ON TRANSITIVE CONTACT AND CR ALGEBRAS

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ABSTRACT. We consider locally homogeneous CR manifolds and show that, under a condition only depending on their underlying contact structure, their CR automorphisms form a finite dimensional Lie group.

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INTRODUCTION

In the past years some of the Authors introduced and investigated the notion of CR algebra (see [2, 14]) to describe the local structure of homogeneous CR manifolds. The understanding of local models is important e.g. for applying the method of E. Cartan to describe the differential invariants of CR structures. A key point is to find under which conditions the infinitesimal automorphisms of the structure form a finite dimensional Lie algebra. A CR structure can be defined by the datum of a smooth involutive *complex* distribution. The real parts of its vectors define a *real* distribution. A strong version of the condition that the CR manifold is not foliated by CR submanifolds of lower dimension is that this real distribution is a (generalised) *contact* distribution, i.e. that its iterated commutators span the full tangent space. The strong interplay between CR and underlying contact structures

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was clearly exploited in the work of N. Tanaka (see [23, 24]). He considered, at each point, the nilpotent \mathbb{Z} -graded real Lie algebra m canonically associated to a contact structure. To describe the infinitesimal automorphisms, one needs to consider *extensions*, or prolongations \mathfrak{G} of m. In this setting, they can be described recursively in terms of derivations of m. When a *CR* structure is imposed on the contact distribuiton, finite dimensionality of the maximal prolongation is, for this \mathbb{Z} -graded model, equivalent to the fact that the vector valued Levi form has trivial kernel. Thus Cartan's method applies, via Tanaka's theory, to the case where the associated \mathbb{Z} -graded Levi-Tanaka algebras (cf. [13]) are isomorphic at all points and the Levi form is nondegenerate.

The idea of introducing *CR* algebras in [14] originated from the observation that many interesting homogeneous examples of *CR* manifolds lead to infinite dimensional Levi-Tanaka algebras, their Levi forms having nontrivial kernels. An obvious generalization of the nondegeneracy condition is to require that the iterated Levi forms have a trivial kernel. This condition, that was called *weak nondegeneracy* in [14] and is equivalent to the notion of *(Levi) k-nondegeneracy* used by other authors (see e.g. [5]), is indeed equivalent to the fact that the corresponding *CR* manifold is not the total space of a *CR* fibration with complex fibres. The differential invariants for *CR* manifolds of hypersurface type in real dimension 5 satisfying the notion of *k*-nondegenericity have been so far studied by several authors with different techniques (see e.g. [9, 15, 12, 18, 16]). Further developments in higher dimensions appeared in [21, 19]. A theory of invariants for 2-nondegenerate *CR* hypersurfaces in arbitrary dimension, modeled on Tanaka's approach, has been recently developed by Porter and Zelenko in [20].

In this paper we address the question on finite dimensionality of the full group of the automorphisms of CR manifolds of arbitrary CR dimension and CR codimension, whose CR structure is locally homogeneous. Weak nondegeneracy is a much more restrictive condition that the one we found to guarantee the finite dimensionality of the Lie algebra of infinitesimal CR automorphisms. In fact, our criterion only involves the underlying contact structure. We found this fact very interesting. Indeed, it is preliminary to an approach where this (generalized) contact structure is a priori given as a characteristic of the manifold on which the addition of a CR structure is meant to modelling different geometrical or physical situations. Our condition was called *ideal nondegeneracy* in [14], where the fact that it was a sufficient criterion for the finite dimensionality of the maximal extension was correctly stated; however, in the proof given there there was a gap that we fill here in §6.

Our proof of the existence of maximal extensions of *CR* algebras relies on a review of the classical work on transitive geometry (see e.g. [6, 7, 10, 11]), allowing us to substitute formal power series to the canonical construction of Tanaka in the \mathbb{Z} -graded case. Our discussion is restrained mostly at a purely algebraic level. Thus, for a better understanding of the geometrical

significance of our results, we refer the reader to [4] for a thorough introduction to *CR* and homogeneous *CR* manifolds.

Let us briefly describe the contents of the paper.

1 collects some general notions we thought relevant for the exposition. Contact and transitive pairs and triples and *CR* algebras are defined, not restraining to finite dimensionality. We explicitly required that the (possibly infinite dimensional) Lie algebras involved have a topological structure, although this structure is implicitly defined by the requirement that the isotropy subalgebra is closed and has finite codimension. We also list various nondegeneracy conditions that will be investigated in the later sections.

In §2 we construct a canonical descending chain of subspaces which is associated to a contact pair to explain *contact nondegeneracy*.

An analogous construction in \$3, characteristic of *CR* algebras, describes weak *CR*-nondegeneracy. We show by an example that it is in fact a much more restrictive condition than nondegeneracy of the underlying contact structure.

We found convenient to explain in §4 the way the abstract contact triples of §1 relate to actual homogeneous contact manifolds, to motivate our later use of transitive contact geometry.

In §5 we introduce graded Lie algebras and the finiteness criterion of Noburu Tanaka.

By using Tanaka's criterion, we prove in §6 our main result, which states that *CR* algebras whose corresponding contact triple is nondegenerate are finite dimensional.

In §7 we deal with the general construction of the representation of transitive contact pairs by structures involving vector fields with formal power series coefficients. This is the main tool in the *transitive geometry* of [11]: in this purely algebraic setting a germ of homogeneous space is substituted by a topological Lie algebra in which the isotropy subalgebra is closed and has finite codimension. This describes a situation in which the values of the infinitesimal automorphisms of the structure span the full tangent space at a point.

In the final §8 we utilize transitive geometry to construct maximal extensions of *CR* algebras. Then the result of §6 yields the theorem that locally homogeneous *CR* manifolds with a nondegenerate underlying contact structure have a finite dimensional Lie algebra of infinitesimal *CR* automorphisms and hence, in particular, their *CR* automorphisms make a finite dimensional Lie group.

1. Definitions and preliminaries

In this section we introduce some notions which are relevant for an *infinitesimal* description of homogeneous (generalised) contact manifolds and of various geometrical structures that can be defined on them. We are particularly interested in partial complex structures, and *CR* algebras (see [14]) fall in this realm. A *topological Lie algebra* is a Lie algebra g_0 over a topological field k with a fixed structure of topological Hausdorff vector space for which the Lie product is continuous. We say that g_0 is *linearly compact* if the intersection of any family of affine subspaces of g_0 having the finite intersection property has a nonempty intersection. (For details, see e.g. [7]).

In the following we assume that g_0 is real and denote by g its complexification. Conjugation in g is always understood with respect to the real form g_0 . For a \mathbb{C} -linear subspace \mathcal{L} of g we set

(1.1)
$$\tilde{\mathcal{L}}_0 = \{ \operatorname{Re}(Z) \mid Z \in \mathcal{L} \}, \qquad \tilde{\mathcal{L}}_0 = \mathcal{L} \cap \bar{\mathcal{L}} \cap \mathfrak{g}_0.$$

- **Definition 1.1.** A *contact pair* is the pair (g_0, \mathcal{L}_0) consisting of a linearly compact topological real Lie algebra g_0 and a closed linear subspace \mathcal{L}_0 of g_0 having a finite dimensional complement in g_0 and spanning g_0 as a Lie algebra.
 - A contact C-pair is the pair (g₀, L) consisting of a linearly compact topological real Lie algebra g₀ and a closed complex linear subspace L of g such that (g₀, L̃₀) is a contact pair.
 - A *transitive pair* (g₀, h₀) consists of a linearly compact topological Lie algebra g₀ and a closed subalgebra h₀ having finite codimension in g₀ and not containing nontrivial ideals of g₀.
 - A *contact triple* is a triple (g₀, b₀, L₀) such that (g₀, L₀) is a contact pair, (g₀, b₀) a transitive pair, b₀ ⊆ L₀ and [b₀, L₀] ⊆ L₀.
 - A contact C-triple is a triple (g₀, h₀, L), such that (g₀, h₀) is transitive, (g₀, L) is a contact C-pair, h₀ ⊂ L and [h₀, L] ⊆ L.
 - A fundamental almost CR pair is a contact \mathbb{C} -pair (g_0, \mathcal{L}) for which $\check{\mathcal{L}}_0$ is a Lie subalgebra of g_0 and $[\check{\mathcal{L}}_0, \mathcal{L}] \subseteq \mathcal{L}$. The contact triple $(g_0, \check{\mathcal{L}}_0, \check{\mathcal{L}}_0)$ is said to be *associated* to (g_0, \mathcal{L}) .
 - A *fundamental CR algebra* is an almost *CR* pair (g₀, q) such that q is a complex Lie subalgebra of g.

Remark 1.1. In the definition of a *CR* algebra (g_0, q) of [14] it was not required that $(q + \bar{q}) \cap g_0$ generates g_0 as a Lie algebra. When this is not the case and (g_0, q_0) is a Lie algebra associated to a homogeneous *CR* manifold M_0 , then the manifold M_0 can be described, at least locally, as the product $M'_0 \times N_0$ of a *CR* manifolds M'_0 having the same *CR* dimension of M_0 and a totally real N_0 . For many purposes we could reduce to M'_0 , which is a homogeneous *CR* manifold whose *CR* algebra at a point has the same q, while g_0 is substituted by the span of \tilde{q}_0 .

We will use the following notions.

Definition 1.2 (Nondegenracy conditions).

We say that a contact triple (g_0, h_0, L_0) is

- *strictly nondegenerate* if $\{X \in \mathcal{L}_0 \mid [X, \mathcal{L}_0] \subseteq \mathcal{L}_0\} = \mathfrak{h}_0$.
- *nondegenerate* if any ideal of g_0 which is contained in \mathcal{L}_0 is already contained in \mathfrak{h}_0 .

