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Reciprocal teaching and peer discussion in mathematics:
a proposal for lower secondary education

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Part I

Towards a new mathematics education in lower secondary schools

Introduction

Teaching mathematics to lower secondary school students is a complex issue. On the one hand, children are beginning their journey toward maturity; on the other, mathematics is starting to reveal its abstract nature. Despite this, it remains part of a single continuum with other sciences, which are often perceived as less “exact” and more closely connected to observation and experimentation. During this three-year period, mathematics, like other subjects, is taught to all students without distinction — unlike later stages of education where students can choose a more specialized path based on their interests and aptitudes.

In lesson planning, it is therefore essential to consider various approaches that can engage as many students as possible, ranging from those with a keen interest in mathematics to those who may avoid it due to dislike or prejudice. Special attention should be given to students with special educational needs, including those with specific learning difficulties (SpLD), such as dyscalculia, and those with challenges related to socio-economic, linguistic, and cultural factors. At the same time, the world is changing, and teaching and learning tools must evolve and adapt accordingly. Today, school is no longer the sole source of information for children and adolescents, who are exposed to countless cultural stimuli — often contradictory. The role of the school, therefore, is to help make sense of these stimuli and to provide the tools necessary for learning and discerning information. School must also foster the development of methods and strategies that enable students to navigate their educational journeys, not only during their school years but throughout their lives. Education, in fact, can no longer be considered a limited period but must be seen as a lifelong process.

The *Indicazioni nazionali per il curriculum della scuola dell’infanzia e del primo ciclo d’istruzione* [57] (National Guidelines for the Curriculum of Nursery and First Cycle Schools) of 2012 aims to address the need to adapt the teaching-learning process to a constantly evolving society. Nursery school and the first cycle serve as children’s introduction to society and have the dual role of ensuring learning and shaping future citizens. While nursery school focuses on fostering identity and autonomy, the first cycle (ages 6 to 14) emphasizes the acquisition of fundamental knowledge and skills, as well as the languages and codes of our culture, viewed in relation to other cultures and the media. Primary school, in particular, aims to equip students with basic cultural tools. Through the specific languages of various disciplines, it offers students the opportunity to acquire essential knowledge in terms of cognitive, emotional, and social development. In lower secondary school, students are introduced to disciplines as perspectives on reality, methods of understanding, interpreting, and representing the world. However, this approach is effective only when the disciplines are not presented as rigid, isolated entities but are integrated into a broader framework, recognizing that complex problems can be addressed only through the collaboration of different disciplines and cultures. This applies to mathematics as well.

The student profile at the end of the first cycle of education reads:

His mathematical and scientific-technological knowledge allows him to analyze data and facts of reality and to verify the reliability of the quantitative and statistical analyses proposed by others. The possession of rational thought allows him to face problems and situations on the basis of certain elements and to be aware of the limits of statements concerning complex issues that do not lend themselves to univocal explanations. [57]

In other words, mathematics must provide students with the tools to scientifically describe the world around them while fostering the ability to communicate and argue effectively, as well as to understand the viewpoints and reasoning of others. The development of mathematical thinking is a long process in which concepts, skills, and competencies are intertwined, revisited, and reinforced multiple times. Students must learn to recognize recurring patterns, establish analogies with familiar models, select appropriate actions, and link them together to solve problems. This gradual process of mathematization, generalization, and formalization begins to take shape in lower secondary school. While this is an ambitious and challenging goal, research in mathematics education offers numerous insights to help implement it in line with the aims and methods outlined in the National Guidelines.

Considerations of this kind constitute the motivation for this thesis, which contains a review of what makes teaching effective, a description of relevant teaching tools already implemented in Italian schools, and finally the design, implementation and initial experimentation of a specific tool that is new in the Italian educational landscape.

Structure of the thesis

This project began with an exploration of various proposals within the Italian educational context aimed at creating learning environments conducive to acquiring mathematical skills. Part I of this thesis describes several exemplary proposals. Additional proposals that were analyzed can be found in Appendix A.

This exploration raised the question of whether the activities described in Part I — centered on workshop teaching, problem-solving, and promoting argumentation, sometimes through play — are sufficient to ensure students educational success.

Consequently, we explored the meaning of effective teaching, addressing themes such as formative assessment, self-evaluation, metacognition, and reciprocal and dialogic teaching. These reflections form the basis of Part II of the thesis.

Given these premises, we felt it would be useful to design a tool that could help students and teachers understand their position in the teaching-learning process. This led to the development of a research proposal inspired by the work on science teaching by the University of York Science Education Group (UYSEG). This proposal is described in Part III.

Part IV details the phases of the classroom experimentation of our research and presents the results, followed by the conclusions.

Chapter 1

The learning environments and the centrality of the student

The educational landscape in which today's school is immersed is very complex. School learning is only one of the many experiences that young people live, and the school must be able to promote in students the ability to make sense of their experiences, so that each person can develop a conscious and open identity. The National Guidelines refer to the centrality of the individual.

The student is placed at the center of the educational process in all its aspects: cognitive, emotional, relational, physical, aesthetic, ethical, spiritual, and religious. In this perspective, teachers must design and implement their educational projects not for abstract individuals, but for people who live in the here and now, who raise specific existential questions, and who seek horizons of meaning.

According to this perspective, the teacher cannot ignore the emotions that students experience during the learning process and must strive to create conditions that foster well-being in school. According to Daniela Lucangeli, teacher of Developmental Psychology at the University of Padua, when emotions are associated with a cognitive process, they become ingrained in the brain along with the concepts. Positive emotions enhance the lasting memory of what has been learned, whereas a learning system based on the fear of making mistakes or the fear of the teacher or tests creates a cycle of negative emotions that hinder learning [52].

In this context, as Dedò and Sferch state in [35], errors should be considered as natural occurrences that serve as a starting point for understanding, from a metacognitive perspective, the elements that hinder learning. What can a teacher do to promote this view? First of all, teacher can start by setting an example: when a teacher makes a mistake, there should be an effort to engage with students about the error, attempting to analyze the mental process that led to it. This approach encourages students to reflect on their own mistakes to comprehend their origins and causes. It constitutes a crucial metacognitive exercise for students and proves beneficial for teachers as well. Through this analysis, teachers can reassess the severity of errors; what may have seemed like a conceptual mistake might turn out to be a procedural error stemming from incorrect automatisms or a distorted understanding of mathematics. For instance, one might believe that a geometric solution is not valid for an arithmetic problem or vice versa.

In second place, a shift in perspective is necessary regarding evaluation, which should not revolve solely around errors. In a classroom setting, the primary focus should be on evaluating the development of critical thinking and ensuring that the majority of students acquire accurate concepts and attitudes. Positive evaluation should not be contingent solely upon the absence of errors. The correctness of the final result does not always signify an understanding of the problem or the given exercise; it can result from the mechanical application of a procedure. Conversely, correct reasoning and reasonable attempts may have been made that, despite not yielding the desired result, demonstrate a thoughtful engagement with the material [35].

The teacher should have the ability to transform the fear of making mistakes into the right to make mistakes and convey the message that assessment is a valuable tool for personal growth, providing insights into one's individual journey. Teacher feedback, when used consciously, can foster this perspective and enhance the sense of self-efficacy. The emotional aspect is crucial in learning, especially in mathematics, where a negative attitude often underlies poor results. Among the competency goals to be achieved at the end of lower secondary school, the most challenging, in my opinion, is as follows:

the pupil has strengthened a positive attitude towards mathematics through significant experiences and has understood how the mathematical tools learned are useful in many situations to operate in reality [57].

How, then, can the development of a positive attitude be promoted? First, we need to define its characteristics.

According to Rosetta Zan [78], an author of numerous texts on mathematics education with particular focus on difficulties, three factors can be considered to define students' attitudes towards mathematics: emotional disposition (liking/disliking), the vision of mathematics (conceptual vision/procedural vision), and the sense of self-efficacy (success/failure). When even one of these factors is not positive, the disposition towards this discipline cannot be considered favorable.

Working on the conceptual vision of mathematics and cultivating a sufficient sense of self-efficacy, in other words, the feeling of having control over the process and the outcome, positively influences the attitude.

To achieve this goal, the National Guidelines provide valuable support to teachers. They serve as a framework for curriculum planning within individual schools and outline the competency goals to be attained. Simultaneously, they provide guidance on the path to achieving these goals, emphasizing the creation of suitable learning environments.

A learning environment, in this context, is not merely a physical space but rather a setting where students can engage in meaningful learning. The overarching objective is the educational success of all students. Thus, it is essential to start from students' background and to implement activities by considering the differences and features of each student. The school is envisioned as a space for knowledge construction, wherein learning extends beyond an individual process and emerges from comparison and collaboration among peers. Through dialogue, the strengths and limitations of each student can surface, allowing them to identify effective strategies and study methods that align with their characteristics. The student is seen as an active participant, cognizant of their own learning, capable of understanding assigned tasks and goals, and equipped to reflect on the outcomes of their actions, whether successes or failures.

Within the Italian landscape of research in mathematics education, numerous projects align with the National Guidelines, drawing explicit reference to approaches such as the workshop method, games, problem-solving, and discussion.

1.1 Mathematics workshop

A fundamental element in establishing an effective learning environment is the concept of a workshop, primarily seen as

the moment in which the student is active, formulates hypotheses and checks their consequences, plans and experiments, discusses and argues his own choices, learns to collect data, negotiates and constructs meanings, leads to temporary conclusions and new openings the construction of personal and collective knowledge [57].

When referring to a laboratory in the context of mathematics, it is more about a method of working. Teaching through workshops involves starting with a problem or question that piques curiosity, presenting a challenge for both the student and the teacher. For students, it is a challenge that makes them active participants in the learning process. They formulate hypotheses, engage in discussions with classmates, and reflect on results and their significance. For teachers, conducting workshops demands excellent planning and observation skills, necessitating a shift from a procedural approach to a more creative one. This creative approach encourages student questions, comparisons, and values not only the final product but also partial contributions.

Content selection is also critical. Adapting school curricula across disciplines to meet societal expectations is a challenge. Rather than identifying specific skills needed for various professions, which may be unknown, the approach should expose students to a broad range of mathematical concepts. Activities should focus on developing critical thinking, reasoning, creativity, autonomy, and responsibility — essential skills for accessing knowledge beyond mathematics. Teachers must choose meaningful content while fostering and activating these skills.

In the context of mathematical workshops, it is not the presence of equipment or the use of specific materials that define them, but rather the approach taken. In small groups, students analyze questions, invent paths to follow, make trial-and-error attempts, and may encounter inconclusive or incorrect reasoning before finding the right path. Once a result is achieved, it is reviewed and negotiated within the group to generalize and communicate to others. These steps mirror the scientific method, forming the basis for the work of every proficient researcher [36]. Making mistakes is an integral part of the learning process, providing students with opportunities to reflect on obstacles encountered. Simultaneously, teachers can leverage these mistakes to plan subsequent activities.

The active engagement of students is a hallmark of group work, with the teacher playing a direct role. Initially, the teacher must contribute to fostering a positive atmosphere. Throughout the activity, the teacher may assume various roles based on the situation. S/he can function as an external observer, collecting information without interference, or acts as a tutor, posing questions to stimulate reflection or alter perspectives. When necessary, the teacher can model effective problem-solving by explaining thoughts and reasoning on the board, guiding students along the right path or backtracking when needed [64].

Hands engagement is another common aspect of the math workshop. Manipulating real objects helps clarify abstract concepts. Our minds often grasp and appreciate concepts better when, in addition to hearing about or seeing representations, we construct and touch models that illustrate them. Emma Castelnuovo's mathematics teaching emphasizes this idea, stating that "geometry is seeing with the mind's eye". Recognizing that the mind is less democratic than the hands, she developed numerous ways to teach mathematics using simple materials like string, rubber bands, and iron bars. This approach provides everyone, including those with difficulty handling abstraction, the opportunity to understand even the most challenging concepts [50].

This mathematics approach begins with the tangible but delves into deeper concepts. By manipulating a movable material, students transition from qualitative to quantitative observations, explore correspondences, analyze borderline cases, and even engage in reasoning by the absurd. This process allows a shift from the concrete to abstraction, although it is not always straightforward. The artifact, being inherently different from the abstract concept it aims to represent, can pose a challenge. The clarity of a concept in our minds facilitates its projection onto a real object. However, this transition may not be automatic for those unfamiliar with the abstract concept, risking the attribution of characteristics to the abstract that belong solely to the concrete object. When planning a workshop activity, teachers need to consider this possibility and emphasize to students that the object is merely a tool for grasping the idea [36].

The use of materials does not overshadow language and reasoning. Concepts emerge and solidify through the interaction between hands and the brain, between reasoning and construction. In a society where youth often believe creation is limited to the virtual world, the engagement of hands and the body fosters a form of intelligence that remains essential [51].

In workshop activities, students showcase their abilities and psychic/intellectual faculties. The teacher observes, reflects, experiments, and gathers data before advancing in teaching. As stated by Emma Castelnuovo in [26], the classroom transforms into a mathematics teaching laboratory. Research, statistics, and experiments with students guide teachers in deciding what to propose, but the path and evolution of the activity closely depend on the class's response. The laboratory is the school, and not everything that occurs during a lesson is easily translatable to paper.

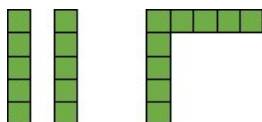
Adopting the teaching methods introduced by Emma Castelnuovo is the conviction underlying the decisions made by the *Officina matematica* of *Cenci's casa-laboratorio*. Since 2002, the workshop has been an annual event lasting three days, where teachers actively participate, taking on the role of students. The learning process is derived from the interactions among the participants rather than the specific activities chosen. This philosophy leads to the choice of not disseminating the training materials externally, as simply reading the “instructions for use” would not be enough to ensure their effectiveness.

1.1.1 Drawing numbers

The heart of the mathematical workshop is the student, and every student in the class must be an active participant. When planning a workshop, it is important to start with a simple question so that all students, including the weaker ones, can participate and feel involved. However, at the same time, the activity must be mathematically meaningful and, as much as possible, open up new perspectives. It should be like a room with “low floor, high ceiling, and wide walls” [65].

As an example we consider an activity proposed by Antonella Castellini, a lower secondary school teacher and trainer in mathematics education [24].

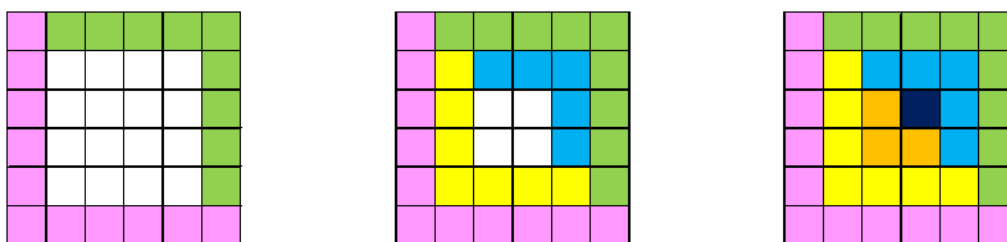
The activity begins by providing the students with two strips of paper, each consisting of 5 squares. Students are asked to arrange the strips at a right angle, superimposing one square: in this way, they construct a gnomon. The question is: how many squares are there in the composite figure?



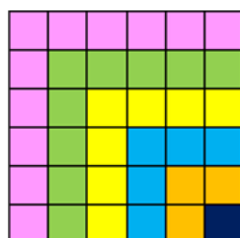
The same activity can then be repeated with strips of 6 squares. What number is found by forming the gnomon?

The question is very simple and accessible to all, it is accessed through a “low floor,” but one can reach a “high ceiling.” Continuing with the examples, at some point, the students will realize that the numbers are all odd because they are formed by subtracting 1 from an even number. Indeed, there are two strips, and regardless of their length (let’s assume it is n), together they form an even number ($2n$). In the overlap, one square is covered, so it must be subtracted from the total ($2n-1$); thus, we obtain the predecessor of an even number, which is certainly odd.

At this point, we can propose arranging two gnomons to form a square, as shown in the first figure below. The inner space will always be a square, and one can imagine enclosing it with two smaller gnomons (as depicted in the second figure). The process can continue until a degenerate gnomon, co-consisting of a single square, is reached (third figure).



The construction shows us that a square with side length n is formed by the sum of odd numbers and, by changing the position of the gnomons, it is easier to realize that they are the first n odd numbers. That is, for any square: $n^2 = 1 + 3 + 5 + \dots + (2n - 1)$.



For example:

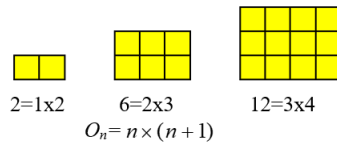
$$\begin{aligned} 16 &= 1 + 3 + 5 + 7, \\ 49 &= 1 + 3 + 5 + 7 + 9 + 11 + 13, \\ 100 &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19. \end{aligned}$$

This result is not at all trivial and helps us understand how the workshop walls are “wide walls”. It opens a window, in fact, onto number theory, which is the branch of mathematics that studies the integers.

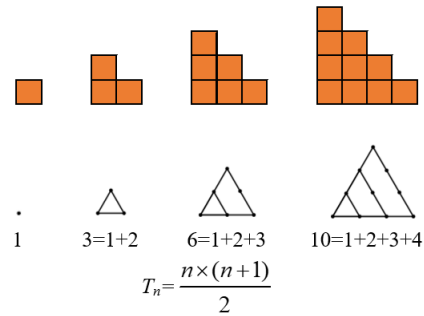
The study of numbers as such, independently of being associated with a measurement or a quantity, began over 2500 years ago with the Pythagoreans, and even today there are many open problems.

Similarly to the gnomon representing an odd number, the Pythagoreans had identified families of numbers and figurate numbers: for example, in addition to square numbers, seen in the previous activity, there are also oblong numbers, triangular numbers, polygonal numbers (pentagonal, hexagonal, etc.).

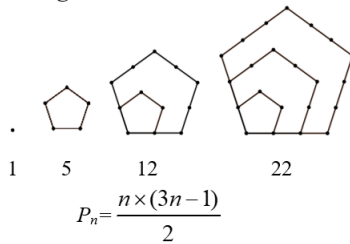
Oblong numbers



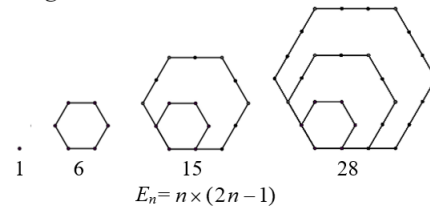
Triangular numbers



Pentagonal numbers

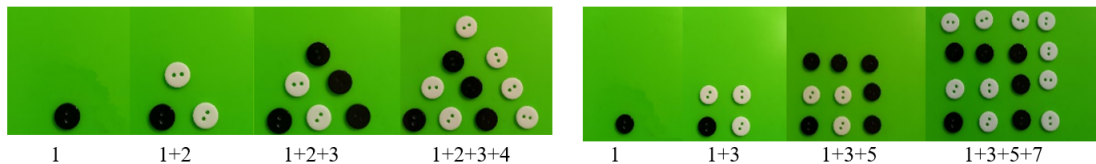


Hexagonal numbers

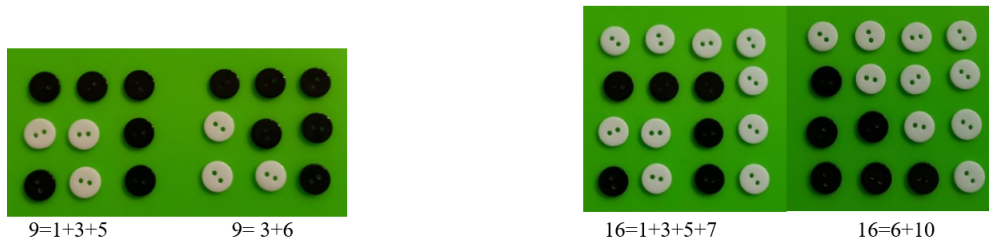


In an accessible, creative, and stimulating learning environment, students can use visual demonstrations to highlight the relationships between different types of numbers.

