one has that $0 \leq f_\mu(x) < f_\mu(1/2) = \mu/4 < 1/2$ and therefore the sequence $(f_\mu^{on}(x))_{n\geq 1}$ is increasing or decreasing according to whether $f_\mu(x) < x_2^*$ or $f_\mu(x) > x_2^*$; in both cases such a sequence converges to x_2^*.

   □

4. If $\mu = 2$ we can apply the same reasoning of the case $\mu < 2$ to prove that the basin of attraction of x_2^* is $(0, 1]$.

Moreover, since

$$x_2^* = 1 - \frac{1}{\mu} = \frac{1}{2} \quad \text{and} \quad f_2'\Big(\frac{1}{2}\Big) = 0, \tag{1.41}$$

the fixed point x_2^* is superattractive and the sequence $(f_2^{on}(x))_n$ converges to x_2^* very fast (more than geometrically).

5. If $2 < \mu < 3$ then $x_2^* = 1 - (1/\mu) > 1/2$ and

$$f_\mu'(x_2^*) = 2 - \mu \in (-1, 0) \tag{1.42}$$

hence the fixed point x_2^* is an attractor but the iterates oscillate around it.

6. If $\mu = 3$ then $x_2^* = 1 - \dfrac{1}{\mu} = \dfrac{2}{3}$ and $f_\mu'(x_2^*) = 2 - \mu = -1$. It can be proved that the fixed point x_2^* is still an attractor.

If $\mu \in]3, 4[$ then $f_\mu'(x_2^*) < -1$. As well as the fixed point x_1^*, the fixed point x_2^* has become a repeller (see Def. 1.19). However, if we consider the second iterate $f_\mu^{\circ 2}(x)$, a fixed point of it (i.e. a periodic point of period 2) is the zero of $f_\mu^{(o2)}(x) - x$. Since

$$\begin{aligned} f_\mu^{(o2)}(x) - x &= \mu f_\mu(x)(1 - f_\mu(x)) - x \qquad (1.43) \\ &= \mu[\mu x(1-x)][1 - \mu x(1-x)] - x \\ &= x(\mu - \mu x - 1)(\mu^2 x^2 - \mu^2 x - \mu x + \mu + 1) \\ &= \big[f_\mu(x) - x\big](\mu^2 x^2 - \mu^2 x - \mu x + \mu + 1) \end{aligned}$$

a fixed point of $f_\mu^{\circ 2}(x)$ is either a fixed point of $f_\mu(x)$ or a zero of the quadratic polynomial

$$\mu^2 x^2 - \mu(\mu + 1)x + \mu + 1. \tag{1.44}$$

The discriminant of the polynomial (1.44)

$$\mu^2(\mu+1)^2 - 4\mu^2(\mu+1) = \mu^2(\mu+1)(\mu-3) \tag{1.45}$$

is positive since $\mu > 3$. Hence the said polynomial (1.44) has two real roots:

$$p_{2\pm} = \frac{1}{2} + \frac{1}{2\mu} \pm \frac{1}{2\mu}\sqrt{(\mu+1)(\mu-3)} \qquad \in (0,1). \tag{1.46}$$

To check if $p_{2\pm}$ are attractors or repellers, we need to compute the derivative of $f_\mu^{\circ 2}(x)$ in these points. This can be done in two different but equivalent ways.

First method. Computing the derivative of $f_\mu^{\circ 2}(x)$ we have

$$(f_\mu^{\circ 2})'(x) = \mu^2(1-2x)(2\mu^2 x - 2\mu x + 1) \tag{1.47}$$

then replacing $x = p_{2\pm}$, after some calculation one gets:

$$(f_\mu^{\circ 2})'(p_{2\pm}) = -\mu^2 + 2\mu + 4. \tag{1.48}$$

Second method. Using the chain rule one obtains

$$(f_\mu^{\circ 2})'(x) = \big[f_\mu\big(f_\mu(x)\big)\big]' = f_\mu'\big(f_\mu(x)\big)\, f_\mu'(x). \tag{1.49}$$

Since $f_\mu(p_{2+}) = p_{2-}$ and $f_\mu(p_{2-}) = p_{2+}$, from the above identity we get

$$\begin{aligned} (f_\mu^{\circ 2})'(p_{2\pm}) &= f_\mu'(p_{2+})\, f_\mu'(p_{2-}) \\ &= \mu^2(1-2p_{2+})(1-2p_{2-}) \\ &= \mu^2\big(1 - 2(p_{2+}+p_{2-}) + 4p_{2+}p_{2-}\big) \end{aligned} \tag{1.50}$$

From Eq. (1.44) we get

$$p_{2+} + p_{2-} = (\mu+1)/\mu \qquad \text{and} \qquad p_{2+} \cdot p_{2-} = (\mu+1)/\mu^2 \tag{1.51}$$

thus we get

$$(f_\mu^{\circ 2})'(p_{2\pm}) = -\mu^2 + 2\mu + 4. \tag{1.52}$$

Hence $(f^{o2})'(p_{2\pm})$, as a function of μ, is decreasing in the interval $[3,4]$ and has value 1 for $\mu = 3$ and value -1 for $\mu = \mu_1 := 1 + \sqrt{6} = 3.449499\ldots$

7. If $3 < \mu < \mu_1$, as we have already shown, the two fixed points are repelling for f while the two periodic points of period two are attracting for f^{o2}.

   For $\mu > \mu_1$ the periodic points of period two become unstable (repelling) and 4 periodic points of period four appear. These points are stable (attracting) for $\mu < \mu_2 = 3.54409\ldots$ and unstable for $\mu > \mu_2$.

   Iterating this procedure, one can construct a sequence $(\mu_n)_n$ such that for $\mu > \mu_n$ a cycle of order 2^n appears.

8. Regarding the limit case $\mu = 4$, by direct calculation one has

$$f_\mu^{\circ 3}(x) - x = \big(f_\mu(x) - x\big)(64\,x^3 - 112\,x^2 + 56\,x - 7)(64\,x^3 - 96\,x^2 + 36\,x - 3) \tag{1.53}$$

The two polynomials of degree 3 have three real roots in $(0,1)$ namely Numerical values are

$$x_1 = 0.1882550991\ldots, \qquad x_2 = 0.6112604670\ldots, \qquad x_3 = 0.9504844340\ldots$$

for the first polynomial and

$$x_4 = 0.1169777784\ldots, \qquad x_5 = 0.4131759112\ldots, \qquad x_6 = 0.9698463104\ldots$$

for the second. These are two cycles of period 3 and by using Sarkovsky's Theorem 1.3.3, there exists orbits of any period.

In Fig. 1-3 it is shown that when μ is 1 or 2 and $x_2^* > 0$ is stable, the graph intersects the 45^o line only once. In that case $x_2^* > 0$ is a fixed point of both f^{o1} and f^{o2}. When μ is larger than the bifurcation value for the flip bifurcation, the graph of f^{o2} intersects the 45^o line three times i.e. at the unstable fixed point at the period-2 fixed points (see Fig. 1-3 lower graph on the right).

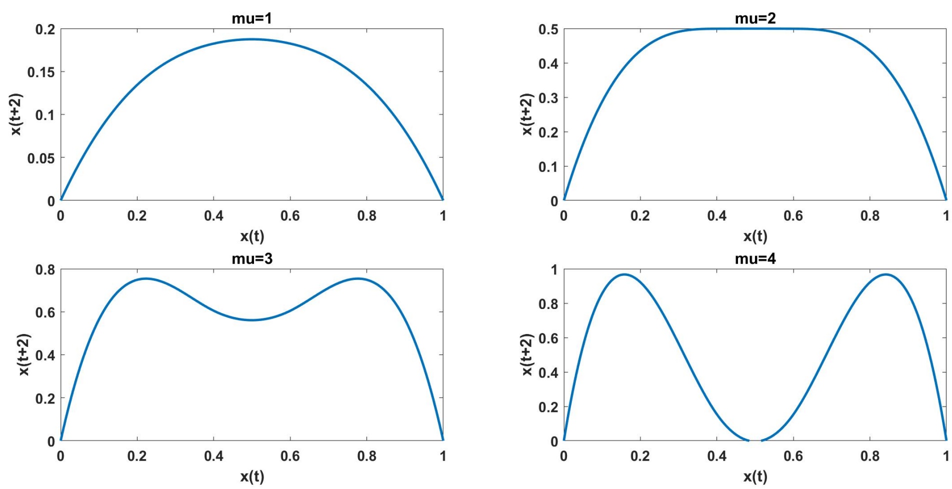


Figure 1-3: Second iteration for the Logistic Map.

1.4.2 Feigenbaum universal constant for the Logistic Map

M.J. Feigenbaum showed [63] that in the Logistic Map the sequence $(\mu_n)_n$ of period-doubling bifurcation values follows the rule

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta_F \approx 4.6692\ldots \tag{1.54}$$

where δ_F is a universal constant and it is present in many one-dimensional non-invertible maps (that is why the Logistic Map is a paradigmatic example for chaos). Therefore if two successive bifurcation values are known, the next bifurcation can be approximately computed from Eq. (1.54). Finally, it can be shown that the limit point of the period-doubling sequence is $\mu_c \approx 3.5699\ldots$ (see Fig. 1-4).

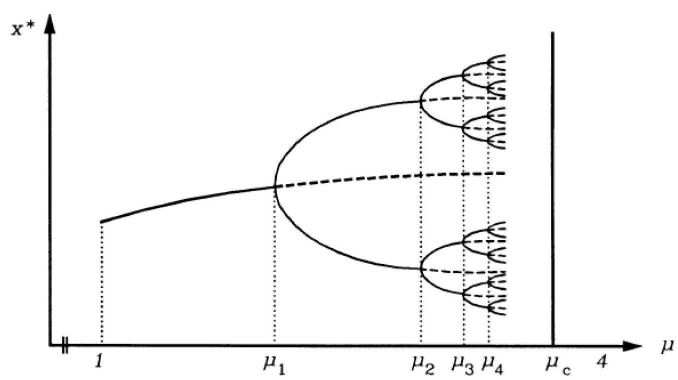


Figure 1-4: Period doubling for the Logistic Map. Ref. H.W. Lorenz [113].

1.4.3 Schwarz derivative for the Logistic Map

Since

$$f_\mu^{'} = \mu - 2\mu x, \quad f_\mu^{''} = -2\mu, \quad f_\mu^{'''} = 0 \tag{1.55}$$

for the Schwarz derivative 1.24 of the Logistic Map one has

$$f^S(x) = -\frac{6}{(1-2x)^2} \tag{1.56}$$

which is $< 0\,\forall\, x \neq 1/2$.

Hence by Theorem 1.4 there exist at most three attracting periodic orbits.

1.4.4 Application of the Singer's theorem to the Logistic Map

In this paragraph we want to show that for the function $f_\mu(x) = \mu x(1-x)$, with $\mu > 1$ it is easy to check that the four conditions of Singer's Theorem 1.5 are all satisfied. Indeed

1. f_μ is infinitely differentiable,
2. The derivative is $f_\mu^{'}(x) = \mu(1-2x)$ and therefore for $c = 1/2$ one has:

$$f_\mu^{'}(x) \begin{cases} > 0 & \forall x < c \\ = 0 & \text{for} \quad x = c\,, \\ < 0 & \forall x > c \end{cases}$$

3. The origin for f_μ is a repeller since $f_\mu(0) = 0$ and $f_\mu'(0) = \mu > 1.$,
4. $f^S(x)_\mu = -\dfrac{6}{(1-2x)^2} < 0 \quad \forall\, x \in I \setminus \{c\}$ (see Paragraph 1.4.3).

1.4.5 Cobweb or Verhulst diagram for the Logistic Map

A *cobweb* or *Verhulst diagram* is a way to visualize the behaviour of a dynamic system under repeated application of a map [28].

The diagram consists of a diagonal $x = y$ line and a curve representing $y = f(x)$. The algorithm [2] is:

- Find the point on the function curve with coordinates $x_0, f(x_0)$.
- Plot horizontally across from this point to the point on the diagonal line with coordinates $f(x_0), f(x_0)$.
- Plot vertically from the point on the diagonal to the function curve that has coordinates $f(x_0), f(f(x_0))$.
- Repeat from step 2 as required.

Figure 1-5 shows the cobweb diagram for the Logistic Map.

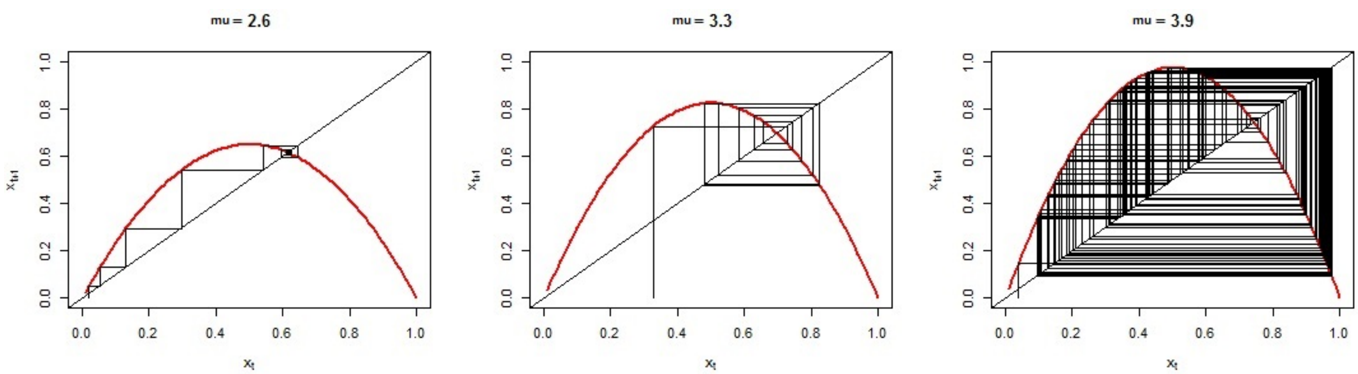


Figure 1-5: Cobweb diagram for the Logistic Map. Source [4].

1.5 Cycles and limit cycles

In this Section we consider a continuous time dynamic system described by the state transition function $\phi(t, \tau, x)$., from now on, definitions and results are taken from R. Devaney [52], H. W. Lorenz [113] and S. Sternberg [175] unless differently specified.

Definition 1.26 (Limit cycle)**.** A limit cycle (see Fig. 1-6) is a closed orbit Γ for which there exists a tubular neighbourhood $U(\Gamma)$ [129] such that for all $x \in U(\Gamma)$ one has

$$\lim_{t \to +\infty} d(\phi(t, \tau, x), \Gamma) = 0, \tag{1.58}$$

where we have set

$$d(y, \Gamma) = \inf_{z \in \Gamma} |y - z|. \tag{1.59}$$

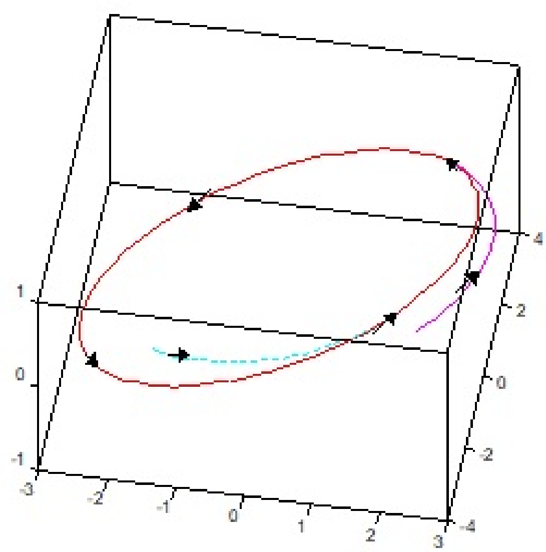
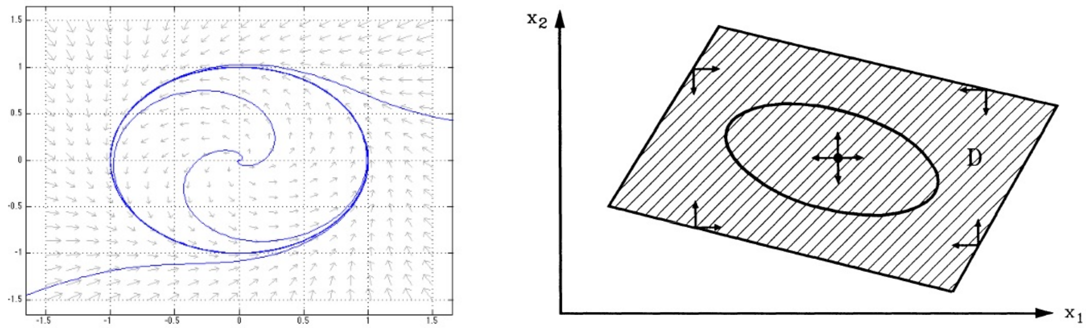


Figure 1-6: A limit cycle.

In order to establish the existence of limit cycles, in the two-dimensional case, we can refer to the following theorem by Poincaré and Bendixon.

Theorem 1.10 (Poincaré-Bendixon [113])**.** *Let D be a non-empty, compact (i.e. closed and bounded) set of the plane not containing fixed points of a C^1 vector field $\boldsymbol{f}$ from D to $\mathbb{R}^2$ and let $\gamma \subseteq D$ be an orbit of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Then either γ is* *a closed orbit or γ asymptotically approaches a closed orbit (i.e. there exists a limit cycle in D).*

Figure 1-7: Convergence to the limit cycle. On the boundary of D, the vector field points inwards the set, therefore once a trajectory enters in D it will stay forever.



(a) A system with a stable limit cycle in a vector field. Ref. Caltech [36]

(b) Limit cycle in a compact set D. Ref. H.W. Lorenz [113].

The limitations of Theorem 1.10 are related to finding a suitable set D and to the fact that it is valid only in two dimensions. For example suppose that there exists a compact set $D \subset \mathbb{R}^3$ with the vector field pointing inwards to D (see Figure 1-7b) and that there is a unique unstable equilibrium. Nevertheless it is possible that no closed orbit exists because a trajectory can arbitrarily wander in $\mathbb{R}^3$ without neither intersecting itself nor approaching a limit set (see. Figure 1-8).

When we deal with a linear system of differential equations (as expressed in matrix form)

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \tag{1.60}$$

its solution with initial point $\boldsymbol{x}_0$ for $t = 0$ is $e^{tA}\boldsymbol{x}_0$. Therefore we get to know its behaviour by studying the eigenvalues of the matrix A. The Hartman-Grobman Theorem 1.11 explains the behaviour about the fixed points for nonlinear systems

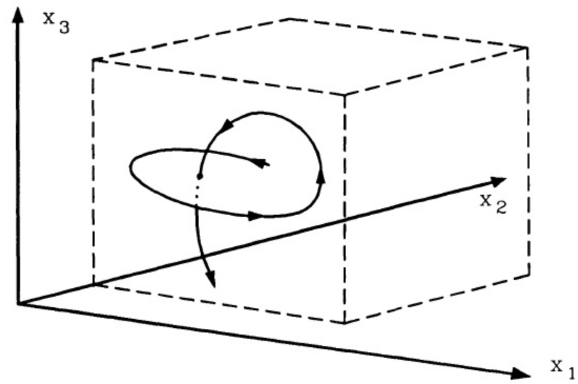


Figure 1-8: In R^3 Poincaré-Bendixon is invalid. Ref. H.W. Lorenz [113].

by a linearisation around the equilibrium. Before that, we need to introduce the following definitions as in Zimmerman [201].

Definition 1.27 (Homeomorphism)**.** A function $h : X \to Y$ is a *homeomorphism* between X and Y if it is continuous and bijective (one-to-one and onto function) with a continuous inverse denoted h^{-1}.

Remark 1.11. An homeomorphisms means that X and Y have similar structure and that h (resp. h^{-1}) stretches and bends the space but does not tear it.

Definition 1.28 (Diffeomorphism)**.** A function $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ is called diffeomorphism of class C^k if it is surjective (onto) and injective (one-to-one), and if the components of $\boldsymbol{f}$ and its inverse have continuous partial derivatives up to the $k - th$ order with respect to all variables.

Definition 1.29 (Embedding)**.** An *embedding* is a homeomorphism onto its image.

Definition 1.30 (Topological conjugacy)**.** Given two maps, $f : X \to X$ and $g : Y \to Y$, the map $h : X \to Y$ is a topological semi-conjugacy if it is continuous, bijective and $h \circ f = g \circ h$, with $\circ$ function composition. In addition, if h is a homeomorphism between X and Y, then we say that h is a topological conjugacy and that X and Y are homomorphic.

Definition 1.31 (Hyperbolic fixed point)**.** In the case of continuous time dynamic system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{1.61}$$

a *hyperbolic fixed point* is a fixed point $\boldsymbol{x}^*$ for which all the eigenvalues of the Jacobian matrix

$$D\boldsymbol{f} = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_1} f_2 & \dots & \partial_{x_1} f_n \\ \partial_{x_2} f_1 & \partial_{x_2} f_2 & \dots & \partial_{x_2} f_n \\ \vdots & & & \\ \partial_{x_n} f_1 & \partial_{x_n} f_2 & \dots & \partial_{x_n} f_n \end{pmatrix} \tag{1.62}$$

calculated in $\boldsymbol{x}^*$ have a non-zero real part.

Theorem 1.11 (Hartman-Grobman)**.** *Let $\boldsymbol{f}$ be C^1 on some $E \subset \mathbb{R}^n$ and let $\boldsymbol{x}^*$ be a hyperbolic fixed point that without loss of generality we can assume $\boldsymbol{x}^* = \boldsymbol{0}$. Consider the non-linear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with flow $\phi(t, 0, \boldsymbol{x})$ and the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, where A is the Jacobian $D\boldsymbol{f}(\boldsymbol{0})$. Let $I_0 \subset \mathbb{R}, X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ such that X, Y and I_0 each contain the origin. Then there exists a homeomorphism $H : X \to Y$ such that for all initial points $\boldsymbol{x} \in X$ and all $t \in I_0$*

$$H(\phi(t, 0, \boldsymbol{x})) = e^{tA} H(\boldsymbol{x}) \tag{1.63}$$

holds. Thus the flow of the non-linear system is homeomorphic to e^{tA} (i.e. to the flow of the linear system).

1.6 Embedding dimension

Let us consider the map

$$x_{k+1}^i = f_i(\mathbf{x}_k), \qquad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, \ldots, n \tag{1.64}$$

where the variable x^i is not directly observable.

Definition 1.32 (Time series)**.** For the observable variable

$$\bar{x}_k^i = h(x_k), \tag{1.65}$$

we denote with $\{\bar{x}_k^i\}_{k=1}^T$ the *time series* of observations.

The embedding dimension is a statistical measure which indicates the smallest dimension required to embed an object (as for instance a chaotic attractor) [109] and it is defined as follows.

Definition 1.33 (Embedding dimension)**.** Let us consider the last m element of observations as arranged in the vector ${}_{[m]}\bar{\boldsymbol{x}}_T^i = \{\bar{x}_T^i, \bar{x}_{T-1}^i, \ldots, \bar{x}_{T-m+1}^i\}$ in the observed time series and let us repeat the grouping for each $\bar{x}_k^i$ in the descending order of time $t = T, \ldots, 1$ by dropping the remaining $m-1$ elements.
If m denotes the *embedding dimension* this results in the m-dimensional vectors where

$$\begin{aligned} {}_{[m]}\bar{\boldsymbol{x}}_T^i &= \{\bar{x}_T^i, \bar{x}_{T-1}^i, \ldots, \bar{x}_{T-m+1}^i\} \\ {}_{[m]}\bar{\boldsymbol{x}}_{T-1}^i &= \{\bar{x}_{T-1}^i, \bar{x}_{T-2}^i, \ldots, \bar{x}_{T-m}^i\} \\ &\vdots \\ {}_{[m]}\bar{\boldsymbol{x}}_m^i &= \{\bar{x}_m^i, \bar{x}_{m-1}^i, \ldots, \bar{x}_1^i\}. \end{aligned} \tag{1.66}$$

Remark 1.12. The vector ${}_{[m]}\bar{\boldsymbol{x}}^i_T$ is also called *m-history* and describes a point in an m-dimensional space, where the coordinates are the delayed observed values $\{\bar{x}^i_T, \bar{x}^i_{T-1}, \dots, \bar{x}^i_{T-m+1}\}$. The sequence $\{{}_{[m]}\bar{x}^i_k\}^T_{k=m}$ of points forms a geometric object in this space.

Remark 1.13. F. Takens (1981) [179] showed that if:

1. the variables x^i of the true dynamical system are located on an attractor (i.e. there are no transients),

2. the functions $g(x)$ in the true dynamical system and the observation function $h(x)$ are smooth,

3. $m > 2n - 1$,

the sequence $\{\bar{x}^i_t\}^T_{t=m}$ is topologically equivalent to the object generated by the true dynamical system described by Eq. (1.64).

1.6.1 Time lag

Instead of considering the m element of observations as arranged in the vector

$${}_{[m]}\bar{\boldsymbol{x}}^i_T = \{\bar{x}^i_T, \bar{x}^i_{T-1}, \dots, \bar{x}^i_{T-m+1}\},$$

one may take the vector

$${}_{[m]}\bar{\boldsymbol{x}}^i_{T-\tau} = \{\bar{x}^i_T, \bar{x}^i_{T-(1+\tau)}, \dots, \bar{x}^i_{T-(m-1+\tau)}\},$$

by sampling the time as follows $t = T, T-(1-\tau), \dots, \tau$. This vector is called *delayed* or *time lagged* vector and the delay τ corresponds to the spacing between the observations.

Mutual information The delay τ is determined in a way that the values ${}_{[m]}\bar{\boldsymbol{x}}_T^i$ and ${}_{[m]}\bar{\boldsymbol{x}}_{T-\tau}^i$ are "sufficiently independent to be useful as coordinates in a time-delay vector but not so independent as to have no connection with each other at all" [161]. To this end, it might be useful recurring to the mutual information as defined by

Definition 1.34 (Mutual information [47])**.** Let us consider two jointly discrete random variables X and Y, the *mutual information* is the double sum:

$$\mathrm{I}(X;Y) = \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p_{(X,Y)}(x,y) \log\left(\frac{p_{(X,Y)}(x,y)}{p_X(x)\, p_Y(y)}\right), \tag{1.67}$$

where $p_{(X,Y)}$ and p_X, p_Y denote, respectively, the joint distribution and the marginal probability mass functions of X and Y.

1.7 Chaos

As previously mentioned the Logistic Map exhibits an irregular behaviour that, according to the following definitions, we call chaotic.

Definition 1.35 (Closure)**.** Let S be a subset of $\mathbb{R}^n$. The *closure* of S is the set of points x such that every open ball centred at x contains a point of S and it is denoted as $\overline{S}$.

Definition 1.36 (Dense set)**.** Let be $D \subset S$; D is *dense* in S if $\overline{D} = S$.

Example 1.10. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 1.37 (Topological transitivity)**.** The map $f : S \to S$ is said to be *topologically transitive* if for any pair of open sets $U, V \subset S$ there exists $n \in \mathbb{N}$ such that $f^{on}(U) \cap V \neq \emptyset$.

Remark 1.14. The idea is that a topologically transitive map has points that move under iteration from one arbitrarily small neighbourhood to any other. This means that the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map.

Definition 1.38 (Sensitive Dependence on Initial Conditions - SDIC)**.** $f : S \to S$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in S$ and any neighbourhood N of x, there exists $y \in N$ and $n \geq 0$ such that $\mid f^{on}(x) - f^{on}(y) \mid > \delta$.

Remark 1.15. Sensitive dependence on initial conditions for a map means that if there exist points arbitrarily close to x, at least one of those will eventually move away from x by at least δ under iteration of f. Such a behaviour may magnify small errors caused by round-off errors in computations.

Example 1.11. The Logistic Map possesses sensitive dependence on initial conditions for $\mu > 2 + \sqrt{5}$.

A popular definition of chaos is

Definition 1.39 (Chaos - Devaney [52])**.** The map $f : S \to S$ is said to be *chaotic* on S if

1. f is topologically transitive
2. The set of the periodic points is dense in S.
3. f has sensitive dependence on initial conditions

Banks et al. (1992) [15] have shown that the first two conditions are sufficient for defining chaos when S is a not a finite set.

Theorem 1.12 (Banks et al. [15]). *If the map $f : S \to S$ is topologically transitive and is dense in S, then f has sensitive dependence on initial conditions.*

Now we are in position to restate Theorem 1.7.