A fundamental almost *CR*-pair (g_0, L) is

- *strictly nondegenerate* if $\{Z \in \overline{\mathcal{L}} \mid [Z, \mathcal{L}] \subset \mathcal{L} + \overline{\mathcal{L}}\} = \mathcal{L} \cap \overline{\mathcal{L}}$.
- weakly non-degenerate if there is no almost *CR* pair (g_0, \mathcal{L}') with $\mathcal{L} \subsetneq \mathcal{L}' \subseteq \mathcal{L} + \overline{\mathcal{L}}$.
- *contact nondegenerate* if the associated contact triple (g₀, L̃₀, Ĩ₀) is nondegenerate.

Remark 1.2. For a fundamental almost *CR* pair, strict nondegeneracy is equivalent to strict nondegeneracy of the associated contact triple and implies weak nondegeneracy, which in turn implies contact nondegeneracy.

2. Descending chain of a contact pair

Given a contact pair (g_0, \mathcal{L}_0) , we construct a descending chain of \mathbb{R} -linear subspaces of g_0

$$(2.1) \qquad \cdots \supseteq \mathcal{G}_{-h} \supseteq \mathcal{G}_{1-h} \supset \cdots \mathcal{G}_{-1} \supseteq \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \mathcal{G}_h \supseteq \mathcal{G}_{h+1} \supseteq \cdots$$

defining by recurrence

(2.2)
$$\begin{cases} \mathcal{G}_{-1} = \mathcal{L}_0, \\ \mathcal{G}_h = \mathcal{G}_{h+1} + [\mathcal{G}_{h+1}, \mathcal{G}_{-1}], & \text{if } h < -1, \\ \mathcal{G}_h = \{X \in \mathcal{G}_{-1} \mid [X, \mathcal{G}_{-1}] \subseteq \mathcal{G}_{h-1}\} & \text{if } h \ge 0. \end{cases}$$

Since, by assumption, \mathcal{L}_0 is a subspace of finite codimension that generates g_0 as a Lie algebra, there is a nonnegative integer μ such that $\mathcal{G}_{-\mu} = g_0$. Indeed the ascending chain of subspaces

$$\{0\} = (\mathcal{G}_{-1}/\mathcal{G}_{-1}) \subseteq (\mathcal{G}_{-2}/\mathcal{G}_{-1}) \subseteq \cdots (\mathcal{G}_{-h}/\mathcal{G}_{-1}) \subseteq (\mathcal{G}_{-h-1}/\mathcal{G}_{-1}) \subseteq \cdots$$

of the finite dimensional vector space (g_0/\mathcal{G}_{-1}) stabilizes and, by their definition, if $(\mathcal{G}_{-h}/\mathcal{G}_{-1}) = (\mathcal{G}_{-h-1}/\mathcal{G}_{-1})$, then $\mathcal{G}_{-r} = \mathcal{G}_{-h}$ for all r > h. For a contact pair $\bigcup_h \mathcal{G}_{-h} = g_0$, and hence $\mathcal{G}_{-h} = \mathcal{G}_{-h-1} = g_0$ for some h > 0.

Definition 2.1. The smallest nonnegative integer μ for which $\mathcal{G}_{-\mu} = \mathfrak{g}_0$ is called the *depth*, or *kind*, of the contact pair ($\mathfrak{g}_0, \mathcal{L}_0$).

Proposition 2.1. Let (g_0, L_0) be a contact pair, (2.1) the associated descending chain. Set

(2.3)
$$c_0 = \bigcap_{h \in \mathbb{Z}} \mathcal{G}_h.$$

Then:

- (1) All G_h are closed subspaces of \mathfrak{g}_0 .
- (2) c_0 is the largest ideal of g_0 contained in \mathcal{L}_0 .
- (3) (2.1) is a filtration of g_0 .
- (4) For all $h \ge 0$, G_h is a Lie subalgebra of g_0 .

Proof. (1) The closedness of \mathcal{G}_{-1} was assumed in the definition of a contact pair. For $h \ge 0$ the statement follows by recurrence, because each \mathcal{G}_h is an intersection of the inverse images of the closed subspace \mathcal{G}_{h-1} by the continuous linear maps $\mathcal{G}_{-1} \ni X \to [X, Y] \in \mathfrak{g}_0$, for Y varying in \mathcal{G}_{-1} .

For h < -1, the statement is true because all \mathbb{R} -linear subspaces V with $\mathcal{L}_0 \subseteq V \subseteq \mathfrak{g}_0$ are closed in \mathfrak{g}_0 , since \mathcal{L}_0 is closed and has finite codimension in \mathfrak{g}_0 . This completes the proof of (1).

(2) and the fact that G_0 is a Lie subalgebra of g_0 are streightforward consequences of the definitions.

To complete the proof, it suffices to check that (2.1) if a filtration. We begin by checking the commutators of elements belonging to subspaces with negative indices. If h < -1, and we assume that $[\mathcal{G}_0, \mathcal{G}_{h+1}] \subseteq \mathcal{G}_{h+1}$, then

$$[\mathcal{G}_{0}, \mathcal{G}_{h}] = [\mathcal{G}_{0}, \mathcal{G}_{h+1} + [\mathcal{G}_{h+1}, \mathcal{G}_{-1}]]$$

$$\subseteq [\mathcal{G}_{0}, \mathcal{G}_{h+1}] + [[\mathcal{G}_{0}, \mathcal{G}_{h+1}], \mathcal{G}_{-1}] + [\mathcal{G}_{h+1}, [\mathcal{G}_{0}, \mathcal{G}_{-1}]]$$

$$\subseteq \mathcal{G}_{h+1} + [\mathcal{G}_{h+1}, \mathcal{G}_{-1}] = \mathcal{G}_{h}.$$

This implies by recurrence that $[G_0, G_h] \subseteq G_h$ for all $h \leq 0$.

Let now h > 0 and assume that $[\mathcal{G}_0, \mathcal{G}_{h-1}] \subset \mathcal{G}_{h-1}$. By (2.2) we already have $[\mathcal{G}_h, \mathcal{G}_{-1}] \subseteq \mathcal{G}_{h-1}$. Then

$$[[\mathcal{G}_0, \mathcal{G}_h], \mathcal{G}_{-1}] \subseteq [[\mathcal{G}_0, \mathcal{G}_{-1}], \mathcal{G}_h] + [\mathcal{G}_0, [\mathcal{G}_h, \mathcal{G}_{-1}]]$$
$$\subseteq [\mathcal{G}_h, \mathcal{G}_{-1}] + [\mathcal{G}_0, \mathcal{G}_{h-1}] \subseteq \mathcal{G}_{h-1}$$

shows, by recurrence, that $[\mathcal{G}_0, \mathcal{G}_h] \subseteq \mathcal{G}_h$ for all $h \ge 0$.

By (2.2) we have $[\mathcal{G}_h, \mathcal{G}_{-1}] \subseteq \mathcal{G}_{h-1}$ for all integers *h* and this implies that $[\mathcal{G}_{h_1}, \mathcal{G}_{h_2}] \subseteq \mathcal{G}_{h_1+h_2}$ when either $h_1 \leq 0$ or $h_2 \leq 0$. When both $h_1, h_2 > 0$, we can argue by recurrence on $h_1 + h_2$. In fact

$$[[\mathcal{G}_{h_1}, \mathcal{G}_{h_2}], \mathcal{G}_{-1}] \subseteq [[\mathcal{G}_{h_1}, \mathcal{G}_{-1}], \mathcal{G}_{h_2}] + [\mathcal{G}_{h_1}, [\mathcal{G}_{h_2}, \mathcal{G}_{-1}]] \subseteq [\mathcal{G}_{h_1 - 1}, \mathcal{G}_{h_2}]][\mathcal{G}_{h_1}, \mathcal{G}_{h_2 - 1}] \subseteq \mathcal{G}_{h_1 + h_2 - 1}$$

if we assumed that $[\mathcal{G}_{h'}, \mathcal{G}_{h''}] \subseteq \mathcal{G}_{h'+h''}$ when $h'+h'' < h_1+h_2$. This completes the proof of the fact that (2.1) is a filtration and hence of the Proposition. \Box

Lemma 2.2. If \mathfrak{h}_0 is any Lie subalgebra of \mathfrak{g}_0 such that $[\mathfrak{h}_0, \mathfrak{L}_0] \subseteq \mathfrak{L}_0$, then

(2.4) $[\mathfrak{h}_0, \mathcal{G}_h] \subseteq \mathcal{G}_h, \quad \forall h \in \mathbb{Z}.$

In particular, if $(g_0, b_0, \mathcal{L}_0)$ is a contact triple, then all subspaces \mathcal{G}_h of the canonical filtration (2.1) are b_0 -modules.

Lemma 2.3. Let (2.1) be the canonical filtration of the contact pair of a contact triple (g_0, h_0, L_0) . Then,

- (1) $(\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{L}_0)$ is strictly nondegenerate if and only if $\mathfrak{h}_0 = \mathcal{G}_0$.
- (2) (g_0, b_0, L_0) is nondegenerate, if and only if there is a positive integer k such that $G_k \subseteq b_0$.

Proof. Statement (1) follows immediately from the definitions.

To prove (2), we note that, with $c_0 = \bigcap_{h \in \mathbb{Z}} \mathcal{G}_h$ as in (2.3), the nondegeneracy condition can be restated by saying that $c_0 \subseteq \mathfrak{h}_0$. This is equivalent to the fact that the intersection of all subspaces $(\mathcal{G}_h + \mathfrak{h}_0)/\mathfrak{h}_0$ in $\mathfrak{g}_0/\mathfrak{h}_0$ is {0}. The statement follows because these subspaces form a descending chain of vector subspaces of the finite dimensional vector space $\mathfrak{g}_0/\mathfrak{h}_0$.