Once students have identified the “shape” of numbers, they can construct them using simple materials such as colored buttons, highlighting the connection between a figurate number and its successor.

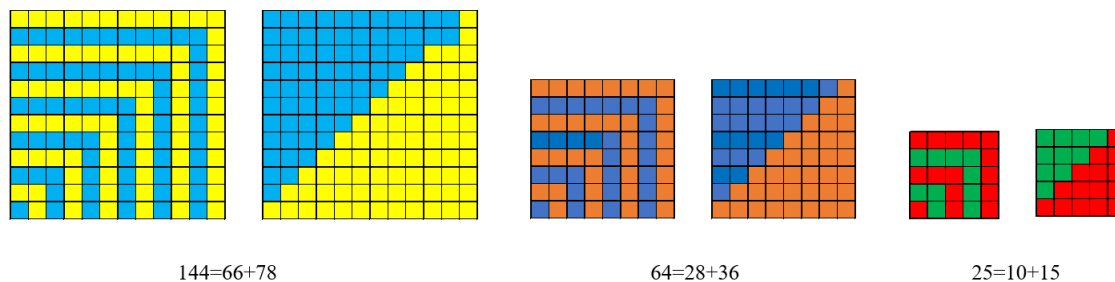


The use of easily movable objects allows students to conduct various experiments without taking too much time. For example, at a certain point, students can be asked to move the buttons in order to visually demonstrate that a square number is always formed by the sum of two triangular numbers.



In the first case, simply move the black button at the bottom left to the center. In the second case, the three white buttons at the bottom left need to be swapped with the three black buttons in the center; to obtain triangular numbers, simply count the number of buttons of each color.

Other examples of square number formed by two triangular numbers:



Through visual demonstrations, students can convincingly justify complex mathematical properties of numbers, even if the arguments are not perfectly rigorous; this can stimulate their curiosity and willingness to continue researching when they have the appropriate cognitive tools.

Visual demonstrations lead students to reflect on arithmetic properties through geometric visualization and can be a valuable aid in memorizing fundamental and meaningful concepts that are often forgotten while applying procedures. An emblematic example is the search for the greatest common divisor (GCD) and the least common multiple (lcm) of a group of numbers: when students begin using procedures that involve decompose numbers into prime factors to find the GCD and lcm, the concept of multiple and divisor is often lost, making it difficult to make sense of what they are doing.

Appendix A.1 presents a proposal [70] regarding the search for the greatest common divisor (GCD) and the least common multiple (lcm) of a group of numbers, using the representation of the product of two numbers as a rectangle.

In Appendix A.1 another activity proposed by Castellini is presented. This is an activity in which students start from a real situation to arrive at a formal algebraic representation.

1.2 Teaching via games

Starting again from the National Guidelines [57]:

In primary school you can use the game, which has a crucial role in communication, in education to respect shared rules, in the development of strategies suitable for different contexts.

More:

The construction of mathematical thought is a long and progressive process in which concepts, skills, competences and attitudes are rediscovered, intertwined, consolidated and developed several times...

Gradually, stimulated by the teacher's guidance and by discussion with peers, the pupil will learn to face problematic situations with confidence and determination...

If the learning environment is intended to be stimulating, highly inclusive, and accessible to all, games are an ideal means of breaking the ice and capturing attention. Pedagogical studies, such as those by Piaget, Vygotsky, and Bruner, indicate that play is an essential element in child development. From early childhood, games facilitate the development of social, relational, and emotional skills, as well as cognitive, expressive, and linguistic abilities. Symbolic and role-playing games in very young children foster the development of representational capacity through imitation and identification with a model. For older children, around 6-7 years old, games can promote the identification of causal, spatial, and temporal relationships. By the age of 12, children can operate on an abstract level, relating mental images, formulating hypotheses, and solving problems through puzzles, strategic games, and logic games.

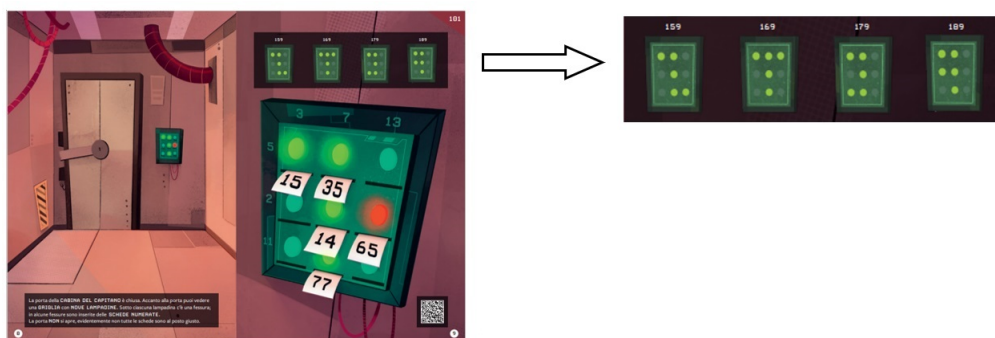
In both games and everyday life, rules must be observed. However, in games, rules are generally more engaging and easily accepted. The act of playing is often considered more important than the outcome, and obstacles or failures are experienced with less anxiety, as they are not perceived as real failures. While the playful dimension is predominant in kindergarten, in subsequent school levels, there is a gradual progression toward the acquisition of learning in various disciplines, which are deepened and organized over time. It is essential to note that learning is not solely an individual process, as collaborative learning forms (such as peer learning, mutual aid, and cooperative learning) play a crucial role.

These considerations suggest that play can be a valuable tool for promoting learning, even for older students. Mathematics is one of the disciplines well-suited for conveyance through play. Primary school textbooks often include puzzles and games, and in recent years, numerous mathematical board games have been introduced to the market, especially targeting primary school students.

The game can serve various purposes. It proves beneficial for introducing a new topic, breaking the routine, and lightening a lesson that some students might find excessively dull. At the conclusion of a learning unit, students can be tasked, perhaps in small groups, with creating a game to present to their classmates, thereby testing their grasp of the material. Alternatively, during an occasional lesson, students can simply enjoy the recreational aspect of mathematics. In many comprehensive schools, there is a practice of encouraging children, on a voluntary basis, to participate in competitions organized by external entities, such as the mathematical games organized by *Pristem (Bocconi University)* [8], or the *Kangourou of Mathematics* [4], or the *Transalpine Mathematical Rally Association* [13]. In these competitions, students individually tackle new and quite complex problems. While stimulating, these competitions may not be considered inclusive, as they tend to attract only the most gifted students. There exists a variety of math games, both competitive and non-competitive. Alongside individual competitions, there are team competitions where, in addition to addressing unusual problems, students must collaborate, each contributing their unique skills. This establishes peer learning, involving discussion, as well as respect and recognition of each other's abilities, forming the foundation for the development of social skills. While competition participation is undoubtedly motivating and often reveals hidden talents not evident in regular class lessons, the emphasis remains on teamwork. Non-competitive games also exist, where there are no winners, and the objective is not to finish before others. Students can be divided into small groups, or the entire class can participate, with the enjoyment derived from playing, attempting, and trying different approaches. In these instances, the reward of the game is not victory but rather the satisfaction of finding a solution.

1.2.1 Non-competitive games

A very interesting proposal for lower secondary school comes from Andrea Maffia and his *Escape books: Destinazione Bellatrix* [53], for grade 6, *La tomba di Akhenaten* [54], for grade 7, and *Il laboratorio del professore* [55], for grade 8. These are books with a non-linear structure in which there is a fantasy setting and an adventure to complete. Each page is marked with a three-digit number; to reach the goal, students must solve puzzles whose solution is indeed a three-digit number indicating the next page to read.



Each puzzle is associated with mathematical content appropriate for the corresponding school age, but puzzles do not follow a predictable order. In other words, students do not know in advance which mathematical concept will be necessary to pass the test. Like the contents, the narrative structure becomes more complex from one book to another. In the second book, for instance, it is possible to follow different paths to reach the final page.

Faced with the problem, students will have to develop the right strategy by recalling concepts already learned or by identifying alternative paths that lead to new knowledge. This activates an informal and free learning process where the role of the teacher is crucial. The teacher must act as a mediator, providing suggestions and/or guidance when students cannot reach a solution on their own, and highlighting the connections between what emerged in the game and the contents of the curricular lesson.

In the puzzle in the following figure, designed for sixth-grade students, for example, the students must calculate the total duration of the itinerary in minutes, convert it into hours to determine how many stops are necessary, and then convert the result back into minutes to obtain the three-digit number that gives access to the next puzzle. If the game follows the explanation in class, it is a simple exercise of conversion between units of measurement, but one that has been given meaning. However, if the topic has not yet been covered, students are given the opportunity to reason based on their daily experience; in case of difficulty, the teacher could suggest thinking about the total duration of a soccer match, taking into account the halftime interval.

ROTTA IMPOSTATA CORRETTAMENTE					
Itinerario determinato: Dubhe ► Phecda					
Phecda ► Alnitak					
Alnitak ► Betelgeuse					
Betelgeuse ► Bellatrix					
Inserire durata complessiva dell'itinerario stabilito					
Si ricorda che per ogni ora di viaggio è necessaria una sosta di 10 minuti					
	ALNITAK	BELLATRIX	BETELGEUSE	DUBHE	PHECDA
ALNITAK	0 min	88 min	73 min	165 min	98 min
BELLATRIX	88 min	0 min	73 min	199 min	173 min
BETELGEUSE	73 min	73 min	0 min	186 min	101 min
DUBHE	165 min	199 min	186 min	0 min	45 min
PHECDA	98 min	173 min	101 min	45 min	0 min

The question in the figure below, addressed to seventh-grade students, focuses on the use of the connectives *or* and *and*, anticipating some simple concepts of elementary logic. It will then be the teacher's task to evaluate, based on the context of the class, how to use the puzzle, perhaps by recalling concepts of union and intersection of sets, for example.



The absence of competition in this proposal is, in my opinion, the most interesting aspect. Mathematical competitions, although very motivating, are often perceived by students as something for the few. Typically, only the most skilled students enjoy putting themselves to the test. A game like this, on the other hand, is certainly more inclusive and confirms the idea that the development of mathematical thinking requires time and reflection. Looking ahead, there is also the hope that getting students accustomed to collaborative play can contribute to fostering a more modern view of scientific knowledge, conceived as a less competitive activity, in which scientists have an interest in sharing their knowledge to achieve a better result.

The lightheartedness and enjoyment of the game, combined with the seriousness and commitment it involves, contribute to the effectiveness and durability of the educational process because it is enjoyable. Neuroscience studies support the idea that positive emotions trigger intense neuroelectric activity in the brain, encouraging the body to seek and reproduce those experiences. Conversely, negative emotions activate low-intensity reactions that linger in the neural circuitry, serving as reminders of potential harm.

Therefore, it is crucial to link learning with positive emotions to instill a desire for knowledge in students. Mathematical games, as well as recreational and informal activities involving beauty, art, and movement, often evoke these pleasant emotions, leaving a lasting imprint in the memory.

In Appendix A.2, additional examples of teaching-via-game are presented.

1.3 Problem solving

A characteristic of mathematical practice is the resolution of problems, which must be understood as authentic and significant questions, linked to daily life, and not just repetitive exercises or questions which can be answered simply by remembering a definition or a rule. [57]

The rationales for incorporating problems into the classroom are diverse. Numerous mathematicians contend that the core of mathematics lies in problem-solving, asserting that the most effective way to comprehend and learn the subject is through engaging with problems [37]. Historically, mathematics originated as a tool to solve problems, defined as challenges that required resolution. Thus, engaging in mathematics implies the act of “creating” it. When students encounter a problem where they don’t know beforehand which knowledge to apply or when there are multiple viable strategies, they are in the process of “creating” their mathematics. Such problems offer opportunities to revisit the same topic repeatedly, delving deeper and examining it from various perspectives. The same problem may serve to introduce a new concept or to revisit one covered in previous years. Problems can function as entry tests, assessing students’ retention of prior knowledge, or they can address gaps identified during other activities.

Problems need not be directly tied to specific topics; instead, they can be presented to encourage students to “reinvent” mathematics by practicing fundamental processes of mathematical thinking, such as reading, analyzing, observing, classifying, conjecturing, arguing, verifying, and describing, among others. By revisiting concepts in an unsystematic manner, students gradually develop an understanding of the genuine place of these concepts within the discipline, moving beyond associating them solely with the curriculum of a particular academic year.

This approach aligns with Jerome Bruner’s spiral teaching, as outlined in [20]. It commences with straightforward concepts that progressively become more intricate over time. Students repeatedly revisit overarching topics introduced initially but with an increasingly comprehensive perspective. Reexamining an initial concept, after an in-depth study, allows for fresh insights and different analyses and representations than those undertaken previously. Consequently, the ability to critically reason and revise initial conclusions is cultivated. The spiral curriculum acknowledges and even welcomes mistakes and dead ends as integral components of the problem-solving process.

Crucial to this problem-solving journey is the acceptance of mistakes as a means for students to reevaluate initial hypotheses and persist in the pursuit of a solution, even if it requires an extended period.

When proposing activities of this nature in the classroom, one essential element that should never be overlooked is time. Students need the opportunity to dedicate sufficient time to reading and understanding the problem, as these steps are crucial for effective time management in the subsequent phases of the solution process [68]. If students are working in groups, it is important to allow groups that have employed different strategies to engage in discussion and share their approaches, particularly if the presented problems have revealed additional insights [68].

The selection of problems to propose is crucial; often, what is presented as problems in textbooks are, in fact, exercises in applying a rule. It is essential to clarify the meaning of a problem in mathematics and what characterizes a “good problem”. A good problem does not merely

involve reproductive processes but also activates productive processes, giving meaning not only to the results but also to the processes and the ability to describe and support them (argumentation) during the evaluation phase [37]. A “good problem” enhances the mathematical skills of those who tackle it and encourages students to engage in the processes that contribute to the development of mathematical thinking.

According to Maria Dedò and Sofia Sabatti [11], for a problem to be educationally effective, it must have certain characteristics, which may be present in whole or in part.

The first characteristic is mathematical significance, that is, the ability to address a foundational concept or a relevant notion in the discipline. These notions do not necessarily need to be approached systematically. Mathematical thinking does not progress in a linear fashion, and revisiting a concept is necessary to increase awareness and competence. A meaningful problem can therefore be one that addresses an important concept in an unstructured way, which will be systematically explored later on.

The presence of stimulus questions in a problem is useful, as they stimulate students’ reasoning and promote the process of metacognition. Questions help students reflect on the solution process and, therefore, recognize and use effective strategies in different contexts, fostering the ability to learn how to learn. This ability is one of the eight key competences of the *European Framework of key competences for lifelong learning*[3].

Another characteristic that makes a problem educationally relevant is the presence of multiple solutions. This dismantles the idea that a problem has only one solution, and encourages peer discussion about different solutions and their acceptability, thereby promoting the metacognitive process of control and verification. Open problem, with various possible explorations and variants, allow for modifications in data or quantities and encourage students to invent related problems. This openness offers opportunities to appreciate and learn from mistakes, fostering more precise interpretations.

While solving a “good problem”, students engage in discussions and argumentation. This serves as a training opportunity for both students and teachers. Teachers can challenge their own certainties, expand their interpretative understanding, and intervene strategically based on students’ explanations. The teacher is also engaged by genuinely paying attention to students’ explanations and, starting from the students’ words, can address difficulties or introduce new questions [37].

Dedò and Sabatti highlight an additional characteristic: memorability. A “good problem” should be memorable. Its narrative context, linked to personal experiences and emotionally stimulating, paradoxical, or ironical aspects, aids retention. Similarly, elements of surprise, such as extremely large or small numbers or simple solutions to complex problems, remain imprinted in memory. However, a balance must be struck between the narrative and the mathematical goal that students need to achieve.

The difficulty of a problem, when the procedure is not known a priori, can also add interest and memorability, requiring solvers to explore new paths.

Below are some proposed problems, which can serve as explanatory examples.

1.3.1 Some examples of *Good problems*

To focus on problem-based teaching, it is essential to choose a good problem. A source of inspiration for teachers can be the questions proposed in mathematical games like the *Kangourou*[4], the *Transalpine Mathematical Rally*[13], and many others available online on websites dedicated to these competitions. Alternatively, teachers can draw inspiration from the questions in the IN-

VALSI tests, which are generally multiple-choice but can be appropriately modified to generate a problem.

An especially interesting resource is the Italian website *Problemi per matematici in erba* [11], edited by Maria Dedò and Sofia Sabatti, which aims to guide teachers in implementing problem-based teaching. The examples below were taken from this website. A description of the website and additional examples of problems can be found in Appendix A.3.

An example of a significant problem is *The Last Digit*, which touches upon the issue of residue classes, a topic that should not be addressed in the first cycle but can be introduced through a problem to stimulate observations and reasoning that will later allow the recognition of simple algebraic structures.

Another example is *Prisms, Pyramids, and More*, in which students are invited to reason about the number of faces, edges, and vertices of a polyhedron, also evaluating whether it is possible to meet some of the proposed requirements. This is a significant problem because it serves as a bridge between different areas of mathematics, linking geometry to arithmetic and algebra.

The last digit

The story

The other day, Chiara was discussing with two of her friends; also present were Alberto, Chiara's older cousin who studies engineering at university, and her aunt, who is a researcher in mathematics. Chiara is still in primary school and is a clever girl who likes playing with numbers; she first told her friends that she (being a true little witch) knows that $9 \times 9 \times 9 \times 9 \times 9$ is a number that ends with 9; and so far, everyone believed her, especially since they verified her claim with a calculator. But she even challenged them further in this way: "You tell me any number, even a very large one, and I can tell you what the last digit of $9 \times 9 \times \dots$ multiplying as many 9s as the number you've told me. And if this isn't magic...!"

Chiara hasn't studied powers yet and she doesn't know how to use letters instead of numbers properly; if she has, she could have phrased her challenge this way: "You tell me any number n , and I can tell you what the last digit of 9^n is."

To her cousin Alberto, this statement seemed like an unbelievable boast (especially because, when the number becomes really large, you can't even verify it with a calculator...!); however, the aunt said that Chiara is right, that the calculation can be done, fairly simply, and that it could also be done with other numbers instead of 9; she even added that the calculator is absolutely useless for this problem.

And you? Have you figured out what Chiara's "trick" is? And why does this trick work?

The questions

What is the last digit of 9^7 ? and of 9^4 ? and of 9^{57} ? and of 9^{2018} ? and of 9^{2019} ?

Chiara's cousin doesn't understand how it works, and Chiara needs to explain it to him. Help her:

if the exponent is ... then 9 raised to that exponent ends in ... if instead the exponent is ... then 9 raised to that exponent ends in ...

Chiara's cousin was really stubborn. Even though Chiara revealed her "trick" to him, he almost got angry: "I don't believe it! In the cases I can verify by doing the calculations with the calculator, it's true, fine, but this doesn't assure us at all that the same thing is always true!"

How would you convince him that it really is so?

And now try playing with some other numbers.

If Chiara had used 5 instead of 9, the problem would have been easier. What is the last digit of 5^{2018} ? what about 5^{2019} ? Could you formulate a general rule about the last digit of any power

of 5? How would you write this rule? Why are you sure it always works?

Could you find another number that behaves like 5, making Chiara's game very easy? Which one? And why does the same phenomenon happen with this number as with 5?

Here is a more complicated case.

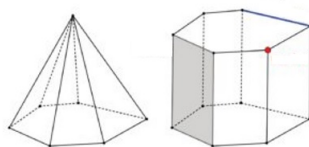
What is the last digit of 7^2 ? and of 7^3 ? and of 7^4 ? and of 7^5 ? and of 7^{57} ? and of 7^{2018} ? and of 7^{2019} ?

To guess what the last digit of 7 raised to any number is, you just need to look at the exponent. But which characteristic of the exponent should we look at?

Alberto continues to say that he doesn't believe it. How do you convince him that things are exactly as Chiara says?

Prisms, pyramids and more

This is a problem you could explore even with your eyes closed, trying to exercise your imagination, and discussing with your fellow students in a way that involves everyone's imagination. However, we think that at some point it will be useful to open your eyes and try to construct some objects.