Theorem 1.13. *(Period three implies chaos - Li and Yorke [111]) If f has a periodic orbit of period three, then f is chaotic.*

1.8 A measure of sensitive dependence on initial conditions

Now let us consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$; the recursive expression

$$\boldsymbol{x}_{N+1} = \boldsymbol{f}(\boldsymbol{x}_N), \tag{1.68}$$

and two initial points $\boldsymbol{x}_0 - \boldsymbol{y_0}$ close to each other such that

$$\boldsymbol{x}_0 - \boldsymbol{y_0} = \boldsymbol{\Delta_0}. \tag{1.69}$$

We denote the iterations from the first to the N-th as

$$\begin{aligned} \boldsymbol{x_1} - \boldsymbol{y_1} &= \boldsymbol{f}^{\circ 1}(\boldsymbol{x}_0) - \boldsymbol{f}^{\circ 1}(\boldsymbol{y}_0) \\ &\vdots \\ \boldsymbol{x}_N - \boldsymbol{y}_N &= \boldsymbol{f}^{\circ N}(\boldsymbol{x}_0) - \boldsymbol{f}^{\circ N}(\boldsymbol{y}_0), \end{aligned} \tag{1.70}$$

their linear approximations as

$$\boldsymbol{x_1} - \boldsymbol{y_1} \approx \frac{d\boldsymbol{f}^{\circ 1}(\boldsymbol{x}_0)}{d\boldsymbol{x}} \boldsymbol{\Delta}_0 \tag{1.71}$$

$$\vdots$$

$$\boldsymbol{x}_N - \boldsymbol{y}_N \approx \frac{d\boldsymbol{f}^{\circ N}(\boldsymbol{x}_0)}{d\boldsymbol{x}} \boldsymbol{\Delta}_0,$$

and by $\Lambda^1, \dots \Lambda^N$ the corresponding N Jacobian matrices evaluated in $\boldsymbol{x}_0$, i.e. $\Lambda^k = \dfrac{d\boldsymbol{f}^{\circ k}(\boldsymbol{x}_0)}{d\boldsymbol{x}}$, $k = 1, \dots, N$.

For all $N \in \mathbb{N}$ let A^N the Jacobian matrix of f^{oN} in x^0 and assume that A^N has n real eigenvalues $\Lambda_1^N, \Lambda_2^N, \dots, \Lambda_n^N$ ordered in such a way that $\Lambda_1^N \geq \Lambda_2^N \dots \geq \Lambda_n^N$.

Definition 1.40 (Lyapunov exponents)**.** The real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ defined by

$$\lambda_i = \lim_{N \to \infty} \frac{1}{N} \log_2(\Lambda_i^N) \tag{1.72}$$

are called *Lyapunov exponents*.

Remark 1.16. Notice that if we consider the trajectories departing from x_0 and y_0, then we have

$$y_N - x_N = f^{oN}(y_0) - f^{oN}(x_0) \simeq A^N(y_0 - x_0) \tag{1.73}$$

Therefore the leading Lyapunov exponent is the rate at which nearby trajectories diverge (see for example [171]), [50]). This indicates how fast predictability of the system is lost. For this reason Lyapunov exponents are often used as a measure of chaos.

Definition 1.41 (Lyapunov spectrum)**.** The set of all Lyapunov exponents is called the *Lyapunov spectrum* and the sign of the Lyapunov exponents determines whether stretching and contracting occur in a dynamical system.

Example 1.12. Let us consider the Logistic Map with parameter $\mu = 2.5$. The corresponding fixed point is $x_2^* = 0.6$ and the product of derivatives in 0.6 is $\prod =$

0.5^N. The Jacobian is simply the derivative therefore $\Lambda^N = 0.5^N$. The unique Lyapunov exponent is $\lambda = \log_2(0.5^N)/N = \frac{N}{N}\log_2\frac{1}{2} = -\log_2(2) = -1$ which means that the orbit rapidly converges to the fixed point. In Figure 1-9 are listed some Lyapunov exponents of the Logistic Map while in Figure 1-10 the Logistic Map and the Lyapunov exponents are plotted versus the μ parameter.

t	x_t	$f'(x_t)$	$\prod_{t=1}^{N} f'(x_t)$	$\lambda(N)$
1	.600	0.799	0.799	-.321
2	.960	3.680	2.944	.778
3	.153	2.771	8.158	1.009
4	.520	0.160	1.307	0.096
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	.262	1.899	$0.178 \cdot 10^7$	0.989
22	.774	2.195	$0.392 \cdot 10^7$	0.995
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
99	.221	2.225	$0.598 \cdot 10^{30}$	0.999

Figure 1-9: Lyapunov exponents of the Logistic Map from H. W. Lorenz [113].

1.9 Measure of an attractor

The calculation of the dimension of chaotic attractors can be based on: 1) the geometry of the attractor) 2) information (e.g. the frequency with which a trajectory visits various parts of the attractor), 3) the dynamic properties of the attractor [73].

1.9.1 Geometry of the attractor

Let us define $B_r(u)$ as the open ball centred in u and with radius r as follows

$$B_r(u) = \{x \in \mathbb{R}^n | \, |x - u| < r\} \tag{1.74}$$

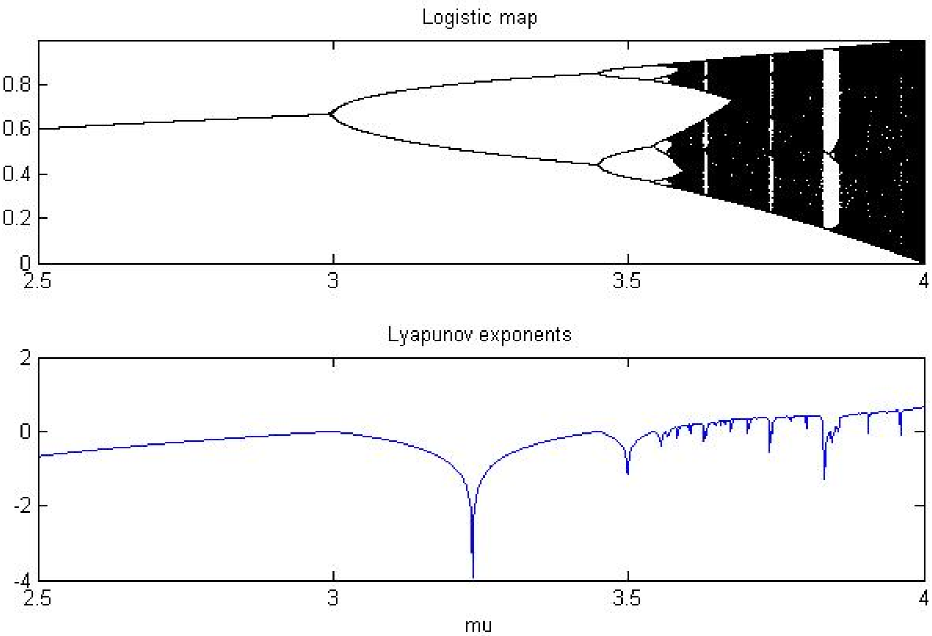


Figure 1-10: Figure on top: Logistic Map versus μ. Figure below: Lyapunov exponents versus μ.

Definition 1.42 (Box-counting dimension)**.** Let us take the minimum set of balls of radius ϵ that can cover the attractor and let us denote $N(\epsilon)$ the minimum number of balls needed. The *box-counting dimension* is

$$D_{BC} = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} \tag{1.75}$$

Example 1.13. D_{BC} is 1 for a line, 2 for an area and $\dfrac{\log 3}{\log 2}$ for the Sierpinski triangle graphically represented in Fig. 1-11.

Non-uniform open balls of radius less than $< \epsilon$ can be used to cover the attractor instead of using balls of the same radius. We start by defining a metric and then we show the corresponding measure.

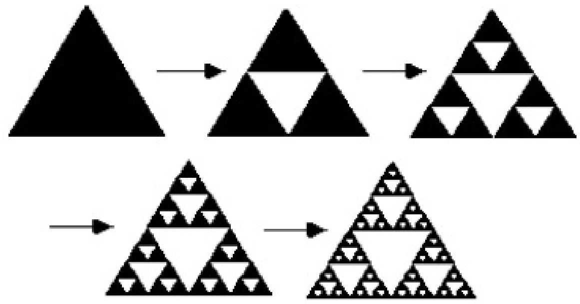


Figure 1-11: Sierpinski triangle. Ref. Devaney [53].

Definition 1.43 (Hausdorff measure)**.** Let us consider a compact set $X \subset \mathcal{R}^n$; for each real number $d \geq 0$, we define the d-dimensional *Hausdorff measure* of X

$$\mathcal{H}^d(X) = \lim_{\epsilon \to 0} \quad \inf_{(u_i),\,(r_i)} \sum_{i=1}^{\infty} r_i^d \tag{1.76}$$

where the infimum is taken on the set of the sequences (u_i) of points of X and of positive numbers (r_i) such that

$$X \subset \bigcup_{i=1}^{\infty} B_{r_i}(u_i) \text{ and } r_i < \epsilon \text{ for all } i. \tag{1.77}$$

Definition 1.44 (Hausdorff dimension)**.** The *Hausdorff dimension* of a compact set X is

$$D_H(X) = \inf \left\{ d > 0 : \mathcal{H}^d(X) = 0 \right\} \tag{1.78}$$

Remark 1.17. The Hausdorff dimension for many types of sets are listed in [3]. For example the set of periodic points of the Logistic Map, with $\mu = \mu_c$ (see Sec. 1.4.2), has Hausdorff dimension 0.538.

Definition 1.45 (Fractal dimension)**.** An object is said to have a *fractal dimension* if its dimension is a noninteger number.

Definition 1.46 (Strange attractor)**.** *Strange attractors* are attractors specific to chaotic systems that possess the following properties:

- They attract trajectories (at least those that start from points close to them).
- They are of the SDIC type, i.e. if ones takes a pair of initial points close to each other, if the trajectories they originate are attracted by the strange attractor, they will diverge more and more with the passage of time.
- Their dimension is fractal.

Remark 1.18. When the attractor of a dynamical system is a small non-integer number, there is an indication that the attractor is strange[113].

1.9.2 Measures of information

The box-counting dimension of Def. 1.42, may be difficult to compute. A diffentent and easier to compute fractal dimension can given, in particular, tor the orbit of a dynamical system [11].

Correlation

Definition 1.47 (Heaviside function)**.** The *Heaviside* or *step function* is defined as follows

$$\mathcal{H}(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.79}$$

Definition 1.48 (Correlation integral [113])**.** The *correlation integral* is a spatial correlation measure aiming to measure the degree of 'kinship' between two different points on the (strange) attractor. This integral is defined for m sub-series of the orbit γ as

$$C(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=m}^{N} \mathcal{H}\left(r- \|_{[m]}\, \bar{\boldsymbol{x}}_m^i -_{[m]}\, \bar{\boldsymbol{x}}_m^j \,\|\right) \qquad \text{with} \quad i \neq j, r \geqslant 0 \tag{1.80}$$

with ${}_{[m]}\bar{\boldsymbol{x}}_m^i$ as defined in 1.6.

An estimator of the correlation integral is the correlation sum.

Definition 1.49 (Correlation function [11])**.** Let $\gamma = \{x_1, x_2, \dots\}$ be an orbit of the map f on $\mathbb{R}^n$. Given $r > 0$, we define the *correlation function* $C(r)$ as

$$C_\gamma(r) = \lim_{N\to+\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \mathcal{H}(r - |x_i - x_j|) \tag{1.81}$$

Hence $C(r)$ is approximately the proportion of pairs of orbit points having distance less than r.

Remark 1.19. Clearly $C_\gamma(r)$ increases from 0 to 1 as r increases from 0 to $+\infty$.

Definition 1.50 (Correlation dimension [73] [75])**.** The *correlation dimension* is defined as

$$D_C(\gamma) = \lim_{r\to 0} \frac{\log(C_\gamma(r))}{\log(r)}, \tag{1.82}$$

if such a limit exists. For this reason if $D_C(\gamma) = d$, for $r > 0$ "small", one has that $C_\gamma(r) \approx r^d$.

Remark 1.20. It can be shown (see [156]) that the correlation dimension defined in Eq. (1.82), is an upper bound of the Hausdorff dimension.

Entropy

Another indicator of the amount of information produced on an attractor is the Kolmogorov-Sinai (KS) entropy (or just Kolmogorov entropy) and it is a useful indicator of chaos. In fact it has been shown, (see for example Sivakumar Berndtsson (2010) [171]), that the entropy value KS converges to a positive value when time series are chaotic .

Definition 1.51 (Kolmogorov-Sinai entropy [114])**.** Let us consider a time series $\{x_t\}_1^T$ and let us partition the phase space into hypercubes with side lengths ϵ and denote the resulting n cubes by c_i for $i = 1, \ldots, n$ (see Fig. 1-12). Let us start with an initial value $x(t_1)$ and repeat the measurements at fixed points in time $(t_1 + \delta), (t_1 + 2\delta), \ldots (T)$. The joint probability that the trajectory starting at $x(t_1)$ is in cube c_1 at time $(t_1 + \delta)$, in cube c_2 at time $(t_1 + 2\delta), \ldots$, and in cube c_n at at the final point (T) is denoted by $\rho_{c_1,c_2,\ldots c_n}$.

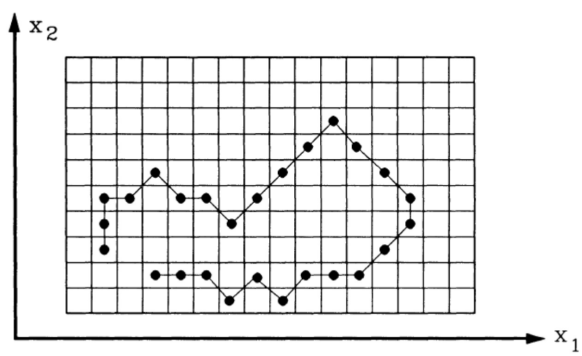


Figure 1-12: A partition of the phase space in the plane with hypercubes. Source H.W. Lorenz [114]

The *Kolmogorov-Sinai entropy* is then defined as

$$KS = -\lim_{\epsilon \to 0} \lim_{T \to \infty} \lim_{\delta \to 0} \frac{1}{T\,\delta} \sum_c \rho_{c_1,c_2,\ldots c_n} \log \rho_{c_1,c_2,\ldots c_n} \tag{1.83}$$

Definition 1.52 (An approximation of the Kolmogorov-Sinai entropy)**.** Since Eq. (1.83) is difficult to compute Grassberger and Procaccia [74] suggested to approximate it trough the correlation integral. In particular, denote with $C^m(\epsilon)$ the correlation integral of a time series with embedding dimension m. Thus the approximation of the Kolmogorov-Sinai entropy is

$$KS_2 = \lim_{m\to\infty}\lim_{\epsilon\to 0}\frac{1}{\delta}\log\frac{C^m(\epsilon)}{C^{m+1}(\epsilon)} \tag{1.84}$$

Remark 1.21. It was shown by Grassberger and Procaccia (1983) [74] that KS_2 is a good approximation of the Kolmogorov-Sinai entropy and that $KS_2 \leq KS$.

Remark 1.22. According to the Pesin's theorem [149] the sum of all the positive Lyapunov exponents gives an estimate of the Kolmogorov–Sinai entropy. So, if $KS > 0$, then the biggest Lyapunov exponent is bigger than zero and the system is chaotic, i.e. by "increasing the embedding dimension the Kolmogorov entropy approaches a finite and positive value" [113].

In order to deal with noise, Jayawardena et al. (2010)[89] have presented a modified KS entropy that is closer to the entropy of the nonlinear system calculated by the Lyapunov spectrum than the general correlation entropy and that is more robust to noise than the KS correlation entropy.

Definition 1.53 (Modified correlation entropy (MCE) - Jayawardena et al. [89])**.** Given the correlation sum computed for two values of the embedding dimension e.g. m and m + 2, the modified correlation entropy is

$$KS_3 = \lim_{m\to\infty}\lim_{\epsilon\to 0}\frac{1}{2\delta}\log\frac{C^m(\epsilon)}{C^{m+2}(\epsilon)} + \frac{1}{2\delta}\log\frac{m - \dfrac{d\log C^m(\epsilon)}{d\log\epsilon}}{m - D_C}. \tag{1.85}$$

Chapter 2

Signal analysis: Spectral analysis, Recurrence Plot and RQA measures

2.1 Spectral analysis

Definition 2.1 (Frequency)**.** The *frequency* ν is the number of occurrences of a repeating event or oscillation per unit of time. Therefore if an event is periodic of period T then

$$\nu = \frac{1}{T}. \tag{2.1}$$

Frequency is measured in Hertz (Hz) (i.e. in cycles per second, cps). Alternatively $\omega = 2\pi v$ denotes the angular frequency.

Definition 2.2 (Spectral analysis)**.** The analysis of a signal in terms of a spectrum of frequencies or related quantities such as energies, eigenvalues, etc. is called *spectral analysis.*

Definition 2.3 (Phase)**.** Let $x(t)$ be a time series or a periodic signal and T be its period that is, the smallest positive real number such that

$$x(t+T) = x(t) \quad \forall\, t. \tag{2.2}$$

Then the phase of $x(t)$ with respect to the initial time t_0 is

$$\varphi(t) = 2\pi \left[\!\!\left[\frac{t - t_0}{T} \right]\!\!\right] \tag{2.3}$$

where $[\![\,\cdot\,]\!]$ denotes the fractional part of a real number.

A sinusoid can be represented mathematically by the Euler's formula, i.e. as the sum of two complex-valued functions:

$$A \cdot \cos(\omega t + \theta) = A \cdot \frac{e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta)}}{2}, \tag{2.4}$$

where i is the imaginary unit, A the amplitude (i.e. the maximum absolute height of the curve), ω the frequency (i.e. how rapidly the function oscillates), θ the phase (i.e. the starting point, in angle degrees, for the cosine wave).

The frequency of the wave measured in Hz is $\omega/2\pi$ or can be denoted as:

$$A \cdot \cos(\omega t + \theta) = \mathrm{Re}\{A \cdot e^{i(\omega t+\theta)}\} \tag{2.5}$$

with $\mathrm{Re}\{\cdot\}$ meaning the real part.

In fact

$$\mathrm{Re}\{A \cdot e^{i(\omega t+\theta)}\} = \mathrm{Re}\{A \cdot \cos(\omega t + \theta) + iA \cdot \sin(\omega t + \theta)\} = A \cdot \cos(\omega t + \theta). \tag{2.6}$$

Definition 2.4 (Phasor)**.** Given a sinusoidal signal represented in the time-domain form as

$$v(t) = A \cdot \cos(\omega t + \theta), \tag{2.7}$$

the *phasor* is the corresponding representation in the frequency-domain form

$$V(i\omega) = A \cdot e^{i\theta} = A\angle\theta \tag{2.8}$$

Therefore a phasor is a "complex number, expressed in polar form, consisting of a magnitude equal to the peak amplitude of the sinusoidal signal and a phase angle equal to the phase shift of the sinusoidal signal referenced to a cosine signal" [155].

Definition 2.5 (Fourier transform)**.** The Fourier transform of a Lebesgue integrable function $f : \mathbb{R} \to \mathbb{C}$ is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i x \xi}\, dx, \tag{2.9}$$

for any real number ξ.

Remark 2.1 (Fourier transform). The Fourier transform is called a representation of the function, in terms of frequency instead of time; thus, it is a frequency domain representation. It is invertible in the sense that $\hat{f}(\xi)$ can be taken back to $f(x)$ therefore, linear operations that could be performed in the time domain have counterparts that can often be performed more easily in the frequency domain. Frequency analysis also simplifies the understanding and interpretation of the effects of various time-domain operations, both linear and non-linear. For instance, only non-linear or time-variant operations can create new frequencies in the frequency spectrum.

Remark 2.2 (Fourier transform). The Fourier transform, applied to a given complex function defined over the real line, returns a frequency spectrum containing all information of the original signal. For this reason the original function can be completely reconstructed through the inverse Fourier transform. However, in order to do so, it is required the preservation of both the amplitude and phase of each frequency component.

Definition 2.6 (Energy spectral density)**.** *Energy spectral density* of a continuous-time signal $x(t)$, describes how the energy of a signal or a time series is distributed with frequency and it is denoted as E_s (unit^2 per second)

$$E_s = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, df \tag{2.10}$$

where

$$\hat{x}(f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} x(t) \, dt, \tag{2.11}$$

is the Fourier transform of the signal and f is the frequency. If the signal is discrete energy is defined as

$$E_s = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.12}$$

Definition 2.7 (Average power)**.** Given a signal $x(t)$ the average power P over all time is:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_0^T |x(t)|^2 \, dt. \tag{2.13}$$

Remark 2.3. A stationary process, may have a finite power but an infinite energy. This because, energy is the integral of power, and the stationary signal continues over an infinite time. For this reason in such cases we cannot use the energy spectral density in Def. 2.6 but we need to introduce the concept of power spectral density.

Definition 2.8 (Amplitude spectral density)**.** In analyzing the frequency content of the signal $x(t)$, one might like to compute the Fourier transform. However, for many signals of interest the Fourier transform does not formally exist. In such a case one can use a truncated Fourier transform where the signal is integrated only over a finite interval $[0, T]$ called as *amplitude spectral density*

$$\hat{x}_{\omega} = \frac{1}{\sqrt{T}} \int_{0}^{T} x(t) e^{-i\omega t} \, dt. \tag{2.14}$$

Definition 2.9 (Power spectral density [154])**.** The *power spectral density* is

$$S_{xx}(\omega) = \lim_{T \to \infty} \mathbf{E}\left[|\hat{x}(\omega)|^2 \right] \tag{2.15}$$

Remark 2.4 (Spectral density). The *spectral density* describes how the energy of a continuous-time signal is distributed with frequency

Definition 2.10 (Spectral density estimation)**.** The spectral density is usually estimated using Fourier transform methods (such as the Welch method) which consist of "sectioning the record, taking modified periodograms of these sections, and averaging these modified periodograms"[188].

Remark 2.5 (Power spectrum). The phasor contains amplitude and phase in polar coordinates. Squared amplitude (or power) is referred to as a *power spectrum* (see Fig. 2-1) and answers the question “How much of the signal is at a frequency"[48]. This because "periodic signals give peaks at a fundamental and its harmonics; quasi-periodic signals give peaks at linear combinations of two or more irrationally related frequencies (often giving the appearance of a main sequence and side-bands); and chaotic dynamics give broad band components to the spectrum. Indeed this later may be used as a criterion for identifying the dynamics as chaotic" [48].

Definition 2.11 (Harmonic frequencies)**.** Let us consider a signal $x(t)$, the set of possible frequencies $\omega_j = j/n$ for $j = 1, 2, \ldots, n/2$ in which it could be decomposed

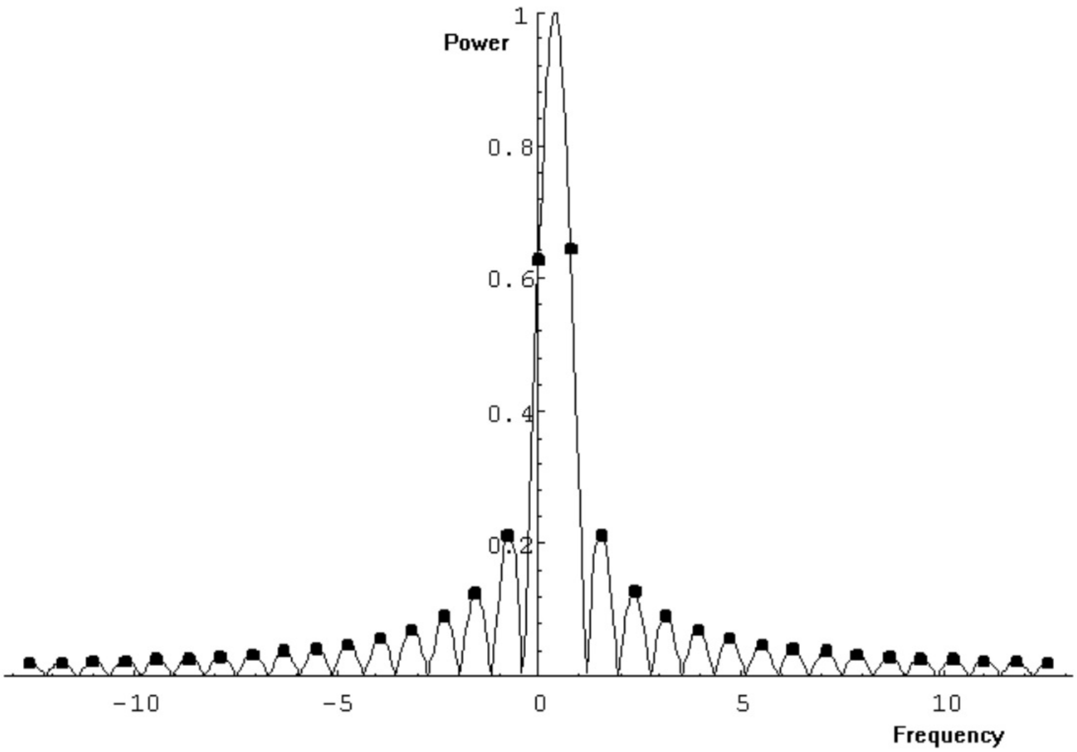


Figure 2-1: Power spectrum of $e^{i(\omega_0 t+\theta)}$ with frequency $\omega_0 = 0.126\pi$ obtained from a time series of length = 8 and with steps $\Delta = 0.25$. Source Ref. [48].

it is called set of harmonic frequencies.

$$x(t) = \sum_{j=1}^{n/2} \left[\beta_1 \left(\frac{j}{n}\right) \cos(2\pi\omega_j t) + \beta_2 \left(\frac{j}{n}\right) \sin(2\pi\omega_j t) \right] \tag{2.16}$$

Remark 2.6. In order to reconstruct the signal $x(t)$ we need to estimate $\beta_1\left(\frac{j}{n}\right)$ and $\beta_2\left(\frac{j}{n}\right)$, $2*n/2 = n$ parameters by the Fourier transform

$$\beta_1\left(\frac{j}{n}\right) = \sum_{t=1}^{n} \frac{n}{2} \cos(2\pi\omega_j t) \qquad j = 1, 2, \ldots, n/2$$

$$\beta_2\left(\frac{j}{n}\right) = \sum_{t=1}^{n} \frac{n}{2} \sin(2\pi\omega_j t) \qquad j = 1, 2, \ldots, n/2. \tag{2.17}$$

Definition 2.12 (Periodogram)**.** For the signal $x(t)$ the *periodogram* P is the plot of

$$P\left(\frac{j}{n}\right) = \hat{\beta}_1{}^2\left(\frac{j}{n}\right) + \hat{\beta}_2{}^2\left(\frac{j}{n}\right) \tag{2.18}$$

versus j/n for $j = 1, 2, \ldots, n/2$.

2.1.1 Power spectrum of the Logistic Map

As in H. W. Lorenz[114] let us assume that a time series $x_i; j = 1, \ldots, n$ of a single variable has been observed at equidistant points in time. The Fourier transform of the series x_i is defined as

$$\overline{x}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j \exp(-2\pi ijk/n), \qquad k = 1, \ldots, n \tag{2.19}$$

It can be shown that the *autocorrelation function*, defined by

$$\psi_m = \frac{1}{n} \sum_{j=1}^{n} x_j x_{j+m} \tag{2.20}$$

can be written in terms of the Fourier transform:

$$\psi_m = \frac{1}{n} \sum_{k=1}^{n} |\overline{x}_k|^2 \cos\Bigl(\frac{2\pi mk}{n}\Bigr). \tag{2.21}$$

Inverting Eq. (2.21), we get

$$|\overline{x}_k|^2 = \frac{1}{n} \sum_{m=1}^{n} \psi_m \cos\Bigl(\frac{2\pi mk}{n}\Bigr). \tag{2.22}$$

The graph obtained by plotting $|\overline{x}_k|^2$ as a function of the frequency $2\pi/n$ is the power spectrum.

Remark 2.7 (Power spectrum interpretation). A power spectrum displaying several distinguishable peaks is a sign of *quasi-periodic* behaviour. Dominating "peaks represent the basic frequencies of the motion, while minor peaks can be explained as linear combinations of the basic frequencies. If the underlying system is discrete, a single peak corresponds to a period-2 cycle, the emergence of two additional peaks to the left and to the right sides of the first peak, respectively, correspond to a period-4 cycle, 7 peaks correspond to a period-S cycle, etc."[114]. If peaks emerge in a continuum the time series is either random or chaotic. In Fig. 2-2 it shown the the power spectrum of the Logistic Map for two different values of the bifurcation parameter μ.