Definition 2.2. The *order of degeneracy* of a contact triple $(g_0, b_0, \mathcal{L}_0)$ is the smallest positive integer *k* for which $\mathcal{G}_k \subseteq b_0$.

We observe that 0 is *strict* nondegeneracy, and *degenerate* corresponds to ∞ -degenerate.

Example 2.4. To a contact pair (g_0, \mathcal{L}_0) we can always associate the triples $(g_0, \{0\}, \mathcal{L}_0)$ and $(g_0/\mathfrak{a}_0, \mathcal{G}_0/\mathfrak{a}_0, \mathcal{L}_0/\mathfrak{a}_0)$, where \mathfrak{a}_0 is the largest ideal of g_0 which is contained in \mathcal{G}_0 . The first one is a contact triple iff g_0 is finite dimensional. The second one is a contact triple provided g_0/\mathcal{G}_0 is finite dimensional.

Remark 2.5. In §4 we will explain how a contact triple is canonically associated to a homogeneous contact manifold, providing in this way a geometrical motivation for Definition 1.1.

It is useful to reformulate the nondegeneracy conditions of §1 in terms of iterated Lie brackets. We define by recurrence

 $\begin{cases} [X_1] = X_1, \\ [X_1, X_2] = \text{Lie product of } X_1 \text{ and } X_2 \text{ in } g, \\ [X_1, \dots, X_k, X_{k+1}] = [[X_1, \dots, X_k], X_{k+1}], \text{ for } k \ge 2, \\ \text{if } X_1, \dots, X_{k+1} \in g, \end{cases}$

Proposition 2.6. A necessary and sufficient condition in order that a contact triple (g_0, b_0, L_0) be k-nondegenerate is that

(2.5)
$$\forall X \in \mathcal{L}_0 \setminus \mathfrak{h}_0, \exists X_0, \dots, X_k \in \mathcal{L}_0 \text{ such that } [X, X_0, \dots, X_k] \notin \mathcal{L}_0.$$

A necessary and sufficient condition in order that a CR algebra (g_0, q) be weakly nondegenerate is that

(2.6) $\forall Z \in \mathfrak{q}, \exists Z_0, \dots, Z_k \in \mathfrak{q} \text{ such that } [\overline{Z}, Z_0, \dots, Z_k] \notin (\mathfrak{q} + \overline{\mathfrak{q}}). \square$

3. Descending chain of a CR algebra

Let $(\mathfrak{g}_0, \mathfrak{q})$ be a *CR* algebra. We already noted that strict nondegeneracy implies weak nondegeneracy. It is well known that the two conditions are not equivalent. Set $\tilde{\mathfrak{q}} = (\mathfrak{q} + \bar{\mathfrak{q}})$ and $\tilde{\mathfrak{q}}_0 = \tilde{\mathfrak{q}} \cap \mathfrak{g}_0$. These are not, in general, subalgebras, but only linear subspaces of \mathfrak{g} and \mathfrak{g}_0 , respectively. To better understand weak *CR* nondegeneracy, we construct recursively descending chains of complex Lie subalgebras and of complex vector subspaces of \mathfrak{g} :

(3.1)
$$\bar{\mathfrak{q}}^{(0)} \supseteq \bar{\mathfrak{q}}^{(1)} \supseteq \cdots \supseteq \bar{\mathfrak{q}}^{(h)} \supseteq \bar{\mathfrak{q}}^{h+1} \supseteq \cdots$$

(3.2)
$$\tilde{\mathfrak{q}}^{(0)} \supseteq \tilde{\mathfrak{q}}^{(1)} \supseteq \cdots \supseteq \tilde{\mathfrak{q}}^{(h)} \supseteq \tilde{\mathfrak{q}}^{h+1} \supseteq \cdots$$

by setting

(3.3)
$$\begin{cases} \bar{\mathfrak{q}}^{(0)} = \bar{\mathfrak{q}}, & \text{for } h = 0, \\ \tilde{\mathfrak{q}}^{(0)} = \mathfrak{q} + \bar{\mathfrak{q}}, & \\ \bar{\mathfrak{q}}^{(h)} = \{ Z \in \bar{\mathfrak{q}}^{(h-1)} \mid [Z, \mathfrak{q}] \subseteq \tilde{\mathfrak{q}}^{(h-1)} \}, \\ \tilde{\mathfrak{q}}^{(h)} = \mathfrak{q} + \bar{\mathfrak{q}}^{(h)}, & \text{for } h \ge 1. \end{cases}$$

Lemma 3.1. We have

 $(3.4) \qquad \bigcap_{h\geq 0}\bar{\mathfrak{q}}^{(h)} = \bigcap_{h\geq 1} \{Z\in\bar{\mathfrak{q}} \mid [Z,W_1,\ldots,W_h]\in\tilde{\mathfrak{q}}, \ \forall W_1,\ldots,W_h\in\mathfrak{q}\}.$

Proof. Let us denote by \mathcal{A} , \mathcal{B} the left and right hand side of (3.4), respectively. Since $[\bar{\mathfrak{q}}^{(h)},\mathfrak{q}] \subset \mathfrak{q} + \bar{\mathfrak{q}}^{(h-1)}$ for all h > 0, we obtain that $\mathcal{A} \subseteq \mathcal{B}$.

Vice versa, if $Z \in \bar{q}$ does not belong to \mathcal{A} , then there is a positive integer h with $Z \notin \bar{q}^{(h)}$. This means that there is $W_1 \in q$ such that $[Z, W_1] \notin q + \bar{q}^{(h-1)}$. If h = 1, this suffices to show that $Z \notin \mathcal{B}$. If h > 1, write $[Z, W_1] = Z_1 + W'_1$ with $Z_1 \in \bar{q}$ and $W_1 \in q$. Since, by assumption, $Z_1 \notin \bar{q}^{(h-1)}$ we can find $W_2 \in q$ such that $[Z_1, W_2]$, and hence also $[Z, W_1, W_2]$, does not belong to $q + \bar{q}^{(h-2)}$. Iterating this argument, we show that there are $W_1, \ldots, W_h \in q$ such that $[Z, W_1, \ldots, W_h] \notin q + \bar{q}$ and therefore Z does not belong to \mathcal{B} . This completes the proof.

Lemma 3.2. All $\bar{\mathfrak{q}}^{(h)}$ are complex Lie algebras and

(3.5)
$$q' = q + \bigcap_{h \ge 0} \overline{g}^{(h)} = \bigcap_{h \ge 0} \widetilde{q}^{(h)}$$

is the largest complex Lie subalgebra satisfying

$$(3.6) \qquad \qquad q \subseteq q' \subseteq q + \bar{q}.$$

Proof. Let us show first that the $\bar{q}^{(h)}$'s are Lie subalgebras. This holds true for h = 0, because the conjugate \bar{q} of q with respect to the real form g_0 is a complex Lie subalgebra of g. Assume that we already know that $\bar{q}^{(h)}$ is a Lie algebra for some $h \ge 0$. If $Z_1, Z_2 \in \bar{q}^{(h+1)}$, we have $[Z_1, Z_2] \in \bar{q}^{(h)}$ because $\bar{q}^{(h+1)} \subseteq \bar{q}^{(h)}$ and, by our inductive assumption, $\bar{q}^{(h)}$ is a Lie algebra. Moreover,

$$[[Z_1, Z_2], \mathfrak{q}] \subseteq [Z_1, [Z_2, \mathfrak{q}]] + [Z_2, [Z_1, \mathfrak{q}]] \subseteq [Z_1, \bar{\mathfrak{q}}^{(h)} + \mathfrak{q}] + [Z_2, \bar{\mathfrak{q}}^{(h)} + \mathfrak{q}]$$
$$\subseteq [Z_1, \bar{\mathfrak{q}}^{(h)}] + [Z_1, \mathfrak{q}] + [Z_2, \bar{\mathfrak{q}}^{(h)}] + [Z_2, \mathfrak{q}] \subseteq \bar{\mathfrak{q}}^{(h)} + \mathfrak{q},$$

shows that $[Z_1, Z_2] \in \bar{\mathfrak{q}}^{(h+1)}$. Clearly the right hand side of (3.5) is a Lie subalgebra of \mathfrak{g} with $\mathfrak{q} \subseteq \mathfrak{q}' \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$. By Lemma 3.1 it is the maximal complex Lie subalgebra of \mathfrak{g} containing \mathfrak{q} and contained in $\mathfrak{q} + \bar{\mathfrak{q}}$. In fact it contains $\mathfrak{l} \cap \bar{\mathfrak{q}}$ for all complex Lie subalgebras with $\mathfrak{q} \subseteq \mathfrak{l} \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$. \Box

Proposition 3.3. The following are equivalent

- (1) (g_0, q) is weakly CR nondegenerate;
- (2) $\bigcap_{h \ge 0} \overline{\mathfrak{q}}^{(h)} = \mathfrak{q} \cap \overline{\mathfrak{q}};$ (3) $\bigcap_{h \ge 0} \overline{\mathfrak{q}}^{(h)} = \mathfrak{q}.$

Sequences (3.1) and (3.2) can be used to measure weak nondegeneracy. Let the *lenght* of a descending chain of vector spaces

$$V_0 \supseteq V_1 \supseteq \cdots \supseteq V_h \supseteq V_{h+1} \supseteq \cdots$$

be the smallest integer v such that $V_h = V_v$ for all $h \ge v$. By Lemma 3.2 we have the statement:

Proposition 3.4. *The sequences* (3.1) *and* (3.2) *have the same lenght* v *and all their terms with indices smaller than* v *are different.* \Box

Definition 3.1. We say that (g_0, q) is (v-1)-nondegenerate if the descending chains (3.1) and (3.2) have finite lenght v.