In the figure you see a pyramid (on the left) and a prism (on the right). In the prism, we have highlighted a vertex (in red), an edge (in blue), and a face (in gray). Could you then imagine, or build:

(a) a pyramid with 5 faces? a prism with 5 faces?

(b) a pyramid with 9 vertices? a prism with 9 vertices? a pyramid with 15 edges? a prism with 15 edges?

(c) a polyhedron that has the same number of faces and vertices?

(d) a polyhedron with 20 edges? a polyhedron with 21 edges? one with 12 edges? one with 99 edges? one with 100 edges? one with 11 edges?

(e) a polyhedron with 6 faces, all triangular? a polyhedron with 11 faces, all triangular? a polyhedron with 12 faces, all triangular?

Warning: some of these constructions (to be precise, three) are impossible.

We then ask you to:

(a) describe (in writing) the polyhedra you imagined (or constructed), as if you had to explain what they are to a friend who cannot see them;

(b) say which are the three requests that are impossible to satisfy. Can you also explain why they are impossible?

Although these two problems belong to different categories (a narrative problem and an abstract context problem), they both include stimulating questions that offer points for reflection not only to reach the final solution but also to motivate the students to argue their points.

In both cases, the questions have an increasing level of difficulty, and to give the correct answer, especially to demonstrate that something is impossible, one must engage in reasoning that implies a deep understanding of the context. For example, to assert that the polyhedron with

11 triangular faces does not exist, one must be aware that each edge is common to two faces. This awareness is reached when one is forced to respond to an impossible request: when constructing a polyhedron using models, for example, two faces are automatically joined by the edge they have in common, but having to explain this implies a different cognitive process.

This is how a problem can also activate a metacognitive process in the solvers.

The problem *Rhombuses on squared paper*, which we pose below, for example, challenges students to overcome the strong stereotype of the "standard" representation according to which a rhombus is such only when drawn with diagonals parallel to the lines of the graph paper. By tackling a problem like this, one can acquire the flexibility needed to recognize other shapes in different contexts and situations.

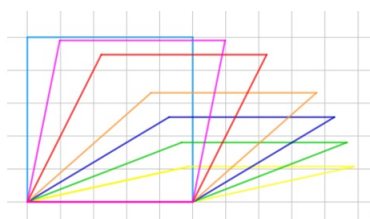
Rhombuses in squared paper

How many different rhombuses with a side of exactly 5 squares can you draw on squared paper so that all their vertices are at the intersections of the squares?

Hint: remember that the diagonal of a rectangle with sides of 3 squares and 4 squares is exactly 5 squares long.

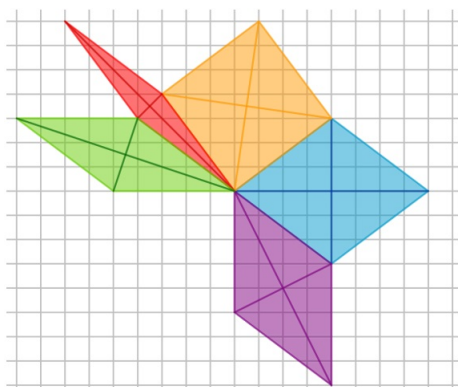
Draw, on squared paper, all the rhombuses you have found.

In the problem, there are two requests that need attention: the rhombuses with a side of 5 squares must have vertices coinciding with the intersections of the squares on the paper, and they must be different. If you do not consider the first request, you can draw infinite rhombuses, one for each value the acute angle ($0^\circ < \alpha \leq 90^\circ$) can take.



Adding the constraint of the vertex at the intersection of the graph paper reduces the possibilities to 5.

Consequently, 5 different rhombuses can be obtained. One is the square with a side of 5 squares, the other 4 correspond to the combinations produced by the Pythagorean triple 3, 4, 5.



The second question goes to the heart of the problem: reflecting on what makes two rhombuses equal or different from each other. To reach the conclusion, students might use different solving strategies, reasoning about the size of the angles or the length of the diagonals, for example.

Below is an example of an open problem with multiple solutions.

The pastries

For a charity sale, Luisa volunteered to prepare some pastries, and with the ingredients she had at home, she managed to make 38 of them.

She's thinking about how to transport them and discovers that they fit perfectly in those egg trays designed for exactly 6 eggs because each pastry is roughly the size and shape of an egg. She decides to use the egg trays as trays because she has many at home, and in this way, the pastries won't get ruined.

Luisa plans to take some of her egg trays filled with pastries to the market and keep some pastries at home for her little brother Marco, who loves them.

How many full trays can Luisa take to the charity sale market?

How many pastries are left for Marco?

You may have noticed that there are multiple ways to answer these questions.

Which of your answers allows Luisa to take the maximum number of full egg trays to the market?

Is there an answer where Marco doesn't get any pastries at all? If your answer to this last question is no, what is the minimum number of pastries that are left for Marco?

This problem aims to focus the attention of the students on the meaning and properties of quotient and remainder in the division of natural numbers. However, the problem request is not just to find the maximum number of full egg trays; by asking to leave some pastries for the little brother, it aims to explore all possible solutions, which can be more or less generous towards Marco.

You can ask the students to represent the equations that express the solutions found in the form:

$$38 = 6q + r$$

where q is the number of full trays and r is the number of pastries left for Marco.

Among all pairs of numbers q and r that satisfy the equation, only one, where $0 \leq r < 6$ and q is the largest possible, truly expresses the meaning of dividing (in $\mathbb{N}$) 38 by 6.

In the expression

$$38 = 6 \times 6 + 2$$

there are the four terms of the division: dividend, divisor, quotient and remainder. Using these equations also refers to a fundamental aspect of teaching first-cycle mathematics, the correct use of “=” symbol.

By varying the number of available spots in the egg trays or the total number of pastries, the problem opens up to other scenarios such as residue classes, for example.

Some of the proposed problems start from a final situation and ask us to reconstruct the initial conditions. This is a useful backward process because it forces students to engage in an intuitive and discovery-based reasoning: predicting or making plausible a result for which a more rigorous explanation or demonstration will be given later. This approach encourages the emergence of creative solving strategies that trigger discussion and comparison among students.

An example of a problem that can be solved with a backward approach and different solving strategies is *Victoria regia*.

Victoria regia

There is a round lake in South America. Every year, on June 1st, a Victoria regia flower appears right in the middle of the lake (its stem rises from the bottom, and its petals rest on the water like those of a water lily).

Every day, the area occupied by the Victoria regia doubles, and finally, on June 30th, it covers the entire lake, the petals detach, and the seeds sink into the lake.

On which day of June does the area occupied by the Victoria regia correspond to half the surface of the lake?

An interesting aspect of this problem is that, by reasoning backward, one realizes that only five days before the end of the month, the lake is covered by the flower for about 3%; therefore, the Victoria regia takes 25 days to cover 3% of the lake and only 5 days to complete the remaining 97%. This way of growing, which is far from the linear growth model to which we are more accustomed to, generates a surprising effect that impresses the concept of exponential function on the students' memory.

The *Victoria regia* problem is, therefore, also an example of what it means for a problem to be memorable: an answer that surprises because it is far from common sense leaves an impression on memory, even more so if the students initially got it wrong.

1.4 Argumentation

The preceding paragraphs repeatedly referred to discussions among students explaining their positions and the importance of teaching argumentation. Argumentation is one of the goals outlined in the 2012 National Guidelines at the conclusion of the initial education cycle [57]:

The student produces arguments based on the theoretical knowledge acquired ... He supports his own beliefs, bringing adequate examples and counterexamples and using concatenations of statements; agree to change your mind by recognizing the logical consequences of a correct argument.

The development of mathematical thinking is a gradual process encompassing knowledge, skills, and competencies that intricately intertwine and evolve. To facilitate this, it is crucial for students to learn to reflect on what they are learning and how they are learning, adopting a metacognitive approach.

Engaging in argumentation not only aligns with the objectives set by national guidelines but also transforms classroom practices into formative assessments. When students articulate how they arrived at a correct or incorrect answer, it becomes easier for both the teacher and student to grasp what has been learned and provide appropriate feedback. Providing a reasoned answer entails taking responsibility for one's opinion, and comparing responses with peers facilitates reflection. However, fostering argumentation requires more than just explaining one's answer; it necessitates designing activities that enable students to formulate and validate conjectures.

The "Language and Argumentation" working group, led by Francesca Morselli from the mathematics department at the University of Genoa, proposed a definition of argumentation and considered the essential elements for effective argumentation. Argumentation, in a general sense, involves a series of arguments supporting an assumption. While an argument can be a sentence, in mathematics, it may also be represented by expressions, numerical data, or drawings constituting a rationale for or against a specific proposition or opinion.

To engage in argumentation, one must possess knowledge of the argument’s content and understand how to manage the steps of reasoning and their logical and linguistic concatenation. Therefore, it is essential to gradually cultivate a set of attitudes, values, and logical-linguistic resources to internalize the cultural values inherent in argumentation.

The “Language and Argumentation” group chose to initiate the discussion based on the model presented by Stephen Toulmin in [69]. The model was originally designed for evaluating the validity of legal arguments but has proven to be valuable in various other contexts.

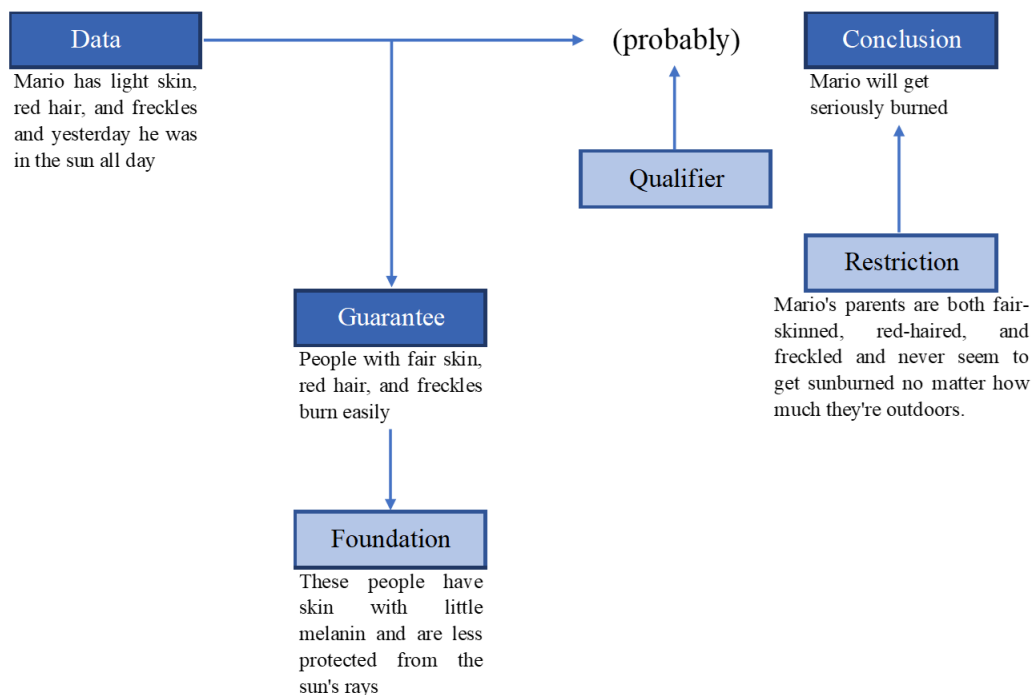
The fundamental concept is that a successful argument must offer a solid justification. Toulmin outlines six interconnected components in an argument, with three being essential. These basic components include *data*, *conclusion*, and *guarantee*.

The *conclusion*, also referred to as the thesis, is the statement whose merit needs to be established. *Data* or evidence is a fact invoked to support the claim, and the *guarantee* is a statement that authorizes the connection from the data to the conclusion.

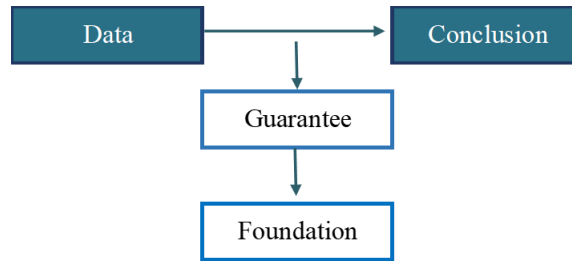
In more intricate arguments, the use of the other three elements - *foundation*, *qualifiers*, and *restrictions* - may also be necessary .

By *foundation*, we refer to supporting statements that lend greater credibility to the guarantee. *Qualifiers* are words or phrases expressing the speaker’s level of confidence or certainty about the statement, while *restrictions* are statements acknowledging legitimate limitations that can be applied to the thesis.

For example:



Alternatively, a simplified model structure can also be employed:



Argumentative validation occurs through the comparison of hypotheses, strategies, and theses. To facilitate this, it is essential to engage students in open-ended problems, susceptible to various interpretations and solution strategies. It is also important to encourage pupils to compare and collaborate in selecting solution strategies and in becoming aware of and explaining recurring errors.

Students, either individually or in small groups, provide answers that are then shared and analyzed by the entire class. The collective analysis and reflection on the answers enable a gradual refinement of argumentative techniques and serve as a formative evaluation.

In accordance with Black and William's definition in [19] :

Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction.

In the provided table by Black and William [19], key strategies for formative assessment are outlined. It is evident that the actors involved in formative assessment extend beyond just teachers; students play a crucial role not only in relation to others but also in self-assessment.

	Where is the learner going?	Where is the learner right now?	How to get there?
Teacher	1. Clarifying learning intentions and criteria for success	2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding	3. Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	4. Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	5. Activating students as the owners of their own learning	

When a student provides a well-reasoned answer, it enables both the teacher and classmates to delve into the student's thought process, facilitating effective feedback. Simultaneously, the student takes responsibility for their own opinions and learning.

According to Morselli, Cusi, and Sabena as mentioned in [58], the feedback strategies employed not only serve as effective tools for fostering class discussions but also act as powerful instruments for implementing various formative evaluation strategies.

During discussions, when a teacher directs attention to a student's contribution, it implies that

it should be scrutinized in more detail. This approach transforms the student into a valuable resource for their classmates. Moreover, requiring a well-founded explanation for an answer prompts students to focus on the essential characteristics of a mathematical argument, such as correctness, clarity, and completeness. This emphasis aims to enhance their skills in these aspects. Additional guidance from the teacher encourages students to self-regulate, contemplating the use of mathematical tools and thinking strategies. This process enables them to construct a repository of knowledge ways of interpreting a text or a representation, and problem-solving strategies, which they can draw upon when confronted with new challenges. Simultaneously, students are prompted to analyze their problem-solving approaches, identifying both strengths and issues associated with their methods.

Below is an example of how the strategies identified in the previous table can be implemented.

1.4.1 Enhancing argumentation through space-time graphs

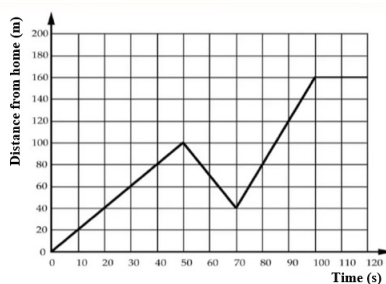
An interesting activity that also shows the interdisciplinary nature of argumentation is the activity conducted by Monica Testera of the *Istituto Comprensivo* of Carcare (Savona). This activity born from the collaboration between Francesca Morselli and Annalisa Cusi and Cristina Sabena of the Department of Philosophy and Educational Sciences of the University of Turin, which is part of the European FaSMEd project, designed to improve the mathematics learning of struggling students with the help of technology. The project involved an experiment with the *connected classroom*, that is, a software that allows the exchange of worksheets and answers between the teacher and students, who work in small groups, and then project the answers to the Interactive Whiteboard (IWB) to be analyzed and compared during the collective discussion. This experiment was inspired by the MARS project (Mathematics Assessment Resource Service), specifically regarding space-time graphs [59].

The project starts with an introductory activity using motion sensors that translate the students' physical movements into graphs visible in real-time on the IWB.

In this way, students familiarize themselves with graphical representation and give them meaning, eventually reaching the point where they can make hypotheses about the type of graph that can be obtained from a certain type of movement. They become able to infer the real movement from the graph.

The activity then continues with the presentation of a problem:

Every morning, Tommaso walks along a straight road, from his house to the bus stop, which is 160m away from his home. The following graph describes how he traveled the route yesterday.



What happened between the seconds 50 and 70? How did you determine your answer?

To guide the answer, additional questions can be provided, such as:

After how many seconds did Tommaso reach the bus stop?

Did he cover exactly 160m? Why?

The activity then continues with slightly different exercises: for instance, a graph is provided along with three possible stories to match it, constructed by the teacher playing with potential misunderstandings that might arise from reading the graph. Alternatively, a story is provided, and the students must construct the time-distance graph.

For each question, small group discussions and reflections with the entire class are encouraged, serving as means to convey the content.

The opinions of the students are collected regarding the mathematical, metacognitive, and emotional aspects both during the activity (e.g., which graph is correct? Was it difficult to understand the text of the problem? How do you feel after reading the text of the problem?) and at the end of the activity (e.g., is this result correct? Is this argument complete? What have you learned? Did you enjoy the activity?).

During the experimentation, conducted in the presence of university researchers acting as observers, students' responses were collected and analyzed.

The teacher starts the class discussion (point 2 of the table), and a student answers the first question regarding Tommaso's journey in this way:

Tommaso was walking along the road until he met his friends who asked him to go to school with them, extending the journey. So, he covered more than 160 meters.

A classmate corrects his group's response (point 5), saying:

We wanted to add that he was walking towards the bus but turned back to his friends, and then they all went to the stop because that downward segment is to indicate that... he follows his friends.

At this point, the teacher intervenes to invite the second student to clarify the answer (point 3): *Following his friends... and when you say, 'so he covered more than 160 meters to get to the bus stop', what do you mean?*

Answer:

That from 0 to 160 is a precise straight line, but having encountered an obstacle, he extended the journey because being a point...

At this point, there is an intervention by a classmate (point 4) who provides an empirical argument referring to the initial activity with the sensors:

But nobody knows that the graph ... I mean..., it's not like the other time when going forward it was straight and instead when going back it went down.

Teacher (point 3):

'Nobody knows', what?

Answer:

That the graph changes direction when... for example, if Tommaso moves away, the graph moves forward and rises, if instead he approaches, the graph goes down.

Now the observer and the teacher leverage the students' interventions as resources for each other to provide feedback.

Student:

But, in my opinion... I mean, we wrote something like covering several meters, but if... it's not that... it doesn't say that the graph changes direction when he moves away or approaches...

Observer:

So you're saying -we said it because we remembered what we saw the other time, but is it true that we should interpret it this way here too?- is this your doubt?... Who tells us that we can say that when the graph goes up, it means he is moving away?

Another student:

We say it because it's written ... on the y-axis ... 'distance expressed in meters'.

Teacher:

So?

Student:

So, the distance from home and the closeness ... so looking at the graph, you can understand that he moves away and then you can also see the time, I mean, in 50 seconds he moves away 100 meters and then in 20 seconds he approaches by 60 meters ..."

The student has acted as a resource and provided a theoretical argument (meaning of the axes).

Using the Toulmin model [69], it can be observed how starting from the data (the space-time graph) leads to the answer (Tommaso returns back between seconds 50 and 70), citing as a guarantee the fact that the graph grows downwards. Initially, the foundation is the empirical activity with the sensors; a student states "... *when you were going forward, it was straight, and when you turned back, it went down...*". Then, thanks to the intervention of a second student who acts as a resource for his classmate, the theoretical foundation of the meaning of the axes is reached (... *it's written on the y axis... the distance from home expressed in metres*").

Appendix A.4 presents additional examples of activities aimed at developing students' argumentative skills.

1.5 Learning environment and persistent difficulties

Based on the aforementioned considerations, it is evident that creative teaching and discovery play a crucial role in fostering a positive attitude towards mathematics. The emotional aspect significantly shapes each individual's perception of mathematics. This leads to the question: what impact does this teaching approach have on students with low achievement in mathematics? Recent studies indicate encouraging results in such cases. When addressing low achievement in mathematics, a distinction needs to be made between persistent *difficulties* arising from non-cognitive factors and specific learning *disabilities*. Typically, the demarcation line is drawn based on percentiles derived from diagnostic test results. However, challenges arise due to variations in standards among different nations, uncertainties regarding the choice of content (primarily written calculation), and concerns about intervention strategies.