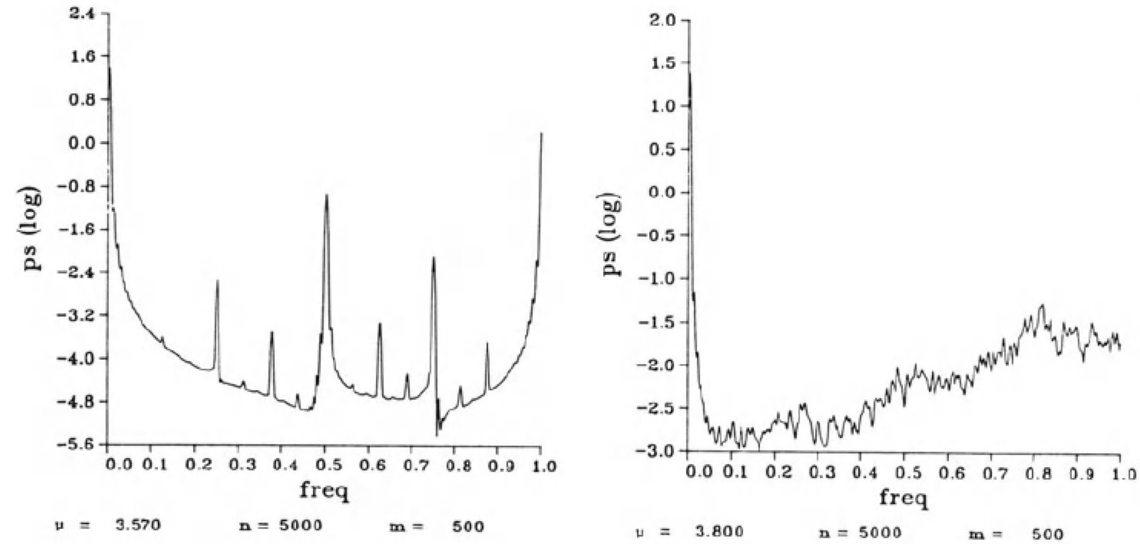


Figure 2-2: Power spectrum of the Logistic Map. Figure on the left displays regular peaks corresponding to the period doubling bifurcations. Figure on the right shows the power spectrum in the chaotic region where it is not possible to isolate dominating frequencies. Source H. W. Lorenz [114].

2.2 Recurrence Plot

Let x_i be the orbit of a dynamical system and let us consider the so called *delayed vectors* denoted as

$$\boldsymbol{x}_i = (x_i, x_{i+1}, ..., x_{i+(m-1)}). \tag{2.23}$$

where m is the embedding dimension (see 1.33).

Fixed a $\epsilon > 0$, for all coordinates (i, j) we can define the function

$$R_{i,j}(\varepsilon) = \mathcal{H}\left(\varepsilon - \parallel \boldsymbol{x}_i - \boldsymbol{x}_j \parallel\right) \quad i, j = 1, \ldots, N. \tag{2.24}$$

Definition 2.13 (Recurrence Plot [55])**.** A *Recurrence Plot* (RP) is a matrix of dots in a $N \times N$ square where the coordinates (i, j) are displayed if $R_{i,j}(\varepsilon) = 1$ i.e. the distance between $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is less than ϵ.

Therefore the RP of $\boldsymbol{x}_i \approx \boldsymbol{x}_j$ shows, for a given t, the indices of times at which a phase space trajectory visits the same area in the phase space. The diagonal is called Line Of Identity (LOI) and vertical segments represent phase space trajectories which remain in the same phase space region for some time, while diagonal lines represent trajectories which run parallel for some time. Thus the RP enables us to investigate the m-dimensional phase space trajectory through a two-dimensional representation of its recurrences. Large scale structures in RP can be classified as homogeneous, periodic, drift and disrupted (*see* Fg. 2-3). Small scale structures (isolated dots, diagonal lines and vertical/horizontal lines and rectangular regions) are the basis of a quantitative analysis of the RPs.

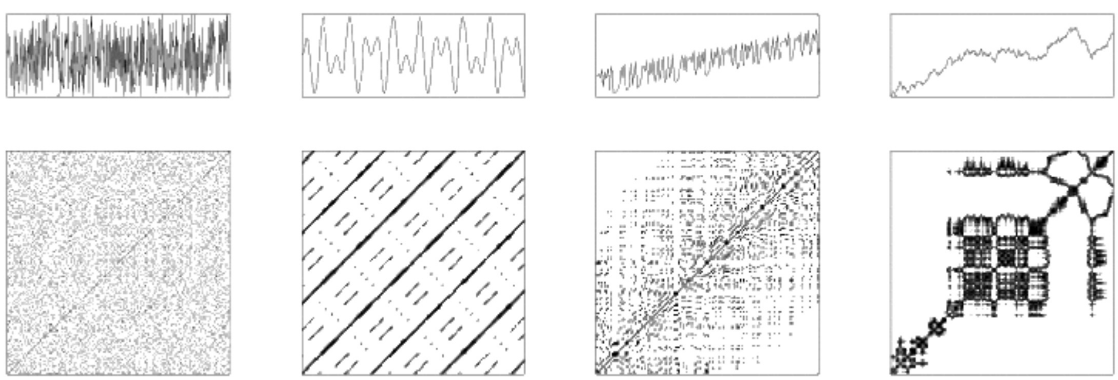


Figure 2-3: Recurrence plots coupled with its time series. From left to right it is shown the time series and the related RP for a white noise, a harmonic oscillation with two frequencies, the Logistic Map and data from an auto-regressive process. Source [5].

2.3 Recurrence Quantification Analysis

Recurrence is defined as the ability of a dynamical system to return to the proximity of the initial point in phase space, and Recurrence Quantification Analysis (RQA) was developed to understand the behaviour of the phase space trajectory of dynamical systems [119].

2.3.1 RQA measures

In the RQA the following measures can be defined:

S.1 Recurrence (REC), i.e. the density of recurrence points in a recurrence plot (RP). This measure counts those pairs of points whose spacing is below a predefined cut-off distance. Its value is a function of the periodicity of the systems: the more periodic the signal dynamics, the higher the REC.

S.2 Determinism (DET) measures the number of diagonals and indicates the duration of stable interactions which is represented, graphically, by the recurrence points in the RP whose forming lines are parallel to the line of identity (LOI). However it must be noted that high values of DET "might be an indication of determinism in the studied system, but it is just a necessary condition, not a sufficient one" (Marwan [117]).

S.3 Maximal deterministic line (MAXLINE) measures the length of the said line found in the computation of DET. According to Eckmann et al. (1987) [55] line lengths on RP are directly related to the inverse of the largest positive Lyapunov exponent therefore small MAXLINE values are "indicative of randomlike behavior". "In a purely periodic signal, lines tend to be very long, so

MAXLINE is large" [122]. Last but not least, there is a positive probability that white noise processes can have a high MAXLINE, although this is unlikely.

S.4 Entropy (ENT) is the Shannon entropy measured in bits because of the base-2 logarithm (which are the bins over the diagonals). "ENT quantifies the distribution of the diagonal line lengths. The larger the variation in the lengths of the diagonals, the more complex the deterministic structure of the RP" [122].

S.5 Trend (TREND) is the regression between the density of recurrence points parallel to the LOI and its distance to the LOI. As TREND measures how quickly a recurrence point departs from the main diagonal, it aims to detect nonstationarity.

S.6 Laminarity (LAM), analogous to DET, measures the number of recurrence points which form vertical lines and indicates the amount of laminar phases (intermittency) in the system.

S.7 Trapping time (TT) measures the average length of the vertical lines, therefore showing how long the system remains in a specific state.

Remark 2.8. With regard to the RP, points on the LOI are excluded from the measures S.1, S.2 and S.3 because they are trivially recurrent. REC, DET, ENT, MAXLINE and TREND are sensitive to parallel trajectories along different segments of the time series. LAM and TT are able to find chaos-chaos transitions. The ratio of determinism is represented by the lengths of diagonal lines.

2.3.2 RQE correlation index

We start from the definition of a rolling window because, as mentioned in Webber, [119] "one of the most useful applications of recurrence quantifications is to examine long time series of data using a small moving window traversing the data. For example, in retrospective studies it is possible to study subtle shifts in dynamical properties just before a large event occurs".

Definition 2.14 (Rolling window)**.** Let us set $\mathcal{I} = \{1, \ldots, n\} \subseteq \mathbb{N}$ and, for each $(k, i) \in \mathbb{N}^* \times \mathbb{N}^*$ with $k < n$ and $i \leq n - k + 1$.

A discrete time *rolling or sliding window* is

$$\mathcal{I}_{k,i} = \{i, \ldots, i + k - 1\} \, (\subset \mathbb{N}) \tag{2.25}$$

where k and i are, respectively, the size and the window's index.

Remark 2.9. It can be noted that the number of windows of size k, as defined in Eq. (2.25), is $q = n - k + 1 (\geq 2)$.

Definition 2.15 (Recurrence quantification epoch [198])**.** When a time series is divided into a series of *windows* or *epochs* of smaller length, the resulting RQA on those multiple sub-series it is called *Recurrence Quantification Epoch (RQE).*

Remark 2.10. When performing the RQE it can happen that some windows may overlap. For example Webber [186] partitioned a time series of 227,957 points in shorter windows (or epochs), each 1.024 seconds long and "adjacent windows were offset by 256 points (75% overlap), fixing the time resolution to 256 ms".

Definition 2.16 (Sampling)**.** For each $(k, i, l) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^*$ we denote $S^l_{k,i}$ the l-th RQA measure of the the epoch

$$S_{k,i} = \{S_t \mid t \in \mathcal{I}_{k,i}\}. \tag{2.26}$$

Definition 2.17 (RQE correlation index)**.** For each $l \neq m$, we denote $\rho^{l,m}$ the Spearman's correlation coefficient between $S_{k,i}^{l}$ and $S_{k,i}^{m}$.

Therefore there are $p = \binom{L}{2}$ pairs of correlations $\rho^{l,m}$ and $q \times p$ pair of epoch correlations $\rho_{k,i}^{l,m}$ so that the product

$$P(\ RQE\ )_{k,i} = \prod_{\substack{l,m=1 \\ l \neq m}}^{L} (1 + \rho_{k,i}^{l,m}) \tag{2.27}$$

can be defined as the *RQE correlation index* for the rolling window $\mathcal{I}_{k,i}$ and it varies between 0 and 2^p.

Definition 2.18 (RQE absolute correlation index)**.** The product

$$P_{abs}(\ RQE\ )_{k,i} = \prod_{\substack{l,m=1 \\ l \neq m}}^{L} (1 + |\rho_{k,i}^{l,m}|) \tag{2.28}$$

can be defined as the *RQE absolute correlation index* for the rolling window $\mathcal{I}_{k,i}$ and it varies between 1 and 2^p.

2.3.3 RQE correlation index on a sample signal

In order to test whether the aforementioned correlation index can help to understand the changes in a times series, we start with a known signal. Let us simulate a random signal distributed as $\varepsilon \sim \mathcal{N}(\mu, \sigma^2)$ and let us change its mean and variance as in Table 2.1 and shown in Fig. 2-4.

Now let us apply the RQA on both the original and the transformed signal by using the parameters in Table 2.2.

The resulting correlation for the original signal and the final signal are displayed

Table 2.1: Perturbed random signal according to a given μ and σ^2. For example for the first interval, 100 points have been randomly generated from a $N(0,1)$ distribution. For the second interval, 40 points have been randomly generated from a $N(1,1)$ distribution, and so on.

N.	Interval		μ	σ^2
1	0	100	0	1
2	101	140	1	1
3	141	200	1	4
4	201	280	4	4
5	281	300	4	6
6	301	400	-5	6
7	401	420	2	6
8	421	500	2	1
8	501	560	0	1
9	561	600	0	3
10	601	700	1	3

Table 2.2: RQA parameters of the perturbed signal.

Embedding	10
Radius	80
Line	5
Shift	1
Epoch	50
Distance	Meandist, Euclidean
Numb. of epochs	642

in Figure 2-5 and Figure 2-6 respectively. It is worth noting that the RQE absolute correlation (2.28) is able to detect 9 of the 10 intervals appearing in Table 2.1 (one being clouded by the windowing filtering).

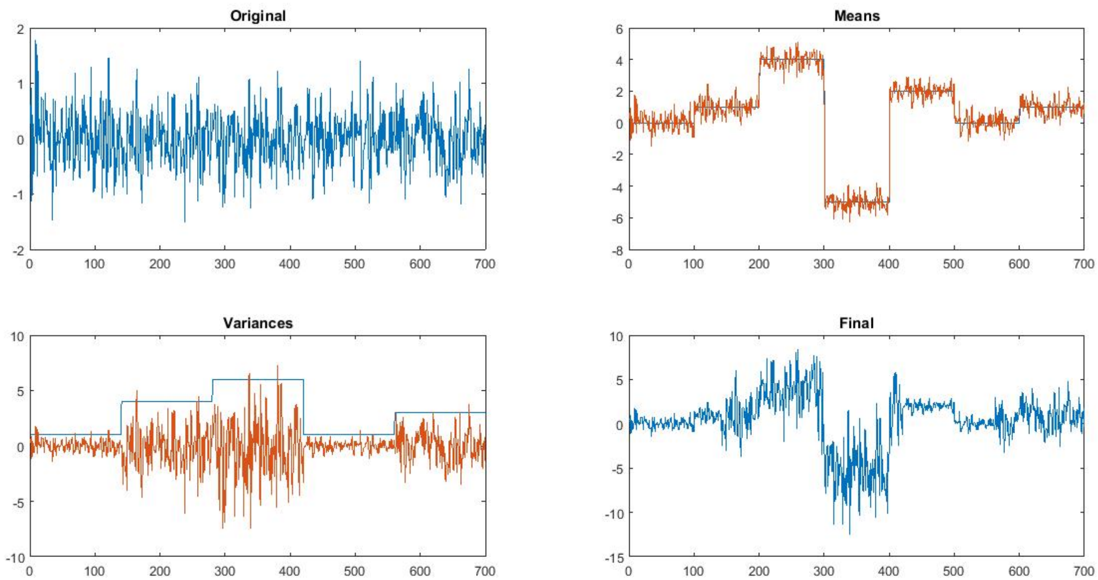


Figure 2-4: Clockwise: original sample signal $N(0,1)$, sample signal with changes in mean, sample signal with changes in variance and resulting final signal with changed mean and variance.

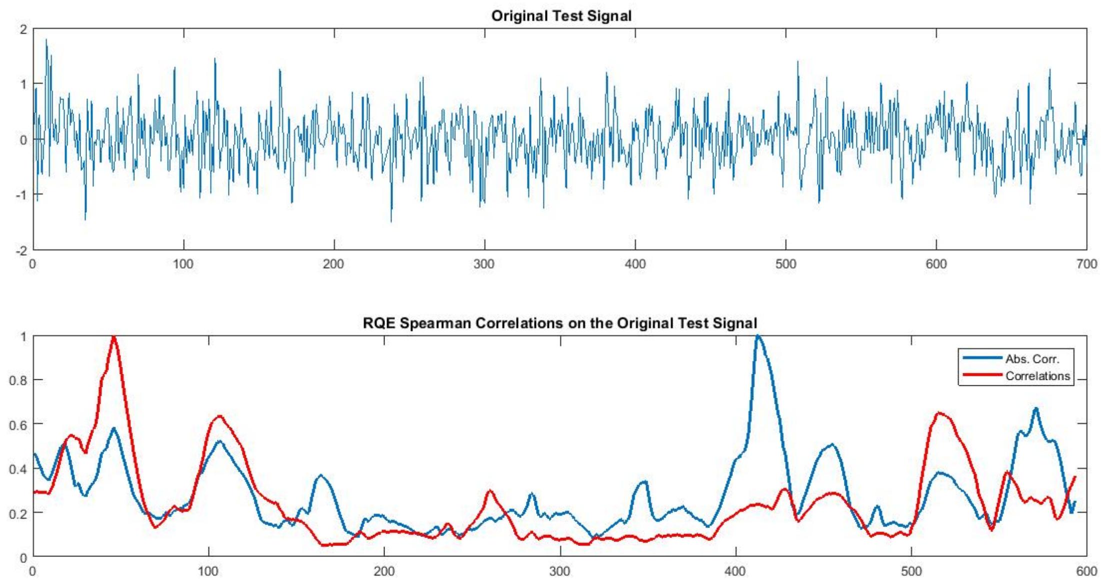


Figure 2-5: Spearman correlations (below) versus the original test signal (above). RQE absolute correlation (in blue) is displayed next to correlation (red). Difference in the x-axis numbering between the picture above and below, is due to the windowing mechanism.

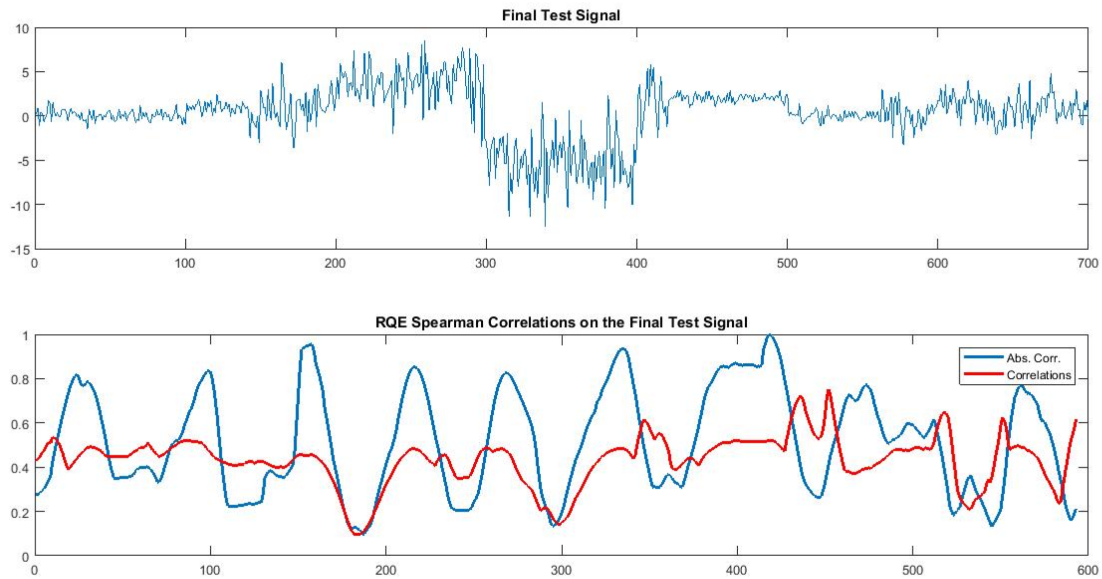


Figure 2-6: Spearman correlations (below) versus the final test signal (above). RQE absolute correlation (in blue) is displayed next to correlation (red). See how the RQE correlation calculated as in Equation 2.28 is closer than the other and it is able to detect more finely changes in the times series. Difference in the x-axis between the picture above and below, is due to the windowing mechanism.

Part II

Non-linearities in economics

Chapter 3

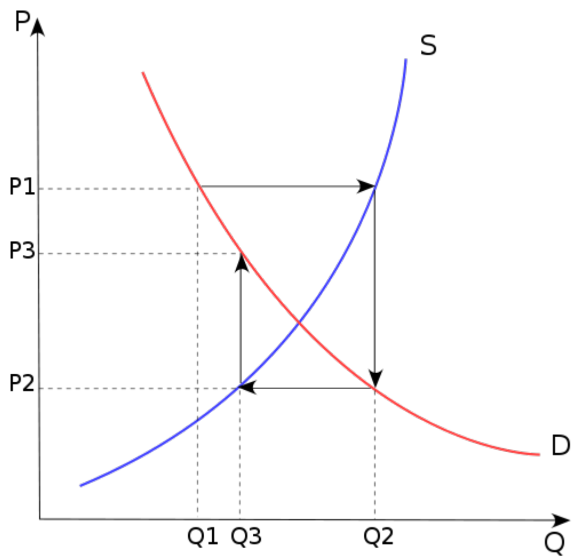
On business cycles and growth

In this Chapter we seek to find out why economics has been concerned with non-linear dynamics Sec. 3.1, the role of the business cycle in economics Sec. 3.2, how RQA can be of help in analysing business cycles Sec. 3.3 and growth Sec. 3.4.

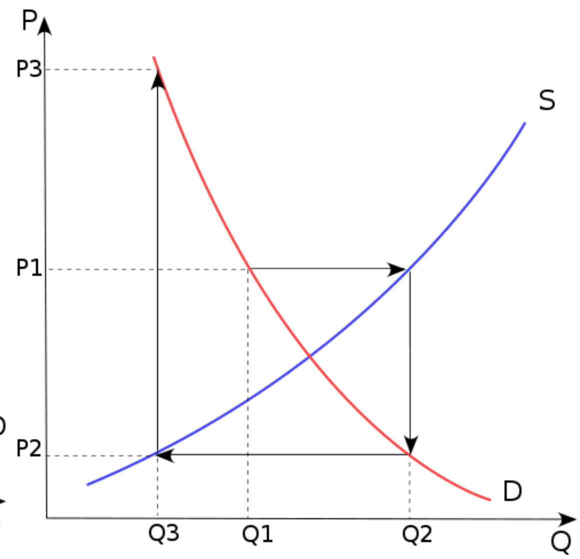
3.1 Non-linearities in economics

The dynamic analysis of non-linear phenomena in economics dates back to 1887 with the first cobweb model (see Fig. 3-1), in which demand and supply would adjust to the market equilibrium with some time lag. Regularly recurring cycles were spotted in the market prices [22] who found that producers base their current output on the price they observe in the market during the previous year. This behaviour can be found for example in agriculture because of the lag between planting and harvesting (for the history of the model and the related theorem see Ezekiel [58]).

One of the main problems of the cobweb dynamic consisted in the complexity of the analysis that it involved; in fact, even Keynes, in his General Theory [97],



(a) Cobweb model, convergent case. Source [1].



(b) Cobweb model, divergent case. Source [1].

Figure 3-1: Fluctuation of prices and quantities in the cobweb model.

did not venture into a dynamic analysis, but he limited himself to explaining why the system headed towards a point of static equilibrium. Only some of those who inherited his legacy such as Harrod, in modelling economic growth, tried to embed the Keynesian analysis into a dynamic perspective so that equilibrium or imbalance situations were generated endogenously (for more details see Part III).

3.2 Business cycles

3.2.1 Background and definition

The pioneering work of Burns and Mitchell (1946)[34] gives what is the considered the classic definition of business cycles

Business cycles are a type of fluctuation found in the aggregate economic activity of nations that organize their work mainly in business enterprises: a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle.

This definition is useful here for:

1. the creation of composite leading, coincident, and lagging indexes based on the consistent pattern of comovement among various variables over the business cycle (e.g. see Shishkin [170]).

2. the identification within the business cycles of separate phases or regimes.

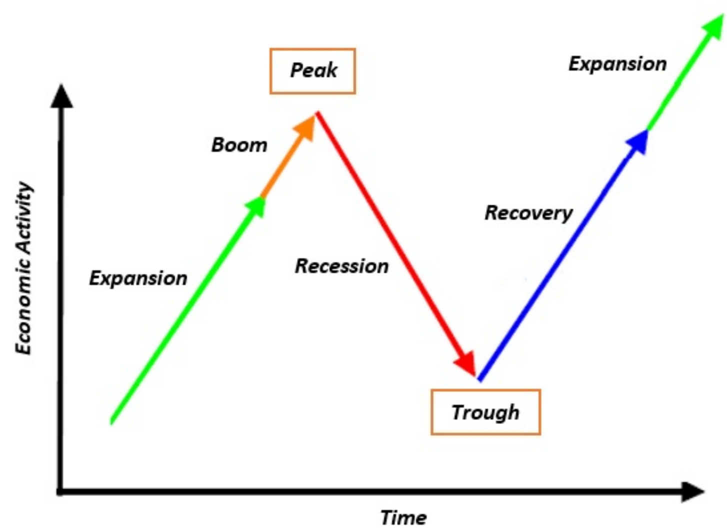


Figure 3-2: Business cycle phases where recession (through) follows expansion (peak).

The National Bureau of Economic Research (NBER)[132] defines a recession as "a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales. A recession begins just after the economy reaches a peak of activity and ends as the economy reaches its trough. Between trough and peak, the economy is in an expansion. Expansion is the normal state of the economy; most recessions are brief and they have been rare in recent decades".

With regard to the cycle Schumpeter [168] mentioned four stages connecting production, stock exchange, public confidence, demand, interests rates and prices:

1. expansion (increase in production and prices while interests rates are relatively low)

2. crisis (stock exchanges crash and multiple bankruptcies of firms occur)

3. recession (drops in both prices and production and rise of interests rates)

4. recovery (stocks prices recover because of the fall in prices and incomes)

In addition Schumpeter suggested that each business cycle has its own typology according to the periodicity, so that a number of cycles were named after their discoverers (see Table 3.1, for a review refer to Korotayev and Sergey (2010)[98]).

Table 3.1: Business cycles taxonomy sorted by length. From left to right: type of business cycle, name of the scholar who identified it, year in which the cycle was identified, time span of the cycle.

Name	Scholar	First studied	Length of cycle
Inventory cycle	Kitchin	1923	From 3 to 5 years
Fixed investment cycle	Juglar	1862	From 7 to 11 years
Demographic cycle	Kuznets	1930	From 15 to 25 years
Technological cycle	Kondratiev	1935	From 45 to 60 years

Fig. 3-2 depicts the four stages of a business cycle: (1) expansion ; (2) *boom period* in which aggregate demand rises much more than aggregate output and this overheats the economy bringing it close to its production ceiling; (3) recession; (4) recovery when economy restarts to grow after a *through*. The vertical distance between the peak and the trough is called *specific cycle amplitude*.

3.2.2 Literature and investigation on the root causes of business cycles

Theories on business cycles (for a review see Hillinger (1992)[83], Zarnowitz (1992)[197], Mullineux (1990)[130] and Cooley (1995)[46]) study volatility of economies and may differ from each other depending on:

1. Their ability to explain cycles without having to rely on outside forces/shocks. They are called *endogenous*. By contrast *exogenous* business cycle theories require the intervention of the above mentioned forces/shocks.

2. The assumption of a general equilibrium framework (neoclassical theories) or the assumption of market imperfections and/or disequilibrium (Keynesian theories).

3. The possibility of attributing cycles to real shocks or monetary shocks, or excess/lack investment or consumption.

4. The derivation of business cycles starting from the interaction of individuals (micro-funded theories) or by considering the aggregate variables as a whole (macro-funded theories).

Until the Keynesian revolution, classical and neoclassical explanations were the mainstream of economic cycles; but after that revolution neoclassical macroeconomics was largely spurn. Starting from 1980s there has been a resurgence of neoclassical approaches, the real business cycle (RBC) theory being the most important of them (see the seminal paper of Kydland and Prescott (1982) [104]). The main assumption of the neoclassical approach, is that individuals and firms respond optimally all the time. Hence public intervention, at the best, has no effect on the economy whereas most of the time has a negative effect. Even slumps represent, given the situation, the optimal solution. The idea behind this approach is that governments should focus on long-term growth instead of focussing on stabilization. RBC differs in this way from Keynesian economics and monetarism. These two other approaches relate recessions to some failure of the market. On the contrary, RBC explains business cycle fluctuations with real shocks such as innovations. The success of RBC relies on the fact that it can mimic many measurable business cycle properties. This notwithstanding RBC models still have some issues notably in interpreting the Solow "residuals", i.e. the part of growth which is not explained by capital accumulation and labour force expansion. Arguably Solow defined the aforementioned residuals as "a measure of our ignorance" whereas RBC describes them as the part of growth which is explained by technical progress.