Remark 3.5. Strict nondegeneracy is 0-nondegeneracy, while weak nondegeneracy is v-nondegeneracy for some $v < +\infty$.

Example 3.6. For *CR* algebras, contact is a weaker notion than weak nondegeneracy. A score of examples can be obtained by considering real orbits in complex flag manifolds (see [1, 3]) whose *CR* algebras are fundamental, but not weakly nondegenerate. We give here a simple example, consisting of the minimal orbit M_0 of **SU**(1, 5) in the complex flag manifold *M* consisting of triples (ℓ_2 , ℓ_3 , ℓ_4) of complex 2, 3, 4-planes in \mathbb{C}^6 with $\ell_2 \subset \ell_3 \subset \ell_4$. A point of M_0 is a flag in the **SU**(1, 5)-orthogonal of an isotropic line. Let us give the explicit matrix representation. We define $g_0 \simeq \mathfrak{su}(1, 5)$ and $q, q' \subset \mathfrak{sl}_6(\mathbb{C})$ by

Then (g_0, q) is not weakly nondegenerate, because $q \subsetneq q' \subset q + \overline{q}$, while $(g_0, \check{q}_0, \tilde{q}_0)$ is a contact triple which is nondegenerate because $\mathfrak{su}(1, 5)$ is simple and therefore does not contain nontrivial ideals.

4. Homogeneous contact structures

Let G_0 be a Lie group, acting transitively on a smooth manifold M_0 . Fix a point p_0 of M_0 . The injective quotient of

$$\pi: \mathbf{G}_0 \ni x \to x \cdot \mathbf{p}_0 \in M_0$$

yields the idenfication $M_0 \approx \mathbf{G}_0 / \mathbf{H}_0$ of M_0 with the quotient of \mathbf{G}_0 by the stabilizer \mathbf{H}_0 of \mathbf{p}_0 . A \mathbf{G}_0 -equivariant contact structure on M_0 is the datum

of a constant rank distribution \mathscr{L}^* on M_0 , which is invariant for the left translations by elements of \mathbf{G}_0 :

$$x \cdot \mathscr{L}^* = \mathscr{L}^*, \quad \forall x \in \mathbf{G}_0.$$

The pullback $\mathscr{L} = \pi^*(\mathscr{L}^*)$ is a left-invariant distribution on \mathbf{G}_0 generated by a subspace \mathcal{L}_0 of left-invariant vector fields on \mathbf{G}_0 , containing the Lie algebra \mathfrak{h}_0 of \mathbf{H}_0 . Moreover, the vector subspace

$$L_{p_0} = \{ \Theta_{p_0} \mid \Theta \in \mathscr{L}^* \} \subseteq T_{p_0} M_0$$

must be invariant for the differential at p_0 of the translations by elements of \mathbf{H}_0 . This yields

$$\operatorname{Ad}(x)(\mathcal{L}_0) = \mathcal{L}_0, \quad \forall x \in \mathbf{H}_0,$$

which also implies that $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$ for the Lie algebra \mathfrak{h}_0 of \mathbf{H}_0 .

Vice versa, il \mathcal{L}_0 is an Ad(\mathbf{H}_0)-invariant linear subspace of the Lie algebra \mathfrak{g}_0 of \mathbf{G}_0 , then the push-forward on M_0 of the distribution on \mathbf{G}_0 generated by the left-invariant vector fields corresponding to the elements of \mathcal{L}_0 is a smooth distribution $\mathscr{L}^* = \mathcal{L}_0^*$ on M_0 , which is invariant by the \mathbf{G}_0 -translations on M_0 .

Assume now that we do not know a priori that M_0 is a homogeneous space, but we are given a constant rank distribution \mathcal{L}^* on M_0 and a Lie algebra g_0^* of smooth vector fields on M_0 which leave \mathcal{L}^* invariant: this means that $[g_0^*, \mathcal{L}^*] \subseteq \mathcal{L}^*$.

We say that g_0^* is *transitive* at p_0 if

$$\{X_{p_0}^* \in T_{p_0}M_0 \mid X^* \in \mathfrak{g}_0^*\} = T_{p_0}M_0.$$

Let $g_0 = g_0^{* \text{opp}}$, where the superscript "*opp*" means that, if we denote by *X* the element of g_0 corresponding to the vector field *X*^{*} of g_0^* , then

$$[X, Y]^* = -[X^*, Y^*], \quad \forall X, Y \in \mathfrak{g}_0.$$

With $L_{p_0} = \{ \Theta_{p_0} \mid \Theta \in \mathscr{L}^* \}$, let us set

$$\mathfrak{h}_0 = \{ X \in \mathfrak{g}_0 \mid X_{p_0}^* = 0 \}, \quad \mathcal{L}_0 = \{ X \in \mathfrak{g}_0 \mid X_{p_0}^* \in L_{p_0} \}.$$

Proposition 4.1. If \mathfrak{g}_0^* is transitive, then $(\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{L}_0)$ is a transitive contact triple.

Proof. The quotient $\mathfrak{g}_0/\mathfrak{h}_0$ maps injectively into $T_{\mathfrak{p}_0}M_0$ and therefore is finite dimensional. Let $X \in \mathcal{L}_0$ and $Y \in \mathfrak{h}_0$. Then we can find a vector field Θ , vanishing at \mathfrak{p}_0 , such that $X^* + \Theta \in \mathcal{L}_0^*$. Then

$$[Y^*, X^* + \Theta] = [X, Y]^* + [Y^*, \Theta] \in \mathcal{L}_0^*.$$

Since $[Y^*, \Theta]$ vanishes at p_0 , this means that $[X, Y]_{p_0}^* \in L_{p_0}$, showing that $[Y, X] \in \mathcal{L}_0$.

5. \mathbb{Z} -graded Lie Algebras and a Tanaka's theorem

We will use the following criterion ([24, §11]):

Proposition 5.1 (N.Tanaka). Let

(5.1)
$$\mathfrak{G} = \bigoplus_{h \ge -\mu} \mathfrak{G}_h$$

be a \mathbb{Z} -graded real Lie algebra, with $\dim_{\mathbb{R}}(\mathfrak{G}_h) < +\infty$ for all $h \in \mathbb{Z}$, having finitely many summands with negative index. Assume that \mathfrak{G} is transitive: this means that

$$\{\eta \in \mathfrak{G}_h \mid [\eta, \mathfrak{G}_{-1}] = \{0\}\} = \{0\}, \text{ if } h \ge 0.$$

Then a necessary and sufficient condition for \mathfrak{G} to be finite dimensional is that

(5.2)
$$\mathfrak{G}' = \{\eta \in \mathfrak{G} \mid [\eta, \mathfrak{G}_h] = \{0\} \text{ for } h \le -2\}$$

be finite dimensional.

Let us comment on this criterion. In the following, we assume that \mathfrak{G} is transitive.

Clearly, \mathfrak{G}' is a \mathbb{Z} -graded Lie subalgebra of \mathfrak{G} and

(5.3)
$$[\mathfrak{G}'_h,\mathfrak{G}_{-1}]\subseteq\mathfrak{G}'_{h-1}, \ \forall h\in\mathbb{Z}.$$

Given real vector spaces V, W, let $\mathcal{M}^h(V, W)$ denote the space of W-valued *h*-multilinear forms on V and $Symm^h(V, W)$ the subspace consisting of those which are symmetric.

We define a map $\eta \to \eta_h$ of \mathfrak{G} into $\mathcal{M}^h(\mathfrak{G}_{-1},\mathfrak{G})$ by associating to each $\eta \in \mathfrak{G}$ the multilinear form

$$\eta_h(\xi_1, ..., \xi_h) = [\eta, \xi_1, ..., \xi_h], \text{ for } \xi_1, ..., \xi_h \in \mathfrak{G}_{-1}.$$

We also consider the alternate \mathfrak{G}_{-2} -valued bilinear form on \mathfrak{G}_{-1} :

(5.4)
$$\omega(\xi_1, \xi_2) = [\xi_1, \xi_2], \text{ for } \xi_1, \xi_2 \in \mathfrak{G}_{-1}$$

and set

(5.5)
$$\mathfrak{sp}(\omega) = \{T \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{G}_{-1}) \mid \omega(T(\xi_1), \xi_2) + \omega(\xi_1, T(\xi_2)) = 0\}.$$

Lemma 5.2. For each $h \ge 0$, the maps

1

(5.6)
$$\begin{cases} \mathfrak{G}_0 \ni \eta \longrightarrow \eta_1 \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{G}_{-1}), & for \ h = 0, \\ \mathfrak{G}_h \ni \eta \longrightarrow \eta_h \in \mathcal{M}^h(\mathfrak{G}_{-1}, \mathfrak{G}_0), & for \ h > 0, \end{cases}$$

are injective.

(5.7)
$$\begin{cases} \mathfrak{G}'_0 \subseteq \{\eta \in \mathfrak{G}_0 \mid \eta_1 \in \mathfrak{sp}(\omega)\}, & for \ h = 0, \\ \mathfrak{G}'_h \subseteq \{\eta \in \mathfrak{G}_h \mid \eta_k \in Symm^k(\mathfrak{G}_{-1}, \mathfrak{G}_{h-k})\}, & for \ h, k > 0. \end{cases}$$

Proof. The fact that the maps in (5.6) are injective is a straightforward consequence of transitivity.