A recent model proposed by Karagiannakis, Baccaglini-Frank, Papadatos [49] introduces tests in four domains: basic numerical skills, memory, logical reasoning, and visual-spatial abilities. This model helps identify the specific area where difficulties are most concentrated, facilitating the development of targeted and effective intervention strategies.

In the absence of such interventions, there is a tendency to overestimate the incidence of *disability* compared to mere *difficulty*. Teaching methods that involve imitation of algorithms often rely heavily on verbal memory, posing challenges for dyslexic students or those with limited cognitive resources. Conversely, approaches that do not prescribe pre-established strategies and require productive effort enable long-term memory storage, leading to more significant improvements, particularly in weaker students [48].

These findings challenge the common belief that less cognitively proficient students should avoid problem-solving and instead memorize algorithmic rules to overcome their limitations. On the contrary, everyone should have the opportunity to engage in problem-solving tasks. Using diverse channels for accessing and producing information, promoting generalization processes, and encouraging conjectures and argumentation can prevent learning difficulties and promote a more inclusive teaching environment.

It is crucial to maintain a delicate balance, ensuring that the productive effort required in addressing challenging problems serves as a facilitator rather than an obstacle to learning. Once again, it is the teacher's responsibility to observe students, evaluate the overall progress of the class, and make adjustments as needed. Assessing implemented skills, both mathematical and social, over medium to long-term durations provides a more comprehensive measure of effectiveness than evaluating individual activities.

Part II

Effective teaching

Chapter 2

The role of teachers in effective learning

In Part I, we have seen some of the educational proposals in the Italian panorama aimed at lower secondary school students. Each of the proposals presented, as well as many other examples that have not been reported here, contribute to creating a suitable learning environment in the classroom.

Many factors can influence the outcome of education. What a student brings in terms of skills, attitudes, family background, and social context is something the teacher has little control over. However, what the teacher can and should influence, in my opinion, is the quality of teaching. This refers to the characteristics of instructional actions that have a stronger correlation with positive outcomes in student education. Keeping in mind that there is no absolute ideal teacher, many studies have focused on teacher effectiveness.

In particular, John Hattie [45] conducts a comprehensive meta-analysis in 2009, examining the impact of over 130 factors on school learning. The factors were analyzed quantitatively, using a statistical index called effect size (ES), and qualitatively, by reporting which conditions tend to influence the outcomes of educational interventions. In [46] Hattie further expands the database, encompassing over 1200 meta-analyses, not only focusing on what works but especially on what works best. He asks, in other words, what makes teaching effective. Observing the distribution of over 50,000 effect sizes, Hattie concludes that almost all interventions have a positive impact. However, to identify a noteworthy positive effect, it is necessary to establish a minimum level. He identifies an ES of 0.40 as a threshold for effectiveness and as a starting point for evaluating the impact that teachers have on students.

According to Hattie [46], the most important actors in the educational process are teachers.

The differences between highly effective and low-effective teachers primarily concern their attitudes and expectations when making decisions about key aspects of teaching – i.e. what to teach and at what level of difficulty – and their awareness of the progress of students and the effect of their teaching.

This statement would seem to be in contrast with one of the pillars of the National Guidelines which talks about the centrality of the student, but that is not the case.

The student's learning remains central, because the teacher's action must be based on awareness of the extent of this learning, through a continuous exchange of feedback.

The expert teacher not only knows how to identify the most effective ways to present his/her discipline, but monitors learning by seeking feedback from students and, in turn, responds appropriately on the basis of the information received. Therefore, it is important that student learning is visible. Visible to teachers, because they learn from their own teaching, and visible to students, who learn to be teachers of themselves.

Visible learning combines, rather than opposes, teacher-centered teaching and student-centered learning and knowledge.[46]

When teachers assess whether learning is happening they can significantly intervene to change the direction of the learning and make progress happen. To do this, the teacher must:

- be aware of the learning goals;
- know when a student is reaching them;
- have awareness of the prior knowledge with which the student approaches the task;
- master the content so that s/he can provide meaningful and challenging experiences that allow for some form of progressive development.

These considerations are perfectly in line with the National Guidelines [57] regarding evaluation, according to which

Evaluation precedes, accompanies and follows the curricular paths. Activate the actions to be undertaken, promote the critical assessment of those completed. It assumes a prevalent training function, accompanying learning processes and stimulating continuous improvement.

Therefore, we have to distinguish between *assessment* or assessment *for* learning and *evaluation* or assessment *of* learning. The Italian National Guideline emphasizes the focus on assessment for learning as a systematic process of documenting and using empirical data to measure knowledge, skills, attitudes, and beliefs. By supporting assessment, teachers try to improve the students' path towards learning. On the other hand, when we talk about assessment of learning in education, we focus on grades and intend to answer the question: "What's been learned?". Evaluation is product-oriented while assessment is process-oriented.

According to the information presented in [46], providing teachers with formative assessment has a very high impact (ES=0.90) on teaching effectiveness, as does providing feedback (ES=0.75). Similarly impactful are class discussions (ES=0.82), reciprocal teaching (ES=0.74), and teaching metacognitive strategies (ES=0.69).

In the upcoming sections, we will delve into these aspects.

2.1 Formative assessment

At this point, it is useful to clarify the different types of evaluation and assessment. Castoldi [27] identifies five types of assessment/evaluation:

1. Predictive (orientation): it has the aim of predicting the most suitable future study pathway for a student (for example orientation tests).
2. Diagnostic: it takes place in the initial phase of the training process and has the purpose of analyzing the characteristics of a student's at the beginning of the path that s/he will have to complete.
3. Formative: aims to provide feedback to the learner and teacher.
4. Summative: it takes place in the final phase of a training course (educational module, school year, etc.) and has the purpose of verifying the results achieved by the student.
5. Certification: it has the purpose of providing a mark attesting the results achieved by the student at the end of study pathway to be used for his/her choice of subsequent studies or his/her entry into the world of work.

In [38] D'Intino highlights how the the most common type of assessment in Italian schools is summative, often expressed with a qualitative mark or with a number. This prevailing form of assessment often risks overshadowing the others, especially the formative one, by confusing the numerical grade with the feedback that should be given to both the pupil and the teacher. Numerical marks as feedback on a student's test can be difficult to read in educational terms for many reasons. For example, oversimplification results in the same numerical grade or tasks that highlight totally different difficulties; often, in the absence of an in-depth description of the performances, the mark is experienced by the student as a judgment on the person and not on the task; sometimes pupils and parents focus on the final result without considering other aspects; mistakes are seen as something to be avoided and students themselves tend to hide their own difficulties in order to avoid a lower mark. Ferretti [39] affirms this is not the way to go if the primary aim of assessment is to improve student learning; the observations and evaluations of the teachers must, therefore, give students an indication on the next learning step to be tackled.

By clearly defining the evaluation criteria and the learning objectives related to the task, it is possible to shift the student's attention, so that the evaluation is not perceived as a judgment on the person but as an indicator of the learning level, also counteracting fear of mistakes, which become a step to accept, know and exploit to improve.

D'Intino [38] concludes that in this way assessment becomes formative and, at the same time, a useful tool for the teacher to:

- understand in more detail the subject-related difficulties of the students;
- draw useful information to best support their students and to plan teaching;
- reflect on the discipline and on one's teaching beliefs with the aim of regulating teaching;
- make the characteristics of his/her own assessment method increasingly clear to him/herself and to the students.

Our next aim is to better define formative assessment and examine it critically. We provide a response to the critique through two examples.

We have already talked about the fact that formative assessment should be directed towards the process rather than the product of learning.

This type of assessment must be formative for both the pupil and the teacher, as the pupil needs to know what the next learning step will be and the teacher can get useful evidence to proceed in his/her work agenda. So formative assessment cannot be expressed by a grade or a judgment nor can it be confused with summative assessment.

Black and Wiliam [19] give a definition that starts from relating the three spheres acting in the education process: the intentions of the teacher (the teacher's agenda), the needs of the students (internal world of each student) and the teacher-student relationship.

During the education process students and teachers show evidence that allows the teacher to adapt his/her educational offer with the students learning needs. In particular, Black and Wiliam state:

Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited.

In this definition, by instruction we mean the combination of teaching and learning and it emerges that the assessment agent is not only the teacher but also the student him/herself and his/her peers. Another important aspect to consider here is the "decision making" based on the evidence gathered: the attitudes and intentions of the parties involved, as well as the collection of evidence are very important, but it is equally important that these signals are correctly interpreted and used and produce effects that really improve learning.

However, sometimes, even the best-designed interventions do not translate into learning for all students, and what works in a context may not lead to the same result in another context. This is part of the unpredictability of learning, which is determined by many factors acting simultaneously. The teacher must have a goal before starting the lesson, but cannot fully assume the responsibility for learning because s/he cannot learn instead of the pupil. The teacher can design situations that promote the autonomy of the students, taking into account their needs as much as possible, but s/he cannot be limited to merely assisting because s/he would abdicate his/her role. It is necessary to mediate between the need to guide the lesson and the need to make the pupils masters of their learning.

Teachers have to make decisions on many occasions, often quickly. Such decisions may be better grounded if they are based on evidence from their students' results or on a comparison with the results of other students (such as those in the previous year) or with the information elicited in previous lessons. In elaborating the response to an imperfect output from the student, the teacher must interpret the student's contribution to understand what it reveals in terms of motivation or thought and choose the optimal response based on his/her knowledge and experience.

This point raises another important issue in educational evaluation highlighted by Bennett in [17] regarding the effectiveness of formative assessment. Bennet claims that there is no clear scientific evidence on the effectiveness of formative assessment as a facilitator of student learning: the literature shows that the benefits can vary greatly between different subpopulations of students. Above all, these differences suggest we should not consider the practice of formative assessment effective in general, but we should relate it to the context and also to the discipline. If the teacher does not know how to interpret the signals because s/he does not adequately know

the discipline or has not been sufficiently trained to interpret the students' answers, formative assessment cannot be effective.

In [17], Bennet claims that the domain dependency issue is crucial because, in order to be effective, formative assessment needs to fit into the set of knowledge, processes and strategies related to competence in a specific domain. An inexperienced teacher in the cognitive domain of interest is less likely to know what questions to ask, what to look for in the students' performance, what inferences to make about their knowledge, and what actions to take to improve instruction. Teachers need substantial knowledge to effectively implement formative assessment in classrooms. Acquiring this knowledge requires time and also useful classroom materials that model the integration of pedagogy, domain knowledge and docimological knowledge. It should not be forgotten, in fact, that formative assessment is still assessment. Therefore, within its conceptualization, there should also be relevant measurement principles. This does not mean distorting formative assessment but including, rather than ignoring, relevant principles.

Such an approach implies that formative assessment needs to be integrated into the curriculum. In the following example we see how formative assessment materials can be built around the key concepts and underlying processes of a discipline, particularly in mathematics.

2.1.1 The FaSMEd project as a promotion of formative assessment strategies

In Part I, subsection 1.4.1 we explored how formative assessment is used to pursue the learning objective of improving students' argumentative skills. The activities related to the FaSMEd project [58] are characterized by a dual approach: on one hand, promoting argumentation through formative evaluation; on the other hand, using argumentation as a tool for formative evaluation, due to its ability to make the students' thinking "visible".

In [58], lower secondary school students engaged with the study of space-time graphs through the use of digital technologies provided by connected classroom. Specifically, in subsection 1.4.1 we discussed how class discussions, orchestrated by both the teacher and the observer, can promote the students' argumentative skills.

Let us now consider the project from another perspective, that is, the use of the connected class and argument to prompt students to activate formative assessment strategies.

The activities planned in [33] are aimed at encouraging students to reflect on their teaching-learning processes at both the metacognitive and affective levels.

The theoretical framework is based on three fundamental aspects outlined in [19]. The first aspect is to consider as protagonists of the teaching-learning process all the actors involved (the teacher, the student and the classmates) in the belief that, although in many cases the decisions will be made by the teacher, the assessment agent is not only the teacher, but also the peers, or the individual learner.

Secondly, in [75], reference is made to different formative assessment strategies, which can be summarized in five main categories:

- A. *Clarifying/understanding/sharing* learning intentions and evaluation criteria;
- B. *Engineering* effective classroom *discussions*, questions, and learning tasks;
- C. *Providing feedback* that moves learners forward;
- D. *Activating students as instructional resources* for one another;
- E. *Activating students as the owners* of their own learning.

As a third aspect, the opportunities offered by technology through a connected classroom are considered, namely:

- *Send and show*, when technology is used to support communication and to activate class discussions;
- *Process and analyze*, when technology is used to analyze the data and information collected during the lessons;
- *Create an interactive environment*, when technology is used to create a shared working environment, where students work individually or collaboratively, or a learning environment in which to explore mathematical content.

The activities in this project [32] are based on several worksheets divided into four types:

1. *problem worksheets* introducing a problem and asking one or more questions involving the interpretation or the construction of the representation (verbal, symbolic, graphical, tabular) of the mathematical relation between two variables (e.g. interpreting a time-distance graph);
2. *help worksheets* aimed at supporting students who face difficulties with the problem worksheets by making specific suggestions (e.g. guided questions);
3. *poll worksheets* prompting a choice between various options for answering a question;
4. *discussion worksheets* prompting a focused discussion.

The lesson begins with the teacher sending a problem worksheet to students divided into pairs or small groups who then in turn send the answer to the teacher. At this point the teacher can decide to send help sheets if necessary. Once the answers have been received by the teacher, they can be displayed via smart board and used in different ways: the answers can be analyzed one at a time, or similar answers can be compared simultaneously to evaluate analogies and differences or, again, many different answers can be projected by asking the students to group them according to certain criteria.

Similarly, help sheets can be displayed and used in two different ways: students who have not received them can reflect on the possible role played by the help provided; the others can discuss how they used the help, making their reasoning explicit.

Poll worksheets also have great potential because they allow one to get answers from all students and not just those who are less fearful of expressing their opinions. Depending on when they are used during the class, they allow the teacher to investigate, for example, the degree of self-reported confidence in the understanding of a certain topic or in previous knowledge. Alternatively, a projected survey result can be used to prompt reflection and initiate a discussion to introduce a new topic, practice a particular skill, or highlight a misconception.

One of the objectives in [32] was to verify how connected classroom technology contributes to fostering formative assessment strategies, with specific attention to effect at making the students' thinking visible.

First of all, it was observed that the exchange of worksheets between teachers and pupils, especially where the help worksheet were provided, contributed to activating formative assessment

strategy C. The discussion in class, orchestrated by the teacher to comment on the open answers or on the results of the surveys, activates strategies B and C. Students are led to carefully read, discuss and compare the answers, thus becoming, at the same time, masters of their own learning (strategy E) and a resource for their peers (strategy D). Another aspect highlighted is the potential of argumentation in term of formative assessment.

The research in [31] has highlighted the great potential of these tools; however, it has also been noted that the effectiveness of these means greatly depends on the ability of the teacher, who must monitor the progress of students while keeping alive their motivation and ability to reflect. For this reason, another strand of research has studied the interventions related to providing feedback intentionally implemented by teachers (FA strategy C).

The authors analyzed the videos of the class discussions, and also obtained additional information from the notes that the observer and the teacher took during the activities. In particular, the authors focused on the following questions:

1. What are teacher's strategies that may foster FA strategy C?
2. What level of feedback is provided?
3. What are the effects of feedback strategies in terms of activation of other FA strategies such as D and E?

According to Hattie's definition [47], feedback is considered "as information provided by an agent regarding aspects of one's performance or understanding". Feedback can come from different subjects such as the teacher, a parent, the pupil him/herself, a classmate or even the textbook or experience. It is information thanks to which the student can confirm, add, rearrange the initial information both in terms of content, of cognitive strategies, and of metacognitive knowledge. According to Hattie and Timperly [47], four types of feedback can be distinguished, based on the domains they refer to:

- *feedback about the task*, aimed at focusing attention on issues related to the interpretation of the assignment text or the correctness of the response provided;
- *feedback about processing the task*, relating to the processes necessary to understand and effectively face the task;
- *feedback about self-regulation*, focused on the individual's ability to self-monitor and consciously direct his/her own actions;
- *feedback about the self as a person*, covering issues relating to the assessment of the individual and including emotional aspects.

Effective feedback should allow the student to answer three fundamental questions:

- Where am I going? (or, what are the goals of the activity?);
- How am I doing? (or, what progress have I made towards my goal?);
- Where to next? (or what is the next step to take? what should I do to improve/progress?)

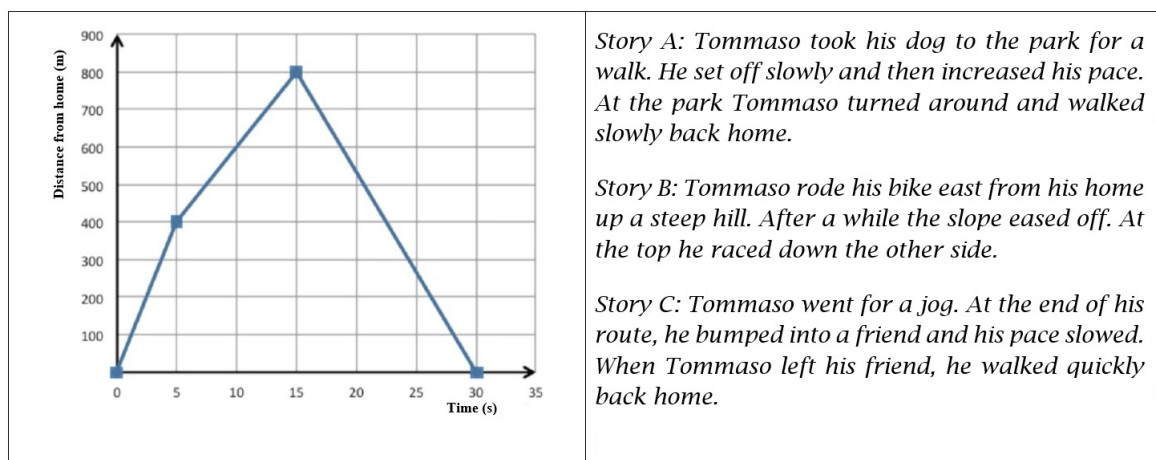
In [31] research led to identify typical strategies employed by the teacher to provide feedback, that is the FA strategy C. in subsection 1.4.1. Five strategies particularly emerged:

- *revoicing*: the teacher repeats a particular intervention by a student in order to focus the attention of the class on it. Often the teacher also uses intonation to emphasize certain key words in the sentences s/he repeats.
- *rephrasing*: the teacher reformulates a student's intervention with the dual objective of focusing the attention of the class on it and to ensure that it is better clarified and made explicit. The teacher activates this strategy when she believes that a useful intervention is not clear enough.
- *rephrasing with scaffolding*: the teacher, in addition to reformulating a pupil's intervention, adds elements that can support the students in tackling the problem under examination.
- *relaunching*: the teacher reacts to a student's intervention by asking a related question. In this way, the teacher provides implicit feedback on the student's intervention, suggesting that it may be an interesting observation, or highlights the problematic aspects, highlighting the need to correct it.
- *contrasting*: the teacher concentrates on two or more interventions that constitute contrasting positions with reference to the same topic under discussion.

The first three strategies show the class that the student's intervention is important, therefore they constitute implicit feedback; at the same time, they can activate strategy D (activating students as resources for each other), because the student becomes a resource for others. The relaunching and comparison further activate strategy D (activating students as resources for each other) and E (activating the student's responsibility for their own learning). When a student has to defend his/her own opinion and compare it with that of his/her peers, s/he becomes more aware of what and how s/he is learning.

In [31] the authors point out that during class discussion the teacher's feedback can evolve: first it refers generically to the task but, when the teacher focuses his/her attention on characteristics of a mathematical argument, it becomes feedback on the performance of the task. To clarify the conclusions reached by the study, let us analyze a specific case.