Hence the identification of the root cause of economic fluctuations varies between schools of thoughts. The Keynesian/post-Keynesian view is that behind the cycles aggregate demand is inherently unstable therefore, unless governed, it can reach levels below or above full employment. This interpretation created a lively debate among econometricians that had to model and measure economies. For example Timbergen (1939)[181], (1951)[182] and Frisch (1933)[67] who maintained that the economy is intrinsically stable and cycles are the effect of exogenous shocks. In particular Timbergen, following Slutsky (1927), (1937)[172], modelled the "economy as a system of stochastically disturbed difference equations, the parameters of which could be estimated from actual time series"[116]. Similarly, according to Frisch, cycles are the effect of delays in new capital, spurred by the increased consumer demand. This would cause recurrent but temporary oscillations in output absorbed in two or three cycles. Schumpeter identified in innovation and creative destruction the factors that deviate the economy from Walrasian equilibrium: "capitalism is by nature a form or method of economic change and not only never is but never can be stationary" [167]. In this context "imperfections" must be intended as those perturbations of the Walras equilibrium that lead to booms because of high profits made by frontrunners. This ends when more and more entrepreneurs copy the strategy of the pioneer firms and, therefore, a greater competition depresses the business margins up to forcing foreclosures. At this point a depression starts and the market is cleaned of unprofitable firms. This equilibrium is maintained until technological or other innovations lay the basis for another expansion. Phelps (1969) [151] and Lucas (1972) [115] explain business cycles on the ground of incomplete information, given that "key economic decisions on pricing, investment or production are often made on the basis of incomplete knowledge of constantly changing aggregate economic conditions. As a result, decisions tend to respond slowly to changes in economic fundamentals, and small or temporary economic shocks may have large and long-lasting effects on macroeconomic aggregates" [81]. The so-called Austrian School [184] [59] claims that a sustained period of low interest rates leads to an excessive creation of credit and then to an unstable imbalance between savings and investments. From this point of view a recession (or "credit crunch") is caused by the need of re-establishing the equilibrium. In other terms, monetary shocks influence "relative prices, such as the term structure of interest rates, systematically altering profit rates across economic sectors. Resource use responds to those changes, generating a cyclical pattern of real income. The divergence of the interest rate structure, from the previous and unchanged time preferences, means that the expansion is unsustainable and must end in recession" [6]. The RBC theory [104] [46], which was the mainstream view until the financial crisis of 2008, assumes that markets are perfect. This implies that business cycle, in itself, is the efficient response to exogenous changes in the real economy. Other theories such as the so-called "debt deflation" [64] [124] [23], which gained momentum after 2008, contends that over-indebtedness may lead to liquidations, fall of bank assets, credit crunch and finally to a recession.

Given the different interpretations of business cycles, implications in terms of control of undesired consequences such as unemployment, inflation, etc. differ too. From a control theory [196] point of view the economy may be seen as a dynamical system in which the state is limitedly known and the observations contain noise, or which is chaotic by nature [25], [195], [174], [146] [136]. Therefore, to it applies the research on controlling stochastic dynamic systems, e.g. Kushner (1967)[103], Guo and Wang (2010)[77], Fleming and Rishel (2012)[65], as well as the research on controlling chaotic dynamical systems, e.g. Romieras et al. (1992)[157], Grebogi and Laib (1997)[76], Calvo and Cartwright (1998)[37], Pettini (2005)[150]. More recently noise, coupled with incomplete information, has been approached in terms of static (Bashkirtseva et al. (2017)[19], Bashkirtseva (2017)[18]) or dynamic (Bashkirtseva (2017c)[17]) feedback regulators. However the way in which this can be empirically applied in terms of economic policy has not been resolved yet.

3.2.3 Detecting non-linearities in data

To make the matter even more complicated, economic time series are short because of low sampling frequency. For example, data such as the aggregate capital stock is available only on annual basis, "some prewar U.S. output and price series are only available for benchmark years which may be a decade apart" [54]. For this reason it is a common practice interpolating data to increase data frequency. However, as explained by Dezhbakhsh and Levy (1994)[54], conventional methods are not able to detect stationary processes because "segmented linear interpolation of a stationary process leads to varying moments that may be viewed as an indication of non-stationarity in a conventional sense". Therefore the suggestion is to analyse those "series in the context of periodic time-series models rather than by conventional methods" [54]. To date there are different approaches to finding the periodicity of a signal. Time-frequency representation and wavelet transformation, spectral representation, Fourier analysis, etc. On the other hand, RQA has proved its ability in detecting non-linear behaviour or chaos in several fields (e.g. acoustic emission [153]) as well as in discovering feasible precursors of catastrophic events triggered by Earth's crust phenomena [199].

In fact the ability of RQA to predict catastrophic changes is because RQA is based upon the change in correlation structure of the observed phenomenon that is known to precede the actual event in many different systems ranging from physiology and genetics to economics.

Gorban et al. (2010)[71] studied the behaviour of systems approaching a critical transition by many experiments and observations of groups of humans, mice, trees, grassy plants, and financial time series. They observed that even before obvious symptoms of crisis appear, correlation increases, and, at the same time, variance (and volatility) increases too. More specifically, with regard to finance, their case study of the thirty largest companies from the UK stock market within the period 2006-2008 supports the hypothesis of increasing correlations during a crisis and, therefore, that correlation (or equivalently determinism) increases when the market goes down (respectively decreases when it recovers). Along this line Orlando and Zimatore (2017)[144] defined the so called RQE correlation index and they have shown, on a test signal, that it is able to detect regimes' changes. Moreover, by computing the RQE correlation index on USA GDP data [134], they have found that it may help in anticipating recessions [145].

3.3 Recurrence quantification analysis on the business cycles

As already mentioned there is a debate in the literature whether the dynamics of an economy is chaotic or stochastic, and whether shocks are endogenous or exogenous (e.g. RBC theory, Austrian School, Neo-Keynesian economics, etc.). Most studies concentrated on financial time series (e.g. stock indices) because of accessibility of data, frequency and length and found co-existence of stochastic and chaotic behaviour [120], [121]. This thesis, with an extensive analysis on macroeconomic data (i.e. consumption, investment, capital and income), aims to investigate two issues. The first is the applications of recurrence plots, and their quantitative description provided by RQA, to dynamical regimes of business time series. The second is whether RQA can give some indications on the nature of trade cycles as well as on the nature of macroeconomic variables and the economy.

RQA applications to economics and finance are not widespread, and started later than in other fields [198], [49], [95], [41], [126]. The interest in RQA by economists stemmed from the world financial crisis of 2007-2010 which was not anticipated by a large part of economic literature [102]. In fact, the majority of economists, basing their models on standard equilibrium, implicitly assumed that "economies are inherently stable and that they only temporarily get off track" Colander et al (2009)[44]. Moreover, the paradigm of the rational representative agent, "largely ignored" [44] the risk of new financial products and of the interconnections of markets. Therefore RQA applied to economics was seen as a potential "tool for the revealing, monitoring, analysing and precursoring of financial bubbles, crises and crashes" Piskun and Piskun (2011)[152]. Fabretti and Ausloos (2005)[60] found examples in financial markets where RQA could detect a difference in state and recognize the critical regime such that a warning before a crash (in their case 3 months in advance) can be given. Along this line Addo et al. (2013)[8], looking for signals anticipating financial crises, highlights "the usefulness of recurrence plots in identifying, dating and explaining financial bubbles and crisis" and claims that the findings from the data analysis with recurrence plots "shows that these plots are robust to extreme values, non stationarity and to the sample; are replicable and transparent; are adaptive to different time series and finally, can provide better chronology of financial cycles since it avoids revision of crisis dates through time". Strozzi et al. (2007b)[177] studying the Nordic Spot Electricity Market Data confirm that determinism and laminarity detect "changes more clearly than standard deviation and then they provide an alternative measure of volatility". Moloney et al. (2010)[126] investigating arbitrage-free parity theory for the Credit Default Swaps (CDS) and bond markets questioned the assumption of a stable equilibrium "which is central to the arbitrage-free parity theory". In addition they found evidence of deterministic structures in the data and that "market is being trapped at certain levels" where "equivalence being trapped for a period of time is a characteristic of a nonlinear system (not a periodic or a random system)".

3.4 Growth

As mentioned by Salvadori (2003) [162] "economic growth was central in classical political economy from Adam Smith to David Ricardo, and then in its 'critique' by Karl Marx, but moved to the periphery during the so-called 'marginal revolution'. John von Neumann's growth model and Roy Harrod's attempt to generalise Keynesian principle of effective demand to the long-run re-ignited interest in growth theory. Following the publication of papers by Robert Solow and Nicholas Kaldor in the mid-1950s, growth theory became one of the central topics of the economics profession until well into the early 1970s. After a decade of dormancy, since the mid-1980s, economic growth has once again become a central topic in economic theorising. The recent theory is called 'endogenous growth theory', since according to it the growth rate is determined from within the model and is not given as an exogenous variable".

While Kaldor's theory influenced the academic debate on business cycles, Harrod inspired Solow who, with his seminal paper "A Contribution to the Theory of Economic Growth" (1956) [173], set the basis of modern growth theory. However, recent research based on a thorough reading of Harrod's theory [24, 78], challenges Solow's interpretation "which ultimately dominated the profession's view of Harrod" [78]. The idea that the Harrod model "implied a tendency toward progressive collapse of the economy" and that he invoked a fixed-coefficients production function" has "little to do with the problem of long-run growth as Solow understood it, but instead addressed medium-run fluctuations, the "inherent instability" of economies" [78].

There are several reasons why in this thesis we are dealing with the Harrod's model. First of all, Harrod, through Solow's interpretation, contributed to the foundation of modern growth theory. Secondly, the Harrod model provides a dynamic framework and some guidelines to policy-makers in terms of supply-side policies. In fact they should consider the combination of investment, technological change, population growth, unemployment and aggregate demand. Another reason is that, in his framework, the warranted rate of growth is not a single moving equilibrium, but a “highly unstable” one. This is called Harrod’s knife-edge instability or the Instability Principle.

Similarly but coming from a different angle (i.e. static analysis and microeconomic foundations of macroeconomic dynamics), Leijonhufvud defines the notion of a stability corridor as a time-path in which economic activities "are reasonably well coordinated" [108]. Moreover the "the system is likely to behave differently for large than for moderate displacements from the "full coordination" time-path. Within some range from the path (referred to as "the corridor" for brevity), the system’s homeostatic mechanisms work well, and deviation-counteracting tendencies increase in strength. Outside that range these tendencies become weaker as the system becomes increasingly subject to "effective demand failures". If the system is displaced sufficiently "far out", the forces tending to bring it back may, on balance, be so weak and sluggish that for all practical purposes he Keynesian "unemployment equilibrium" model is as sensible a representation of its state as economic statics will allow. Inside the corridor, multiplier-repercussions are weak and dominated by neoclassical market adjustments; outside the corridor, they should be strong enough for effects of shocks to the prevailing state to be endogenously amplified. Up to a point, multiplier-coefficients are expected to increase with distance from the ideal path. Within the corridor, the presumption is in favor of "monetarist" policy prescriptions, outside of it in favor of "fiscalist". Finally, although within the corridor market forces will be acting in the direction of clearing markets, institutional obstacles of the type familiar from the conventional Keynesian literature may, of course, intervene to make them ineffective at some point. Thus, a combination of monopolistic wage-setting in unionized occupations and legal minimum-wage restrictions could obviously cut the automatic adjustment process short before "equilibrium employment" is reached"[108].

Both views, macroeconomic and dynamic (by Harrod) and static and microfounded (by Leijonhufvud) converge the idea of "existence of thresholds at the start of the mechanisms that are at work" [105]. Therefore the idea of dynamically unstable multiple equilibria or the alternative Harrod's suggestion of a Leijonhufvud's "corridor stability" in our opinion is worth being explored. Especially because, whereas in the 1970s and the 1980s unemployment and stagflation discarded those theories, in the 20th century "in the leading Western economies there have been prolonged periods when more saving would have been beneficial, and others with every appearance of inadequate effective demand" [57]. As the Harrod's model is one of the few able to predict that, "it still deserves serious attention" [57].

Chapter 4

The Kaldor model

In this Chapter we provide some background related to the Keynes multiplier and the multiplier–accelerator model (Sec. 4.1.1), which is instrumental to introduction of the literature on the Kaldor model (Sec. 4.1.2). The illustration of the Kaldor model as described by the author concludes (Sec. 4.2).

4.1 Background

4.1.1 Keynes multiplier and Hansen–Samuelson model

According to Keynes, income Y depends on a parameter m (called Keynes multiplier) times investment I. This multiplier is derived from the marginal propensity to consume ζ. The dynamic relationship between Y and I is

$$Y_t = mI_t = \frac{1}{(1-\zeta)} I_t. \tag{4.1}$$

An explanation of business cycles is the positive (resp. negative) acceleration due to the effect of income variation on capital accumulation. This model is based on the Keynesian approach and it was first described by P. Samuelson, who credited H. Hansen for the inspiration [163]. The resulting "multiplier–accelerator" model (also known as Hansen–Samuelson model) is

$$Y_t = 1 + \zeta(1+\beta)Y_{t-1} - \zeta\beta Y_{t-2} \tag{4.2}$$

with β representing the sensitivity of investments to changes of consumption C

$$I_t = \beta[C_t - C_{t-1}]. \tag{4.3}$$

The second order linear difference Eq. (4.2) displays different solutions depending on the roots of the equation and on the relationship between the parameters [130].

4.1.2 Literature on Kaldor business cycle model

Among economic models, one of the most fruitful applications in the field of chaotic phenomena is the one worked out by Kaldor in 1940 for the business cycle [92]. The author's intention, contrary to the traditional Keynesian multiplier-accelerator concept, was to explain from a macroeconomic viewpoint the fundamental reasons for cyclical phenomena. However Kaldor did not formalize mathematically his model but gave a qualitative description which prompted out authors to firstly specify the equations and, secondly, to find out under which conditions abnormal behaviour could be produced or under which conditions bifurcations or even chaos could be generated.

In his work Morishima (1959)[128] along with Yasui (1953)[194] and Ichimura (1955)[85], [84], were the first in formalizing mathematically Kaldor's ideas and investigating the existence, stability and uniqueness of limit cycles in a nonlinear trade cycle model. Hicks (1950)[82] and Goodwin (1951)[70] furhter studied the model in the form a system of second order, nonlinear, differential equations. In particular Goodwin (1951) built his model on the imbalance between the actual and the desired capital stock and found that without technological progress the equilibrium is unstable and that the economy can oscillate inside a limit cycle. With technological progress, instead, there is no equilibrium and recessions are shorter than expansions (which is in line with the stylized facts on business cycles). Kalecki (1966)[94] suggested to divide the investment process into three steps, the first being the decision, the second the time needed for the production and the last the delivery of capital goods. In such a way the dynamics of capital stock in the economy is described by a non-linear difference-differential equation which exhibits a complex behaviour (including chaos) and, as a result, oscillations of capital induce fluctuations of other economic variables. Rose (1967)[158] introduced the Poincaré -Bendixson Theory for two-dimensional autonomous system and Chiarella (1990)[42] modified the Goodwin model by introducing a model of monetary dynamics with an adaptive expectation of inflation. In this model the velocity of money circulation is a nonlinear function of expected inflation.

Thirty years after the original formulation, Chang and Smith (1971)[40] reanalysed the model and they proved that:

- The necessary and sufficient conditions enunciated by Kaldor in order to determine that a cycle was established were neither necessary nor sufficient. Instead, these conditions stressed that the onset of the cycle and its amplitude were subordinated to the values of the following parameters:

- The velocity of adjustment α;

- The initial perturbations;

- The position of functions I and S with respect to each other (see Fig. 4-1 and 4-2).

- Some additional hypotheses to those adopted by Kaldor are necessary for a cycle to arise. The above-mentioned hypotheses are conditions for the existence of a point of stable equilibrium - an attractor - for each trajectory.

- Based on the hypothesis explained by them they were able to determine the onset of a limit cycle by applying the Poincaré -Bendixson theorem.

Krawiec et al. (1999)[100], (2001)[101] introduced a time delay in their specification of the Kaldor-Kalecki model. Their model admits a limit cycle. Moreover the persistence of cycles in the linear approximation depends crucially on the delay parameter and, additionally, on both the speed of adjustment and the initial disturbances. Finally they noticed that preserving the condition of a s-shaped investment function is not necessary for creating a limit cycle if the mechanism of time delay is introduced into the model. Finally Orlando (2016)[136], (2018)[137] formalized a model in which the investment and consumption are represented by a hyperbolic tangent instead of the usual periodic arctangent. Moreover he proved that his model displays a chaotic behaviour.

4.2 The Kaldor model

Let us to formalize the Kaldor model as follows:

$$
\begin{aligned}
\dot{Y}_t &= \alpha(I_t - S_t) \\
\dot{K}_t &= I_t - \delta K_t
\end{aligned}
\tag{4.4}
$$

where Y_t, I_t, S_t, K_t defines respectively income, investment, saving and capital at time t. In Eq. (4.4) α is the rate by which the output responds to excess investment $I - S$ and δ represents the depreciation rate of capital K.

In addition let us assume with Kaldor that

$$
\begin{aligned}
&I_Y > 0, \quad I_K < 0, \\
&S_Y > 0, \quad S_K > 0.
\end{aligned}
\tag{4.5}
$$

In order to explain the dynamics of I and S, Kaldor suggested that $I = I(Y, K)$ and $S = S(Y, K)$ are non-linear functions of income and capital. This is graphically shown in Fig. 4-1 where the curves $I(Y)$ and $S(Y)$ cross each other in three points A, B and C. Those points correspond to three different equilibrium level as defined by the equality $I = S$.

If production is low at Y_A, that is the level of income corresponding to the equilibrium A, there will be an excess of capacity. This will absorb the increase of aggregate demand and, as consequence, there will be no or little investment. In the case in which the economy is running high, e.g. $Y = Y_C$, the capacity is saturated, hence the cost of an additional unit of capital is increasing. On the other side the yield of investments is decreasing as the more rewarding initiatives have been already

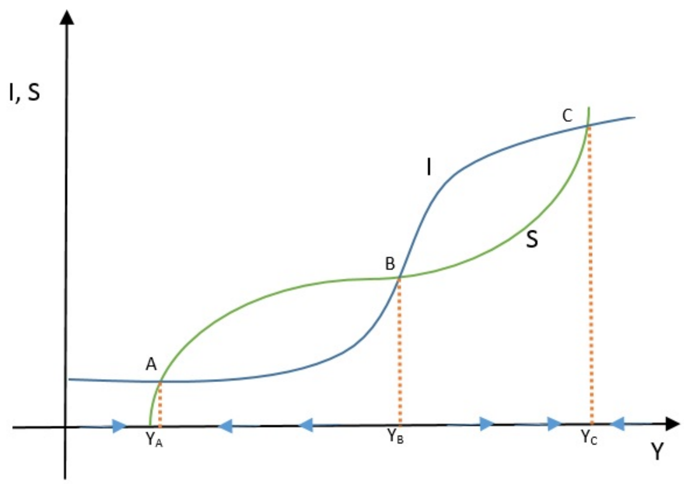


Figure 4-1: Investment (blue curve) and saving (green curve) versus income (x-axis). Equilibrium is for the level of income corresponding to $I = S$. For example if income lays between Y_B and Y_C, the imbalance between investment and saving pushes the economy towards a higher level of income until when $I = S$ in C.

funded. This explains the non-linearity of investments.

Savings rates are assumed to be high for both low and high levels of output. The reason is that for $Y = Y_A$ income is almost completely consumed and families have presumably depleted their finances. For this reason an increase of income would be likely to be directed to restore some savings. For $Y = Y_C$ consumption is already high hence additional income will be saved.

Given the shape of I, and S he income corresponding to the equilibrium is Y_A, Y_B or Y_C. Whilst Y_A and Y_C are stable, Y_B is not (on the left savings exceeds investments and on the right vice versa).

For Kaldor the business cycle is caused by capital accumulation. For example let us assume that $Y = Y_C$ and that I depends on K in such a way that $\frac{dI}{dK} < 0$. In such a case the stock of capital increases and then the marginal productivity of capital declines and so does the investments curve I.

When the production is high this brings down prices which implies that more income can be saved. This $\frac{dS}{dK} > 0$ i.e. the savings curve shifts upwards.

This process has the effect to move down Y_C and up Y_B (see Fig. 4-2) and it will continue until the two points will meet and the S and I curves will be tangent at that point. To the left the ensuing point of equilibrium is for $Y = Y_A$ which represents a crash for the economy.

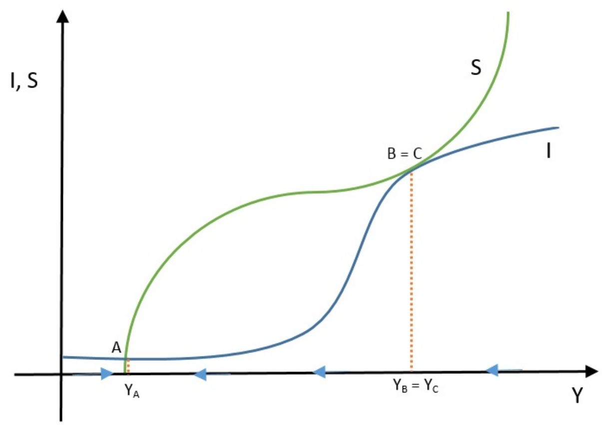


Figure 4-2: Representation of investment I, and saving S - dynamic analysis: because of declining productivity investment shift downwards and the ensuing price's reduction move saving upwards.

Chapter 5

The Harrod's model

As mentioned in the Introduction, Sec. III, the objective of this thesis is twofold: to provide a personal specification of a business cycle model within the Kaldor-Kalecki framework (see Chapter 6) and to choose a chaotic specification of the Harrod-Domar model [174] to prove that a) real data could be obtained by a suitable calibration of model's parameters, b) the calibrated model confirmed theoretical predictions [138].

In this Chapter we illustrate the aforementioned chaotic specification of the Harrod's model that will be tested then in Chapter 8.

5.1 A mathematical specification of the Harrod's model

As already mentioned, Kaldor and Harrod laid down the basis for modern theory on growth and cycles. In particular, Kaldor suggested that growth depends on income distribution and that the shifts between wages and profits determine the savings ratio. Therefore equilibrium is achieved when G_n (the rate of growth required for a full employment) equates G_w (the warranted rate of growth). Keynes argued that in the short run, through the multiplier, more demand (e.g., investments, public spending, exports) translates into an increase in output. Harrod shares the same view with regard to the short run but "notes that investment not only induces production through the multiplier, but also simultaneously expands capacity. On this basis he shows that investment is sustainable only if it is self-consistent, and for this to hold it must follow a particular growth path which he calls the warranted path" [169]. In other terms, in Harrod's view, it is the discrepancy between the natural rate of growth (G_n), the warranted rate of growth (G_w) and the actual one (G), that generates instability. This instability could be lessened when the economy is open to foreign trades.

5.1.1 The Harrod knife-edge

According to Harrod, "for a country in which G_w is tending to exceed G_n, there is by consequence a chronic tendency to depression (because G cannot exceed G_n), a positive value of the balance of trade expressed as a fraction of income (i.e., the net export rate) may be beneficial" [79]. Therefore, Harrod "predicts that incompatibilities between long-term saving and investment opportunity are all but certain to cause prolonged unemployment (which will be structural where G_n exceeds G_w and demand deficient where G_w exceeds G_n) with persistent inflation in addition wherever long-term saving is inadequate for the natural rate of growth" [57]. In terms of public policy 'the difficulties may be too great to be dealt with by a mere anti-cycle policy" [80] hence government should increase public investment when $G_w > G_n$ or, conversely, seek to generate more long term savings when $G_w < G_n$.

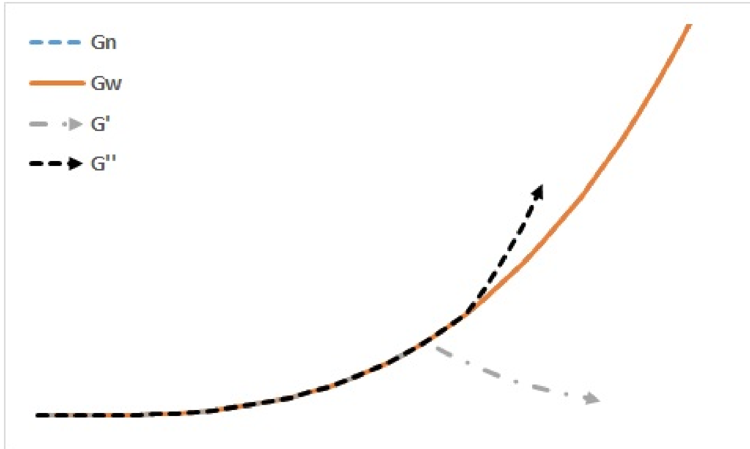


Figure 5-1: The Harrod knife-edge or unstable equilibrium. When $G = G_n = G_w$ there is sustainable full employment. A departure from that may lead to recession (G') or booming periods (G').

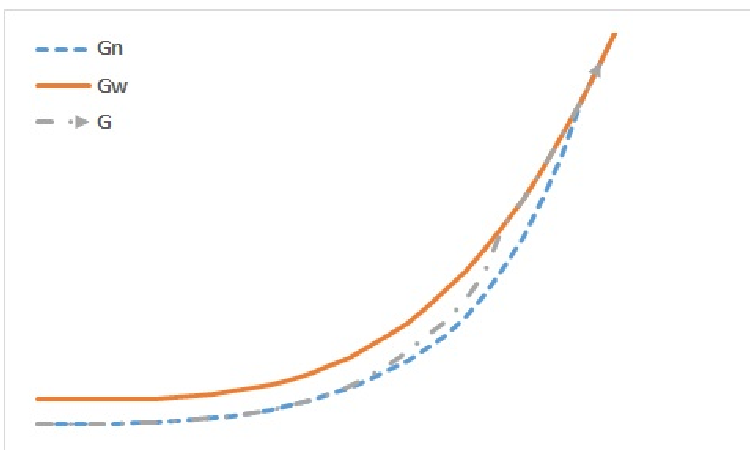


Figure 5-2: Supply side policy to raise the natural growth path. When $G = G_w < G_n$ there is a permanent unemployment equilibrium. Policy-makers may employ supply side policies in order to increase both: the actual growth and G and the natural growth G_n.

5.1.2 Discussion

The model we are testing [174] claims that Harrod's speculation holds true only for a specific set of parameters and with positive net exports coupled with competitiveness in foreign markets. In those specific conditions regular cycles in the long period can be achieved. In the following we list some variables/equations that will be used in the ensuing part where some assumptions will be made and new variables will be identified.

Table 5.1: List of variables in the Harrod model

Variable	Description
I_j	Ex-ante investment including both equipment and desired inventory stocks
I	Ex-post investment including both equipment and effective inventory stocks
S	Ex-post saving
E	Exports
M	Imports
$X = E - M$	Balance of trade
Y	Effective demand
$S/Y = \Sigma$	Share of income saved
$x = X/Y$	Ratio of balance of trade to income (or simply the net export rate)
$I/Y = \Sigma - x$	Share of income invested
$G = \dot{Y}/Y$	Actual rate of growth of domestic income
Y_e	Expected demand
$C_r = I_j/\dot{Y}_e$	Desired capital coefficient[a]
$C = I_j/\dot{Y}$	Actual capital coefficient[b]
$G_w = \dot{Y}_e/Y$	Warranted expected rate of growth of aggregate demand
G_n	Technical progress (rate of growth)
G_f	Rate of growth of foreign demand
ϕ	Sensitivity of the difference between actual and warranted relative changes of demand

[a] "The requirement for new capital divided by the increment of output to sustain which the new capital is required"[79].
[b] "The increase in the volume of goods of all kinds outstanding at the end over that outstanding at the beginning of the period divided by the increment of production in the same period"[79].

As in [174] we assume that:

(A) The desired capital is an increasing function Φ of the difference between the current and the expected change of demand i.e.,

$$C_r = \Phi\left(\frac{\dot{Y} - \dot{Y}_e}{Y}\right) = \Phi\left(G - G_w\right) \tag{5.1}$$

such that $\Phi > 0$. So that, ex-post, at the equilibrium $C_r = C^*$, $I_j = I$, $\Phi(0) = C^* > 1$, $G = G_w$ (or equivalently $Y = Y_e$). Denoted $\phi > 1$ as a reaction parameter representing how sensitive are firms to discrepancies between actual and warranted relative changes of demand, the linearisation of ((5.1)), in $G - G_w$, can be expressed as

$$C_r = \Phi\left(G - G_w\right) = \left[C^* + \varphi(G - G_w)\right] \tag{5.2}$$

(B) According to Alexander [10], changes in the growth rate of income depends on the difference between ex-ante and ex-post investments, that is

$$U = I_j - I = C_r\dot{Y}_e - I, \tag{5.3}$$

so that dividing by Y and considering that $I/Y = (S - X)/Y = \Sigma - x$, we have that the relative gap $u = U/Y$ can be written as

$$u = U/Y = I_j/Y - I/Y = C_r G_w - (\Sigma - x) = \Phi\left(G - G_w\right)G_w - \Sigma + x. \tag{5.4}$$

Therefore $\dot{G}$ can be expressed as a function F of u with F increasing (resp. decreasing) with u and if we assume F to be linear we obtain

$$\dot{G} = F(u) = F\left(\Phi\left(G - G_w\right)G_w + \Sigma - x\right) = \alpha\left\{\left[C^* + \varphi(G - G_w)\right]G_w - \Sigma + x\right\}, \tag{5.5}$$

with $0 < \alpha < 1$ because investment changes in the productive capacity make investment sticky.