If $\eta \in \mathfrak{G}_0$ and $[\eta, \mathfrak{G}_{-2}] = 0$, then

 $0 = [\eta, [\xi_1, \xi_2]] = [[\eta, \xi_1], \xi_2] + [\xi_1, [\eta, \xi_2]] = \omega(\eta_1(\xi_1), \eta_2) + \omega(\xi_1, \eta_1(\xi_2))$ shows that $\eta_1 \in \mathfrak{sp}(\omega)$. In the same way, if $\eta \in \mathfrak{G}$ and $[\eta, \mathfrak{G}_{-2}] = 0$, we obtain that

 $[\eta, \xi_1, \xi_2] = [[\eta, \xi_1], \xi_2] = [[\eta, \xi_2], \xi_1] = [\eta, \xi_2, \xi_1], \ \forall \eta \in \mathfrak{G}', \ \forall \xi_1, \xi_2 \in \mathfrak{G}_{-1}.$ This gives at once that $[\eta, \xi_1, \xi_2, \xi_3] = [\eta, \xi_2, \xi_1, \xi_3]$ for $\eta \in \mathfrak{G}'$ and $\xi_1, \xi_2, \xi_3 \in \mathfrak{G}_{-1}$, while

$$0 = [\eta, [\xi_1, \xi_2, \xi_3]] = [\eta, [[\xi_1, \xi_2], \xi_3]] = [[\eta, [\xi_1, \xi_2]], \xi_3] + [[\xi_1, \xi_2], [\eta, \xi_3]]$$
$$= [[\xi_1, [\eta, \xi_3]], \xi_2] + [\xi_1, [\xi_2, [\eta, \xi_3]]] = -[\eta, \xi_3, \xi_1, \xi_2] + [\eta, \xi_3, \xi_2, \xi_1]$$

shows that also $[\eta, \xi_1, \xi_2, \xi_3] = [\eta, \xi_1, \xi_3, \xi_2]$ for $\eta \in \mathfrak{G}'$ and $\xi_1, \xi_2, \xi_3 \in \mathfrak{G}_{-1}$. This yields symmetry on the triples. Arguing recursively on *k*, we obtain, for all k > 3,

$$0 = [[\eta, \xi_1, \dots, \xi_{k-2}], [\xi_{k-1}, \xi_k]]$$

= $[\eta, \xi_1, \dots, \xi_{k-1}, \xi_k] - [\eta, \xi_1, \dots, \xi_k, \xi_{k-1}].$

This shows that $[\eta, \xi_1, \ldots, \xi_{k-1}, \xi_k] = [\eta, \xi_1, \ldots, \xi_k, \xi_{k-1}]$. Thus η_k stays invariant under the transposition (k - 1, k). By the recursive assumption, it is also invariant under the transpositions (j - 1, j) for $2 \le j < k$ and thus is invariant under the full permutation group of $\{1, \ldots, k\}$.

Example 5.3. Let V be a real vector space of finite dimension n, viewed as a degree (-1)-homogeneous Abelian real Lie algebra. Then its maximal Levi-Tanaka extension is isomorphic to the \mathbb{Z} -graded Lie algebra \mathfrak{P} of vector fields with polynomial coefficients in \mathbb{R}^n , the grading being defined by

(5.8)
$$\mathfrak{P}_{h} = \left\{ \sum_{i=1}^{n} p_{i}(x) \frac{\partial}{\partial x_{i}} \middle| p_{i} \in \mathbb{R}_{h+1}[x_{1}, \dots, x_{n}] \right\}, \quad h \ge -1.$$

Here $\mathbb{R}_{h+1}[x_1, \ldots, x_n]$ denotes the vector space of homogeneous polynomials of degree (h + 1) in the x_1, \ldots, x_n variables.

If n = 2m and V has a complex structure, then the Levi-Tanaka extension of $V \oplus gl_{\mathbb{C}}(V)$ is isomorphic to the \mathbb{Z} -graded complex Lie algebra $\mathfrak{P}^{\mathbb{C}}$, of homolorphic complex vector fields with holomorphic polynomial coefficients, with the gradation defined by

(5.9)
$$\mathfrak{P}_{h}^{\mathbb{C}} = \left\{ \sum_{i=1}^{m} p_{i}(z) \frac{\partial}{\partial z_{i}} \middle| p_{i} \in \mathbb{C}_{h+1}[z_{1}, \dots, z_{n}] \right\}, \quad h \ge -1.$$

Here $\mathbb{C}_{h+1}[z_1, \ldots, z_n]$ denotes the vector space of homogeneous holomorphic polynomials of degree (h + 1) in the z_1, \ldots, z_n variables.

Proof. The fact that \mathfrak{P} is a maximal transitive extension of $\mathfrak{P}_{-1} \simeq V$ is a consequence of the fact that $[\mathfrak{P}_h, \mathfrak{P}_{-1}] = \mathfrak{P}_{h-1}$ for $h \ge 0$.

Analogously, when V has a complex structure, $\mathfrak{P}^{\mathbb{C}}$ is a maximal transitive extensions of $V \oplus \mathfrak{gl}_{\mathbb{C}}(V) \simeq \mathfrak{P}_{-1}^{\mathbb{C}} \oplus \mathfrak{P}_{0}^{\mathbb{C}}$ because $[\mathfrak{P}_{h}^{\mathbb{C}}, \mathfrak{P}_{-1}^{\mathbb{C}}] = \mathfrak{P}_{h-1}^{\mathbb{C}}$ for all $h \ge 0$.

6. A FINITNESS CRITERION FOR CR ALGEBRAS

We recall that the contact triple associated to a fundamental *CR* algebra (g_0, q) is

$$(\mathfrak{g}_0, \check{\mathfrak{q}}_0, \tilde{\mathfrak{q}}_0) = (\mathfrak{g}_0, \mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{g}_0, (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}_0).$$

We consider the canonical filtration (2.1) of the contact pair (g_0, \tilde{q}_0) and the corresponding \mathbb{Z} -graded Lie algebra

(6.1)
$$\mathfrak{G} = \bigoplus_{h \in \mathbb{Z}} \mathfrak{G}_h, \text{ with } \mathfrak{G}_h = \mathcal{G}_h / \mathcal{G}_{h+1}.$$

Denote by $\pi_h : \mathcal{G}_h \to \mathfrak{G}_h = \mathcal{G}_h / \mathcal{G}_{h+1}$ the projections onto the quotients.

Lemma 6.1 (Partial complex structure). *There is a unique complex structure J on* \mathfrak{G}_{-1} *, defined by*

(6.2)
$$J(\pi_{-1}(X)) = \pi_{-1}(Y) \text{ iff } X + iY \in q.$$

The operator $J \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{G}_{-1})$ satisfies

$$(6.3) [J(\xi_1), J(\xi_2)] = [\xi_1, \xi_2], [J(\xi_1), \xi_2] + [\xi_1, J(\xi_2)] = 0, \quad \forall \xi_1, \xi_2 \in \mathfrak{G}_{-1}.$$

Proof. To show that J is well defined, we need to verify that, if $X, Y \in \mathfrak{g}_0$, $X \in \mathcal{G}_0$ and $X + iY \in \mathfrak{q}$, then $Y \in \mathcal{G}_0$. If $X' \in \mathcal{G}_{-1}$, then we can find $Y' \in \mathcal{G}_{-1}$ such that $X' + iY' \in \mathfrak{q}$. Then

(6.4)
$$[X + iY, X' + iY'] = [X, X'] - [Y, Y'] + i([X, Y'] + [Y, X']) \in \mathfrak{q}.$$

Since by assumption both [X, X'] and [X, Y'] belong to \mathcal{G}_{-1} , then also [Y, X']and [Y, Y'] belong to \mathcal{G}_{-1} . This shows that $[Y, \mathcal{G}_{-1}] \subset \mathcal{G}_{-1}$, and then $Y \in \mathcal{G}_0$. Formula (6.4) holds in general for $X + iY, X' + iY' \in \mathfrak{q}$, yielding (6.3).

Lemma 6.2. The \mathfrak{G}_{-2} -valued form (5.4) is nondegenerate, i.e.

$$\xi \in \mathfrak{G}_{-1}, \quad \omega(\xi, \xi') = 0, \ \forall \xi' \in \mathfrak{G}_{-1} \Longleftrightarrow \xi = 0.$$

Lemma 6.3. Let $(\mathfrak{g}_0, \mathfrak{q})$ be a CR algebra and (2.1) the associated \mathbb{Z} -filtration. assume that there is a nonnegative integer k such that $\mathcal{G}_k \subset \mathfrak{h}_0 = \mathfrak{q} \cap \overline{\mathfrak{q}} \cap \mathfrak{g}_0$. Then $\mathfrak{G}'_{2k+1} = \{0\}$.

Proof. By the assumption, any Y of G_k belongs to q and therefore

 $[Y, Z_1, \ldots, Z_{k+2}] \in \mathfrak{q}, \quad \forall Z_1, \ldots, Z_{k+2} \in \mathfrak{q}.$

In the complexification of \mathfrak{G} , this yelds the equation

 $[\theta, \xi_1 + iJ(\xi_1), \dots, \xi_{k+2} + iJ(\xi_{k+2})] = 0, \quad \forall \theta \in \mathfrak{G}_k, \quad \forall \xi_1, \dots, \xi_{k+2} \in \mathfrak{G}_{-1},$

as $\pi_{-2}(q) = \{0\}$, because q is contained in the complexification of \mathcal{G}_{-1} .

Let now $\eta \in \mathfrak{G}_{2k+1}$. Fix $\xi \in \mathfrak{G}_{-1}$ and consider, for $0 \le h \le 2k+3$, the (2k+4) vectors of \mathfrak{G}_{-2} :

$$v_h = [\eta, \xi_1, \dots, \xi_{2k+3}]$$
, with $\xi_i = J(\xi)$ for $i \le h$ and $\xi_i = \xi$ for $h < i \le 2k+3$.