The teacher, using connected classroom technology, sends students a problem worksheet where a graph and three possible stories are presented:



The question is: *What is the story that this graph represents? Justify your answer.*

Students work in pairs and send their written answers to the teacher's computer, as soon as

they feel ready.

The students' choice of option B allows to detect the typical error that is made in the interpretation of distance-time graphs, that is interpreting the graph as a "drawing of a hill". However, it should be emphasized that the fact that a hill is introduced is not the main reason why story B is not the one represented by the graph. The main reason why story B should be discarded is that it implies that the distance from the origin (the house) increases over time, while the third section of the graph represents a return to the origin.

A student explains this answer by saying:

Tommaso climbs the hill, the first piece is a steep ascent, the second is always uphill, but less steep. When he gets to the top, Tommaso goes down and comes home.

At this point another student says:

This is a graph, it is not the drawing of the hill, thus giving a feedback about the task to other students. He activates himself as instructional resource for his classmates (strategy D), providing feedback about the task (strategy C).

Then the teacher *revoices* and *relaunches*. S/he poses another question linked to a previous intervention to deepen students' understanding:

It is not the drawing of the hill; what does the graph represent?

At the same time, s/he gives implicit feedback to the author of the intervention suggesting that it is useful. The second student continues his reasoning and says:

The journey of one boy.... and anyway they told that it is not possible to do 1600 meters in half an hour, we already said it last time [he refers to the lesson with motion sensors], it is a graph, it doesn't have to be really real... really near to reality" .

This intervention, which may seem confusing, is actually interesting. First, the student says that an ascent of 1600 m cannot be walked in half an hour, providing an empirical justification, which is not adequate for solving the problem. Then he corrects himself and says that we need to take into account the structure of the graph that models the situation. He gives feedback (strategy C) about the processing of the task, pointing out that the justification must not rely on empirical arguments but on the interpretation of the task. The teacher's relaunching is efficient in turning student's former intervention, which provided feedback about the task, into meaningful feedback about the processing of the task.

This example highlights the crucial role played by the teacher in supporting formative assessment processes within class discussions. The teacher activates these strategies *intentionally*. Since the feedback provided by the teacher is sometimes implicit, it is not always understood by the students and the teacher must use different strategies to have an effect.

Therefore, it is important that teachers are trained in formative assessment practices.

2.1.2 The Project Comenius "FAMT&L – Formative Assessment in Mathematics for Teaching and Learning"

We consider a second case study, a training project for teachers of pupils aged 11-16, to promote the proper use of formative assessment in mathematics.

In order for formative assessment to be a facilitator for learning mathematics, it is crucial that the teachers are aware of the potential of the tool they are using and know how to put it into practice. Since the practice and culture of evaluation in schools is still often anchored to traditionalist models, the need for *in-service* teacher training has been highlighted.

The European project LLP Multilateral Project Comenius (2013-2016) - FAMT&L (Formative

Assessment in Mathematics for Teaching and Learning) was born with the aim of promoting good formative assessment practices in mathematics teaching for students aged 11-16. The five member partners are *Alma Mater Studiorum*, University of Bologna in Italy, the University of Applied Sciences and Southern Switzerland - SUPSI, the University of Cergy-Pontoise in France, the University of Cyprus and InHolland University in the Netherlands. The FAMT&L project has been carried out through several phases:

1. analysis of teacher training needs
2. design and implementation of training arrangements
3. intervention and verification of results.

Initially, investigations were conducted into the beliefs and daily actions of teachers relating to the practices they considered formative. At this stage, questionnaires were drawn up for a group of teachers, who voluntarily undertook a training course, and questionnaires for students (about 450) of various secondary schools in the Emilia Romagna region. Alongside the survey, the participants in the project had some of their classes recorded in videos.

The aim was to analytically understand the beliefs and classroom practices of mathematics teachers and to identify their training needs.

The sample of students and teachers is non-significant, so the initial survey does not allow for any type of generalization, but it has nevertheless highlighted that, although teachers attach great importance to formative assessment, it is used less than expected. This is confirmed both by the students' answers [40] and by the analysis of the videos of the lessons [72]. Consequently, in order to improve the skills of mathematics teachers in the use of formative assessment and to create a training model based on effective methodologies to develop a culture of formative assessment, the following have been designed and implemented:

- a specific training curriculum (implemented in each of the five countries involved);
- materials and tools for training (specifically a web repository for video analysis in the classroom).

For the design of these courses, an attempt was made to answer the following research questions [41]:

- What does it mean to support teachers in a training process oriented towards authentic formative assessment practices in the classroom?
- What are the most effective training strategies? What strategies enable real change in the teachers' beliefs and practices?
- What training tools support the teachers' reflective thinking?

According to Vannini [72], in order to change his/her teaching practices, a teacher must find authoritative feedback within educational or formative experiences, able to undermine their beliefs and propose new coherent structures of attitudes. Often, the theory acquires meaning when it finds a connection with the personal and professional history of practicing teachers. To adopt new beliefs and abandon old ones, the real advantage of the new practices has to be

evident. Therefore the authors decided to use video analysis for the training and change of practices.

In [42] video analysis strategies are considered particularly useful for the construction of new representations. Through the video analysis, the teacher in training can observe himself and the class situation asynchronously. The beliefs present in teachers' minds can be compared with the empirical data (the recorded traces) and restructure or evolve. Furthermore, the group comparison with colleagues leads each teacher to play the role of observed and observer, looking at the practices of others but also looking at himself from the outside. Describing these behaviors, seeing their aspects in detail, trying to rethink them in an improved way are very valuable training opportunities for teachers.

During the pilot course, teachers reflected and shared beliefs and opinions in small and large groups at first, then were led to identify and recognize, among others, formative assessment situations in the classroom. Specifically, three types of teacher behavior have been highlighted:

- how the teacher engages students, with the use of questions that create opportunities for student to intervene and answer,
- how s/he uses modeling behavior,
- how s/he communicates the learning feedback.

Throughout the process, individual and group analysis and critical reflection forms were filled in, and each trainee constantly compiled a personal diary.

The periodic analysis of these observations allowed the authors to develop and test an internationally shared observation and video analysis tool - a structured grid - to analyze evaluation practices in mathematics.

The indicators in the grid refer to different dimensions, each inherent in a specific phase of the formative assessment:

- sharing evaluation criteria,
- administering different types of assessment tests,
- correcting the tests in the classroom and using error in a formatively way,
- communicating feedback.

In the next phase, systematic observation studies were carried out on a larger sample and, with the videos collected and analyzed, web repositories were created. This is in fact the key tool of the whole project.

But how is it possible to assess the effectiveness of this training course?

What are the expected changes in teachers and how can they be highlighted?

Changes can be expected at the level of knowledge, of specific skills, but also at the level of metacognitive elements such as beliefs, motivations and the sense of self-efficacy, which can often heavily affect the success of a training path.

To evaluate the effect of the training intervention, researchers decided to consider variables related to the perceptions, beliefs and attitudes of the teachers involved; they chose to monitor how gradually, during the training course, teachers felt they were mastering knowledge and skills on evaluation, how confident they felt in recognizing and analyzing formative evaluation

situations in the classroom and in arguing the effectiveness of this evaluation function. In terms of meta-competence, educational convictions and sense of personal self-effectiveness were also analyzed. The fairly short duration (six weeks) and the lack of institutional agreement to verify the acquired knowledge with an examination test or to observe the practices in class did not allow researchers to make a different choice. As internal validation, however, two measurements were prepared:

- an end-of-course focus group, useful for exploring, in more qualitative terms and with greater depth, the perceptions of competence of trainee teachers;
- a collection of self-documentation (video or audio-recorded) by teachers on the formative assessment practices they carried out in their own classes at the end of the training course.

With these premises, the research shows that the training intervention, specifically characterized by the use of video analysis, had a positive effect on each of the variables considered as (albeit partial) indicators of competence.

In the learning perceptions questionnaire, participants were asked to rate their perception of competence on each of the following indicators from 1 to 5 (from low to high):

- knowledge of the theoretical framework on formative assessment;
- explanation of why formative assessment is a highly effective intervention;
- ability to analyze a classroom situation about assessment with predetermined indicators;
- knowledge of how to organize a classroom situation in which formative assessment is integrated in lesson planning.

The comparison between the answers provided at the beginning and at the end of the course, indicates a significant increase in the sense of self-efficacy in almost all the dimensions considered. These results, although entirely exploratory and linked only to self-perceptions of competence, are nonetheless pertinent, since the most relevant objective to be pursued in a training course is to expect a change in practice.

After a month from the end of the course, each trainee presented to the group a portfolio of a formative assessment experiment carried out in their own class. The methodological clarity and consistency with the theoretical indications of those presentations suggest that there was an effective, positive evolution of the evaluation practices of these teachers.[42]

Chapter 3

Dialogic and reciprocal teaching for effective learning

As Hattie [46] states, for teaching to be effective, learning must be visible. Using formative assessment is one way to make learning visible.

In examples provided in subsections 2.1.1 and 2.1.2 we saw that formative assessment is linked to students' metacognition and is based on dialogue as an essential tool for learning. By comparing and listening to students' answers, the teacher can obtain information and adapt his/her teaching to the students' needs and to understand the effect s/he has on his/her students.

Students receive guidance and suggestions from teachers, but at the same time, students' interventions become a resource for their peers. As students become more proficient, the role of the teacher becomes more nuanced, limited to observing and intervening when necessary. This gradual shift of responsibility is one of the characteristics of reciprocal teaching described by Palincsar and Brown in [62].

Reciprocal teaching emerged as a tool to improve reading comprehension. According to this model, the teacher reads in an initial phase a portion of the text, demonstrating how to use concrete tools such as formulating questions or generating summaries. Then, the teacher asks individual students to do the same, based on their current abilities. Students may rework or comment on another student's intervention, suggest additional questions, request clarifications on materials they haven't understood, or help others understand misunderstandings. Throughout this process, the teacher supports students by providing appropriate feedback tailored to each student's level.

In [66] Rosenshine and Meister describe reciprocal teaching in this way:

... the practice becomes a dialogue: one student asks questions, another answers, and a third comments on the answer; one student summarizes and another comments on or helps to improve the summary; one student identifies a difficult word and the other students help to infer the meaning and give reasons for the inferences they made. The emphasis throughout is on cooperative effort by teacher and students to bring meaning to the ideas in the text, rather than merely restating the words. In addition, during the dialogue, students are provided instruction in why, when, and where such activities should be applied to new text.

Hattie in [46] notes that, in this way, the *dialogic class* is activated, described by R. Alexander

[14] as one whose essential characteristics are:

- collectivity, because the learning tasks are performed together;
- reciprocity, because we listen to each other and share ideas;
- support, because you are not afraid of negative repercussions in case of misconceptions;
- cumulation, because the development of the final idea accounts for various contributions;
- purpose, because teachers plan activities with learning objectives and success criteria in mind.

J. Bennet *et al.* in [16] analyze the role of small group student discussions in teaching experimental science. To this end they considered several factors:

- focus of the discussion topic;
- group structures and interactions;
- negotiation and agreement on meaning through discussion;
- effects on understanding;
- factors promoting effective discussion to enhance understanding.

The authors point out that the common point of effective discussion is not related to a particular topic, but to the fact that any discussion requires students to start from evidence to support a particular hypothesis, theory or point of view.

Positive leadership is also necessary: at least one member of the group must act to promote discussion in order to reach common understanding. Sometimes, in fact, there may be factors that negatively influence understanding: for example, it may happen that students are more inclined to discuss how to carry out the task rather than the content, or that students who are better at supporting their beliefs induce the group to reach a common idea more because of social reasons than because of the content itself.

Tasks that involve an internal conflict (dissimilarity between the members of the group or in understanding or in views) or external (the teacher proposes activities in which a group presents itself with conflicting opinions), give better results in terms of understanding. Moreover, the discussion is more fruitful when the data to discuss are provided by the teacher, who also selects a limited set of concepts to delimit the content of the discussion. Understanding increases if students are directed in constructing arguments.

The above highlights that it is useful for teachers and students to be guided and supported to make small group discussions effective.

Considering that, as stated in [46], classroom discussion and reciprocal teaching are two of the most impactful factors for teaching effectiveness, we clarify their meaning by analyzing peer instruction, which incorporates both dialogic and reciprocal teaching.

3.1 Peer instruction

The term *Peer Instruction* (PI) was introduced by Eric Mazur in 1991. Mazur devised a simple method to engage his physics students at Harvard University in the teaching process. In [56], Mazur describes his method to transform conventional lectures into more interactive sessions. After a brief lecture, students are asked to individually respond to a conceptual question and vote using a flashcard or a personal response system commonly known as a “clicker”. If the majority of students answer incorrectly, the teacher asks them to persuade their neighbors that they have the correct answer. After peer discussion, students are asked to vote again. Finally, the teacher explains both the correct and incorrect answers. Schematically, the following phases can be distinguished:

1. Question posed
2. Students given time to think.
3. Students record individual answers.
4. Students convince their neighbors (peer discussion)
5. Students record revised answers
6. Feedback to teacher: tally of answers
7. Instructor’s explanation of correct answer

In [73] Vickrey et al. analyze various studies on the use of PI that evaluated the effect of this pedagogical practice in various STEM disciplines, especially at the university level. From this review, it emerged that, in general, the use of PI improves students’ abilities to solve complex problems and apply acquired knowledge to solve new problems. Additionally, it is generally appreciated by both students and teachers. In particular, in [73], the authors highlight the favorable aspects and limitations of the various steps of the model proposed by Mazur.

The first aspect to consider is the quality of the question (step 1). Choosing a good question is fundamental to achieve positive effects. To facilitate conceptual understanding, it is important that the question has an explicit pedagogical purpose. It has also emerged that the benefits of peer discussion are more evident when the questions are conceptual rather than recall or algorithmic, and that improvements are more apparent with difficult questions.

Another aspect considered is individual reflection (steps 2 and 3). In some studies, it has emerged that not all teachers use Mazur’s model in all its parts, but in some cases, they move to peer discussion immediately after posing the question, skipping steps 2 and 3. The studies by Nicol and Boyle [60] and Nielsen *et al.* [61] have highlighted that individual reflection of students before peer discussion enhances its quality because students are forced to identify an answer and reflect to argue during the subsequent discussion. On the contrary, starting directly with discussion leads less confident students to passively accept others’ answers. Therefore, steps 2 and 3 should not be skipped.

Regarding step 3, no significant differences were highlighted using clickers or low-tech voting systems like colored cards. However, the use of clickers allows for the immediate display the distribution of the answers as a bar graph. Some studies have shown that viewing the graph after the initial vote prompts students to change their answer to the most common one. This bias is more pronounced for difficult questions and could influence the learning gain between the

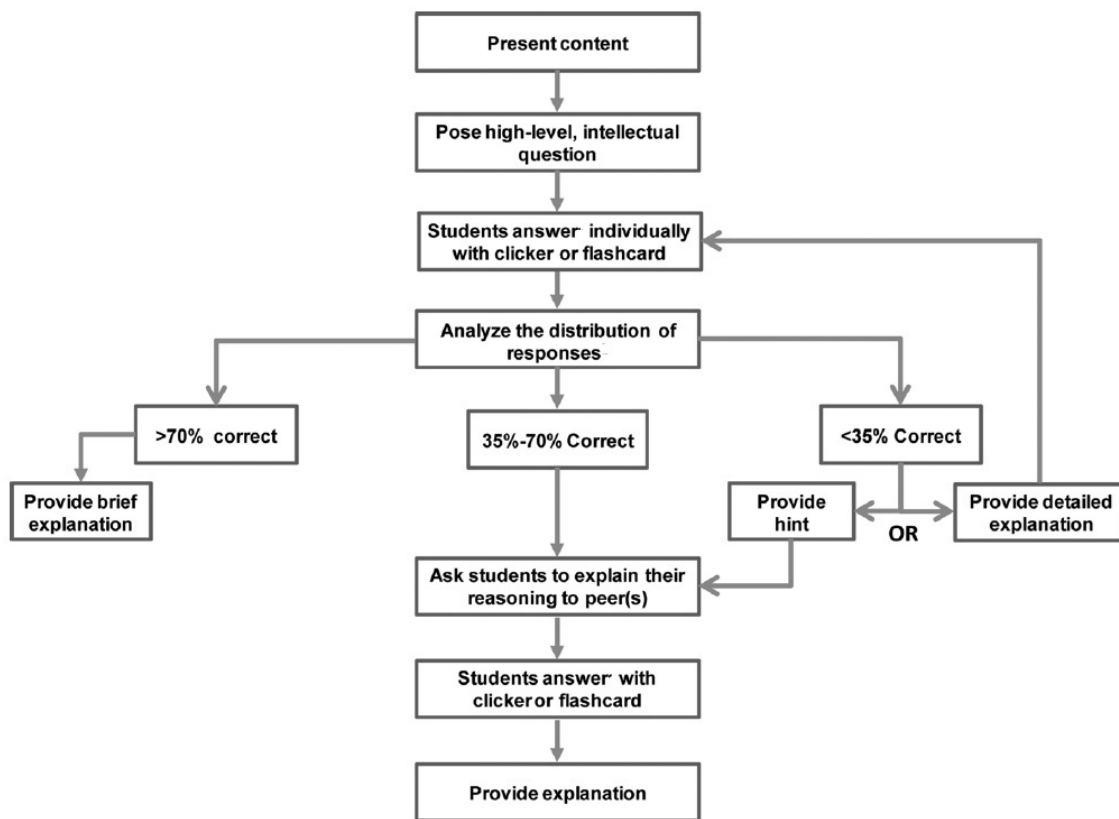
first and second votes. Vickrey *et al.* argue that further research is needed to fully evaluate the effect of displaying the response graph after the initial vote.

A crucial step of peer instruction is peer discussion (steps 4 and 5). Some studies have focused on when to involve students in the discussion. Crunch and Mazur [30] assert that the greatest learning gain occurs when the percentage of correct responses after the first vote ranges from 35% to 70%. Below 35%, it is possible that the concept requires further explanation or hint. Above 70%, the teacher should directly move on to commenting on the response without discussion. Although peer discussion generally has a positive impact on student learning, caution must be exercised not to overestimate students' understanding. Peer interactions, as well as the quality of discussions, are not easily defined, and some students may change and provide correct answers even if they have not fully understood the underlying concepts. Therefore, evaluating the impact of PI solely based on the increase in correct answers after the revote could be misleading.

The last aspect to consider is the role of collecting responses and the critical reflection of teachers on correct or incorrect answers (steps 6 and 7). Although the main feature of PI is peer discussion, the collection of responses and the critical reflection of teachers on correct or incorrect answers (steps 6 and 7) play a very important role in improving student learning. When teachers explain the answers at the end of the cycle, emphasizing their significance or providing prompts to set up the discussion, greater learning gains are achieved.

Based on these considerations, a revised model for peer instruction is proposed in [73]. This model could, on one hand, guide teachers in the use of this practice and, on the other hand, inform researchers in the design of protocols that measure the fidelity of teachers in implementing PI.

In the figure, the phases of this revised model are schematically depicted.



As seen in the diagram, the role of the teacher is crucial for peer instruction as well as for formative assessment.

To conclude the discussion on effective teaching, below we describe a project created by the Science Education Group at the University of York (UYSEG) for science teaching, where, in our opinion, peer instruction is implemented along with formative assessment, all while keeping educational content at the forefront.

3.1.1 An example from experimental sciences: the BEST Project

Best Evidence Science Teaching (BEST) [2] is a large online collection of free resources for biology, chemistry, physics and Earth science for secondary school. The resources are designed to help teachers test and consolidate students' understanding of key concepts in science to facilitate the progression of knowledge.

The resources were developed by the University of York Science Educational Group (UYSEG), directed by Judith Bennet. The project started in 2016 with the preparation and publication of resources for the 11-14 age group, then expanded to the 14-16 age group. Resources for the 7-11 age group is being planned.

The BEST project consists in the design of specific resources, with the aim of translating the general principles of educational research into everyday practice. The BEST resources provide, for each concept, sequenced steps for learning progression, diagnostic questions and response activities.