(C) The saving rate varies over time depending on unforeseen differences between technical progress and rate of growth and on income fluctuations:

$$\dot{\Sigma} = \varepsilon \left(G_n - G_w\right) + \delta \dot{G}_w, \tag{5.6}$$

where the parameters ϵ and δ represent the sensitivities and the variable $\dot{G}_w$ describes the economic cycle.

(D) Let us introduce the mapping Ψ that we assume to be linear. Moreover let us denote with $\zeta, \sigma, \mu > 0$ the sensitivities of the balance of trade to foreign rate of growth, technical progress and domestic growth rate respectively. Thus we assume that changes in the ratio of the trade balance depend on G_f, G_n and G as follows

$$\frac{\dot{x}}{x} = \Psi(G_f, G_n, G) \quad \text{with} \quad \frac{\partial \Psi}{\partial G_f} > 0, \ \frac{\partial \Psi}{\partial G_n} > 0 \, \text{and} \, \frac{\partial \Psi}{\partial G} < 0. \tag{5.7}$$

We can assume that the mapping Ψ is linear and by denoting the sensitivities $\zeta, \sigma, \mu > 0$ of the balance of trade to foreign rate of growth, technical progress and domestic growth rate, Equation ((5.7)) can be rewritten as

$$\frac{\dot{x}}{x} = \Psi(G_f, G_n, G) = \left(\zeta G_f + \sigma G_n - \mu G - m\right), \tag{5.8}$$

where $m > 0$ implies that $Y(G_f, 0, 0) < 0$ i.e., a constant domestic production without technical progress has a negative effect on the balance of trade or, equivalently, $\zeta G_f - m < 0$.

(E) The expected rate of change of aggregate demand is linear with γ to the difference between G and G_w i.e.,

$$\dot{G}_w = \gamma(G - G_w), \tag{5.9}$$

where $\gamma \geq 1$ denotes how quick the expected rate of growth adjusts to the actual growth.

(F) The dynamics of technological progress is described by a continuous, increasing non-linear function of share of income saved and devoted to investments

$$G_n = G_n(\Sigma) = \beta(\xi - \Sigma)\Sigma, \quad \text{with} \quad \beta > 1 \quad \text{and} \quad 0 < \xi < 1. \tag{5.10}$$

Therefore the Harrod's dynamic can be written as [174] as

$$\begin{aligned} \dot{G} &= \alpha \left\{ \left[C^* + \varphi(G - G_w) \right] G_w - \Sigma + x \right\} \\ \dot{\Sigma} &= \varepsilon \left(G_n - G_w \right) + \delta \dot{G}_w \\ \dot{x} &= \left(\zeta G_f + \sigma G_n - \mu G - m \right) x. \end{aligned} \tag{5.11}$$

By replacing on it Eq. (5.9) and (5.10) we obtain the following specification we want to test

$$\begin{aligned} \dot{G} &= \alpha \left\{ \left[C^* + \varphi(G - G_w) \right] G_w - \Sigma + x \right\} \\ \dot{G}_w &= \gamma(G - G_w) \\ \dot{\Sigma} &= \varepsilon \left[\beta(\xi - \Sigma)\Sigma - G_w \right] + \delta\gamma(G - G_w) \\ \dot{x} &= \left[\zeta G_f + \sigma\beta(\xi - \Sigma)\Sigma - \mu G - m \right] x. \end{aligned} \tag{5.12}$$

where: α, γ, ε, β, δ, ζ, σ, and μ, are the parameters that will be calibrated in Chapter 8.

Part III

A contribution to economic theory

Chapter 6

A new form of Kaldor-Kalecki model on business cycles

Goodwin [70] was of the opinion that "economists will be led, as natural scientists have been led, to seek in nonlinearities an explanation of the maintenance of oscillation"; for this reason, we have investigated economic cycles as being generated by non-linear differential equations showing that the proposed model represents a chaotic behaviour.

The main goal of this Chapter is to study chaotic behaviour within Kaldor's framework as described in Chapter 4. In the following, we suggest an alternative to the usual models available in literature which shows chaotic dynamics, and adheres to Kaldorian specifications. This is obtained by declaring that the investment and saving functions, I and S respectively are: nonlinear, regular, increasing (or at least not decreasing) while capital and income are growing. This will be achieved by new specification of the functions describing the investments and consumption as variants of the hyperbolic tangent function rather then the usual arctangent prosed by the author of this thesis in References [136] and [137]. This is original work.

6.1 The model

In Chapter 4 we presented the Kaldor model in continuous time. Here we discuss the following discrete time version

$$\begin{cases} Y_{t+1} - Y_t = \alpha(I_t - S_t) = \alpha[I_t - (Y_t - C_t)] \\ K_{t+1} - K_t = I_t - \delta K_t, \end{cases} \tag{6.1}$$

where Y, I, S, K define respectively income, investment, saving and capital. In Eq. (6.1) α is the rate by which the output responds to excess investment and δ represents the depreciation rate of capital. As seen in Chapter 4, Kaldor suggested that $I = I(Y, K)$ and $S = S(Y, K)$ are non-linear s-shaped functions of income and capital.

Our proposed modifications are based on the considerations. First of all, it should be noted that the difference in timing between consumption and investments reflects the process of observation, decision (including financing) and actual investment. Therefore we can suppose that investment I_t at time t is proportional to a certain level of capital K_{t-1} at time *t-1* according to a factor which is a function of the difference $(K^d_{t-1} - K_{t-1})$ between desired capital K^d and owned capital K, i.e.

$$I_t = K_{t-1} \cdot f_1(K^d_{t-1} - K_{t-1}). \tag{6.2}$$

Similarly, consumptions C_t at time t can be taken proportional to income Y_t at the same time through a factor which is a function of the difference $Y^d_t - Y_t$ between the desired Y^d and current income Y, i.e.

$$C_t = Y_t \cdot f_2(Y_t^d - Y_t). \tag{6.3}$$

Concerning the form of the function $f_1(K^d - K)$, it should be noted that the desired stock of capital depends on factors such as expected profit rate, expected level of output, etc. which can be linked to Y. In fact, for example, the simple accelerator model assumes that K is proportional to the level of production Y as follows: $I = kY$ (where k is a parameter called "capital-output ratio"). The flexible accelerator model, instead, assumes that investments are modelled as $I = s(K_{t-1}^d - K_{t-1})$, where $s \in (0, 1]$ is a coefficient representing the speed of adjustment. In the context of our model we assume that $0 < f_1 < r$ for some $r < 1$ (where r is a fraction of K_{t-1}).

As to the function $f_2(Y^d - Y)$, it represents the fraction of income Y which is consumed; hence it is reasonable to assume that there exists a constant $c > 0$ such that $c < f_2 < 1$ everywhere.

f_1 and f_2 are clearly increasing (as higher is the difference between what is desired and what is owned and higher is f) we will model the according the specifications that $I = I(Y, K)$ and $C = C(Y, K)$ but, as required by Kaldor, we need to impose the sigmoidality [160]. In order to do so, most of papers on the Kaldor model, have used the *arctg* function (see Fig. 6-1).

However, it is well known that the *arctg* function has some contraindications from a numerical point of view (see Bradford and Davenport (2002)[31], Collicott (2012)[45], Walter (2010)[185], Gonnet and Scholl (2009)[69]). This form is not commonly used in natural sciences for modelling growth and it diminishes its usefulness as, for example, it prevents a connection to some other theories as the classic Solow-Swan growth model. For the above mentioned reasons we suggest to use two variants

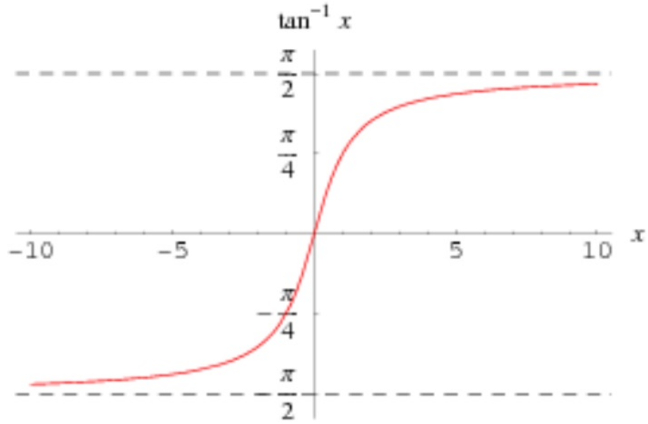


Figure 6-1: Graph of the inverse function. Source Wolfram [192]

.

of the *hyperbolic tangent*, namely:

$$f_1(z_1) = \rho \frac{exp(2z_1/\tau_1)}{exp(2z_1/\tau_1)+1} \quad \text{and} \quad f_2(z_2) = \frac{exp(2z_2/\tau_2)+c}{exp(2z_2/\tau_2)+1}, \tag{6.4}$$

so that $f_1(z_1)$ goes to 0 as $z_1 \to -\infty$ and tends to ρ as $z_1 \to \infty$ whereas $f_2(z_2)$ goes to c as $z_2 \to -\infty$ and tends to 1 as $z_2 \to \infty$. τ is the parameter controlling the slope of the function (see Fig. 6-2).

So far we specified the functions but we still need to identify their arguments. We said that what counts for investment decisions is K^d and Y^d. As it is not possible to exactly know the desired values K^d and Y^d, we can derive them from the actual behaviour of economic agents as follows:

- As mentioned above the desired stock of capital is associated to Y and K, hence it is legitimate to describe the difference $K^d - K$ as a function of the difference between the relative income variation (defined as $\Delta Y/Y$) and the

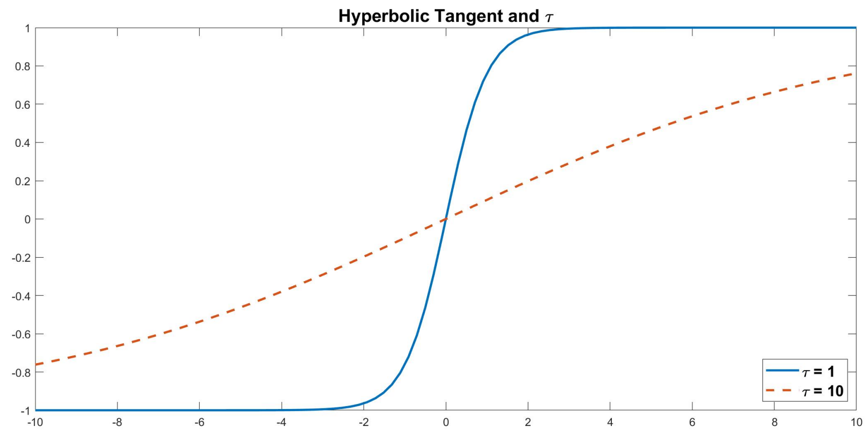


Figure 6-2: Graph of the hyperbolic tangent showing how the parameter τ determines the knee.

relative capital variation (defined as $\Delta K/K$), i.e.

$$K_t^d - K_t = g_1 \left(\frac{Y_t - Y_{t-1}}{Y_{t-1}} - \frac{K_t - K_{t-1}}{K_{t-1}} \right). \tag{6.5}$$

If, for instance, the income variation (at time *t -1*) has been 3%, a smaller capital growth (still at time *t-1*) could be interpreted by entrepreneurs as a need to adapt the stock of capital to economic growth (similarly to the Kalecki assumption (1935)[93] that the saved part of profit is invested and the capital growth is due to past investment decisions).

- Analogously we can describe $Y^d - Y$ as a function of the difference between the relative income variation and the relative consumption variation, i.e.

$$Y_t^d - Y_t = g_2 \left(\frac{Y_t - Y_{t-1}}{Y_{t-1}} - \frac{C_t - C_{t-1}}{C_{t-1}} \right). \tag{6.6}$$

Hence we are assuming that the change in consumption is a kind of barometer of the state of health of the economy, so that, for example, in the case of high inflation families revise and reduce their consumption. In other words, we want to give an account here of the variation in purchasing power due to inflation or to those depressive effects of the business cycle (reduction of workers' contractual power, downsizing followed by dismissals, etc.) or the expansive effect (rise of real wages due to increased workers contract power because entrepreneurs need to expand production, easier access to credit, and so on).

Concerning the form of the functions g_1 and g_2, let us notice that:

1. It is reasonable to assume that the difference $x = \Delta Y/Y - \Delta K/K$, has an upper bound $k > 0$,

2. For $x = \Delta Y/Y - \Delta K/K > 0$ the difference $K^d - K = g_1(x)$ increases with x and tends to $+\infty$ (respectively $-\infty$) as $x \to k$ (respectively as $x \to 0^+$),

3. For $x = \Delta Y/Y - \Delta K/K \leq 0$ we can assume that $K^d - K = g_1(x)$ has a constant negative value,

4. The same remarks can be made for the difference $y = \Delta Y/Y - \Delta C/C$ and $Y^d - Y = g_2(y)$

Hence we shall assume that $g_1 = g_2 = g$ has the form (see Fig. 6-3)

$$g(x) = \begin{cases} h & \text{for all} \quad x \leq 0 \\ \\ -\log((k/x)^s - 1) & \text{for all} \quad x \in (0, k] \end{cases} \tag{6.7}$$

with $h < 0$ such that f_1 is close to zero. It should be noted that the shape of the function g is an approximation of the value function of Kahneman and Tversky (see

Kahneman and Tversky 1979[91]) where on the left the function is assumed to be flat.

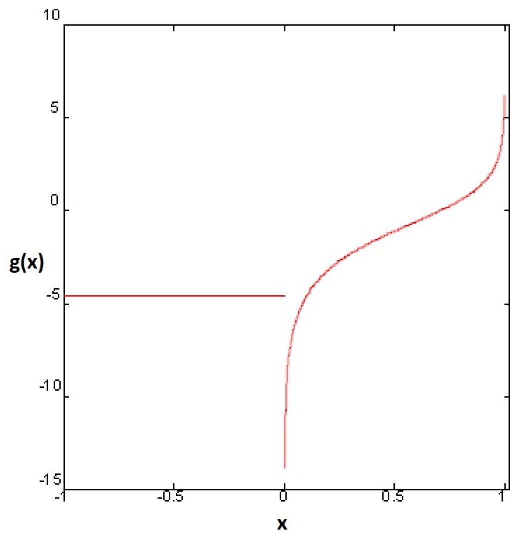


Figure 6-3: Graph of $g(x)$

.

To sum up, the proposed Kaldor model is

$$\begin{cases} Y_{t+1} - Y_t = \alpha \left[f_1 \left(g \left(\frac{Y_{t-1} - Y_{t-2}}{Y_{t-2}} - \frac{K_{t-1} - K_{t-2}}{K_{t-2}} \right) \right) + \right. \\ \qquad\qquad \left. f_2 \left(g \left(\frac{Y_t - Y_{t-1}}{Y_{t-1}} - \frac{C_t - C_{t-1}}{C_{t-1}} \right) \right) - Y_t \right] \\ \\ K_{t+1} - K_t = f_1 \left(g \left(\frac{Y_{t-1} - Y_{t-2}}{Y_{t-2}} - \frac{K_{t-1} - K_{t-2}}{K_{t-2}} \right) \right) - \delta K_t \end{cases} \tag{6.8}$$

These equations, together with (6.4) contain the parameters $\alpha, \delta, \tau_1, \tau_2, \rho, ,k$ with the following meaning:

1. α is the coefficient showing the savings adjustment speed with regard to inves-

tment. Its reciprocal in physics is called delay and measures the time necessary for the adjustment. In other terms, if we assume that the time of reference 1 is equal to a year, then the savings will adjust themselves to the investment over this period. Therefore if $\alpha = 1/2$ this means to assert that six months are sufficient for the verifying of the aforesaid adjustment. In general terms, we would assume that α lies between $1/4$ but less than 2 because such an interval would be economically reasonable.

2. δ is a percentage that determines the fixed capital which is lost during the productive process (due to obsolescence or actual consumption). In the model, when running simulations, we have chosen values between 3% and 6%.

3. τ determines the f function "knee". It is, therefore, a measure of the reactivity of the function to the variation in its argument: as τ grows the function is less steep. We have tried values between 1 and 20.

4. ρ measures the maximum possible level (in capital terms) of investment. This value changes according to the economic system (pre-industrial, industrial, post-industrial) and the type of investment (i.e. high or low capital intensity). A reasonable choice led me to adopt the (0,0.2] interval preferring values around $\rho = 0.16$ for the depiction of our type of economy.

5. $c = 1 - \hat{c}$ represents the average level of consumption. Its complement to 1, multiplied by actual income, determines the minimum level of consumption, therefore it is also called the "base" level. $\hat{c}$, similarly to ρ, is very sensitive to the type of economy in question, but in our context economically admissible values are assumed to be between 40% and 80%.

6. The k parameter changes according to the economic development. For instance it could be that the percentage variation of Y - K (respectively -C) can be low (for developed/less volatile economies) or high (for developing/more volatile economies). It has been experimentally observed that if the maximum admissible intervals of the aforesaid variations are between -1 and +1 (respectively between -2 and 2 the output varies between -0.23 and +0.23 (respectively -1.2 to +1.2).

In each of the above mentioned cases - for a quite broad range of parameters - we found that the model shows a chaotic dynamics. In addition the system can also be used for modelling lagging perturbations or shocks (see Sec. 6.1.1).

Figures 6-4 and 6-5 illustrate two examples of the system's dynamic which differ only because a different initial condition.

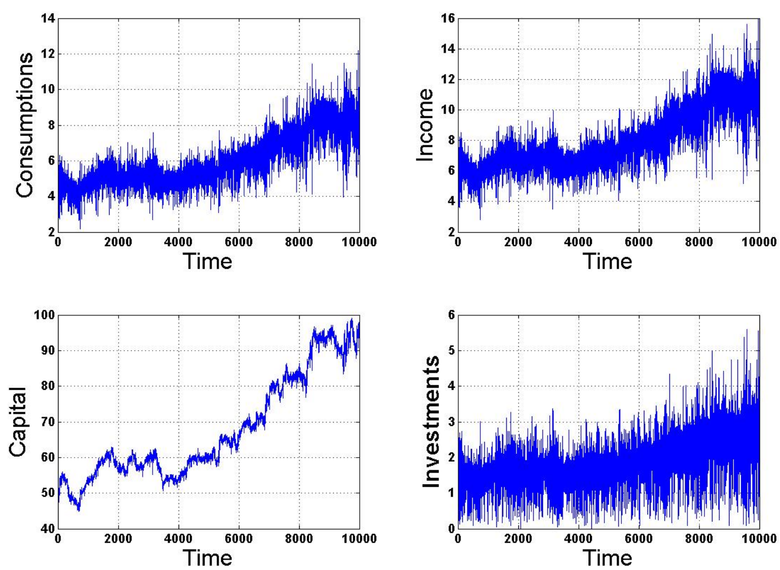


Figure 6-4: A simulation displaying a steady growth. $Y = 4, I, S, K$

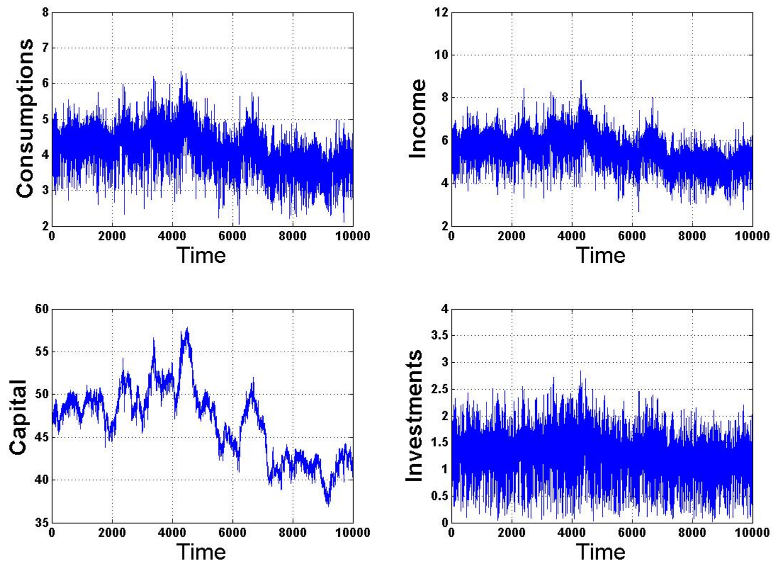


Figure 6-5: A simulation displaying a steady fall. Y, I, S, K

6.1.1 Shocks in the economy

It is worth noting that using the proposed model it is possible to produce a shift that reflects the real situation mentioned by Kaldor, i.e. at a certain stage of the economy some factors exist which take place cumulatively and have the effect of shifting the saving and investment functions in one direction or the other. In the model this was achieved by operating a shift on f by adding or subtracting a value (i.e. the shock) to the argument. The shifting operator works when the capital or the income changes negatively and it has the effect of helping the system to recover from a crisis.

6.1.2 Consumptions, savings and economic recessions

The idea that an increase in the disposable income (possibly through fiscal stimulus such as tax rebates) automatically translates into an increase in the aggregate demand can be erroneous as it neglects to consider the state of health of the economy and therefore the confidence in it.

In fact, if confidence in the economy is low, it could be that people may reduce their consumption during the recessions years: consumers will continue the process of deleveraging (they use the money to pay off debt and save more) because of uncertainty in the future.

For the above mentioned reason in the suggested model we linked the change in consumption to the change of income as follows (see (6.6))

$$w = \frac{\triangle\, Y}{Y} - \frac{\triangle\, C}{C} \tag{6.9}$$

which we believe describes correctly the behaviour of consumption.

6.2 Proof of the chaotic behaviour of the model

Up to now the sensitive dependence on initial conditions and the irregular trend of variables over time has only been showed graphically. This kind of evidence is not sufficient to prove the chaoticity of a system. Therefore we must use some numerical techniques in order to have a better insight of the possible chaotic nature of the system (6.8). Specifically, we will report the results obtained by spectral analysis as well as the calculation of the correlation integral, the Lyapunov exponents, the Kolmogorov entropy and the embedding dimension discussed in Chapters 1 and 2.

6.2.1 Tools

Results shown in this Section were obtained with RRChaos [166] or MATLAB ver. 8.5.0.197613 (R2015a).

6.2.2 Spectral analysis

As already mentioned, spectral analysis extracts the spectral content from a time series by decomposing it into different harmonics with different frequencies. and, by doing that, it identifies the contribution of each harmonic to the overall signal (see for example Stoica and Moses (2005) [176]).

Spectral analysis may help identify chaos for a given time series to discover hidden periodicities in data. Yet, spectral analysis cannot distinguish if a signal is chaotic or stochastic, therefore this technique does not deliver a conclusive answer as observed by Moon (1987) [127]) and (McBurnett (1996) [123]). However, as the proposed model is by construction deterministic, the spectral analysis can definitively help in understanding whether the system shows chaotic dynamics.

As an example the Figures 6-6, 6-7 and 6-8 show the cobweb diagram, the orbit and the periodiograms for different Logistic Maps.

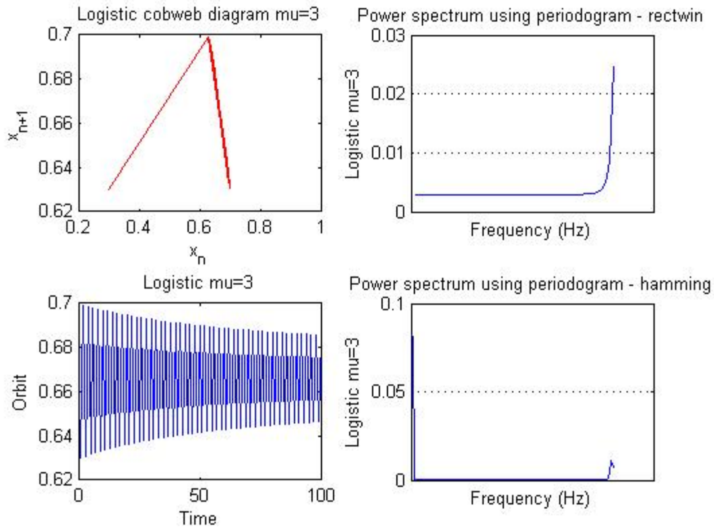


Figure 6-6: Cobweb diagram and periodograms for the Logistic Map (mu=3).

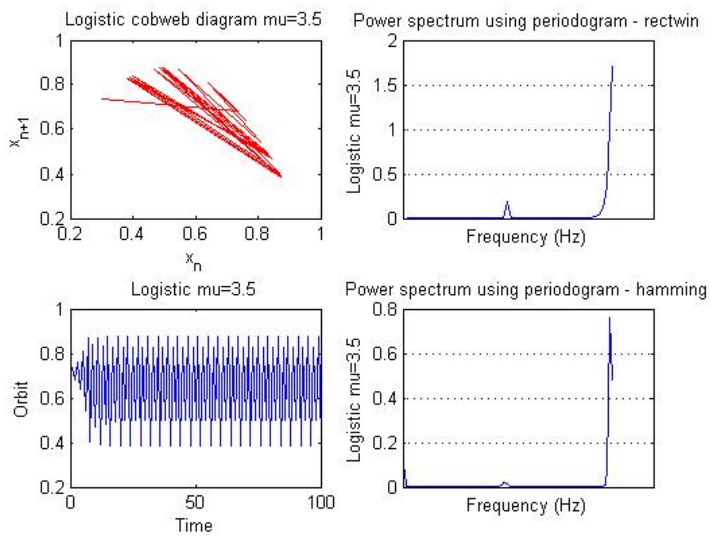


Figure 6-7: Cobweb diagram and periodograms for the Logistic Map (mu=3.5).

Regarding our Kaldorian system, we ran a simulation in order to show that for the generated time series there is no peak that clearly dominates all other peaks. Figure

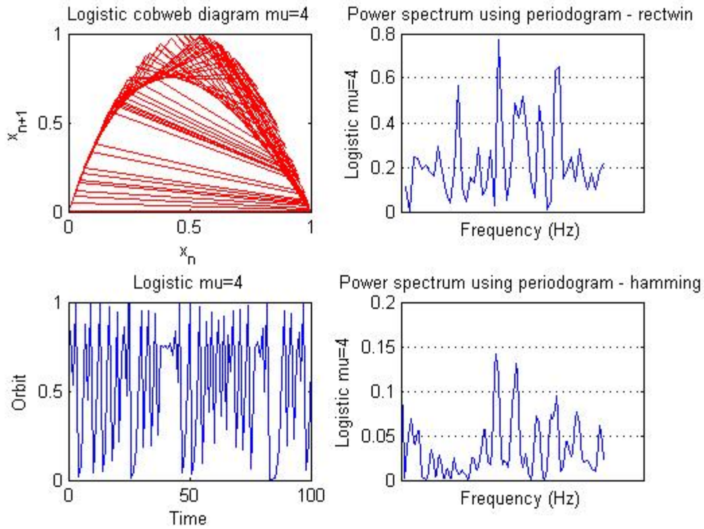


Figure 6-8: Cobweb diagram and periodograms for the Logistic Map (mu=4).

6-9 and 6-10 show the behaviour of the power spectrum for the macroeconomic variables C, K, I, Y with rectangular and Hamming windows, respectively. The presence of several frequency peaks suggests that irregular orbits (chaos) can be identified in the proposed model.