For each choice of ξ_1, \ldots, ξ_{k+1} in \mathfrak{G}_{-1} , the multi-commutator $[\eta, \xi_1, \ldots, \xi_{k+1}]$ belongs to \mathfrak{G}_k . Thus, setting, for each integer *h* with $0 \le h \le k + 1$, $\xi_i^h = J(\xi)$ if $1 \le i \le h$ and $\xi_i^h = \xi$ if $h < i \le k + 1$, the real and imaginary parts of

 $[\eta, \xi_0^h, \dots, \xi_k^h, Z_1, \dots, Z_{k+2}] = 0$, with $Z_1 = \dots = Z_{k+2} = \xi + iJ(\xi)$,

yield (2k + 4) linear combinations of v_0, \ldots, v_{2k+3} which sum to zero. These can be written in the form

(6.5)
$$(v_0, v_1, \dots, v_{2k+3}) \cdot M_k = 0,$$

where M_k is a real $(2k + 4) \times (2k + 4)$ matrix M_k whose columns are the coefficients of the real and imaginary parts of the polynomials $t^h(t - i)^{k+2}$, for $0 \le t \le (k + 1)$.

It is easy to check that these polynomials form a basis of the (2k + 4)dimensional \mathbb{C} -vector space $\mathbb{C}_{2k+3}[t]$ of polynomials of degree less or equal to (2k + 3).

In fact their \mathbb{C} -linear span contains the (2k + 4) polynomials $(t - i)^{k+h+2}$ and $(t + i)^{k+h+2}$, for $0 \le h \le k+1$. These are linearly independent and hence form a basis of $\mathbb{C}_{2k+3}[t]$. Indeed, let a_h, b_h be complex coefficients for which

$$f(t) = \sum_{h=0}^{k+1} (a_h(t-i)^{k+h+2} + b_h(t+i)^{k+h+2}) = 0.$$

Let $r \le k + 1$ and assume that we already know that $a_h = 0$ and $b_h = 0$ if h > r, this being obviously the case when r = (k + 1). Then

$$0 = \frac{i}{2 \cdot (2k+r)!} \frac{d^{2k+r-1}f(-i)}{dt^{2k+r-1}} = a_r, \quad 0 = \frac{-i}{2 \cdot (2k+r)!} \frac{d^{2k+r-1}f(i)}{dt^{2k+r-1}} = b_r$$

shows that also $a_r = 0$ and $b_r = 0$. By recurrence, this proves that all coefficients a_h, b_h must be zero and thus the claimed linear independence of the polynomials $(t-i)^{k+h+2}, (t+i)^{k+h+2}$.

Hence M_k is nondegenerate and (6.5) tells us that all vectors v_0, \ldots, v_{2k+3} are zero. In particular $v_0 = 0$ and therefore we proved that

$$[\eta, \underbrace{\xi, \dots, \xi}_{(2k+3) \text{ times}}] = 0, \quad \forall \xi \in \mathfrak{G}_{-1}$$

Since, by Lemma 5.2, the multilinear \mathfrak{G}_{-2} -valued form η_{2k+3} is symmetric in its arguments, it follows by polarization (see e.g. [25, p.5]) that

$$[\eta, \xi_1, \dots, \xi_{2k+3}] = 0 \text{ for all } \xi_1, \dots, \xi_{2k+3} \in \mathfrak{G}_{-1}.$$

Since ω is nondegenerate, this yields

 $[\eta, \xi_1, \dots, \xi_{2k+2}] = 0 \text{ for all } \xi_1, \dots, \xi_{2k+2} \in \mathfrak{G}_{-1}$ and hence, by transitivity, $\eta = 0$.

Thus we obtain, using also [14, Theorem 10.2],

Theorem 6.4. A CR algebra (g_0, q) for which the associated contact triple $(g_0, \tilde{q}_0, \tilde{q}_0, \tilde{q}_0) = (g_0, q \cap \bar{q} \cap g_0, (q + \bar{q}) \cap g_0)$ is nondegenerate is finite dimensional.

We will prove in §8 that, under the assumptions of Theorem 6.4, (g_0, q) has a *maximal extension* and that this is finite dimensional. To this aim we will generalise, in §7, the construction of §4, by a procedure similar to that of [7, 11].

7. TRANSITIVE PAIRS AND GENERALISED CONTACT DISTRIBUTIONS

7.1. Vector fields with formal power series coefficients. Let *V* be a finite dimensional vector space. The space \mathcal{F} of formal power series associated to *V* is the infinite direct sum

(7.1)
$$\mathcal{F} = \sum_{h\geq 0} Symm^h(V) = \left\{ \sum_{h=0}^{\infty} \alpha_h \middle| \alpha_h \in Symm^h(V) \; \forall h \right\},$$

where $Symm^h(V) = Symm^h(V, \mathbb{R})$ are the real-valued homogeneous multilinear symmetric forms of degree *h* (cf. § 5). The coefficient α_0 is the *value at* 0 of $\sum_{h=0}^{\infty} \alpha_h$. With the standard operations, \mathcal{F} is a local ring, whose maximal ideal \mathcal{F}_0 consists of formal power series vanishing at 0.

Each vector v of V defines a derivation D_v on \mathcal{F} , whose action on each summand $Symm^h(V)$ is described by

(7.2)
$$\begin{cases} D_{v}\alpha = 0 \text{ if } \alpha \in Symm^{0}(V) \simeq \mathbb{R}, \\ D_{v}\alpha = h \cdot v \rfloor \alpha \in Symm^{h-1}(V), \text{ if } \alpha \in Symm^{h}(V), \text{ for } h > 0, \\ \text{i.e. } (D_{v}\alpha)(v_{1}, \dots, v_{h-1}) = h \cdot \alpha(v_{1}, \dots, v_{h-1}, v), \forall v_{1}, \dots, v_{h-1} \in V. \end{cases}$$

The set $\mathcal{D}er(\mathcal{F})$ of derivations of \mathcal{F} is the left \mathcal{F} -module $\mathcal{F} \otimes_{\mathbb{R}} V$ generated by *V*. Thus any derivation X^* is a formal series

(7.3)
$$X^* = \sum_{h=0}^{\infty} X_h^*, \text{ with } X_h^* \in Symm^h(V, V) = Symm^h(V) \otimes_{\mathbb{R}} V.$$

Denote by $\mathcal{D}er_0(\mathcal{F})$ the Lie subalgebra $\mathcal{F}_0 \otimes V$ of derivations vanishing at 0.

7.2. The case of Lie groups. Let G_0 be a real Lie group. We recall that the left and right invariant vector fields on G_0 coincide with the infinitesimal generators L_X and R_X of the one-parameter groups

 $\mathbb{R} \times \mathbf{G}_0 \ni (t, x) \to x \cdot \exp(tX) \in \mathbf{G}_0, \quad \mathbb{R} \times \mathbf{G}_0 \ni (t, x) \to \exp(tX) \cdot x \in \mathbf{G}_0,$

of diffeomorphisms of G_0 , respectively. We have the commutation rules

(7.4)
$$\begin{cases} [L_X(v), L_Y(v)] = L_{[X,Y]}(v), \\ [R_X(v), R_Y(v)] = -R_{[X,Y]}(v), \\ [L_X(v), R_Y(v)] = 0, \end{cases} \quad \forall X, Y \in \mathfrak{g}_0.$$

The exponential map is a diffeomorphism of an open neighbourhood of 0 in its Lie algebra g_0 onto an open neighborhood of the identity, defining a local chart. In order to determine the Taylor series expansions of L_X , R_X in in these coordinates, it is convenient to consider the identities

$$\exp(tX) \cdot \exp(v) = \exp(v + t \cdot R_X(v) + 0(t^2)),$$

$$\exp(v) \cdot \exp(tX) = \exp(v + t \cdot L_X(v) + 0(t^2)),$$

for $v, X \in g_0$. Since the maps $X \to L_X$ and $X \to R_X$ are linear, we obtain

$$\exp(tX) \cdot \exp(v) = (\exp(-v) \cdot \exp(-tX))^{-1} = (\exp(-v - t \cdot L_X(-v) + 0(t^2))^{-1}$$
$$= \exp(v + t \cdot L_X(-v) + 0(t^2))$$

showing that

$$R_X(v) = L_X(-v).$$

We can obtain the Taylor series expansions of L_X and R_X in the g_0 -coordinates from the Baker-Campbell-Hausdorff formula (see e.g. [8]):

(7.5)
$$\begin{cases} L_X(v) = \sum_{h=0}^{\infty} b_h(\mathrm{ad}(v))^h(X), \\ R_X(v) = \sum_{h=0}^{\infty} (-1)^h b_h(\mathrm{ad}(v))^h(X) \end{cases}$$

where the coefficients b_h are defined by

$$\sum_{h=0}^{\infty} b_h t^h = \frac{t}{1 - \exp(-t)} = 1 + \frac{t}{2} + \frac{t^2}{12} + \frac{t^3}{48} + \cdots$$

7.3. Homogeneous spaces. Let $M = \mathbf{G}_0 / \mathbf{H}_0$ be a homogeneous space, with base point $p_0 = [\mathbf{H}_0]$. Fix a linear complement *V* of \mathfrak{h}_0 in \mathfrak{g}_0 . The map

(7.6)
$$V \times \mathbf{H}_0 \ni (v, x) \longrightarrow \exp(v) \cdot x \in \mathbf{G}_0$$

restricts to a diffeomorphism of the product of a neighborhood of $\{0\} \times \mathbf{H}_0$ onto an open neighbourhood of the identity in \mathbf{G}_0 . We use the projection on *M* of $\exp(v)$ to define coordinates near p_0 . In fact, if $\varpi : \mathbf{G}_0 \to M$ is the natural projection, with $\varpi(x) = x \cdot p_0$, then $v \to \varpi(\exp(v))$ is a diffeomorphism of an open neighbourhood of 0 in *V* onto an open neighbourhood of p_0 in *M*. Let \mathcal{F} be as in §7.1.