The sequenced steps suggest how key concepts can be introduced and sequenced to build under-

standing of the big ideas of science. Teachers can use these sequences to plan the curriculum, or they can use BEST resources by incorporating them into their own exiting planning. In particular, 15 big ideas are selected, divided into three groups: biology, chemistry and Earth science, and physics (5 for each discipline).

Biology	Chemistry & Earth Science	Physics
○ The cellular basis of life ○ Heredity and life cycles ○ Organisms & their environments ○ Variation, adaptation & evolution ○ Health and disease	○ Substances and properties ○ Particles and structure ○ Chemical reactions ○ Earth's atmosphere ○ Dynamic earth	○ Matter ○ Forces and motion ○ Sound, light and waves ○ Electricity and magnetism ○ Earth in space

In turn, big ideas are divided into key concepts, organized into teaching topics. On the site, for each group there are maps that illustrate the key concepts and present possible routes for progression through a five-year curriculum 11-16. An example of a physics map is shown below.

PHYSICS (AGE 11-14)				
BIG IDEA PMA: MATTER	BIG IDEA PFM: FORCES AND MOTION	BIG IDEA PSL: SOUND, LIGHT AND WAVES	BIG IDEA PEM: ELECTRICITY AND MAGNETISM	BIG IDEA PES: EARTH IN SPACE
Objects are made of particles with mass. Understanding particles helps us to design our world.	Forces make things change. Understanding forces helps us to predict and control physical change.	Waves radiate information. Understanding waves helps us to communicate.	The everyday world is largely a consequence of electrical charge. Understanding electricity and magnetism helps us develop technology to improve lives.	Understanding the uniqueness of the Earth and the vastness of space gives us perspective and awe.
	Topic PFM1 Forces Key concepts: PFM1.1 What forces do PFM1.2 Describing forces PFM1.3 Balanced and unbalanced forces PFM1.4 Friction PFM1.5 Energy stores and transfers	Topic PSL1 Sound and light Key concepts: PSL1.1 Production and transmission of sound PSL1.2 Characteristics of light		Topic PES1 Solar system and beyond Key concepts: PES1.1 Planets and the solar system PES1.2 Gravity PES1.3 The night sky, stars and galaxies
Topic PMA1 Heating and cooling Key concepts: PMA1.1 Temperature PMA1.2 Heating and cooling PMA1.3 Thermal conduction PMA1.4 Thermal store of energy		Topic PSL2 How we see Key concepts: PSL2.1 The 'passive eye' model of vision PSL2.2 Seeing in colour		
	Topic PFM2 Moving by force Key concepts: PFM2.1 Describing speed PFM2.2 Motion graphs PFM2.3 Changing motion PFM2.4 Drag	Topic PSL3 Making images Key concepts: PSL3.1 The ray model of light to explain images PSL3.2 Refraction and lenses		Topic PES2 Earth and sun Key concepts: PES2.1 Days and seasons
	Topic PFM3 More about force Key concepts: PFM3.1 Mass and weight PFM3.2 Hidden forces PFM3.3 Turning effects		Topic PEM1 Simple electric circuits Key concepts: PEM1.1 Making circuits PEM1.2 Electric current PEM1.3 Voltage PEM1.4 Static electricity	
		Topic PSL4 Waves Key concepts: PSL4.1 Waves on water and ropes PSL4.2 A wave model of sound	Topic PEM2 More electric circuits Key concepts: PEM2.1 Resistance PEM2.2 Parallel circuits	
Topic PMA2 Floating and sinking Key concepts: PMA2.1 Floating, sinking and density PMA2.2 Pressure in fluids PMA2.3 Convection			Topic PEM3 Magnets and electromagnets Key concepts: PEM3.1 Magnetic fields PEM3.2 Electromagnets	

A similar map for the 14-16 age group, with the same *big ideas*, is also given.

As can be seen from the map, each big idea is then organized into topics and key concepts are identified within each of these.

For example, for the big idea *Matter*, the series of key concepts have been organized into topics as follows.

11-14:

Topic PMA1 Heating and cooling Key concepts: 1.1 Temperature 1.2 Heating and cooling 1.3 Thermal conduction 1.4 Thermal store of energy	Topic PMA2 Floating and sinking Key concepts: 2.1 Floating, sinking and density 2.2 Pressure in fluids 2.3 Convection
---	--

14-16:

Topic PMA3 Energy of moving particles Key concepts: 3.1 Transfer of energy by conduction 3.2 Specific heat capacity 3.3 Specific latent heat	Topic PMA4 Particle explanations Key concepts: 4.1 Density 4.2 Pressure	Topic PMA5 Nuclear physics Key concepts: 5.1 Atomic nuclei 5.2 Radioactive decay 5.3 Ionizing radiation 5.4 Radioactive half-life
---	---	---

For each key concept, the focus is on what needs to be learned, through three types of resources:

- a toolkit with appropriately sequenced phases for learning progression;
- diagnostic questions to reveal common preconceptions and misunderstandings;
- responsive activities to challenge misunderstandings and encourage conceptual development.

As an example, below is the toolkit relating to the key concept *Temperature* within the topic *Heating and cooling*:

Key concept (age 11-14) PMA1.1: Temperature

Progression toolkit: Temperature

Learning focus	Temperature is a measure of the average speed at which the particles in a substance or material are moving.				
As students' conceptual understanding progresses they can:	CONCEPTUAL PROGRESSION				
	Predict the temperature of different materials that are all in thermal equilibrium with the room. P	Predict the temperature reached by mixing samples of water that are all at the same temperature. P	Describe the arrangement and movement of particles in a substance in the solid and liquid states.	Describe the changes in particles of a substance or material when its temperature is changed.	Explain the changes in volume of solids and liquids when their temperature is changed.
Diagnostic questions	Three balls	More water	Particle model SL	Pie tin particles	A cup of tea
Response activities	Water and sand		Particle diagram of a liquid	Ouch!	
				Expansion model	

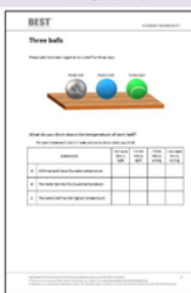
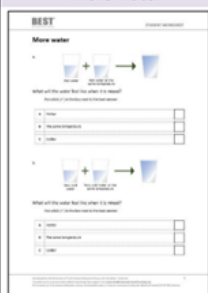
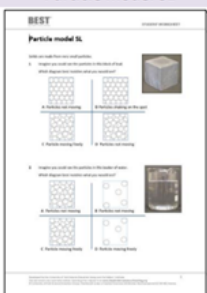
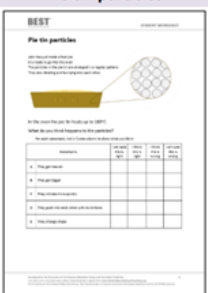
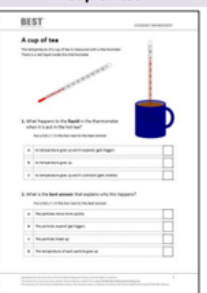
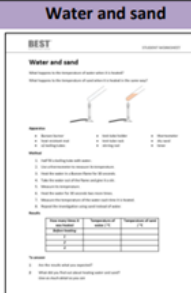
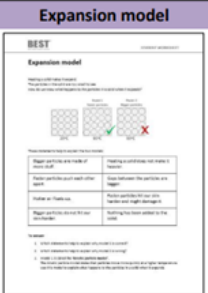
Key:

P

Prior understanding from earlier stages of learning

B

Bridge to later stages of learning

Three balls  Confidence grid	More water  Simple multiple choice	Particle model SL  Simple multiple choice	Pie tin particles  Confidence grid	A cup of tea  Two-tier multiple choice
Water and sand  Application - practical	Particle diagram of a liquid  Critique a representation	Ouch!  Talking heads	Expansion model  Critique a representation	

In the progression toolkit, the following are also highlighted:

- concepts that must constitute a prerequisite for achieving understanding of the key concept (indicated with P);
- the bridge concepts that lead to the subsequent phases of learning (indicated with B).

As can be seen, diagnostic sheets and response activities are provided.

Diagnostic questions aim to highlight what students know and understand, but, at the same time, they can reveal the presence of preconceptions and misconceptions that can hinder learning. The questions are delivered as simple worksheets, and therefore then are accessible to all, in different formats (simple and two-tier multiple-choice questions; talking heads; confidence grids; explanation stories) that are well suited to cooperative learning in class.

They are a source of formative assessment because the student does not just have to give an answer to show whether or not s/he knows the topic. S/he is also invited to indicate how much s/he trusts the answer s/he has chosen and to give a scientific explanation of his/her choice. In this way a good question also provides indications on possible common misunderstandings, or even on problematic conceptions derived from everyday life, which are supported by the student and which could prevent progression in the development of the required level of scientific understanding.

Worksheets are typically short, to-the-point questions, focused on a single idea, and quick to complete. They can be completed individually with pen and paper or in small groups, or they can be projected to the whole class on the IWB. Answers can be given by show of hands or with a voting system. The important aspect is sharing the results and the attention that the teacher must give to the answers and their motivations, encouraging discussion and metacognition.

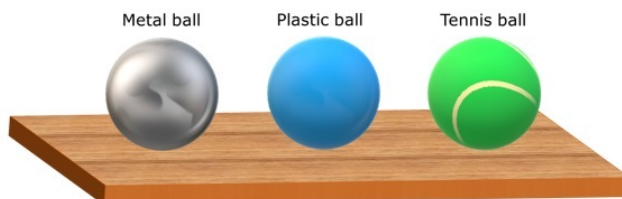
In preparing the diagnostic questions, possible misunderstandings of the students are kept in mind; consequently, even if there is one correct answer, the others must not be so obviously

wrong as to be easily discarded. In this way the questions can actually be diagnostic, because they are revealing.

Here is an example of diagnostic question worksheet.

Three balls

These balls have been together on a shelf for three days.



What do you think about the temperature of each ball?

I am sure this is right	I think this is right	I think this is wrong	I am sure this is wrong
---	---------------------------------------	---------------------------------------	---

A All three balls have the same temperature

--	--	--	--

B The metal ball has the lowest temperature

--	--	--	--

C The tennis ball has the highest temperature

--	--	--	--

BEST
Best Evidence Science Teaching

Developed by the University of York Science Education Group and the Salters' Institute.
This question or activity may have been edited. Download the original from www.BestEvidenceScienceTeaching.org
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Each student worksheet is accompanied by teacher notes. There are references to research conducted on common and everyday misconceptions that can hinder scientific learning on which the diagnostic questions were built. Teachers are also advised on how to use the questions in a formative way: the teacher notes suggest what the answers might reveal about students' misunderstandings and how to proceed.

An example of teacher notes is shown below.

Diagnostic question

Three balls

Overview

Learning focus:	Temperature is a measure of the average speed at which the particles in a substance or material are moving
Observable learning outcome:	Predict the temperature of different materials that are all in thermal equilibrium with the room
Question type:	Confidence grid
Key words:	Temperature

P

PRIOR UNDERSTANDING

This diagnostic question probes understanding of ideas that are usually taught at age 5-11, to aid transition from earlier stages of learning.

What does the research say?

Students aged 11-12 are often able to use and read a thermometer to take temperature readings, but they often make judgements about the temperature of an object based more on the materials it is made from rather than on the temperature of its surroundings. When a piece of metal and a piece of wood are picked up, both at room temperature, the metal feels colder. This gives some students the belief that they are at different temperatures. It can also lead to the misunderstanding that some materials can be heated and others cannot, which is perhaps reinforced by ideas about thermal conductors and thermal insulators. Most teaching schemes take these ideas for granted. (Erickson and Tiberghien, 1985)

Ways to use this question

Students should complete the confidence grid individually. This could be a pencil and paper exercise, or you could use an electronic 'voting system' or mini white boards and the PowerPoint presentation.

If there is a range of answers, you may choose to respond through structured class discussion. Ask one student to explain why they gave the answer they did; ask another student to explain why they agree with them; ask another to explain why they disagree, and so on. This sort of discussion gives students the opportunity to explore their thinking and for you to really understand their learning needs.

Differentiation

You may choose to read the questions to the class, so that everyone can focus on the science. In some situations it may be more appropriate for a teaching assistant to read for one or two students.

Equipment

For the class:

- A metal ball, a plastic ball and a tennis ball and an infra-red thermometer (Optional)

Expected answers

A: All three balls have the same temperature is correct and the other answers are wrong.

How to respond - what next?

The balls are each at the same temperature as their surroundings because they have had sufficient time to reach a thermal equilibrium.

The metal ball feels cold to touch and some students may attribute this to it being at a lower temperature. The reason it feels colder is a challenging idea that is considered in the BEST key concept: PMA1.2 Heating and cooling. (Energy is conducted away from the thermal store of the hand quickly by the metal.)

Students who do not think that all the balls are each at the same temperature may be unsure as to whether the tennis ball or the plastic ball has the highest temperature. This may be because they are not sure which one feels the warmest.

If students have misunderstandings about whether the balls are each at the same temperature, it can help to demonstrate that balls of different materials that are in thermal equilibrium all have the same temperature, by measuring their temperatures with an infra-red thermometer. The following BEST 'response activity' is a student practical that could be used in follow-up to this diagnostic question:

- Response activity: Water and sand

Acknowledgments

Developed by Peter Fairhurst (UYSEG).

Images: Peter Fairhurst (UYSEG).

References

Erickson, G. and Tiberghien, A. (1985). Heat and Temperature. In Driver, R., Guesne, E. & Tiberghien, A. (eds.) *Children's Ideas In Science*. Milton Keynes and Philadelphia: Open University Press.

Most diagnostic questions are associated with a response activity that aim to clarify misunderstandings or enhance students' comprehension of key concepts.

Similar to the diagnostic questions, the response activities in the BEST collection are available in various formats. They can take the form of structured hands-on activities or involve group

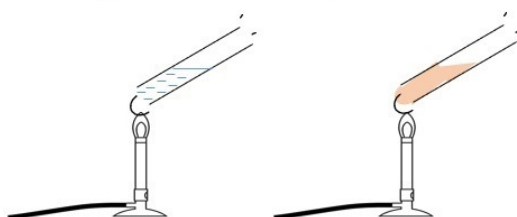
discussions, either in pairs or small groups. The shared goal among these activities is that they conclude with a group discussion, enabling students to compare and articulate their thoughts, thus fostering a deeper scientific understanding.

As an example, following the *Three balls* diagnostic question, it may be essential to strengthen the understanding with the subsequent response activity.

Water and sand

What happens to the temperature of water when it is heated?

What happens to the temperature of sand when it is heated in the same way?



Apparatus

Bunsen burner
heat resistant mat
x2 boiling tubes
test-tube holder
test-tube rack
stirring rod
thermometer
dry sand
timer

Method

1. Half fill a boiling tube with water.
2. Use a thermometer to measure its temperature.
3. Heat the water in a Bunsen flame for 30 seconds.
4. Take the water out of the flame and give it a stir.
5. Measure its temperature.
6. Heat the water for 30 seconds two more times.
7. Measure the temperature of the water each time it is heated.

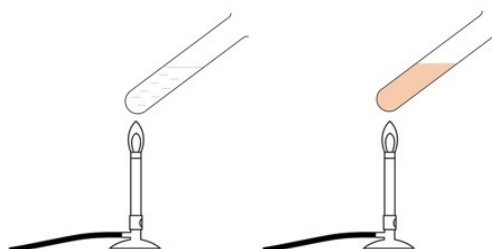
Results

How many times it was heated	Temperature of water / °C	Temperature of sand / °C
<i>Before heating</i>		
1		
2		
3		

Questions

- Are the results what you expected?
- What did you find out about heating water and sand?

Give as much detail as you can



Teacher notes are likewise included for response activities, mirroring the ones offered for diagnostic questions.

Physics > Big idea PMA: Matter > Topic PMA1: Heating and cooling > Key concept PMA1.1: Temperature

Response activity

Water and sand

Overview

Learning focus:	Temperature is a measure of the average speed at which the particles in a substance or material are moving
Observable learning outcome:	Predict the temperature of different materials that are all in thermal equilibrium with the room
Activity type:	Application and practice - practical
Key words:	Temperature, test-tube holder, test-tube rack, stirring rod, boiling tube, Bunsen burner, thermometer

This activity can help develop students' understanding by addressing the sticking-points revealed by the following diagnostic question:

- Diagnostic question: Three balls

P

PRIOR UNDERSTANDING

This activity explores ideas that are usually taught at age 5-11, to aid transition from earlier stages of learning.

What does the research say?

Students aged 11-12 are often able to use and read a thermometer to take temperature readings, but they often make judgements about the temperature of an object based more on the materials it is made from rather than on the temperature of its surroundings. When a piece of metal and a piece of wood are picked up, both at room temperature, the metal feels colder. This gives some students the belief that they are at different temperatures. It can also lead to the misunderstanding that some materials can be heated and others cannot, which is perhaps reinforced by ideas about thermal conductors and thermal insulators. Most teaching schemes take these ideas for granted. (Erickson and Tiberghien, 1985)

Ways to use this activity

This practical activity gives students the opportunity to practise applying their understanding and to clarify their thinking through discussion. To support this, students should complete the practical in pairs or small groups.

Listening to individual groups as they work often highlights any difficulties they might have. These can often be overcome, through a whole class clarification or redirection part way through the activity.

Asking students to report their findings at end of the practical work is a useful check. After a group has fed back, it might be helpful to model an even better answer. You could do this, for example, by asking another group to add to, or clarify, the first observation. Then ask another group to sum up the important part of the observation, and so on.

Differentiation

Providing suitable recording sheets can help some students organise their observations so they can more easily focus on the science. If some students are working with a teaching assistant, then a list of prompt questions for the TA could help to make this activity more purposeful. Some students may benefit from being challenged to plan and organise their own record keeping.

Equipment

For each student/pair/group:

- Bunsen burner
- heat resistant mat
- x2 boiling tubes
- test-tube holder
- test-tube rack
- stirring rod
- thermometer
- dry sand
- timer

Technician notes

Each group will heat half a boiling tube of dry sand.

Thermometers (0-100°C) should be fitted with an anti-roll device (if not integral).

Health and safety

The main risks are burns and broken glassware.

Thermometers easily roll and need to be placed towards the centre of the table.

Boiling tubes are used as they are easy to transfer into a test-tube rack after heating where they can be left to cool. Alternatively the contents can be tipped away and the boiling tube laid on the heat-resistant mat with the test-tube holder still attached.

For each measurement contents should be gently stirred with a stirring rod and not a delicate thermometer bulb.

Practical work should be carried out in accordance with local health and safety requirements, guidance from manufacturers and suppliers, and guidance available from CLEAPSS.

Expected answers

The sand will increase in temperature more quickly than the water, which can be noticed at the beach on a hot day.

Some students are likely to predict that the sand will be resistant to heating. They may think that heating does not raise the temperature of sand because it is made from crushed rock, which is hard and able to withstand heating. Alternatively they may think that because there are no visual changes when sand heats up that it is unchanged in temperature too.

Acknowledgments

Developed by Peter Fairhurst (UYSEG), from an idea by Erickson and Tiberghien (1985)

Images: Peter Fairhurst (UYSEG).

References

Erickson, G. and Tiberghien, A. (1985). Heat and Temperature. In Driver, R., Guesne, E. & Tiberghien, A. (eds.) *Children's Ideas In Science*. Milton Keynes and Philadelphia: Open University Press.

Two important aspects of the BEST project come to light:

- a careful choice of educational content starting from research on learning pathways and on effective sequencing of ideas;

- the extensive use of collaborative structured discussions.

We claim this project matches content knowledge with pedagogy, realizing what Shulman [44] calls Pedagogical Content Knowledge (PCK), a special form of expert teacher knowledge that combines content, pedagogy and student characteristics.

The pedagogical content knowledge is the integration of teachers' pedagogical knowledge (what they know about teaching) and their subject matter knowledge (what they know about what they teach). According to Cochran [28], PCK is what makes the difference between being a teacher and being a scientist:

... an experienced science teacher's knowledge of science is organized from a teaching perspective and is used as a basis for helping students to understand specific concepts. A scientist's knowledge, on the other hand, is organized from a research perspective and is used as a basis for developing new knowledge in the field.