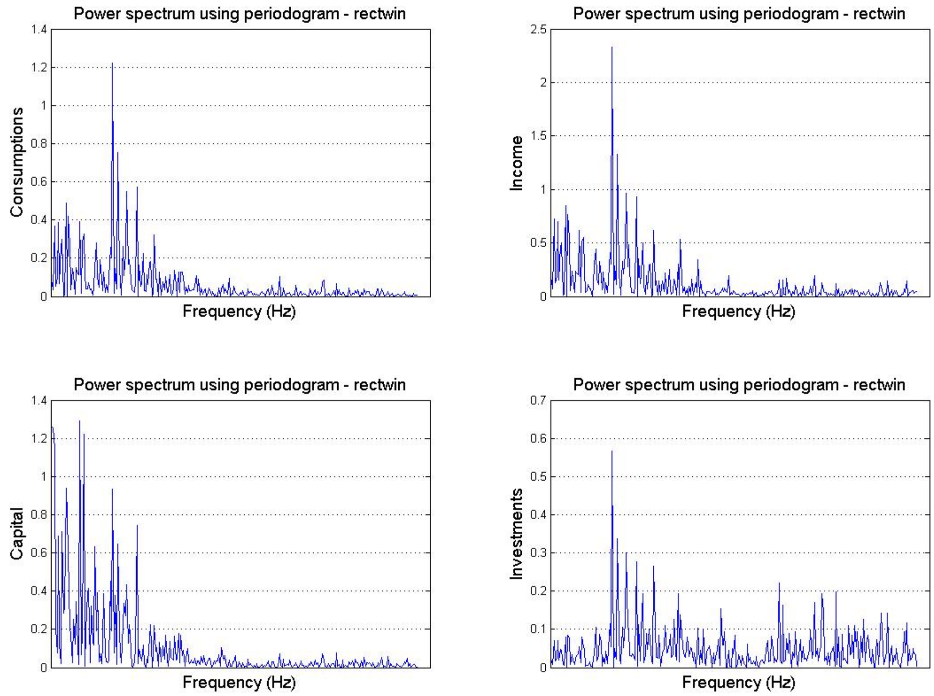


Figure 6-9: Power spectrum with rectangular window for K, I, C, Y. The irregularity of the spectrum hints at the possibility that the series are chaotic.

6.2.3 Correlation integral

As mentioned in Sec. 1.9.2, the correlation integral $C(r)$ of Eq. (1.80), measures the 'degree of kinship' between two different points on the (strange) attractor and it "represents a direct arithmetic average of the pointwise mass function" Theiler (1990)[180].

In Figure 6-11 we can see the value of the correlation integral versus r, and versus its logarithm in Figure 6-12. It can be noted that when $ln(C(r))$ is plotted against $ln(r)$ the slope of the linear section can be interpreted as the dimensionality m of the phase-space orbit within that range of r. Finally the fact that $ln(C(r))$ increases

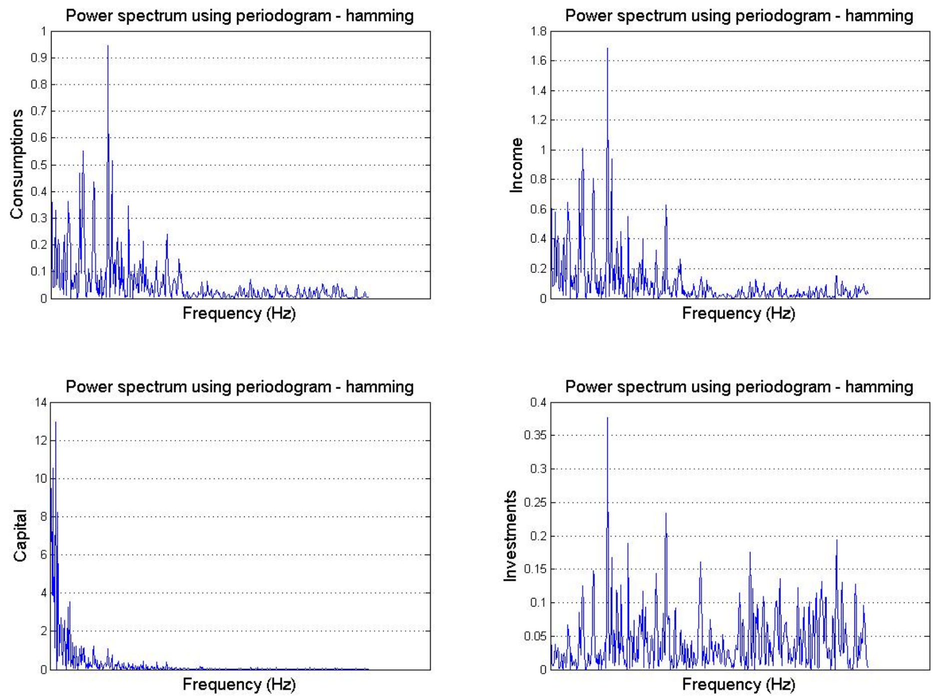


Figure 6-10: Power spectrum with Hamming window for K, I, C, Y. The irregularity of the spectrum shows that the series are chaotic.

regularly confirms that the system is deterministic.

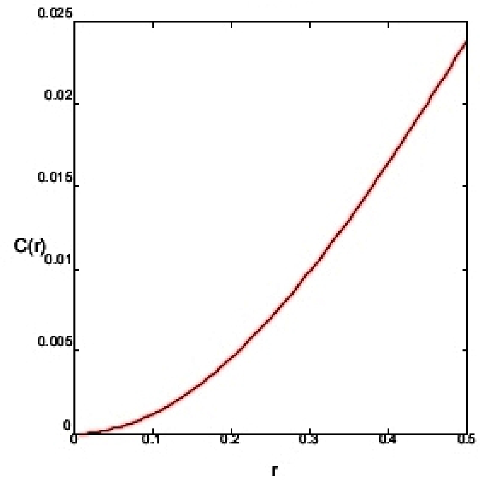


Figure 6-11: Correlation integral trend versus r.

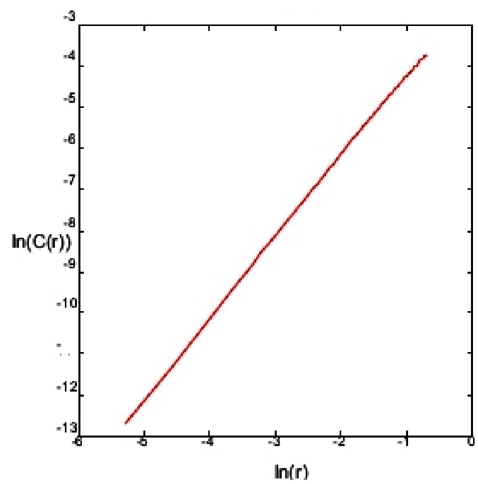


Figure 6-12: Log-log plot where the slope approximates the correlation integral.

6.2.4 Correlation dimension

Another useful notion is the correlation dimension as defined in Eq. 1.50 of Sec. 1.9.2. The correlation dimension is intended to measure the information content "where the limit of small size is taken to ensure invariance over smooth coordinate changes. This small-size limit also implies that dimension is a local quantity and that any global definition of fractal dimension will require some kind of averaging" [180].

In Figure 6-13 it can be seen that the dimension of correlation is noninteger. As D_C is a "more relevant measure of the attractor than D^H because it is sensitive to the *dynamical* process of the coverage of the attractor" (Grassberger and Procaccia (1983 [75]), we can say that the system is fractal [73], [72].

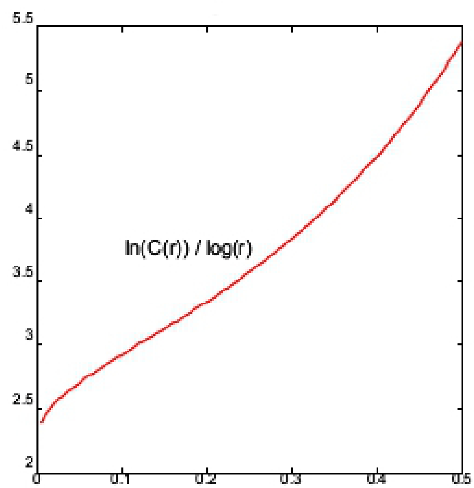


Figure 6-13: Correlation dimension when $r \to 0$.

6.2.5 Lyapunov exponents

Lyapunov exponents are used to measure the rate at which nearby trajectories of a dynamical system diverge (see Def. 1.40).

As a dynamic dissipative system is chaotic if its biggest Lyapunov's exponent is a positive number [113], we have adopted the Wolf algorithm [191], [190] to calculate the biggest Lyapunov exponent. In our simulations, the calculated value has always been positive (see Table 6.1 with the calculated Lyapunov's exponents for 10,000 points time series of C, Y, K and I).

Table 6.1: Lyapunov Exponents

	Min	Max	Mean
Consumptions	6.22	11.399	10.885
Income	12.8338	19.6440	13.3534
Capital	7.3165	14.594	12.999
Investments	5.511	11.969	11.049

6.2.6 Entropy

The Kolmogorov-Sinai KS entropy presented in Sec. 1.9.2 has been added to supplement the above mentioned analysis because KS converges to a positive value when time series are chaotic.

In order to measure KS we used the methodology suggested by J.C. Schouten, F. Takens and C.M. van den Bleek [165], [164] and we found that it was positive (e.g. Kolmogorov entropy = 21.34561, Kolmogorov entropy KML = 3.67711, relative standard error of KML [%] = 0.81194, total number of distances checked = 1,376,660, number of distances found = 15,280).

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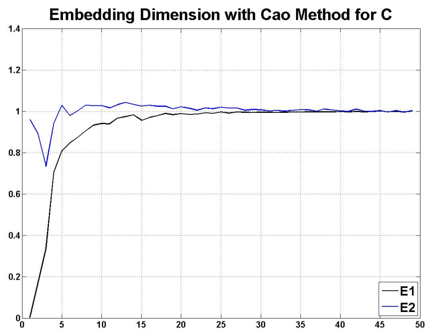
6.2.7 Embedding dimension

As discussed in Sec. 1.33, the embedding dimension is a statistical measure which indicates the smallest dimension required to embed an object (as for instance a chaotic attractor).

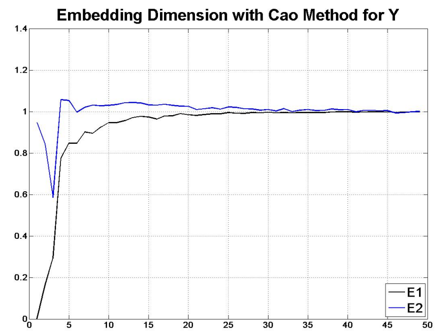
Cao embedding dimension's estimation

In order to compute this quantity, Cao (1997)[38] has suggested an algorithm based on the work of Kennel, Brown and Abarbanel (1992)[96] for estimating the embedding dimension (see [179], [7],[189]) through E1(d) and E2(d) functions, where d denotes the dimension. The function E1(d) stops changing when d is greater than or equal to the embedding dimension staying close to 1. The function E2(d), instead, is used to distinguish deterministic from stochastic signals. If the signal is deterministic, there exist some d such that E2(d)!=1 whilst if the signal is stochastic E2(d) is approximately 1 for all the values (see also [13], [14]).

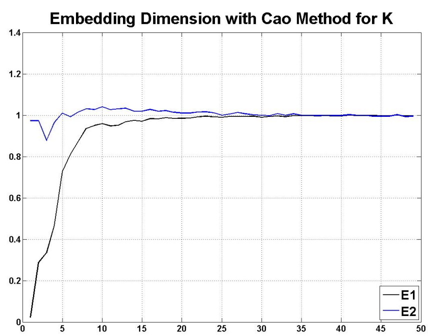
Figure 6-14 illustrates the behaviour of E1 and E2 function for consumption, investment, capital and income of the proposed model. As observed in Cao (2002)[39], when the quantity E2 has values close to 1, with some oscillations for small dimension, the related time series is likely a random series. On the other hand, the presence of oscillatory behaviour away from 1 when embedding dimension are small, implies some weak determinism in the considered time series. As it can be observed from Figure 6-14 related to the proposed model, the quantity E2 is not 1 but approaches this value for $d \geq 10$.



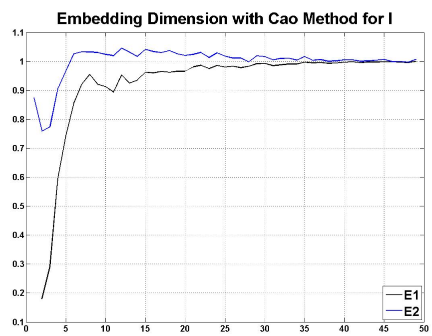
(a) Embedding Dimension for C



(b) Embedding Dimension for Y



(c) Embedding Dimension for K



(d) Embedding Dimension for I

Figure 6-14: Embedding Cao Dimension ($\tau = 1$, data points=10,000).

Symplectic geometry method

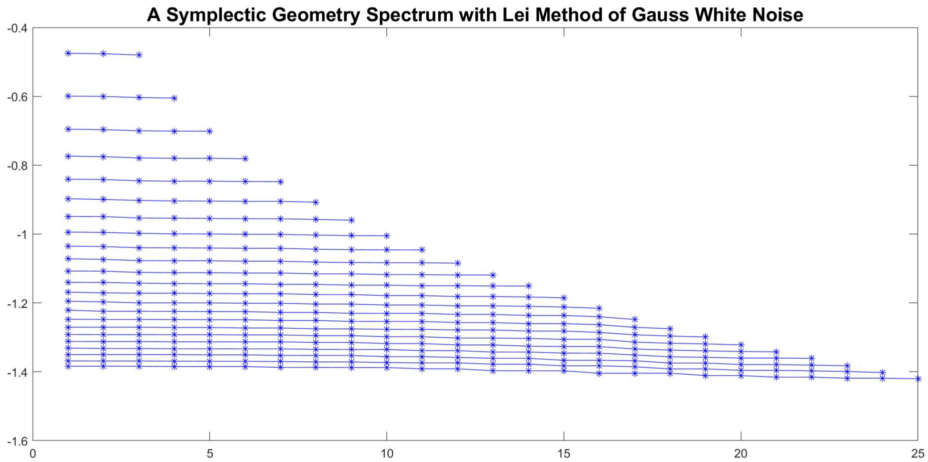
In addition to the Cao's estimation, the symplectic geometry method (see M. Lei, Z. Wang and Z. Feng (2002)[107] H. Xie, Z. Wang and H. Huan (2005)[193], M. Lei and G. Meng(2011)[106]) is used as a consistency check to verify the appropriate embedding dimension from a scalar time series. The symplectic similarity transformation is nonlinear and has measure-preserving properties i.e. time series remain unchanged when performing symplectic similarity transformation. For this reason symplectic geometry spectra (SGS) are preferred to singular value decomposition (SVD) (which is by nature a linear method that can bring distorted and misleading results e.g. see M. Palus and I. Dvorak (1992)[148]).

In Figures 6-15a and 6-15b we show two examples of embedding dimension for a Gaussian white noise and the the Logistic Map respectively, as obtained with the symplectic geometry method.

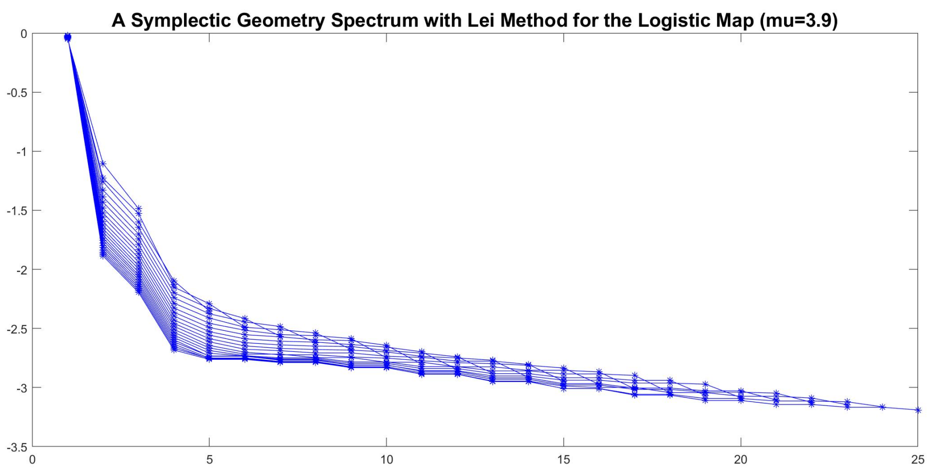
Figure 6-16 depicts the behaviour of the embedding dimension for consumption, investment, capital and income obtained using Symplectic Geometry Spectrum with Lei method. The behaviour of these curves is in accordance with that provided by other well know chaotic system in literature. This is another confirmation of the chaotic behaviour of the proposed model.

6.2.8 Chaotic attractor

As we have repeatedly shown, the system behaves like it was stochastically but we know that it is fully deterministic. In Table 6.2 we have reported the correlation integral versus the embedding dimension for 10,000 points time series of C, Y, K and I. Indeed, it can be observed that the correlation integral doesn't grow with the embedding dimension confirming that the system is deterministic [114].



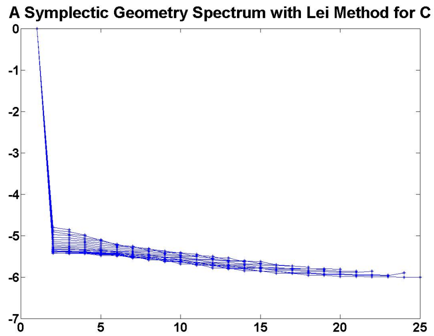
(a) Embedding Dimension for a Gaussian white noise.



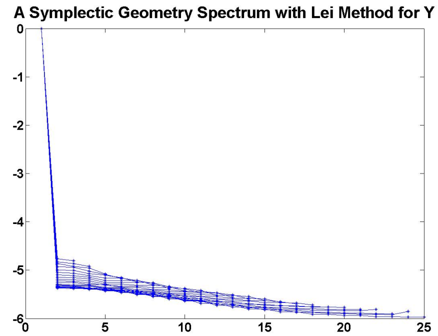
(b) Embedding Dimension for the Logistic Map (mu=3.9).

Figure 6-15: Embedding Dimension Symplectic Geometry Method. Ordinate is $\log \dfrac{\sigma_i}{tr(\sigma_i)}$, abscissa is i). The kink in the figure corresponds to the embedding dimension.

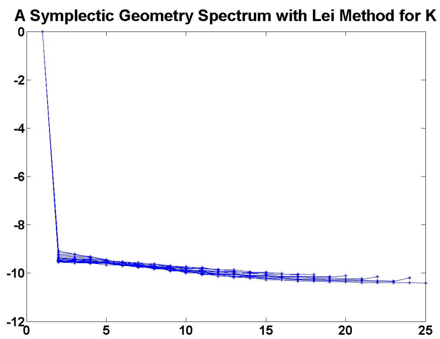
Finally in Figure 6-17 we present the strange attractor for the system obtained with software package RRChaos [166].



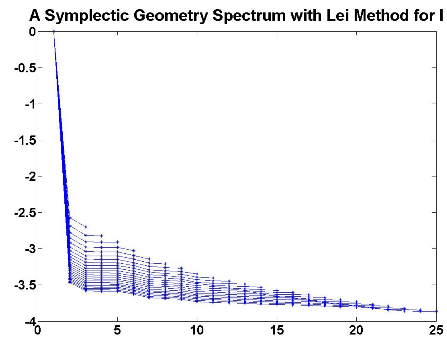
(a) Embedding Dimension for C



(b) Embedding Dimension for Y



(c) Embedding Dimension for K



(d) Embedding Dimension for I

Figure 6-16: Embedding Dimension Symplectic Geometry Method (data points=10,000, ordinate is $\log \dfrac{\sigma_i}{tr(\sigma_i)}$, abscissa is i). The kink in the figure corresponds to the embedding dimension.

6.3 Conclusions

In literature, the usual set-up of the Kaldor's model use the trigonometric investment function *arctg* (see [125], [90], [87], [9], [88], [27], etc.). We have decided, instead, to try a variant of the hyperbolic tangent.

For all the reasons discussed in this Chapter we wish to add further reasons. For example we think that there is no particular justification to prefer the *arctg* whilst there are several to prefer the *tanh*. A further good reason, for example, is that across

Table 6.2: Correlation Integral versus Embedding Dimension

	Embedding Dimension							
Correlation Integral for		2	3	4	5	6	7	8
Consumptions	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797
Income	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797
Capital	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797
Investments	Min	0.053543	0.021604	0.0092081	0.0043155	0.0022121	0.0012546	0.00079854
	Max	0.71333	0.70895	0.70449	0.69998	0.69658	0.69372	0.69127
	Mean	0.30254	0.2599	0.23142	0.21153	0.19686	0.1859	0.17797

multiple sciences, growth and decay are better modelled by an exponential-alike function such as *tanh*. For example in actuarial science and financial mathematics such a law is used for the Gompertz–Makeham law of mortality or compound interests. With regards to economics, in particular, the proposed model could link up to the classic Solow-Swan growth model, in which labour and knowledge are represented by exponential functions. Moreover the function *arctg* tends to its asymptotes quite slowly compared to the hyperbolic tangent whilst we wanted to design a framework in which economic agents can adjust quickly (how quickly depends on parameters as τ) to changes.

We wish to remark that an additional original contribution in our proposed model is the specification of consumption and investment as a function of the difference, respectively, between the growth rates of income and capital, and the growth rates of income and consumption. This has been achieved by considering, *à la* Kalecki, that the investment process has different timing than does consumption, hence the

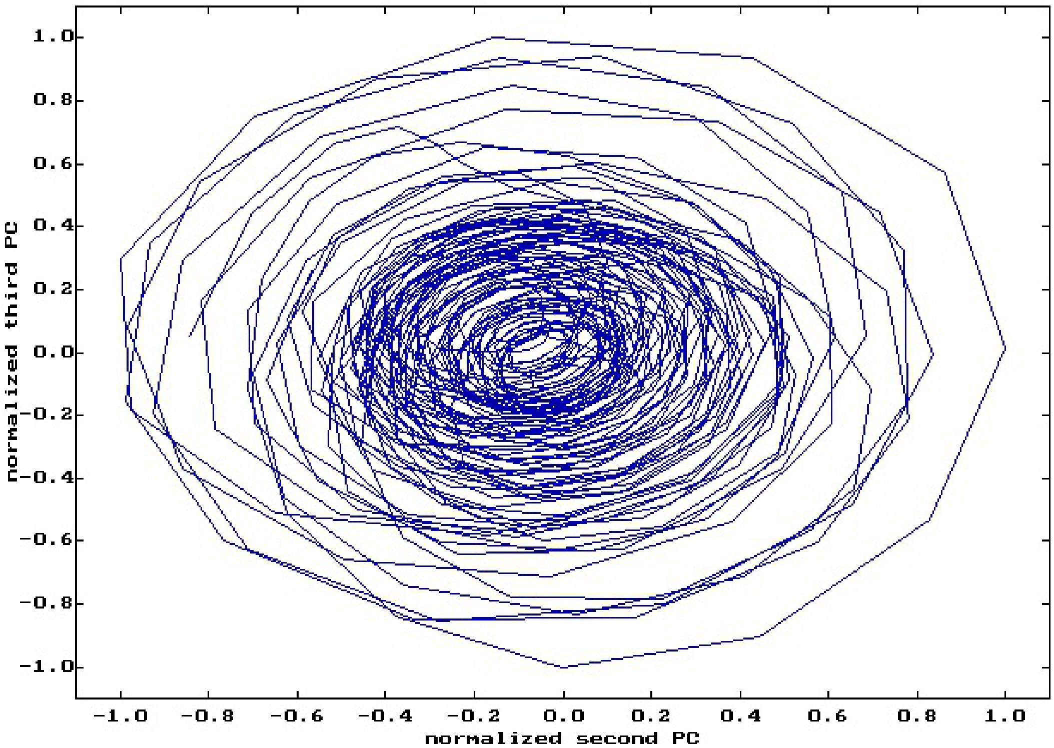


Figure 6-17: Strange attractor.

difference in the considered time lags (see Eq. (6.2) vs. Eq. (6.3)). Last but not least the model can accommodate external perturbations such as shocks by a translation of the argument of the function f (see Sec. 6.1.1).

If anyone thinks of possible future research, we could address the issue of the conditions under which the system generates (presumably asymptotically) regular cycles. Further, the detected non-linear behaviour has some implications which could be potentially considered as being of some interest, e.g. what should be the rules set by regulators (such as the central bank) in a chaotic context? More specifically: If the system is not predictable, not reachable and therefore not observable but nevertheless controllable (for example see [157], [76], [37], [150]), can one set up a system of controls that is able to drive the economy? Are the current instruments of economic policy the correct ones?

Chapter 7

Recurrence Quantification Analysis of Business Cycles

As explained in the Introduction, after having devised a suitable model for business cycles such as that discussed in Chapter 6 and in References [136], [137], our aim is to look for an indicator that could show structural changes in a signal related to chaos [144]. Here we apply RQA and statistical techniques to both: real time series and model generated ones [145]. This to I) find common properties if and where they do exist, II) discover some hidden features of economic dynamics and III) highlight some indicators of structural changes in the signal (i.e. in our case to look for precursors of a crash).

7.1 Material and methods

The variables under investigation are Capital (K), Consumption (C), investment (I) and Income (Y) (see Appendix A). Cyclical swings of economy are typically analysed in terms of the duration or the amplitude between a peak and the succeeding trough [33]. The cycle Peak-Trough-Peak (PTP) can be caused by various factors such as negative shocks in demand, in supply, in price and in credit (i.e. when "financial distress produces sharp discontinuities in flows of funds and spending and when the financial strains include tight monetary policy, much lessened availability of money and credit, sharp rises of interest rates, and deteriorating balance sheets for households, businesses, and financial institutions")[56] as discussed in Chapter 3.

In order to study business cycles and recessions we will apply the RQA on time series extracted from different sources. This because we want to have an extensive set of data, with the highest number of points possible, covering the following dimensions: countries with different development paths (Organisation for Economic Co-operation and Development A.5, Bureau of Economic Analysis A.3), differences in methods for computing the capital (M1, M2 A.4), gross versus net (Levy and Chen (1994) A.4), etc. A further requirement was that, whenever possible, the number of time series should be balanced across variable or dimension.

For the variables K (capital), C (consumption), I (investment) and Y (income) modelled as described in Chapter 6, we collected 55 time series belonging to the following countries: Germany (DEU), Italy (ITA), Korea (KOR), United Kingdom (GBR), Turkey (TUR), Japan (JNP), Spain (ESP) and United States of America (USA). This in order not only to consider the evolution in time but, also, to cover different types of economies (developed vs. developing, stagnating vs. expanding, etc.). Moreover, it is worth noticing that the aforementioned four variables differ in their units of measurement. In fact the "stock" variable K represents the quantity existing at a certain point in time, whereas "flow" variables such C, I and Y are measured over an interval of time. For this reason we balanced the dataset by gathering 41 flow and 14 stock variables.

7.1.1 Data on capital

As mentioned in the Introduction, while financial data are abundant and have many data points, economic time series are not many and data points are very few. This is especially true for capital stock (see Appendix. A.4. In fact aggregate capital stock data is only collected on an annual basis. To analyze quarterly data we referred to time series made available by Levy and Chen (1994) [110] who used Musgrave (1992)[131] for annual capital stock time series, Citicorp (1993)[43] for investments and their price deflator series and U.S. Bureau of Economic Analysis (BEA) for annual depreciation and discard figures.

Levy and Chen calculated quarterly data which is useful for "analyzing the dynamic relationship between aggregate factors of production and output"; in fact "it is preferable to look at the data using quarterly observations because some dynamic phenomena that perhaps take place within the period of a year will not be captured if annual data is used. In addition, from an econometric point of view, use of quarterly data instead of annual data quadruples the sample size which makes empirical statistical inference more reliable" [110].

Adopting the same notation we distinguish between time series in nominal terms from time series in real terms (measured in 1987 dollars) by using the suffix 87 and we denote Segmented Linear Interpolation as M1 and Numerical Iteration as M2 for the following time series: Consumer Durable Goods (CDG), Producer Durable Goods and Equipment (PDG), Non-residential Business Structures (BS).

As explained by Levy and Chen the quarterly capital stock series are constructed using different procedures: "The first is a segmented linear year-to-year interpolation technique. The second technique exploits the dynamic relationship between the capital stock and the corresponding capital investment series and uses annual beginning-of-the year and end-of-the-year capital stock data to estimate the implied quarterly depreciation rates for all three categories of the aggregate capital stock by numerical iteration over the depreciation rates until a convergence is achieved. These depreciation rates are then used along with the quarterly investment and the annual capital stock series to construct quarterly capital stock series" [110].

In our analysis we focus on series built with Method 1 and 2 because, as shown by Dezhbakhsh and Levy[54] "linear interpolation of a trend stationary series superimposes a 'periodic' structure on the moments of the series" whilst, according to Jaeger's (1990)[86] "segmented linear interpolation reduces the size of shock persistence in a difference stationary series".