In analogy with the definitions of L_X and R_X in §7.2, we may introduce a couple of vector fields L_X^* and R_X^* on M, as infinitesimal generators of the one-parameter groups of diffeomorphisms of M, locally defined by

(7.7)
$$\begin{cases} \exp(v) \cdot \exp(tX) = \exp(v + t \cdot L_X^*(v) + 0(t^2)) \cdot \exp(t \cdot H(v) + 0(t^2)), \\ \exp(tX) \cdot \exp(v) = \exp(v + t \cdot R_X^*(v) + 0(t^2)) \cdot \exp(t \cdot H'(v) + 0(t^2)), \\ \text{with } L_X^*, R_X^* \in \mathcal{F} \otimes V, \quad H, H' \in \mathcal{F} \otimes \mathfrak{h}_0. \end{cases}$$

Let us find their formal power series expansions. Set

(7.8)
$$\begin{cases} R_X^*(v) = \sum_{h=0}^{\infty} \mathfrak{x}_h(v), & H'(v) = \sum_{h=0}^{\infty} h'_h(v), \\ \text{with } \mathfrak{x}_h \in Symm^h(V, V), & h'_h \in Symm^h(V, \mathfrak{h}_0). \end{cases}$$

Using (7.5), we obtain

$$\sum_{h=0}^{\infty} (-1)^h b_h(\operatorname{ad}(v))^h(X) = \sum_{h=0}^{\infty} \mathfrak{X}_h(v) + \sum_{h=0}^{\infty} \sum_{r+s=h} b_r(\operatorname{ad}(v))^r(h'_s(v)),$$

yielding by recurrence

(7.9)
$$\begin{cases} X = \mathfrak{x}_0 + h'_0, \\ \mathfrak{x}_{h+1}(v) + h'_{h+1}(v) = (-1)^{h+1} b_{h+1}(\mathrm{ad}(v))^{h+1}(X) \\ -\sum_{r=0}^h b_{r+1}(\mathrm{ad}(v))^{r+1}(h'_{h-r}(v)), \end{cases} \quad (h \ge 0).$$

Let $\pi : \mathfrak{g}_0 \to V$ be the projection along \mathfrak{h}_0 . Equations (7.9) can be used to obtain explicit formulae for \mathfrak{x}_h and h'_h :

(7.10)
$$\begin{cases} \mathfrak{x}_{0} = \pi(X), \\ \mathfrak{h}'_{0} = X - \pi(X), \\ \mathfrak{x}_{h+1}(v) = (-1)^{h+1} b_{h+1} \pi((\mathrm{ad}(v))^{h+1}(X)) \\ -\sum_{r=0}^{h} b_{r+1} \pi((\mathrm{ad}(v))^{r+1}(\mathfrak{h}'_{h-r}(v))), \\ \mathfrak{h}'_{h+1}(v) = (-1)^{h+1} b_{h+1} (\mathrm{ad}(v))^{h+1}(X) \\ -\sum_{r=0}^{h} b_{r+1} (\mathrm{ad}(v))^{r+1}(\mathfrak{h}'_{h-r}(v)) - \mathfrak{x}_{h+1}(v). \end{cases}$$

Since the projection $\mathbf{G}_0 \ni x \to x \cdot \mathbf{p}_0 \in M$ relates R_X^* with the right invariant vector field on \mathbf{G}_0 corresponding to *X*, we obtain:

Lemma 7.1. We have

(7.11)
$$[X^*(v), Y^*(v)] = -[X, Y]^*(v), \quad \forall X, Y \in \mathfrak{g}_0.$$

Analogously, let us set

(7.12)
$$\begin{cases} L_X^*(v) = \sum_{h=0}^{\infty} \mathfrak{y}_h(v), & H(v) = \sum_{h=0}^{\infty} h_h(v), \\ \text{with } \mathfrak{y}_h \in Symm^h(V, V), & h_h \in Symm^h(V, \mathfrak{h}_0). \end{cases}$$

From the equation

$$\sum_{h=0}^{\infty} b_h(\mathrm{ad}(v))^h(X) = \sum_{h=0}^{\infty} \mathfrak{y}_h(v) + \sum_{h=0}^{\infty} \sum_{r+s=h} b_r(\mathrm{ad}(v))^r(h_s(v)),$$

we obtain

(7.13)
$$\begin{cases} X = \mathfrak{y}_0 + h_0, \\ \mathfrak{y}_{h+1}(v) + h_{h+1}(v) = b_{h+1}(\mathrm{ad}(v))^{h+1}(X) \\ -\sum_{r=0}^h b_{r+1}(\mathrm{ad}(v))^{r+1}(h_{h-r}(v)), \end{cases} (h \ge 0).$$

Equations (7.13) can be used to obtain recursive formulae for \mathfrak{y}_h and h_h :

(7.14)
$$\begin{cases} \mathfrak{y}_{0} = \pi(X), \\ h_{0} = X - \pi(X), \\ \mathfrak{y}_{h+1}(v) = b_{h+1}\pi((\mathrm{ad}(v))^{h+1}(X)) \\ -\sum_{r=0}^{h} b_{r+1}\pi((\mathrm{ad}(v))^{r+1}(h_{h-r}(v))), \\ h_{h+1}(v) = b_{h+1}(\mathrm{ad}(v))^{h+1}(X) \\ -\sum_{r=0}^{h} b_{r+1}(\mathrm{ad}(v))^{r+1}(h_{h-r}(v)) - \mathfrak{y}_{h+1}(v), \end{cases}$$

Note that $L_Y^* = 0$ when $Y \in \mathfrak{h}_0$.

To compute the commutator of R_X^* and L_Y^* for a pair $X, Y \in g_0$, we use the infinitesimal description of their flows, which can be obtained from (7.7).

Set Φ_X for the flow of R_X^* and Ψ_Y for the flow of L_Y^* . Then

 $\exp(v + t^2[R_X^*, L_Y^*](v) + 0(t^3)) = \exp(\Phi_X(-t) \circ \Psi_Y(-t) \circ \Phi_X(t) \circ \Psi_Y(t))(v)).$ By lifting the action to **G**₀, we have

$$\exp(\Phi_X(t)(v)) = \exp(tX)\exp(v)\exp(-tH'(v) + 0(t^2)),$$

$$\exp(\Psi_Y(t)(v)) = \exp(v)\exp(tY)\exp(-tH(v) + 0(t^2)).$$

Then we obtain the composition

$$\exp(-tX)\exp(tX)\exp(v)\exp(tY)\exp(-tH(v)+O(t^2))\exp(-tH'(v)+O(t^2))$$
$$\cdot\exp(-tY)\exp(tH(v))\exp(tH'(v))$$

 $= \exp(v) \exp(tY) \exp(-tH(v)) \exp(-tH'(v)) \exp(-tY) \exp(tH(v)) \exp(tH'(v)) + 0(t^{3})$ = $\exp(v) \exp(-t^{2}[Y, (H(v) + H'(v))]) \exp(t^{2}[H(v), H'(v)]) + 0(t^{3}).$

This yields

Lemma 7.2. For $X, Y \in g_0$, we have

(7.15)
$$[R_X^*, L_Y^*](v) = L_{[H+H',Y]}^*(v),$$

where H(v) and H'(v) are described by (7.14) and (7.10), and the right hand side is a composition of formal power series.

This lemma tells us that the *infinitesimal translations* of L_w^* , for a $w \in V$, can be expressed as formal power series whose coefficients are L_Y^* 's for Y in the \mathfrak{h}_0 -module generated by w. In a similar way we also obtain

Lemma 7.3. For $Y_1, Y_2 \in \mathfrak{g}_0$, we have

(7.16)
$$[L_{Y_1}^*, L_{Y_2}^*] = L_{[Y_1, Y_2] + [H_1(v) - H_2(v), Y_1 - Y_2]}^*,$$

where $H_i \in \mathcal{F} \otimes \mathfrak{h}_0$ is defined by the first line of (7.7), with $X = Y_i$, for i = 1, 2.

7.4. **General transitive pairs.** Let us fix a transitive pair. We recall from Definition 1.1 that it is a couple (g_0, b_0) consisting of a linearly compact topological Lie algebra g_0 and of a finite codimensional closed subalgebra b_0 that does not contain any nontrivial ideal of g_0 .