Cochran, DeRuiter, and King [29] revised the initial model by Shulman, also considering two additional components. The first component is teachers' knowledge of students' learning skills and strategies, age and developmental level, abilities, and their prior knowledge. The other component is teachers' understanding of the social, political, cultural, and physical environments in which the classroom operates. PCK goes beyond mere subject matter knowledge and develops over time through teaching experience. When the teacher critically reflects on the subject matter and tailors the material to that specific group of students to whom the information will be taught, this transformation occurs: the teacher, in other words, restructures his/her knowledge for instructional purposes.

We believe that the BEST project helps teachers to facilitate this restructuring. The materials are constructed with consideration of how it is supposed to be learned by students. *Teacher Notes* assist in reflecting on students' prior knowledge and potential age and developmental-related misconceptions. The progression for presenting the content is suggested, but ultimately, it's the teachers who decide which *Diagnostic questions* and *Response activities* to use and how to do so based on their students' characteristics.

A 2021 study [15] investigated how teachers use BEST resources and whether this led to changes in teaching practices.

The case study is based on semi-structured interviews and observations of 12 teachers from the north of England, two of whom worked in the independent sector and the remaining 10 worked in the state-maintained sector.

Some teachers used the diagnostic questions to evaluate the progression of their students' learning during a lesson or a lesson cycle, others used the resources in a formative way, making use of reciprocal and dialogic teaching.

In each case, the teachers involved in the case study incorporated the resources into their own curriculum and said they were able to plan lessons with a greater awareness of what the research says about students' ideas in science.

Furthermore, through the diagnostic questions and response activities, they obtained more evidence about what students understood and, more importantly, why.

The teachers also stated that teacher notes allowed them to feel more in tune with students' responses and to incorporate the students' ideas, making classroom interactions more meaningful by promoting metacognitive discourse.

Learning progressions have also been useful in informing teachers about more or less effective ways of structuring and introducing scientific content.

The authors plan further research on whether the use of resources for a longer period of time can also influence student outcomes.

Part III

Research proposal

Chapter 4

Research project

Our research proposal is inspired by the BEST project of UYSEG; our idea is to create diagnostic worksheets and response activities within the mathematics curriculum for children aged 11-14.

The BEST project in Italy has already influenced educational proposals developed by the Science Group CeSeDi (Centro Servizi Didattici) of the metropolitan city of Turin, led by Professor Marco Falasca. In [43] Group CeSeDi offers insights for constructing a vertical science curriculum from early childhood to secondary education.

The approach centers on reciprocal teaching through well-organized, teacher-guided collaborative groups. Over 20 years, the group has developed cooperative workshops using accessible materials, sharing experiences beyond the group with the wider community. These collaboration involves teachers from various educational levels, on a voluntary basis with no additional payment, exchanging pedagogical content and methods.

Specifically, in [43], the authors present pathways for secondary school that integrate the groups' workshop, features and tools (POE method, intuitive grids, cooperative learning, etc.) with contributions from the University of York Science Education Group (UYSEG), led by Judith Bennet.

The Italian National Guidelines provide a general idea for subject content, but the CeSeDi group uses research from the York group to identify the themes around which to develop a vertical curriculum.

In the same way, following National Guidelines, we selected a topic during the project's first phase, creating sheets with corresponding teacher notes and toolkits for learning progression.

For the selection of topics and the preparation of worksheets, we also considered information obtained from the national INVALSI tests. INVALSI (Istituto Nazionale per la Valutazione del Sistema Educativo di Istruzione e Formazione) is a research institute that conducts periodic and systematic assessments of students' knowledge and skills, as well as the overall quality of the Italian education system. The standardized tests cover three subjects: mathematics, Italian language, and English language. The targeted grades include 2, 5, 8, 10, and 13. In an initial experimental phase, students in grade 6 were also involved.

After building worksheets, we submitted them to teachers and students, formulating three questions:

1. Can these worksheets serve as formative assessment tools, identifying student needs and guiding teachers' future decisions?
2. Are these worksheets effective self-evaluation tools for pupils?
3. Do interventions with the worksheets improve learning of the concepts addressed?

The trial involves students and mathematics teachers from the lower secondary school of the comprehensive institute *F.lli Trillini* in Osimo (Ancona); specifically, second-grade students from the years 2022/2023 and 2023/2024 of the lower secondary school.

Subsequent chapters will detail:

1. worksheets setup
2. stages of testing in class.

4.1 Worksheet setup

The 2012 *Indicazioni Nazionali per il curricolo della scuola dell'infanzia e del primo ciclo di istruzione*[57] categorize the mathematics curriculum into four thematic focal points:

- space and figures
- numbers
- data and predictions
- relationships and functions.

Based on these subdivisions, we decided to organize the topics into subjects, as illustrated in the table below.

THEMATIC FOCAL POINT	SUBJECT	TOPIC
space and figures	plane figures	articulated figures and rigid figures
		corners
		circle
		Cartesian plane
	area and perimeter	area and perimeter of polygon
		area of circle and length of circumference
		Pythagorean theorem
	geometric transformations	symmetry
		translation and rotation
		similarity
	solid figures	polyhedra
		lines and planes in space
		rotation solids
numbers	natural numbers and whole numbers	natural numbers and operations
		multiples and divisors
		powers
		whole numbers and operations
	rational numbers	fractions
		fractions and decimal numbers
		percentages
	real numbers	root extraction
		operations with real numbers
		exponential notation
data and predictions	probability	classical probability
	statistics	statistical data and percentages
		statistical investigation
relations and functions	numbers and letters	literal calculation
		algebraic expressions
		equations and proportions
	relationships and graphs	mathematical laws
		graphs on the cartesian plane
		functions

To begin, we delved into the subject of *areas and perimeters* within the thematic focal point of *space and figures*.

Our focus encompassed three topics: the area and perimeter of polygons, the area of the circle and length of the circumference, and Pythagorean theorem. We identified the key concepts associated with each of these topics.

Similar to the BEST sheets, we employed acronyms to denote subjects, topics and key concepts. The ensuing table illustrates the key concepts identified for the *area and perimeter* subject.

THEMATIC FOCAL POINTS	SUBJECT	TOPIC	KEY CONCEPT
SPACE AND FIGURES	Area and perimeter AP	Areas and perimeters of polygons AP1	Height AP1.1
			Areas of polygons AP1.2
			Variations in areas and perimeters AP1.3
		Area of the circle and length of the circumference AP2	Length of the circumference AP2.1
			Greek π AP2.2
			Area of the circle AP2.3
		Pythagorean theorem AP3	Pythagorean triples AP3.1
			Pythagorean theorem AP3.2

Our project is specifically confined to the topic *areas and perimeters of polygons (AP1)*. As evident from the table we pinpointed three key concepts:

- Height AP1.1
- Areas of polygons AP1.2
- Variations in areas and perimeters AP1.3

For each concept, a learning progression toolkit has been meticulously prepared. Subsequent subsections will delve into a detailed analysis of the key concepts and their corresponding materials.

4.1.1 Key concept: height

Theoretical premises and references

To measure the areas of plane figures, a square serves as the unit of measurement. Specifically, any polygon can be transformed into an equivalent rectangle. The initial step in this transformation is identifying the height in relation to one side of a polygon. Despite the seemingly straightforward nature of height, numerous misconceptions surround it, often stemming from the choices made by teachers. Sbaragli in [67] observes that textbooks commonly define height as a segment that *starts* from a vertex and *falls* perpendicularly onto the opposite side. This definition prompts us to question whether height is truly a segment or a quantity, if it can originate from a vertex and descend onto a side, whether it pertains only to specific figures, and if a polygon has multiple heights or just one. Teachers frequently emphasize that the segment identifying the height terminates at a vertex of the polygon, leading students to exclude segments representing the distance between a vertex and a side that do not share points with the figure's vertices or are partially or wholly outside the polygon. The use of a plumb line to identify height can further confuse verticality with perpendicularity, causing students to overlook a representation of height if the segment is not vertical from the observer's viewpoint.

Considering the outcomes of INVALSI tests, questions involving the concept of height consistently pose the greatest difficulty for students, not only in grade 8 but also for those in grades

10 or 13. This underscores the persistence of misconceptions among a significant number of students [1].

This shows the importance of carefully defining and identifying the heights of a polygon and revisiting this concept multiple times throughout the school curriculum.

Conceptual progression

The conceptual progression begins with recognizing that a polygon has as many heights as sides, and that height can be represented by a segment whose length equals the maximum distance of a point in the polygon to the side designated as the base.

Progression toolkit: height

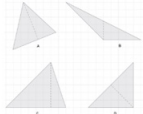
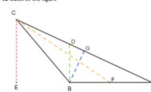
Learning focus	The height of a polygon relative to a side is a segment whose length is the maximum distance between a point on the polygon and the straight line on which the side lies.				
As students' conceptual understanding progresses they can:	CONCEPTUAL PROGRESSION				
	Recognize that there are as many heights as there are sides in a triangle. P	Understand that heights can also be represented by segments not internal to the figure.	Understand that height can be represented by segments perpendicular to the sides or their extensions in the same direction.	Recognize that in a polygon there are as many heights as there are sides and that the heights can be represented by segments perpendicular to the sides or their extensions in the same direction.	Use and distinguish the concepts of perpendicularity, parallelism, horizontality, and verticality. Know the definitions and properties of the main plane figures.
Diagnostic questions	Let's talk about heights!		Triangles and heights	Quadrilaterals and heights	
Response activities			Let's draw heights		Is everything clear?

Key:

P Prior understanding from earlier stages of learning

B Bridge to later stages of learning

To address the key concept height, three diagnostic questions and two response activities have been prepared.

Let's talk about heights! Let's talk about heights!  What do you think about the height of a triangle? Look at these statements, the first statement is true, the others are false. Tick (✓) the option that comes closest to your thinking. <table><thead><tr><th>Statements</th><th>Always true</th><th>Often true</th><th>Sometimes true</th><th>Often false</th><th>Always false</th></tr></thead><tbody><tr><td>A. In a triangle there are three heights.</td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>B. All the heights of a triangle have the same length.</td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>C. A height of a triangle is the perpendicular distance from a vertex to the opposite side.</td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>D. A height of a triangle is the perpendicular distance from a vertex to the line containing the opposite side.</td><td></td><td></td><td></td><td></td><td></td></tr></tbody></table> Confidence grid Let's draw heights Let's draw heights  Material - Quadrilaterals drawn on large sheets (30cm x 20cm) - Ruler (at least 40 cm long) and set square - Stationery (scissors, markers, etc.) Method Form groups of 4 students and place the sheet with the drawing at the center of the group. 1st step Each pupil chooses a side of the quadrilateral and marks it in color, and may also mark the line containing that side. 2nd step Each pupil marks height relative to the side s/he has chosen as a dashed line, in the same color. Observe Each group compares their drawing with those by other groups. Answer - Is the result what you expected? - What can be said in general about the heights of quadrilaterals? - How many heights do you expect to find in a polygon with more than four sides? Application and practice	Statements	Always true	Often true	Sometimes true	Often false	Always false	A. In a triangle there are three heights.						B. All the heights of a triangle have the same length.						C. A height of a triangle is the perpendicular distance from a vertex to the opposite side.						D. A height of a triangle is the perpendicular distance from a vertex to the line containing the opposite side.						Triangles and heights Triangles and heights Observe the triangles.  A. In one of the triangles the dotted segment is NOT a height, in which? Tick (✓) the box next to the correct answer: ☐ A. In triangle A ☐ B. In triangle B ☐ C. In triangle C ☐ D. In triangle D Two-tier multiple choice Is everything clear? Is everything clear? A. Observe the figure.  1A. Which of the following pairs of segments represents two of the heights of triangle ABC? Tick (✓) the box next to the correct answer: ☐ A. CE and CF ☐ B. BD and BG ☐ C. CE and BG ☐ D. CF and BD Two-tier multiple choice
Statements	Always true	Often true	Sometimes true	Often false	Always false																										
A. In a triangle there are three heights.																															
B. All the heights of a triangle have the same length.																															
C. A height of a triangle is the perpendicular distance from a vertex to the opposite side.																															
D. A height of a triangle is the perpendicular distance from a vertex to the line containing the opposite side.																															

Diagnostic question: Let's talk about heights!

The first worksheet, *Let's talk about heights!*, is a diagnostic question designed as an introductory activity aimed at capturing students' prior knowledge through the use of the confidence grid model. Children aged 11-12 have already encountered the concept of height during primary school, and it is important to assess their level of understanding and any potential learning obstacles.

What do you think about the height of a triangle?

Look at these statements.

For each statement, tick (✓) **the** option that comes closest to your thinking.

Statements	I'm sure this is correct	I think this is correct	I think this is wrong	I'm sure this is wrong
A In a triangle there are three heights				
B All the heights of a triangle have the same length				
C A height of a triangle is always internal to the triangle				
D Every height of an acute angled triangle is always internal to the triangle				
E Every height of an obtuse angled triangle is always external to the triangle				

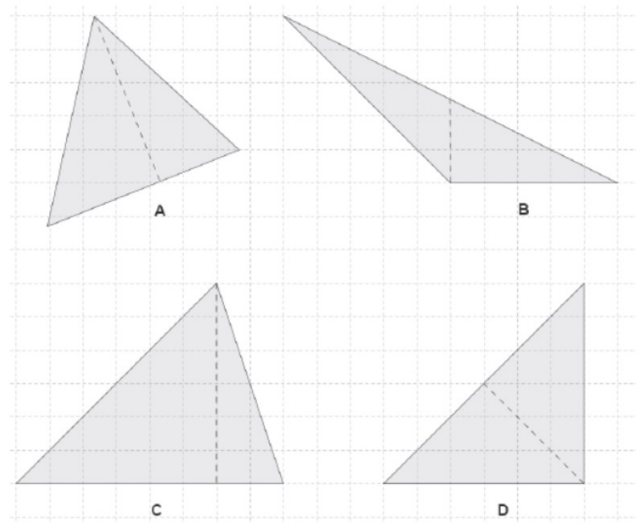
The teacher notes advise using the worksheet by allowing students to reflect and then verbally express their opinions, discussing them first in pairs and then with the entire class. It is important for the teacher not to immediately provide correct answers but to lead the class discussion so that they emerge from the students' contributions. The teacher should highlight significant interventions and ask appropriate questions to help identify correct statements.

Once this preliminary phase, which may take a considerable amount of time, is completed, the students are invited to tackle two diagnostic questions related to the heights of a triangle and a quadrilateral.

Diagnostic question: Triangles and heights

The diagnostic question *Triangles and heights* is inspired by a 2017 grade 5 INVALSI test question [1].

Observe the triangles.



1. In one of the triangles the dotted segment is NOT a height. In which?

Tick (✓) the box next to the correct answer.

- A In triangle A
- B In triangle B
- C In triangle C
- D In triangle D

☐
☐
☐
☐

We have opted for a multiple-choice format and added a question to the original version:

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

- A A height does not divide the base in half
- B A height cannot be oblique
- C A height is not vertical
- D A height is not perpendicular to base

☐
☐
☐
☐

The second question is provided to prompt reflection and help the students articulate their answer. In this age group, not everyone is able to effectively explain the reasoning behind their

choices, and the statements are intended to serve as a starting point to make the students' reasoning explicit.

The options are marked with a color because teachers are advised to collect students' responses using a voting system based on colored cards. The students reflect individually and then show their answer using a colored card indicating the corresponding letter. This system allows the teacher to visualize an overview of the answers and easily determine if there is a variety of responses and if there is consistency. Based on the responses obtained, the teacher does not directly provide the correct answer but initiates a discussion. To encourage everyone's participation, the students are encouraged to share their thoughts with their desk partner, trying to justify their answer. After the discussion in pairs, the class can be asked to vote again to see if there have been any changes; at this point, the teacher can ask some students to justify their answers, initiating a collective discussion.

Diagnostic question: Quadrilaterals and heights

The diagnostic question *Quadrilaterals and heights* follows a similar construction and instructional approach.

Observe the figures

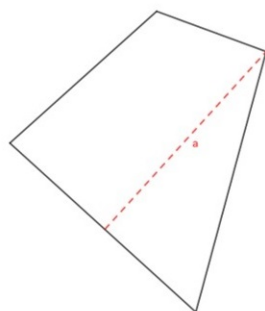


Figure A

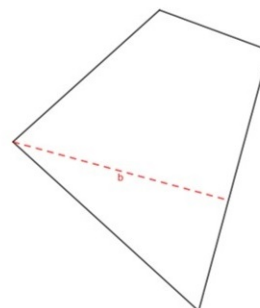


Figure B

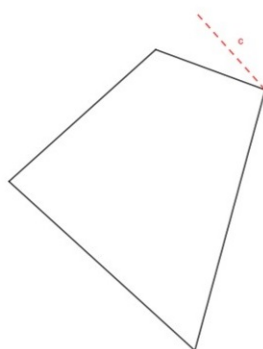


Figure C

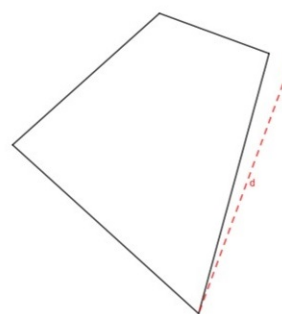


Figure D

1. In which figure is the segment labelled by a letter NOT a height?

Tick (✓) the box next to correct answer.

- | | |
|-------------------|--------------------------|
| A Figure A | ☐ |
| B Figure B | ☐ |
| C Figure C | ☐ |
| D Figure D | ☐ |

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

- | | |
|---|--------------------------|
| A The length of the labelled segment is not the maximum distance | ☐ |
| B The labelled segment is not perpendicular to the side | ☐ |
| C The labelled segment is not inside the polygon | ☐ |

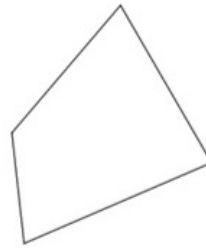
This diagnostic question focuses on the definition of height as a segment whose length is equal to the maximum distance of a point on the polygon from the straight line containing a side. It is possible that students who consider the segments represented in figures A and B incorrect may not recognize that the segments are perpendicular to the side they refer to; therefore, one would expect B as the answer to question 2. Instead, those who choose option D might think that the height must be inside the quadrilateral. The same motivation could also be given by students who identify figure C as the one in which the segment does not represent the height: in this case, however, the associated motivation will not be the correct one (answer A) but option C.

If misunderstandings persist, teachers should activate remedial actions with the means and methods most suited to them (workshop activities, problem solving, traditional lesson etc...).

Response activity: Let's draw heights

The response activity *Let's draw heights* is designed to reinforce the understanding of the concept of height. It is a practical activity to be carried out with ruler and set square in groups of four. Students take turns drawing the height relative to each of the four sides of the quadrilateral. To draw the heights, they must take care in identifying the point (or points) of the figure furthest from the line containing the chosen side, from which the height should be drawn. They must also position the set square correctly so that the height is perpendicular to the reference side.

Let's draw heights



Material

- Quadrilaterals drawn on large sheets (50cm x 70cm)
- Ruler (at least 40 cm long) and set square
- Stationery (pencils, erasers, markers, etc.)

Method

Form groups of 4 students and place the sheet with the drawing at the center of the group.

1st step

Each pupil chooses a side of the quadrilateral and marks it in color, and may also marks the line containing that side.

2nd step

Each pupil marks height relative to the side s/he has chosen as a dashes line, in the same color.

Observe

Each group compares their drawing with those by other groups.

Answer

- 1 Is the result what you expected?
- 2 What can be said in general about the heights of quadrilaterals?
- 3 How many heights do you expect to find in a polygon with more than four sides?