7.1.2 Data on income, investment and saving

Regarding the scope of our analysis we included countries that had very different development paths (see Appendix. A.5) from the Organisation for Economic Cooperation and Development (OECD). OECD data are respectively

- Quarterly GDP Total, Percentage change, Quarterly National Accounts. This indicator is seasonally adjusted and it is measured in percentage change from previous quarter and from same quarter previous year [134].

- Investment (GFCF) Total, Annual growth rate (%). Aggregate National Accounts, SNA 2008 (or SNA 1993): Gross domestic product. Gross fixed capital formation (GFCF) is in million USD at current prices and PPPs, and in annual growth rates [133].

- Saving rate Total. % of GDP. National Accounts at a Glance [135].

7.1.3 Tools

Recurrence Plots, RQA and RQE were discussed in Chapter 2. RQE computes recurrence quantifications on an epoch-by-epoch basis. The RPs shown in the following Figures were obtained with the CRP Matlab Toolbox ver. 5.22,rel. 32 [118]. RQA and RQE were obtained with the package RQA ver. 14.1 [187]. Statistical analysis was carried out by using Systat ver. 10.2 or MATLAB ver. 8.5.0.197613 (R2015a).

7.2 Results and analysis

In this Section we show that, in some cases, early warning signals of dramatic changes (downturns/expansions) can be seen by computing recurrence variables within a moving window (epoch) shifted by a given number of points (delay) throughout the whole sample (which is the RQE). Finally we demonstrate that RQA is a valid technique of investigation as it is able to distinguish between real and nominal data as well as between net and gross time series.

7.2.1 Recurrence Quantification Analysis (RQA)

Recurrence Plot (RP) on USA GDP

In Fig. 7-1 it is shown the recurrence plot of USA GDP% depicted right below its time series. From the RP it is possible to observe the anticipating transitions to turbulent phases. The remarkable result consists of a correspondence between vertical lines in RP (i.e. chaos to chaos transitions) and downturn/upturn periods.

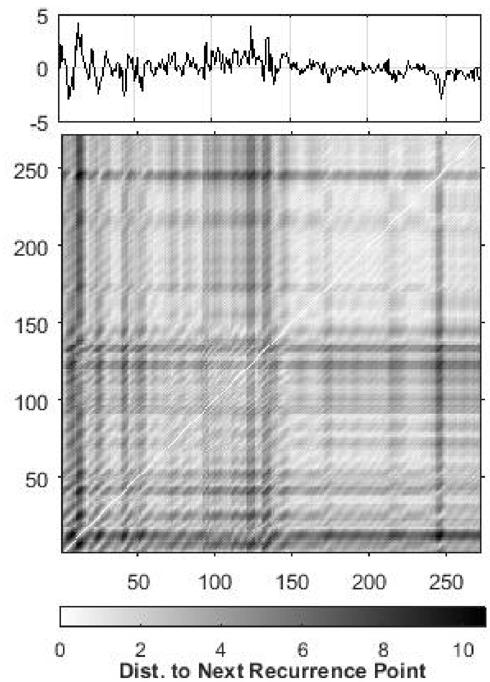


Figure 7-1: Changes in US GDP (above) and its Recurrence Plot (below). Data range: 01-01-1947 - 2016-01-01. ID: A191RP1Q027SBE. Gross Domestic Product, Percent Change from Preceding Period, Quarterly, Seasonally Adjusted Annual Rate. Source: St. Louis Fed, FRED database. Note the alignment between shocks and vertical lines in RP.

RQA on business cycle data

RQA defines the overall complexity of the signal in terms of quantitative indices deriving from RP. Here RQA was carried out on 55 time series from the dataset mentioned above (see Appendices A.3, A.4, A.5) with the following input parameters [187]: time lag (or delay: the spacing between selected input points) = 1; embedding = 10 (embedding dimension: estimated number of dominant operating variables); radius (largest normed distance at and below which recurrent points are defined and displayed) = 80; line (minimum number of sequential recurrent points required to define diagonal and vertical lines) = 5. Time lag = 1 has been chosen because, differently from financial time series, economic time series have few data. Moreover, as specified in Section 1.6.1, quarterly data are independent (they do not suffer of autocorrelation). The method chosen for normalizing vectors in higher dimensional space uses the Euclidean norm and meandist is the method for rescaling the recurrence matrix. It is worth mentioning that the radius has been set with the objective of maximizing the difference among the time series of the whole data set. This happens because when the radius is too large the determinism can saturate, whilst when the radius is too small, few recurrences points in RP could not be describe differences among the time series.

More details will be provided in Paragraph 7.2.3. Here we can anticipate that, with regard to USA quarterly capital data, as reconstructed by Levy and Chen (see Table A.4), no significant differences have been observed between interpolation methods M1 and M2 whilst differences have been found between real and nominal as well as between net and gross time series (see Table 7.1, row 1 where $p > 0.1$ for all the RQA measures, PC1 and PC2).

Recurrence Quantification Epoch (RQE) on business cycle data

To better understand the time evolution of these economic data, RQE analysis was carried out on business cycle time series with the following parameters: window size = 50 points; shift = 1 point; lag = 1; embedding = 10; radius = 80; line = 5. Even in this case, as well as in RP, the drop of DET corresponds to the grey vertical line indicated by FRED DATA (see Fig 7-2).

These results are in line with what was reported by Bastos and Caiados [20]

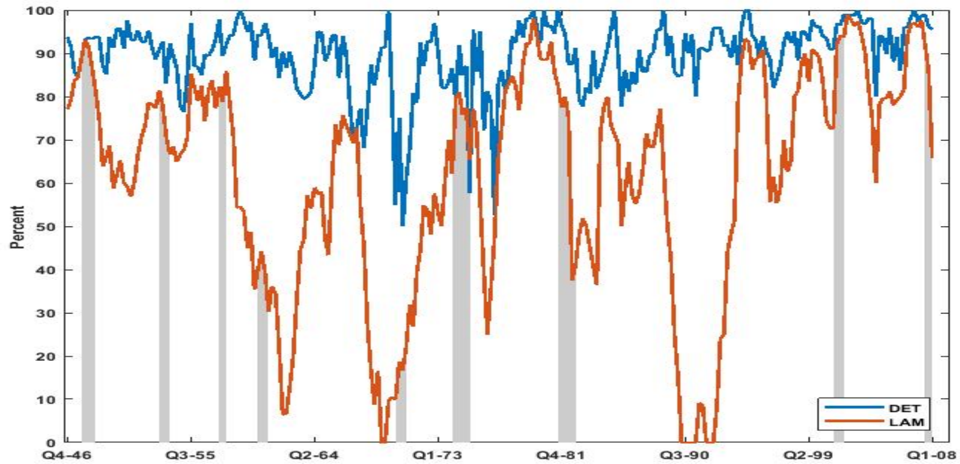


Figure 7-2: Dynamical analysis in the sliding window mode (RQE) where percent of laminarity (LAM) and percent of determinism (DET) refer to the same time series as in Fig. 7-1. Overlapping sliding windows of 50 data points shifted by 1 point (49 data point overlaps) were taken. Variables are plotted in central position in standardized units (su), i.e., after subtracting the average value from absolute values and dividing by standard deviation in each window.

who, comparing 23 stock market indices of both developed and developing countries, found a reduction in DET and LAM during the sub-prime mortgage crisis and even dramatic fall, during the burst of the technology bubble. The latter was also documented in the analysis of the dot-com bubble by Fabretti and Ausloos (2005)[60] and Kousik et al. (2010)[99] where DET and LAM reached the highest values during the bullish period and declined before the bubble burst. Morevover according to Piskun et al. (2010)[152] laminarity (LAM) "is the most suitable measure, sensitive to critical events on markets" where the inverse of laminarity reflects the market volatility (Strozzi et al. (2007a)[178]).

RQE correlation index

In order to understand the limitations of the proposed method and to provide further detail on the power of RQA in in anticipating transitions from laminar to turbulent phases, we resort to the *RQE (absolute) correlation index* described in Orlando and Zimatore (2017)[144] and here discussed in Section 2.3.2.

As shown in [144], while the RQE correlation index 2.28 is able to detect 9 of the 10 intervals in which a random signal was perturbed, the results displayed on Fig. 7-3 are less conclusive.

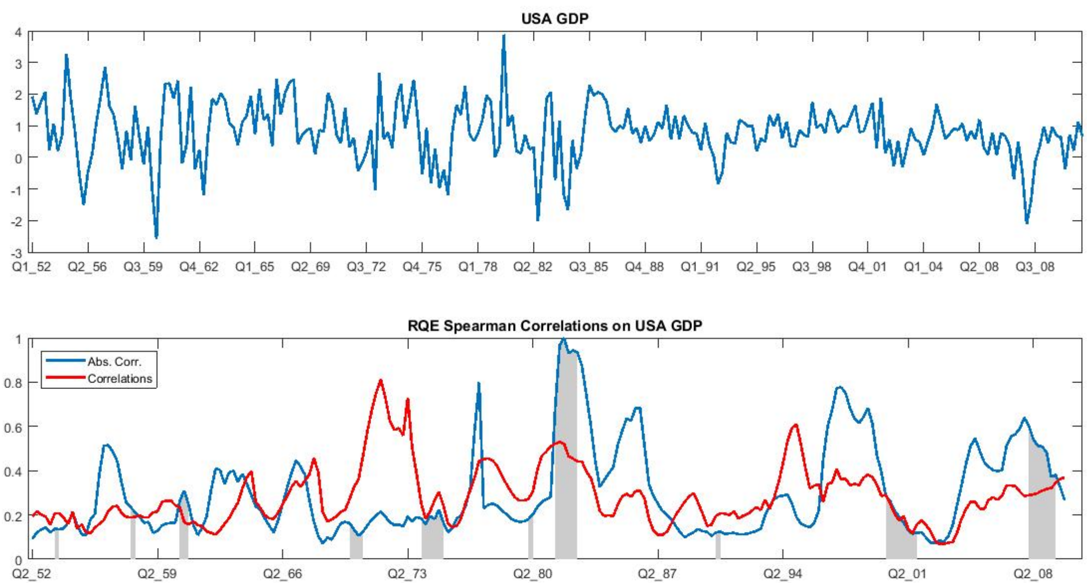


Figure 7-3: Maximum correlations (in blue) between RQE measures versus recession periods (in grey) on the USA GDP [134]. As shown in the figure a change in the index is often linked to a recession. Spearman correlations (below) versus the final test signal (above). RQE absolute correlation (in blue) is displayed next to correlation (red). See how the RQE correlation calculated as in Equation 2.28 is more reactive than the other and it is able to detect more finely changes in the original times series. Difference in the x-axis numbering between the picture above and below, is due to the windowing mechanism. Source Orlando and Zimatore (2017)[144].

7.2.2 Principal Component Analysis (PCA) on RQA

PCA is a multivariate statistical analysis successfully applied to business time series which it minimizes redundant information (Bartholomew et al. (1984)[16]), Zimatore et al. (2002)[200]). In this thesis, we apply PCA to recurrence measures estimated from the aforementioned economic data by taking advantage of the combined use of the two techniques; therefore PCA has been carried out on the main four RQA measures: REC, DET, MAXLINE and ENT. The percentage of total variance explained is respectively: PC1=81%; PC2=11%; PC3=5%.

In Fig. 7-4 for every data series shown in Table A.9, PC1 vs PC2 are depicted. As those are obtained from RQA data, it is possible to observe that RQA preserves some structural differences between income, capital, investment and consumption.

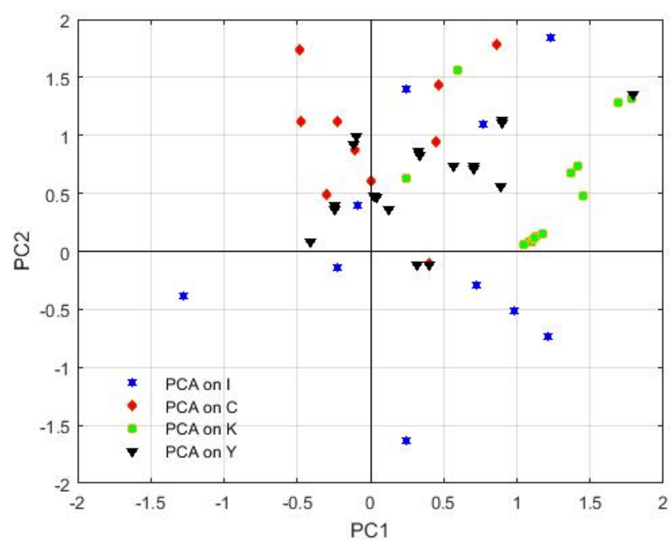


Figure 7-4: Dynamical features of business time series in a principal component space as reported in Appendix A.9. Different symbols (letters) indicate the four macroeconomic variables: I-Investment, C-Consumption, Y-Income, K-Capital. Note how K-capital is clustered.

In order to check whether there are differences among the four macroeconomic variables we applied the Mann-Whitney U test. This test belongs to the class of nonparametric tests and it checks the null hypothesis that a randomly selected value from one sample is less than or greater than a randomly selected value from another sample. In our case and we found (see Table 7.1) that all measures are different with $p < 0.001$.

7.2.3 Statistical analysis on RQA

An additional statistical analysis was performed on the obtained RQA measures. Spearman's correlations among these parameters were estimated. A MANOVA test was conducted for I, C, Y and K variables, for different countries, measures, investment goods, terms and interpolation methods. Data is expressed as means $\pm$ standard deviations.

We anticipate that the results of the tests in following Paragraphs 7.2.3 and 7.2.3 show that RQA is able to capture the difference between stock and flow (MEASURES) as well as the dissimilarities between the four macroeconomic variables (VARIABLES). In addition, the fact that different countries (COUNTRIES) with very different evolutions are not distinguishable, might be an indication of the chaotic nature of economics, as in chaos different paths originate from the same underlying deterministic dynamic. Furthermore the interpolation methods provide equivalent results.

Mann-Whitney Test

The non-parametric Mann-Whitney test was carried out on the RQA measures from the whole data set (see Table A.9).

Table 7.1: Mann-Whitney U Test (p-values)

Row	#	Groups[1]	REC	DET	MAXLINE	ENT	TREND	LAM	TT	PC1	PC2
1	14[2]	Method	0.796	1	0.301	0.439	0.796	0.439	0.439	0.897	0.197
2	55[3]	Variable	0.001	<0.001	<0.001	0.028	0.002	<0.001	<0.001	<0.001	0.026
3	35[4]	Country	0.071	0.253	0.436	0.162	0.157	0.126	0.469	0.253	0.146
4	55[5]	Measure	<0.001	<0.001	<0.001	0.209	0.772	<0.001	0.967	<0.001	0.772

PC1 and PC2 calculated on REC, DET, MAXLINE and ENT

In summary, we observe that (between groups) results in row 1 Table 7.1, fail to reject the null hypothesis that Methods 1 or 2 are equal the distributions of the four variables C, I, Y, K are not the same (see row 2, Table 7.1), the distributions of data belonging to different countries (Germany (DEU), Italy (ITA), Korea (KOR), United Kingdom (GBR), Turkey (TUR), Japan (JNP), Spain (ESP) and USA) do not reject the null hypothesis that they are equal (see row 3, Table 7.1) and, finally, that the distributions of flow and stock variables are different (see row 4, Table 7.1).

Kruskal-Wallis Test

To further confirm the ability of RQA in capturing the differences between macroeconomic variables the following table reports the p-values on RQA of the Kruskal-Wallis Test performed on the time series displayed in Table A.9.

Table 7.2: p-values on RQA from 55 (#C=10, #I=11, #K=14, #Y=20) business time series

Mean	SDev	Mean/SDev	REC	DET	MAXLINE	ENT	TREND	LAM	TT	PC1	PC2
0.012080589	0.000137346	3.33E-07	0.001264944	3.14E-05	2.26E-05	0.027594729	0.002148292	8.61E-08	7.08E-06	8.05E-05	0.00054498

[1] ***Method***: M1, M2; ***Variables***: C, I, K, Y; ***Country***: Germany (DEU), Italy (ITA), Korea (KOR), United Kingdom (GBR), Turkey (TUR), Japan (JNP), Spain (ESP), United States (USA); (Unit) ***Measure***: Flow, Stock; (Time Series').

[2] Number of time series considered by method: 8 M1, 6 M2.

[3] Number of time series considered by variable: 10 C, 11 I, 14 K, 20 Y.

[4] 4 DEU, 4 ITA, 4 KOR, 4 GBR, 3 TUR, 4 JNP, 4 ESP, 8 USA

[5] Number of time series considered by measure: 41 flow, 14 stock.

7.3 Conclusions

So far, in the literature, there are no clear indications whether economic data are chaotic or not. This thesis applied RPs and their quantitative description provided RQA to try to detect subtle but essentially relevant changes in the dynamical regime of business time series. RQA aims at a direct and quantitative assessment of the amount of deterministic structure of time series. Here it was shown that RQA is an efficient and relatively simple tool in non-linear analysis of a wide class of signals. This technique allows for the identification of sudden phase-changes possibly pointing to mechanistically relevant phenomena. Therefore RQA may be suitable to study business cycles and could be used for early detection of recessions (even though some limitations are apparent Sec. 7.2.1).

The results reported so far have revealed the applicability of this methodology to economic time series; especially where other methods may fail because of randomness, non-linearity and non-stationarity of data, and have given new insights into underlying dynamics. In fact both PCA and statistical analysis on RQA seem to validate the technique as macroeconomic variables are clearly distinguishable. In addition RQA seems to confirm that different paths in economic development may originate from the same underlying deterministic dynamic (which is an indication of chaos).

Future research, along this line, will aim to understand whether RQA, applied to both real data and simulations obtained from a nonlinear economic model on the business cycle [136], may present analogies and/or similarities, which could be helpful in understanding the nature of economic dynamics. The implications in terms of control will follow: Given the nature of economy, can one set up a system of controls that is able to steer it?

Chapter 8

An empirical test on Harrod's model

After having illustrated in Chapter 5 the Harrod's model and a chaotic specification of it (see Sec. 5.1, in this Chapter we are going to prove that I) real data could be obtained by a suitable calibration of model's parameters, II) the calibrated model confirms theoretical predictions [138].

8.1 Calibration of the Harrod's model

To test empirically the Harrod model we evaluated the average distance between the historical data series reported in Appendix A and the orbit produced by Eq.(5.12) that starts at time 0 at the same initial point of the data and best fits the time series. This orbit can be found by suitably calibrating the parameters in Eq. (5.12) through a numerical optimization. The aim of the optimization was thus to minimize the square error between the historical time series and the orbit produced by Eq. (5.12) that starts at time 0 at the same initial point of the data series. Mathematically, we want to compute the quantity:

$$D = \frac{1}{286} \sum_{t=0}^{286} \left(d(t) - \frac{1}{\hat{\tau}} \int_{t}^{t+1} \hat{\phi} \left(\hat{\tau}\tilde{t}, P \right) d\tilde{t} \right)^2 , \tag{8.1}$$

where $d(t)$ is the vector that stacks the data of the rate of growth of domestic income, the expected rate of growth of aggregate demand, the share of income saved, and the net export rate for the quarter t ($t = 0$ is the first quarter of 1947, $t = 286$ is the second quarter of 2018). Similarly, P is the vector of the 13 parameters of the model (reported in Table 8.1), and $\hat{\phi}$ stacks the four variables that solve the differential Eq. (5.12) with parameters set in P starting at $\hat{\phi}(0, P) = d(0)$: the integral between t and $t + 1$ allows us to compute the average value of the (continuous) signal over the quarter of interest, to be compared with the data. Note that an additional dummy parameter $\hat{\tau}$ has been added. This parameter permits us to rescale the time of the signal produced by the model, in order to best fit with the time-scale of the data. The optimization variables are the $13 + 1$ parameters of Eq. (5.12) since they have physical meaning only when positive, this adds a set of constrains to be satisfied. Formally speaking, in order to find the best fitting solution, we solve the following constrained optimization problem:

$$\begin{aligned} \min_{P,\hat{\tau}} \quad & \frac{1}{286} \sum_{t=0}^{286} \left(d(t) - \frac{1}{\hat{\tau}} \int_{t}^{t+1} \hat{\phi} \left(\hat{\tau}\tilde{t}, P \right) d\tilde{t} \right)^2 \\ s.t. \quad & \hat{\phi} \left(\hat{\tau} t, P \right) \text{ is a solution of Equation (5.12) with parameters set as } P \\ & \hat{\phi} \left(0, P \right) = d(0) \\ & P \geq 0, \ \hat{\tau} > 0. \end{aligned} \tag{8.2}$$

This problem is solved using the interior point method [26, 35] implemented in the Matlab `fmincon` routine. Since the problem is not convex, the optimization algorithm may converge to a local optimal solution. To better explore the space of the optimal solutions, we introduce a multi-start algorithm: the optimization is then run several times starting from a randomly perturbed sample drawn from a distribution centred in the parameter setting provided in Sportelli and Celi [174].

Since the aim of this thesis is to evaluate the above mentioned version the Harrod model, in Table 8.1 we report the parameters for both the original model and three calibrations we have obtained that present qualitatively different behaviours (together with the value of their distance D for calibration 2, D is computed with $t \leq 6$). The model, calibrated with real data, may display convergence to a long-run equilibrium (calibration 1, Fig. 8-1), divergence (calibration 2, Fig. 8-2) as well as a lightly damped oscillatory behaviour (calibration 3, Fig. 8-3). It is worth saying that the global optimum is obtained with calibration 1. However, with calibrations 2 and 3, we displayed sub-optimal results to provide a context for our results. In fact, qualitatively, we obtain similar values to the ones in [174]. Moreover we agree with this conclusion "when the value of ϵ is large enough, the long period dynamics of the saving rate is such that it can generate an irregular cycle in the system only if the net export rate is very low. On the contrary, starting from positive and meaningful values of the net export rate, the system may simply generate a limit cycle (or at most a double cycle) if a higher ϵ works together with adequate competitiveness on the foreign markets. This is the only formal result consistent with Harrod's intuition that a more moderate cyclical instability can emerge in an open economy compared to a closed one" [174].

Table 8.1: Harrod model parameters.

	Given Model		Calibration		
			Cal. 1	**Cal. 2**	**Cal. 3**
#	**Parameter**	**Given Value/Range**	**Calibrated Value**		
1	α	0.5	0.28	0.29	1.09
2	ϵ	$[0.2, 1.31]$	0.13	0.58	0.52
3	σ	$[2, 4)$	1.42	1.67	2.45
4	G_f	0.03	0.03	0.00	0.54
5	C^*	4	4.00	4.00	3.18
6	β	2.5	2.50	2.50	2.20
7	m	0.07	0.04	0.04	1.23
8	φ	15	15.00	15.00	14.89
9	ξ	0.18	0.18	0.18	0.20
10	μ	1.4	0.78	0.90	2.06
11	γ	1	0.56	0.57	0.36
12	δ	6.2	6.20	6.20	5.94
13	ζ	1.9	1.06	1.09	2.25
Value of D			0.38	0.71	0.55

Original data as provided in [174] and related calibrations. $\bar{G} = max\, G_n$.

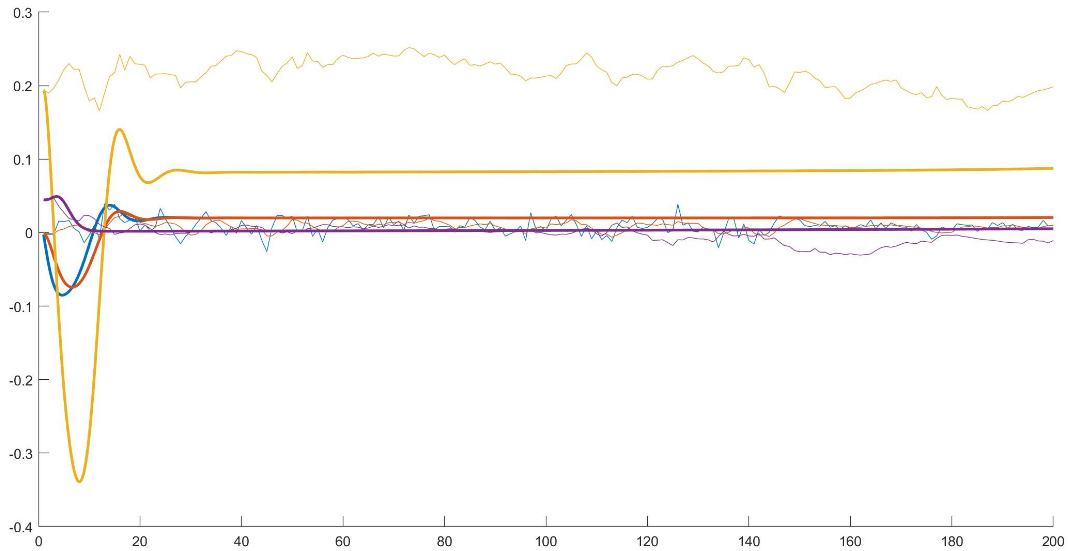


Figure 8-1: Time series obtained with parameters of calibration 1, that displays convergence to the long-run equilibrium. Legend: blue = rate of growth, red = expected rate of growth, yellow = share of saved income, violet: trade to income ratio. Thick line: model, normal line: data.

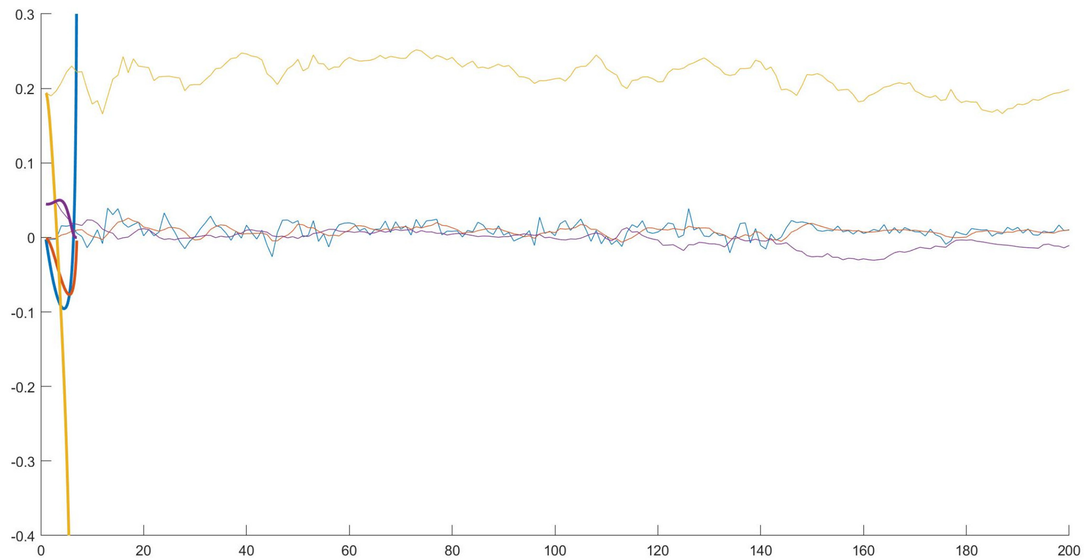


Figure 8-2: Time series obtained with parameters of calibration 2, that displays divergence from the long-run equilibrium. Legend: blue = rate of growth, red = expected rate of growth, yellow = share of saved income, violet: trade to income ratio. Thick line: model, normal line: data.

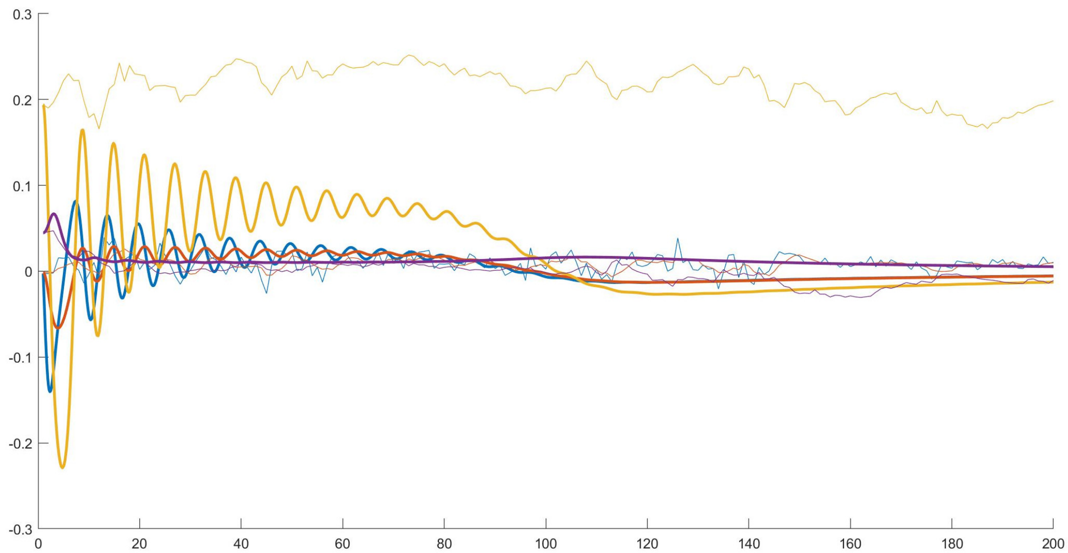


Figure 8-3: Time series obtained with parameters of calibration 3, that displays lightly damped oscillatory behaviour around the long-run equilibrium. Legend: blue = rate of growth, red = expected rate of growth, yellow = share of saved income, violet: trade to income ratio. Thick line: model, normal line: data.