Let us fix a finite dimensional complement V of \mathfrak{h}_0 in \mathfrak{g}_0 . We define $R_X^*(v)$, $L_X^*(v)$, H(v), H'(v) by (7.8), (7.9), (7.12), (7.13), after noticing that to write these formulae it is not needed that \mathfrak{g}_0 be finite dimensional, because the homogeneous summands in the V-coordinates of their Taylor series only involve finite powers of $\mathfrak{ad}(v)$, finite linear combinations, and the projection $\pi : \mathfrak{g}_0 \to V$ along \mathfrak{h}_0 . Set

(7.17)
$$g_0^* = \{R_X^* \mid X \in g_0\}.$$

Theorem 7.4. *If* (g_0, b_0) *is a transitive pair and* V *a linear complement of* b_0 *in* g_0 *, then the map*

(7.18)
$$\mathfrak{g}_0 \ni X \longrightarrow R_X^* \in \mathcal{D}er(\mathcal{F})$$

is injective and defines an anti-isomorphism of Lie algebras between g_0 and g_0^* , with

(7.19)
$$\mathfrak{h}_0 = \{ X \in \mathfrak{g}_0 \mid R_X^* \in \mathcal{D}er_0(\mathcal{F}) \}.$$

The correspondence

(7.20)
$$\mathcal{L}_0 \longrightarrow \mathcal{L}_0^* = \mathcal{F} \otimes \langle L_w^* \mid w \in V \cap \mathcal{L}_0 \rangle$$

is a bijection between the set of \mathfrak{h}_0 -submodules of \mathfrak{g}_0 containing \mathfrak{h}_0 and \mathfrak{g}_0^* invariant left \mathcal{F} -modules of $\mathcal{D}er(\mathcal{F})$. Its inverse is given by

(7.21)
$$\mathcal{L}_0^* \longrightarrow \mathcal{L}_0 = \{ Y \in \mathfrak{g}_0 \mid \pi(Y) = \Theta(0) \text{ for some } \Theta \in \mathcal{L}_0^* \}.$$

Proof. Let $X \in g_0$ and assume that $R_X^* = 0$. We use (7.10). From $\mathfrak{x}_0 = 0$ se obtain that $X \in \mathfrak{h}_0$. Since $b_1 = \frac{1}{2} \neq 0$, the condition that $\mathfrak{x}_1 = 0$ means that $[v, X] \in \mathfrak{h}_0$ for all $v \in V$, and hence $[Y, X] \in \mathfrak{h}_0$ for all $Y \in \mathfrak{g}_0$. In general, we find that $\mathfrak{h}_h(v)$, for each $h \ge 0$, is a multiple of $(\operatorname{ad}(v)^h(X) \text{ and, arguing})$ by recurrence we obtain that $(\operatorname{ad}(Y)^h(X) \in \mathfrak{h}_0$ for all $h \ge 0$ and $Y \in \mathfrak{g}_0$. This yields that actually $\operatorname{ad}(Y_1) \circ \cdots \circ \operatorname{ad}(Y_h)(X) \in \mathfrak{h}_0$ for all h and $Y_1, \ldots, Y_h \in \mathfrak{g}_0$. To show this fact, we argue again by recurrence on h, as the cases of h = 0, 1 are already settled. For h > 1, we note that, for $t_1, \ldots, t_h \in \mathbb{R}$,

$$t_1 \cdots t_h \cdot \operatorname{ad}(Y_1) \circ \cdots \circ \operatorname{ad}(Y_h)(X) = \frac{1}{h!} [\operatorname{ad}(t_1 Y_1 + \cdots + t_h Y_h)]^h(X) + \sum_i 0(t_i^2) + \text{terms of the form } \operatorname{ad}(Y'_1) \circ \cdots \operatorname{ad}(Y'_r)(X) \text{ with } r < h.$$

By the recursive assumption, the coefficient of the monomial $t_1 \cdots t_h$ in the right hand side is an element of \mathfrak{h}_0 and hence $\operatorname{ad}(Y_1) \circ \cdots \circ \operatorname{ad}(Y_h)(X) \in \mathfrak{h}_0$. We proved that the kernel of the map $X \to R_X^*$ is an ideal of \mathfrak{g}_0 contained in \mathfrak{h}_0 . Therefore it is {0} if ($\mathfrak{g}_0, \mathfrak{h}_0$) is a transitive pair.

The conclusion of Lemma 7.1 only depends on the formal definition of R_X^* in (7.8) and (7.9) and therefore is still valid, yielding (7.18).

Also the validity of Lemma 7.2 depends only on the formal definitions of R_X^* and L_Y^* and therefore shows that, when \mathcal{L}_0 contains \mathfrak{h}_0 and $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$, then the left \mathcal{F} -module \mathcal{L}_0^* generated by $\{L_w^* \mid w \in V \cap \mathcal{L}_0\}$ satisfies $[\mathfrak{g}_0^*, \mathcal{L}_0^*] \subseteq \mathcal{L}_0^*$. Vice versa, if \mathcal{L}_0^* is a left \mathcal{F} submodule of $\mathcal{D}er(\mathcal{F})$ with $[\mathfrak{g}_0^*, \mathcal{L}_0^*] \subseteq \mathcal{L}_0^*$, then the set \mathcal{L}_0 of $Y \in \mathfrak{g}_0$ such that $\pi(Y)$ is the value in 0 of a vector field in \mathcal{L}_0^* is a subspace of \mathfrak{g}_0 containing \mathfrak{h}_0 and satisfying $[\mathfrak{h}_0, \mathcal{L}_0] \subseteq \mathfrak{h}_0$. Indeed [X, w] is the value at 0 of $[R_X^*, \mathcal{L}_w^*]$. This yields the correspondence (7.20), completing the proof of the theorem. \Box

We already noted that the map $Y \to L_Y^*$ is linear. In particular, it can be extended by \mathbb{C} -linearity to the case where *Y* belongs to the complexification g of g_0 . Then the second part of the statement of Theorem 7.4 extends to the case of *complex vector distributions*. We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the complexification of \mathfrak{h}_0 .

Theorem 7.5. The correspondence

(7.22)
$$\mathcal{L} \longrightarrow \mathcal{L}^* = \mathcal{F} \otimes \langle L_w^* \mid w \in V \cap \mathcal{L} \rangle$$

is a bijection between the set of \mathfrak{h} -submodules of \mathfrak{g} containing \mathfrak{h} and \mathfrak{g}_0^* invariant left $\mathbb{C} \otimes \mathcal{F}$ -modules of $\mathbb{C} \otimes Der(\mathcal{F})$. Its inverse is given by

(7.23)
$$\mathcal{L}^* \longrightarrow \mathcal{L} = \{Y \in \mathfrak{g} \mid \pi(Y) = \Theta(0) \text{ for some } \Theta \in \mathcal{L}^*\}.$$

8. Extensions

Definition 8.1. We say that a contact triple $(g'_0, \mathfrak{h}'_0, \mathfrak{L}'_0)$ *extends* the contact triple $(g_0, \mathfrak{h}_0, \mathfrak{L}_0)$ if there is an injective homomorphism of real Lie algebras $\phi : g_0 \to g'_0$ such that $\phi(\mathfrak{L}_0) \subseteq \mathfrak{L}'_0, \phi(\mathfrak{h}_0) \subseteq \mathfrak{h}'_0$ and the quotient maps $(g_0/\mathfrak{L}_0) \to (g'_0/\mathfrak{L}'_0)$ and $(g_0/\mathfrak{h}_0) \to (g'_0/\mathfrak{h}'_0)$ induced by ϕ are linear isomorphisms.

We say that a *CR* algebra (g'_0, q') extends the *CR* algebra (g_0, q) if there is an injective Lie algebras homomorphism $\phi : g_0 \to g'_0$, whose complexification we still denote by ϕ , such that $\phi(q) \subseteq q'$ and the induced map on the quotients $g_0/(q \cap \bar{q} \cap g_0) \to g'_0/(q' \cap \bar{q}' \cap g'_0)$ and $(g/q) \to (g'/q')$ are linear isomorphisms.

To deal with extensions, it is convenient to introduce a common Lie algebra in which we can embed both a given Lie algebra and its extension.

Proposition 8.1. Let $(g_0, \mathfrak{h}_0, \mathcal{L}_0)$ be a contact triple. Then there is a maximal contact triple $(g'_0, \mathfrak{h}'_0, \mathcal{L}'_0)$ extending $(g_0, \mathfrak{h}_0, \mathcal{L}_0)$, which is unique modulo isomorphisms.

Let (g_0, q) be a CR algebra. Then there is a maximal CR algebra (g'_0, q') extending (g_0, q) , which is unique modulo isomorphisms.

Proof. The statement follows from Theorems 7.4 and 7.5. If \mathcal{L}_0^* is the \mathcal{F} -module corresponding to \mathcal{L}_0 , we define g'_0^* as the Lie algebra of formal vector fields stabilising \mathcal{L}_0^* and g'_0 equals to its opposite Lie algebra.

Likewise, in the case of a *CR* algebra, we take $g_0^{\prime*}$ to be the stabiliser of $\tilde{\mathfrak{q}}^*$ in $\mathcal{D}er(\mathcal{F})$ and define g_0^{\prime} to be its opposite Lie algebra.

The finiteness result of 6 applies to give informations about the global and local *CR* automorphisms on homogeneous and locally homogeneous *CR* manifolds.

The analytic Lie subgroup of a Lie group G_0 generated by a Lie subalgebra \mathfrak{h}_0 of its Lie algebra \mathfrak{g}_0 may fail to be closed. In this case the pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ is associated to a *locally* G_0 -*homogeneous* manifold, i.e. a smooth open manifold M_0 having the property that the elements of a small open neighborhood of the origin of G_0 act as a transitive group of local diffeomorphisms on an open neighborhood of a base point \mathfrak{p}_0 of M_0 , and \mathfrak{h}_0 is the Lie algebra of the stabilizer of \mathfrak{p}_0 for this action (see e.g. [17, 22]). We can give in an obvious way a notion of *locally* G_0 -*homogeneous CR manifold*, that we employ in the formulation of the following result.

Theorem 8.2. Every contact nondegenerate CR algebra (g_0, q) admits an essentially unique maximal extension (g'_0, q') , which is finite dimensional and is therefore a CR algebra of a locally homogeneous CR manifold whose CR automorphisms form a Lie group of transformations.

Theorem 8.3. Let \mathbf{G}_0 be a Lie group and M_0 a locally \mathbf{G}_0 -homogeneous CR manifold, with associated CR algebra $(\mathfrak{g}_0, \mathfrak{q})$. If $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental and contact nondegenerate, then the local CR automorphisms of M_0 generate a finite dimensional Lie group.

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