8.2 Conclusions

The Harrod's model [79] has the merit of rearranging Keynes's ideas into a dynamic framework with some additional specification on the supply side. In fact "where the warranted growth rate represents an economy's growth path on which aggregate demand and supply remain in balance, the model's natural growth rate reflects the supply of productive resources and the level of technology, the long-run limit to real output growth. The interaction between the warranted and natural growth rates provides a useful perspective for policymaking in today's environmentally-constrained global economy. Also, since the growth of the labor force is built into the natural growth path, the model also helps to clarify policy choices in an economy impacted by immigration" [183]. Therefore "supply-side policies must be developed along with the standard Keynesian demand side policies, and the interactions between the two require disaggregated policies to address specific types of investment, technological change, and demand. That is, it is not generally possible to solve the unemployment problem by simply expanding aggregate demand" [183]. Harrod's theory, and thereof modelisation which built on that [195], [183], [173], [174], may thus be seen as the link between classical economy (that stressed the importance of investment for growth) and the Keynesian approach "primarily concerned with the demand and income generating effect of investment" [51]. In real life this theory was put in practice in India. In fact, the Indian fifth five year plan for the years 1974–1979, was based on a mix of a Harrod macroeconomic model and a Leontief inter-industry model, and it was aimed at achieving both self-reliance and growth. Main priorities on the industrial sectors were the developments of: (i) core industry, (ii) industry for export and diversification, (iii) mass consumption production, (iv) small industry and ancillary industry feeders of large industries. The target growth rate was 4.4% and, as a result, the actual growth rate was 4.8% [51].

Having said that, to remind the importance of the model, this test shows (for a specific set of parameters) that it is possible to find a match between Harrod's suggestions and reality.

Chapter 9

Final Remarks

As mentioned in the Introduction, this Thesis consists of three Parts. Part I and Part II provides mathematical and economic background, while Part III contains the bulk of our research on growth and business cycle.

Intentionally we have left out our research on financial mathematics [143], [139], [140], [142] as this subject matter runs parallel when it comes to assessing market stability, solvency and resilience of financial institutions. Future research will link up them by integrating business cycles into financial models.

Back to the research we have introduced in this Thesis, in Chapter 6 we have proposed a new version of the Kaldor model by using a form of hyperbolic tangent instead of the usual *arctg*. The reasons are numerous and the Chapter gives a full account of them all. Here we only mention that in economics growth models (such as the Solow-Swan model) and in actuarial science and financial mathematics hyperbolic functions are widely used (e.g. Gompertz–Makeham law of mortality or law of compound interests). By considering that the investment process has different timing compared with consumption, we introduced a time delay *à la* Kalecki. Last but not least we have seen that the proposed model can accommodate external perturbations such as shocks or perturbations (repetition).

In Chapter 7 we tried to turn to a fundamental question - is economy stochastic or deterministic? To answer this question we applied recurrence plots and their quantitative description in our research provided by recurrence quantification analysis (RQA). We found out that RQA may be suitable to study business cycles and could be used for early detection of recessions (provided the shown limitations). Moreover, we found out that RQA is able to identify structural characteristics of time series by discriminating flow versus stock variables, real versus nominal data as well as net versus gross time series. Finally, RQA seems to confirm that different paths in economic development can have origins in the same underlying deterministic dynamic (which is an indication of chaos).

In Chapter 8 we have dealt with the Harrod's model [79] because it connects growth and cycle, supply and demand. The fundamental question here is if a chaotic economic model is able to explain reality? Therefore the Chapter is a reality check on the fact that a) real data can be obtained by a suitable calibration of model's parameters, b) the output of the calibrated model matches with theoretical predictions.

Appendix A

The dataset

Time series are taken from a range of sources such as the U.S. Bureau of Economic Analysis (BEA), IMF, the World Bank, Penn World Table by Feenstra et al. [61], Levy and Chen (1994)[110] and the OECD as retrieved from their original dataset or from FRED and explained in details below.

A.1 USA Recessions

Table A.1: USA Recessions

Recessions			
From		To	
Quarter	Year	Quarter	Year
Q4	1948	Q4	1949
Q3	1953	Q1	1954
Q4	1957	Q1	1958
Q3	1960	Q1	1961
Q1	1970	Q4	1970
Q1	1974	Q2	1975
Q1	1980	Q2	1980
Q3	1981	Q4	1982
Q3	1990	Q1	1991
Q2	2001	Q4	2001
Q1	2008	Q3	2009

US. Bureau of Economic Analysis[21]

A.2 World GDP data

For testing Harrod's model, annual world GDP estimate has been retrieved from the Maddison–Penn world table [29, 62], (from 1946 to 1961). This has been linked up with World Bank (`https://data.worldbank.org/indicator/NY.GDP.MKTP.KD.ZG`) and IMF (`https://www.imf.org/external/datamapper/NGDP_RPCH@WEO/OEMDC/ADVEC/WEOWORLD`) data (available from 1961 to 2018). Annual data has been changed into quarterly via the compounding law.

A.3 BEA data

In the following table (TableA.2) we list the time series considered for our analysis on business cycle as retrieved from FRED. Units were transformed in percent change from preceding period (apart from time series n. 3 for which percent change were already taken).

Table A.2: Time series on Consumption, Income and Investment

#	Time series	Data points	Frequency	Data range (from to)	Type	Account code/ID
1	USA PCEC	274	Quarterly	1947-01-01 to 2015-07-01	C	DPCERC1[a]
2	USA DPCER	275	Quarterly	1947-04-01 to 2015-10-01	C	DPCERL1[b]
3	USA GDP	274	Quarterly	1947-01-01 to 2015-07-01	Y	A191RC1[c]
4	USA RGPDI FI	274	Quarterly	1947-04-01 to 2015-07-01	I	A007RL1[d]
5	USA GPDI	274	Quarterly	1947-01-01 to 2015-07-01	I	A006RC1[e]
6	USA RGPDI	275	Quarterly	1947-04-01 to 2015-10-01	I	A006RL1[f]

[a] US. Bureau of Economic Analysis, Personal Consumption Expenditures [PCEC], Seasonally Adjusted Annual Rate, Percent Change, retrieved from FRED, Federal Reserve Bank of St. Louis https://research.stlouisfed.org/fred2/series/PCEC/, January 2, 2016.

[b] US. Bureau of Economic Analysis, Real Personal Consumption Expenditures [DPCERL1Q225SBEA], Seasonally Adjusted Annual Rate, Percent Change, retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/DPCERL1Q225SBEA, June 21, 2016.

[c] US. Bureau of Economic Analysis, Gross Domestic Product [GDP], Seasonally Adjusted Annual Rate, Percent Change, retrieved from FRED, Federal Reserve Bank of St. Louis https://research.stlouisfed.org/fred2/series/GDP/, January 3, 2016.

[d] US. Bureau of Economic Analysis, Real Gross Private Domestic Investment: Fixed Investment [A007RL1Q225SBEA], Seasonally Adjusted Annual Rate, Percent Change, retrieved from FRED, Federal Reserve Bank of St. Louis https://research.stlouisfed.org/fred2/series/A007RL1Q225SBEA/, January 3, 2016.

[e] US. Bureau of Economic Analysis, Gross Private Domestic Investment [GPDI], Seasonally Adjusted Annual Rate, Percent Change, retrieved from FRED, Federal Reserve Bank of St. Louis https://research.stlouisfed.org/fred2/series/GPDI/, January 2, 2016.

[f] US. Bureau of Economic Analysis, Real Gross Private Domestic Investment [A006RL1Q225SBEA], Seasonally Adjusted Annual Rate, Percent Change, retrieved from FRED, Federal Reserve Bank of St. Louis; [A006RL1Q225SBEA], https://research.stlouisfed.org/fred2/series/A006RL1Q225SBEA/, June 27, 2016.

In the following table (Table A.3) we list the time series considered for testing Harrod's model as retrieved from FRED.

Table A.3: BEA time series.

#	Time Series	Data Points	Frequency	Data Range (from to)	BEA Account Code
1	USA SAVE	287	Quarterly	1947-01-01 to 2018-07-01	A929RC1[a]
2	USA GPDIC1	287	Quarterly	1947-04-01 to 2018-07-01	A006RX[b]
3	USA NETEXP	287	Quarterly	1947-01-01 to 2018-07-01	A019RC[c]
4	USA GDPDEF	287	Quarterly	1947-04-01 to 2018-07-01	A191RD[d]
5	USA GPD	287	Quarterly	1947-01-01 to 2018-07-01	A191RC[e]

[a] U.S. Bureau of Economic Analysis, Gross Saving [GSAVE], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/GSAVE`, 20 February 2019.

[b] U.S. Bureau of Economic Analysis, Real Gross Private Domestic Investment (GPDIC1), retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/GPDIC1`, 20 February 2019. U.S. Bureau of Economic Analysis.

[c] Net Exports of Goods and Services (NETEXP), retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/NETEXP`, 20 February 2019.

[d] U.S. Bureau of Economic Analysis, Gross Domestic Product: Implicit Price Deflator (GDPDEF), retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/GDPDEF`, 20 February 2019.

[e] U.S. Bureau of Economic Analysis, Gross Domestic Product (GDP), retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/GDP`, 20 February 2019.

A.4 Levy and Chen data - USA

In the following table (TableA.4 we list the time series on capital reconstructed as described by Levy and Chen (1994) [110].

Table A.4: Time series on Capital - USA

#	Code	Terms	Gross/Net	Goods	Method	Type
1	M1BS87G	Real	Gross	BS	1	K
2	M1CDGG	Nominal	Gross	CDG	1	K
3	M1CDG87G	Real	Gross	CDG	1	K
4	M2BS87G	Real	Gross	BS	2	K
5	M2CDGG	Nominal	Gross	CGD	2	K
6	M2CDG87G	Real	Gross	CGD	2	K
7	M1BSG	Nominal	Gross	BS	1	K
8	M1PDGG	Nominal	Gross	PDG	1	K
9	M2PDG87G	Real	Gross	PDG	2	K
10	M1BS87N	Real	Net	BS	1	K
11	M1CDG87N	Real	Net	CGD	1	K
12	M1PDG87N	Real	Net	PDG	1	K
13	M2BS87N	Real	Net	BS	2	K
14	M2PDG87N	Real	Net	PDG	2	K

data points=175, data range=1948-91

A.5 OECD data

Regarding the scope of our analysis we included countries that had very different development paths. OECD data are respectively quarterly GDP (percentage change)[1], investment (GFCF) (annual growth rate %)[2] and saving rates (% of GDP)[3].

In the following we start, first, by listing the data directly taken from the OECD database (Table A.5, Table A.8) and then, the time series as retrieved from FRED (Table A.6, Table A.7); units were transformed in percent change from preceding period).

Table A.5: Time series on Income

#	Time series	Data points	Data range (from to)	Type	Country	Code/ID
1	Tot Q GDP perc. c. KOR	183	1970 (Q2) - 2015 (Q4)	Y	Korea	TQGDP PC_CHGPP KOR
2	Tot Q GDP perc. c. GBR	243	1955 (Q2) - 2015 (Q4)	Y	United Kingdom	TQGDP PC_CHGPP GBR
3	Tot Q GDP perc. c. ESP	223	1960 (Q2) - 2015 (Q4)	Y	Spain	TQGDP PC_CHGPP ESP
4	Tot Q GDP perc. c. JPN	222	1960 (Q2) - 2015 (Q3)	Y	Japan	TQGDP PC_CHGPP JPN
5	Tot Q GDP perc. c. TUR	222	1960 (Q2) - 2015 (Q3)	Y	Turkey	TQGDP PC_CHGPP TUR
6	Tot Q GDP perc. c. DEU	223	1960 (Q2) - 2015 (Q4)	Y	Germany	TQGDP PC_CHGPP DEU
7	Tot Q GDP perc. c. ITA	223	1960 (Q2) - 2015 (Q4)	Y	Italy	TQGDP PC_CHGPP ITA
8	Tot Q GDP perc. c. USA	275	1947 (Q2) - 2015 (Q4)	Y	USA	TQGDP PC_CHGPP USA

Organization for Economic Co-operation and Development

Quarterly GDP Total, Percentage change, previous period, Q2 1947 – Q4 2015 Source: Quarterly National Accounts, OECD (2016)[134]

[1] Quarterly GDP Total, Percentage change, previous period, Q2 1947 – Q1 2016. Source: Quarterly National Accounts. This indicator is seasonally adjusted and it is measured in percentage change from previous quarter and from same quarter previous year. [134]

[2] Investment (GFCF) Total, Annual growth rate (%), 1951 – 2014. Source: Aggregate National Accounts, SNA 2008 (or SNA 1993): Gross domestic product. Gross fixed capital formation (GFCF) is in million USD at current prices and PPPs, and in annual growth rates.[133]

[3] Saving rate Total, % of GDP, 1970 – 2014. Source: National Accounts at a Glance [135]

Table A.6: Time series on Consumption

#	Time series	Data points	Frequency	Data range (from to)	Type	Account code/ID
1	Private Final Cons. in Korea	179	Quarterly	1970-01-01 to 2014-07-01	C	KORPFCEQDSMEI[a]
2	Private Final Cons. in USA	143	Quarterly	1947-01-01 to 2013-01-01	C	USAPFCEQDSNAQ[b]
3	Private Final Cons. in Italy	94	Quarterly	1991-01-01 to 2014-04-01	C	ITAPFCEQDSNAQ[c]
4	Private Final Cons. in the UK	172	Quarterly	1970-01-01 to 2012-10-01	C	GBRPFCEQDSNAQ[d]
5	Private Final Cons. in Turkey	111	Quarterly	1987-01-01 to 2014-07-01	C	TURPFCEQDSMEI[e]
6	Private Final Cons. in Germany	179	Quarterly	1970-01-01 to 2014-07-01	C	DEUPFCEQDSMEI[f]
7	Private Final Cons. in Spain	78	Quarterly	1995-04-01 to 2014-07-01	C	ESPPFCEQDSMEI[g]
8	Private Final Cons. in Japan	82	Quarterly	1994-04-01 to 2014-07-01	C	JPNPFCEQDSMEI[h]

[a] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in Korea© [KORPFCEQDSMEI], retrieved from FRED, Federal Reserve Bank of St. Louis `https://research.stlouisfed.org/fred2/series/KORPFCEQDSMEI`, June 10, 2016.

[b] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in United States© [USAPFCEQDSNAQ], retrieved from FRED, Federal Reserve Bank of St. Louis `https://research.stlouisfed.org/fred2/series/USAPFCEQDSNAQ`, June 10, 2016.

[c] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in Italy© [ITAPFCEQDSNAQ], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/ITAPFCEQDSNAQ`, June 21, 2016.

[d] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in the United Kingdom© [GBRPFCEQDSNAQ], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/GBRPFCEQDSNAQ`, June 21, 2016.

[e] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in Turkey© [TURPFCEQDSMEI], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/TURPFCEQDSMEI`, June 21, 2016.

[f] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in Germany© [DEUPFCEQDSMEI], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/DEUPFCEQDSMEI`, June 21, 2016.

[g] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in Spain© [ESPPFCEQDSMEI], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/ESPPFCEQDSMEI`, June 21, 2016.

[h] Organization for Economic Co-operation and Development, Private Final Consumption Expenditure in Japan© [JPNPFCEQDSMEI], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/JPNPFCEQDSMEI`, June 21, 2016.

Table A.7: Time series on Investment

#	Time series	Data points	Frequency	Data range (from to)	Type	Account code/ID
1	Tot. Prod. of Inv. Goods for Germany	240	Quarterly	1955-01-01 to 2014-10-01	I	PRMNVG01 IXOB DEU[a]
2	Tot. Prod. of Inv. Goods for Italy	176	Quarterly	1971-01-01 to 2014-10-01	I	PRMNVG01 IXOB ITA[b]
3	Tot. Prod. of Inv. Goods for Spain	200	Quarterly	1965-01-01 to 2014-10-01	I	PRMNVG01 IXOB ESP[c]
4	Tot. Prod. of Inv. Goods for Korea	140	Quarterly	1980-01-01 to 2014-10-01	I	PRMNVG01 IXOB KOR[d]
5	Tot. Prod. of Inv. Goods for Japan	240	Quarterly	1955-01-01 to 2014-10-01	I	PRMNVG01 IXOB JPN[e]
6	Tot. Prod. of Inv. Goods for the UK	188	Quarterly	1968-01-01 to 2014-10-01	I	PRMNVG01 IXOB GBR[f]
7	Tot. Prod. of Inv. Goods for the Euro Area	115	Quarterly	1985-01-01 to 2013-07-01	I	PRMNVG01 IXOB EA17[g]
8	Tot. Prod. of Inv. Goods for Brazil	96	Quarterly	1991-01-01 to 2014-10-01	I	PRMNVG01 IXOB BRA[h]

[a] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for Germany© [PRMNVG01DEQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01DEQ661N`, June 22, 2016.

[b] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for Italy© [PRMNVG01ITQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01ITQ661N`, June 22, 2016.

[c] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for Spain© [PRMNVG01ESQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01ESQ661N`, June 21, 2016.

[d] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for the Republic of Korea© [PRMNVG01KRQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01KRQ661N`, June 22, 2016.

[e] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for Japan© [PRMNVG01JPQ661S], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01JPQ661S`, June 22, 2016.

[f] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for the United Kingdom© [PRMNVG01GBQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01GBQ661N`, June 22, 2016.

[g] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for the Euro Area© [PRMNVG01EZQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01EZQ661N`, June 22, 2016.

[h] Organization for Economic Co-operation and Development, Total Production of Investment Goods for Manufacturing for Brazil© [PRMNVG01BRQ661N], retrieved from FRED, Federal Reserve Bank of St. Louis; `https://fred.stlouisfed.org/series/PRMNVG01BRQ661N`, June 22, 2016.

Table A.8: Time series on Income

#	Code/ID	Data points	Data range (from to)	Type	Country
1	B1_GE GPSA KOR Y	184	1970 (Q2) - 2016 (Q1)	Y	Korea
2	B1_GE GPSA GBR Y	244	1955 (Q2) - 2016 (Q1)	Y	United Kingdom
3	B1_GE GPSA ESP Y	224	1960 (Q2) - 2016 (Q1)	Y	Spain
4	B1_GE GPSA JPN Y	224	1960 (Q2) - 2016 (Q1)	Y	Japan
5	B1_GE GPSA TUR Y	224	1960 (Q2) - 2016 (Q1)	Y	Turkey
6	B1_GE GPSA DEU Y	224	1960 (Q2) - 2016 (Q1)	Y	Germany
7	B1_GE GPSA ITA Y	224	1960 (Q2) - 2016 (Q1)	Y	Italy
8	B1_GE GPSA USA Y	276	1947 (Q2) - 2016 (Q1)	Y	USA
9	B1_GE GPSA ICE Y	224	1960 (Q2) - 2016 (Q1)	Y	Iceland
10	B1_GE GPSA GRE Y	224	1960 (Q2) - 2016 (Q1)	Y	Greece
11	B1_GE GPSA FRA Y	224	1960 (Q2) - 2016 (Q1)	Y	France

B1_GE: Gross domestic product - expenditure approach, GPSA: Growth rate compared to previous quarter, seasonally adjusted, quarterly data[135].

A.6 RQA tables

Table A.9: RQA tables on 55 (10 C, 11 I, 14 K, 20 Y) real time series

#	Table	Row	VARTYPE	Mean	SDev	Mean/SDev	REC	DET	MAXLINE	ENT	TREND	LAM	TT	PC1	PC2
1	1	1	C	1.777	0.81	2.194	29.378	85.402	61	4.294	136.25	61.948	11.65	-0.324	0.482
2	4	1	C	0.036	0.027	1.333	43.097	94.998	79	4.842	-478.416	94.018	19.367	0.834	1.778
3	4	2	C	0.008	0.007	1.143	33.81	87.074	78	4.352	-297.973	72.693	12.338	-0.013	0.606
4	4	3	C	0.002	0.007	0.286	33.161	90.138	51	3.386	-830.171	76.903	11.397	-0.507	1.764
5	4	4	C	0.007	0.011	0.636	35.871	87.623	119	4.602	-269.715	77.533	17.028	0.377	-0.110
6	4	5	C	0.104	0.07	1.486	37.327	93.103	70	4.641	-1033.278	83.873	15.057	0.437	1.432
7	4	6	C	0.012	0.011	1.091	35.559	91.303	77	4.803	-375.587	79.913	16.398	0.421	0.933
8	1	2	C	3.628	3.705	0.979	36.367	83.902	61	4.351	-37.051	66.174	12.537	-0.128	0.871
9	4	7	C	0.012	0.012	1.000	31.714	87.231	46	4.295	-1085.41	62.903	11.415	-0.244	1.118
10	4	8	C	0.003	0.009	0.333	33.79	81.644	32	4.206	-665.148	67.68	18.212	-0.497	1.116
11	1	5	I	1.861	4.474	0.416	31.876	85.859	75	4.4	-28.101	69.081	13.577	-0.112	0.391
12	1	4	I	4.754	9.352	0.508	29.896	83.641	86	4.332	62.648	68.847	12.848	-0.246	-0.143
13	5	8	I	0.014	0.084	0.167	21.259	78.507	62	3.377	-31.291	1.544	6	-1.293	-0.371
14	5	3	I	0.017	0.166	0.102	36.92	92.653	158	4.983	62.794	0.558	9.25	0.962	-0.535
15	5	7	I	0.01	0.106	0.094	46.648	97.998	101	4.926	22.772	-1.000	-1.000	1.202	1.836
16	5	2	I	0.013	0.148	0.088	33.509	95.903	162	5.375	85.122	-1.000	-1.000	1.186	-0.766
17	5	5	I	0.019	0.04	0.475	36.548	91.517	147	4.654	-346.672	82.689	12.85	0.697	-0.303
18	5	4	I	0.031	0.101	0.307	36.625	95.962	92	4.884	-30.778	-1.000	-1.000	0.740	1.082
19	5	6	I	0.004	0.055	0.073	29.899	85.265	166	4.454	-171.536	11.062	8.27	0.229	-1.648
20	1	6	I	6.652	23.279	0.286	36.854	90.155	59	4.554	-16.148	73.575	14.389	0.218	1.399
21	5	1	I	0.014	0.106	0.132	45.291	96.838	196	5.862	6.948	0.71	8	1.989	-0.656
22	2	1	K	0.795	0.213	3.732	40.636	96.762	164	4.646	-177.904	95.871	13.835	1.102	0.128
23	2	2	K	2.063	1.578	1.307	49.091	97.757	126	5.504	-550.28	97.38	32.02	1.664	1.263
24	2	3	K	1.326	0.709	1.870	46.6	97.335	155	4.868	-91.22	97.034	16.142	1.385	0.725
25	2	4	K	0.795	0.218	3.647	40.31	96.296	164	4.614	-187.832	93.601	13.948	1.058	0.077
26	2	5	K	2.065	1.63	1.267	50.133	97.803	126	5.626	-563.075	97.464	34.254	1.759	1.296
27	2	6	K	1.351	0.709	1.906	46.031	96.965	154	4.859	-79.771	95.36	15.629	1.344	0.673
28	2	7	K	1.982	1.262	1.571	32.326	91.168	85	4.536	30.812	81.613	15.054	0.217	0.625
29	2	8	K	-2.227	1.332	-1.672	39.81	92.865	70	4.766	-64.058	85.803	17.326	0.569	1.551
30	2	10	K	0.8	0.208	3.846	41.086	96.852	164	4.703	-199.62	96.006	13.791	1.148	0.147
31	2	11	K	1.351	0.669	2.019	46.282	97.062	164	4.92	-84.862	95.369	16.497	1.429	0.476
32	2	12	K	1.087	0.508	2.140	41.227	95.608	164	4.64	-26.505	95.948	13.481	1.074	0.078
33	2	13	K	0.8	0.214	3.738	40.916	96.152	164	4.645	-197.501	95.159	13.303	1.087	0.098
34	2	14	K	1.102	0.495	2.226	41.636	95.808	165	4.637	-45.891	94.458	13.881	1.123	0.112
35	2	15	K	1.085	0.517	2.099	40.65	95.509	164	4.569	-6.188	94.382	12.881	1.017	0.055
36	1	3	Y	1.718	1.023	1.679	30.557	89.043	81	4.619	-43.274	78.851	16.586	0.101	0.350
37	3	1	Y	1.797	1.794	1.002	30.968	87.728	52	4.468	-374.438	78.095	15.104	-0.138	0.910
38	3	2	Y	0.668	1.037	0.644	34.246	86.134	81	4.403	-79.834	83.786	23.891	0.003	0.477
39	3	3	Y	0.948	1.067	0.888	38.779	92.614	74	5.444	-215.569	90.229	25.694	0.875	1.084
40	3	4	Y	1.043	1.342	0.777	33.352	90.282	68	4.896	-353.221	76.736	15.068	0.316	0.814
41	3	5	Y	1.106	1.975	0.560	41.844	88.807	100	4.905	-12.32	86.814	20.025	0.679	0.705
42	3	6	Y	0.631	1.146	0.551	31.472	79.052	59	4.457	-144.772	63.676	14.243	-0.429	0.071
43	3	7	Y	0.688	1.049	0.656	35.618	84.17	94	5.118	-345.894	78.936	18.107	0.378	-0.146
44	3	8	Y	0.87	1.043	0.834	28.543	85.44	61	4.468	-58.462	72.342	15.107	-0.264	0.347
45	6	1	Y	1.779	1.787	0.996	31.074	88.582	53	4.422	-376.179	78.704	15.211	-0.120	0.991
46	6	6	Y	0.631	1.146	0.551	31.472	79.052	59	4.457	-144.772	63.676	14.243	-0.429	0.071
47	6	11	Y	0.733	1.211	0.605	61.613	93.222	163	4.908	-388.96	92.733	24.88	1.767	1.359
48	6	7	Y	0.685	1.05	0.652	35.392	83.743	93	5.015	-343.31	77.453	17.654	0.299	-0.136
49	6	4	Y	1.042	1.342	0.776	33.322	90.439	68	4.854	-352.012	77.334	14.95	0.301	0.843
50	6	3	Y	0.951	1.068	0.890	38.794	92.772	74	5.434	-215.298	89.832	25.877	0.876	1.103
51	6	2	Y	0.67	1.036	0.647	34.181	86.195	81	4.429	-76.604	83.343	23.523	0.016	0.467
52	6	5	Y	1.104	1.974	0.559	41.879	88.925	100	4.897	-11.964	86.957	19.855	0.681	0.721
53	6	8	Y	0.874	1.048	0.834	28.492	85.82	61	4.436	-59.29	71.393	14.937	-0.267	0.389
54	6	9	Y	0.938	1.872	0.501	36.055	94.132	102	5.23	-361.949	94.077	21.774	0.864	0.536
55	6	10	Y	0.916	2.599	0.352	31.693	92.659	69	5.28	191.97	86.792	35.778	0.546	0.708

Table and row respectively indicate relevant time series considered

PC1 and PC2 are calculated on REC, DET, MAXLINE and ENT, Parameters: LAG=1, EMB=10, EUCLIDEAN DIST=3, MEANDIST=2, RADIUS=40, LINE=5

REC (percent recurrence) = # recurrent points within the set threshold (i.e., RADIUS) / # total points

LAM (percent laminarity) = # recurrent points forming vertical lines / # REC. Reported as -1.000 if # REC=0

TT (trapping time) = mean vertical line length. Reported as -1.000 if # LINE vertical=0

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