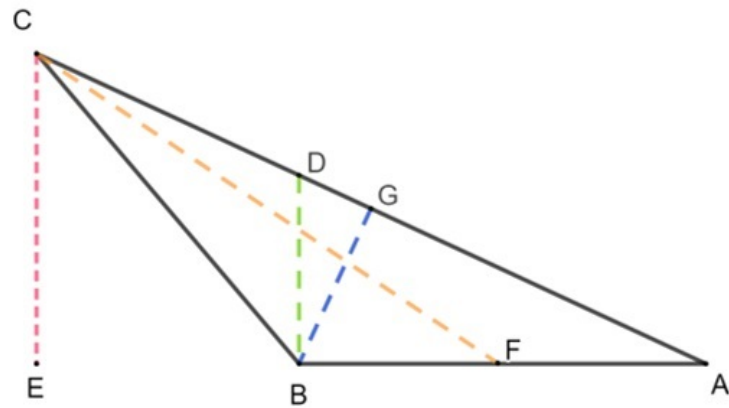
The interaction among students within the group and the final comparison between groups allows students to self-correct and reflect once again on the meaning of height. The final discussion, prompted by questions, enables a generalization of the definition for all polygons.

Response activity: Is everything clear?

The second response activity, *Is everything clear?*, refers to the final stage of the learning progression outlined in the toolkit. Students should be able to use and distinguish between the concepts of perpendicularity, parallelism, horizontality, and verticality.

The worksheet is designed with two multiple-choice questions. The first question is adapted from a question from 2013 grade 6 INVALSI test [1], where students must correctly identify the heights of a triangle among various segments. The second question is constructed similarly but is related to quadrilaterals.

A. Observe the figure



1A. Which of the following pairs of segments represents two of the heights of triangle ABC?

Tik (✓) the box next to correct answer.

A CE and CF

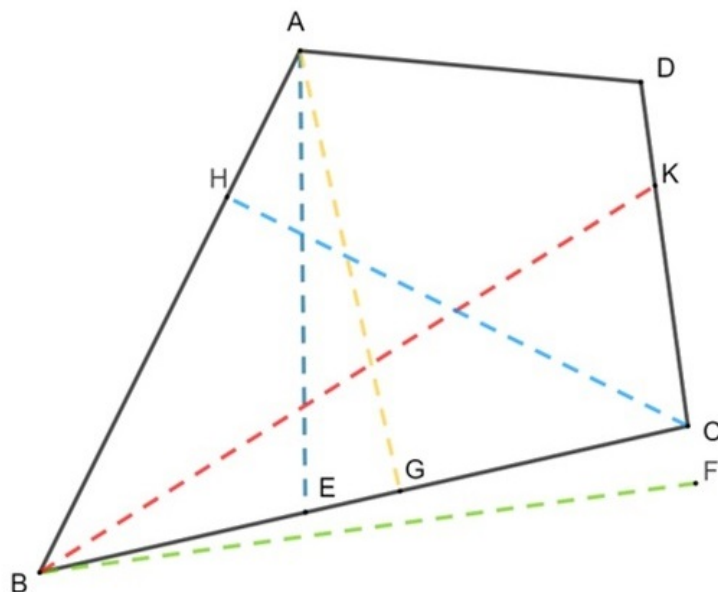
B BD and BG

C CE and BG

D CF and BD

☐
☐
☐
☐

B. Observe the figure



1B. Which of the following triples of segments represents three of the heights of the quadrilateral ABCD?

Tik (✓) the box next to correct answer.

A AE, BK, and CH

B AE, BF, and CH

C BK, BF, and AG

D CH, BF, and AG

☐☐☐☐

The worksheet can also be used as a quick test of whether the concepts have been learned correctly.

4.1.2 Key concept: areas of polygons

Theoretical premises and references

From the analysis of the results of the INVALSI tests for first- and third-year of lower secondary school and second-year of upper secondary school (grade 6, 8 and 10 respectively), it appears that, among the questions on *Space and Figures*, where more than 60% of answers are incorrect, 11 out of 40 concern areas of plane figures. The mistakes arise when the student has to identify equivalent figures or recognize the base of a figure and its relative height [1]. The learning objectives for the third year of lower secondary school, defined in the *Italian National guidelines* [57], include:

to determine the area of simple figures by breaking them down into elementary figures, for example triangles, or by using common formulas.

School textbooks often focus on exercises that involve applying formulas to data that are provided in the statement of the exercise. What is missing, however, are exercises on seeking and identifying elements that allow to calculate areas by observing plane figures.

Conceptual progression

In our proposal, the conceptual progression begins with recognizing equivalent figures and advances to transforming triangles and quadrilaterals into equivalent rectangles. The objective is to bring students closer to a real understanding of mathematical formulas for calculating areas. Being able to transform plane figures into an equivalent rectangle can help internalize the formulas for calculating the areas of polygons without relying solely on memory. In the final step, an activity is proposed to recognize the square as the best unit of measurement for determining the area of a plane figure.

Progression toolkit: areas of polygons

Learning focus	Determine areas of plane figures by decomposing them into elementary figures or using common formulas.				
As students' conceptual understanding progresses they can:	CONCEPTUAL PROGRESSION				
	Identify equidecomposable figures. P	Identify equivalent figures obtained by adding or deleting matching congruent parts. B	Transform triangles and quadrilaterals into equivalent rectangles.	Calculate the area of any polygon.	Identify the appropriate unit of measurement for areas of plane figures.
Diagnostic questions	Odd one out	Equivalent figures	The triangle within the rectangle	Generic polygons	Units of measuring areas
Response activities		Are they equivalent?	Transforming polygons		

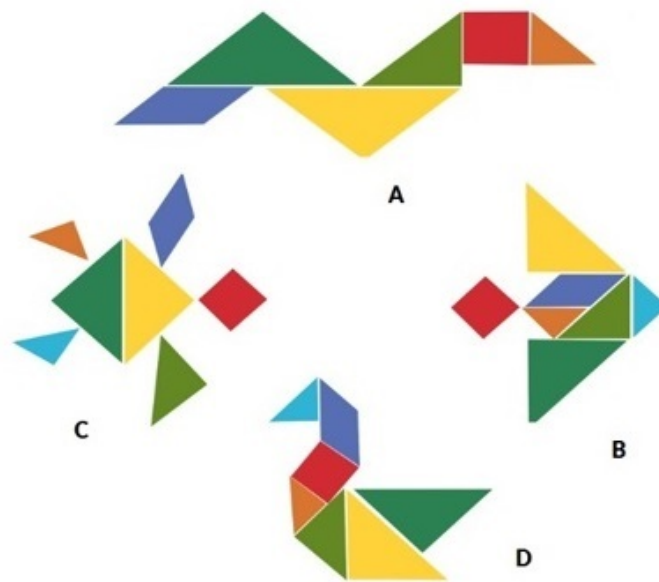
Key:

P Prior understanding from earlier stages of learning **B** Bridge to later stages of learning

Odd one out	Equivalent figures	The triangle within the rectangle	Generic polygons	Units for measuring areas
Odd one out Which of these shapes is the odd one out? Explain the reasons for your answer.	Equivalent figures Read the following statements carefully. The square can be decomposed into 10 squares equivalent to each other. The square can be decomposed into 10 rectangles equivalent to each other. The square can be decomposed into 10 triangles equivalent to each other. The square can be decomposed into 10 rectangles equivalent to each other. The square can be decomposed into 10 squares equivalent to each other. The square can be decomposed into 10 rectangles equivalent to each other. Questions 1. Which statement helps explain why Figure 1 is equivalent to the square? 2. Which statement helps explain why Figure 2 is not equivalent to the square?	The triangle within the rectangle The figure shows a rectangle ABCD along with its diagonals. If the area of the rectangle is known, can you determine the area of the shaded triangle? Tick (✓) the box next to the best answer: A. No, because the four triangles with vertex D are not all equal to each other. ☐ B. No, because it isn't known the dimensions of the rectangle. ☐ C. Yes, because the four triangles are equivalent. ☐ D. Yes, because the triangles four with vertex D are identical. ☐	Generic polygons Observe the figure. If the side of each grid square is 1 m, what is the area of the polygon? A. 22 m² ☐ B. 23 m² ☐ C. 21 m² ☐ D. 24 m² ☐	Units for measuring areas Measuring a quantity means counting how many times a given unit of measurement is contained in the quantity to be determined. For each figure in the table below, consider whether the polygon that has been shaded and is suitable unit to measure area. A. B. C. D. E. F. G. H. I. J. K. L. M. N. O. P. Q. R. S. T. U. V. W. X. Y. Z. AA. AB. AC. AD. AE. AF. AG. AH. AI. AJ. AK. AL. AM. AN. AO. AP. AQ. AR. AS. AT. AU. AV. AW. AX. AY. AZ. BA. BB. BC. BD. BE. BF. BG. BH. BI. BJ. BK. BL. BM. BN. BO. BP. BQ. BR. BS. BT. BU. BV. BW. BX. BY. BZ. CA. CB. CC. CD. CE. CF. CG. CH. CI. CJ. CK. CL. CM. CN. CO. CP. CQ. CR. CS. CT. CU. CV. CW. CX. CY. CZ. DA. DB. DC. DD. DE. DF. DG. DH. DI. DJ. DK. DL. DM. DN. DO. DP. DQ. DR. DS. DT. DU. DV. DW. DX. DY. DZ. EA. EB. EC. ED. EE. EF. EG. EH. EI. EJ. EK. EL. EM. EN. EO. EP. EQ. ER. ES. ET. EU. EV. EW. EX. EY. EZ. FA. FB. FC. FD. FE. FF. FG. FH. FI. FJ. FK. FL. FO. FP. FQ. FR. FS. FT. FU. FV. FW. FX. FY. FZ. GA. GB. GC. GD. GE. GF. GG. GH. GI. GJ. GK. GL. GO. GP. GQ. GR. GS. GT. GU. GV. GW. GX. GY. GZ. HA. HB. HC. HD. HE. HF. HG. HH. HI. HJ. HK. HL. HO. HP. HQ. HR. HS. HT. HU. HV. HW. HX. HY. HZ. IA. IB. IC. ID. IE. IF. IG. IH. II. IJ. IK. IL. IO. IP. IQ. IR. IS. IT. IU. IV. IW. IX. IY. IZ. JA. JB. JC. JD. JE. JF. JG. JH. JI. JJ. JK. JL. JO. JP. JQ. JR. JS. JT. JU. JV. JW. JX. JY. JZ. KA. KB. KC. KD. KE. KF. KG. KH. KI. KJ. KK. KL. KO. KP. KQ. KR. KS. KT. KU. KV. KW. KX. KY. KZ. LA. LB. LC. LD. LE. LF. LG. LH. LI. LJ. LK. LL. LO. LP. LQ. LR. LS. LT. LU. LV. LW. LX. LY. LZ. MA. MB. MC. MD. ME. MF. MG. MH. MI. MJ. MK. ML. MO. MP. MQ. MR. MS. MT. MU. MV. MW. MX. MY. MZ. NA. NB. NC. ND. NE. NF. NG. NH. NI. NJ. NK. NL. NO. NP. NQ. NR. NS. NT. NU. NV. NW. NX. NY. NZ. OA. OB. OC. OD. OE. OF. OG. OH. OI. OJ. OK. OL. OO. OP. OQ. OR. OS. OT. OU. OV. OW. OX. OY. OZ. PA. PB. PC. PD. PE. PF. PG. PH. PI. PJ. PK. PL. PO. PP. PQ. PR. PS. PT. PU. PV. PW. PX. PY. PZ. QA. QB. QC. QD. QE. QF. QG. QH. QI. QJ. QK. QL. QO. QP. QQ. QR. QS. QT. QU. QV. QW. QX. QY. QZ. RA. RB. RC. RD. RE. RF. RG. RH. RI. RJ. RK. RL. RO. RP. RQ. RR. RS. RT. RU. RV. RW. RX. RY. RZ. SA. SB. SC. SD. SE. SF. SG. SH. SI. SJ. SK. SL. SO. SP. SQ. SR. SS. ST. SU. SV. SW. SX. SY. SZ. TA. TB. TC. TD. TE. TF. TG. TH. TI. TJ. TK. TL. TO. TP. TQ. TR. TS. TT. TU. TV. TW. TX. TY. TZ. UA. UB. UC. UD. UE. UF. UG. UH. UI. UJ. UK. UL. UO. UP. UQ. UR. US. UT. UU. UV. UW. UX. UY. UZ. VA. VB. VC. VD. VE. VF. VG. VH. VI. VJ. VK. VL. VO. VP. VQ. VR. VS. VT. VU. VV. VW. VX. VY. VZ. WA. WB. WC. WD. WE. WF. WG. WH. WI. WJ. WK. WL. WO. WP. WQ. WR. WS. WT. WU. WV. WW. WX. WY. WZ. XA. XB. XC. XD. XE. XF. XG. XH. XI. XJ. XK. XL. XO. XP. XQ. XR. XS. XT. XU. XV. XW. XX. XY. XZ. YA. YB. YC. YD. YE. YF. YG. YH. YI. YJ. YK. YL. YO. YP. YQ. YR. YS. YT. YU. YV. YW. YX. YY. YZ. ZA. ZB. ZC. ZD. ZE. ZF. ZG. ZH. ZI.

We use the *Tangram*, a popular school game, to explore the concept of equidisposability of polygons. The game's potential lies in the ability to break down each piece into parts congruent with the two smaller pieces, allowing students to evaluate extension in composite figures without delving into surface measurement. The equivalence between equiextension and equidecomposability in polygons forms the foundation for introducing calculation of areas in subsequent sheets.

Odd one out



Which of these images is the odd one out?

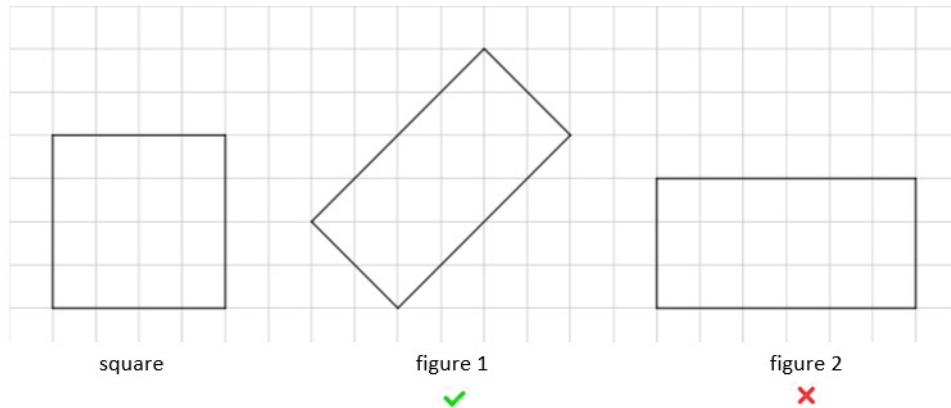
Explain the reasons for your answer.

Diagnostic question: Equivalent figures

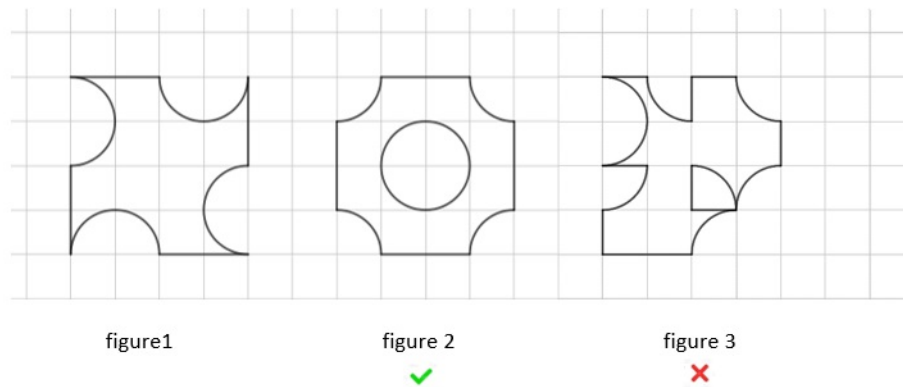
This diagnostic question is designed to promote discussion among students. Students are required to explain the reasons why they assert that two or more figures are or are not equivalent to each other.

Two scenarios are presented in which three figures are compared: two of them are equivalent, and one is not.

1 Observe the image below: Figure 1 is equivalent to the square, Figure 2 is not.



2 The figures below have been obtained by deleting pieces from a square. Figure 1 is equivalent to Figure 2 but not to Figure 3.



To assist students in their reasoning, statements are provided. Students must indicate which statements are useful for explaining the equivalence or non-equivalence of the figures.

For the first situation, the proposed statements are:

The square can be decomposed into 16 squares equivalent to each other.	The square can be decomposed into 32 triangles equivalent to each other.
The square can be decomposed into 4 triangles equivalent to each other.	The square can be decomposed into 8 rectangles equivalent to each other.
The square can be decomposed into 8 triangles equivalent to each other.	The square can be decomposed into 4 rectangles equivalent to each other.
The square can be decomposed into 16 triangles equivalent to each other.	The square can be decomposed into 4 squares equivalent to each other.

For the second situation:

In figure 1, we can count 8 whole squares.	With the missing parts of Figure 1, we can build 2 circles.
Eight quarter circles have been deleted from Figure 1.	Four semicircles have been deleted from Figure 1.

In the first scenario, students are invited to break down the square into equal parts (isosceles or scalene right-angled triangles, rectangles, squares) and to verify the equivalence of the figures by reassembling the parts to construct rectangles 1 or 2. This tiling activity serves as a bridge to introduce units of measurement for area (diagnostic question: *Unit for measuring areas*).

In the second scenario, the focus is on establishing the equivalence of plane figures by deleting congruent parts. The choice of curved sides is meant to make pupils realize that the parts into which the figures are broken down do not necessarily have to be polygons.

This activity, conducted in pairs or small groups, emphasizes discussion. Through discussions, students validate understanding and test explanations.

Providing each group with a shared worksheet to fill out as a group is helpful in encouraging discussion, but every member should be able to report to the class. Listening to the conversations of each group can often give the teacher an idea of how their students think. In the end, each student writes the answer individually in their own notebook; this could help them consolidate their learning.

While the activity may be time-consuming, focusing on this initial phase may facilitate progression in subsequent stages.

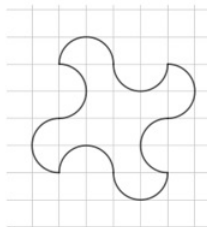
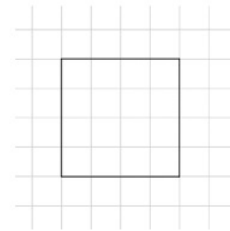
To reinforce acquired knowledge, a follow-up response activity is suggested: *Are they equivalent?*

Response activity: Are they equivalent?

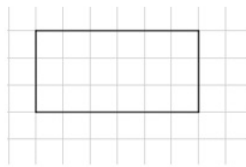
Are they equivalent? is a multiple-choice worksheet, where students identify figures equivalent to a given square.

Are they equivalent?

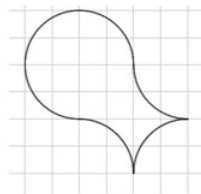
Among the figures below, identify those that are equivalent to the square to the right.



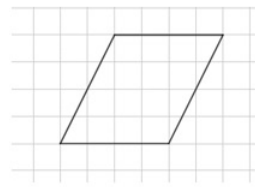
A



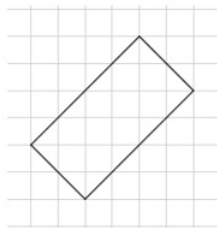
B



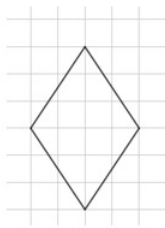
C



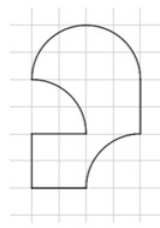
D



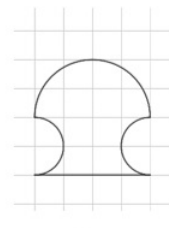
E



F



G



H

Tick (✓) the box next to the answer you think is correct.

- 1 D, E
- 2 A, C, D, E, G
- 3 C, F, G, H
- 4 D, G
- 5 B, D, E, F

☐
☐
☐
☐
☐

We recommend using colored cards as a voting system and giving students the opportunity to compare themselves in pairs and then with the rest of the class. Collective discussion can be organized by asking one student to justify the answer s/he gave and then asking another student

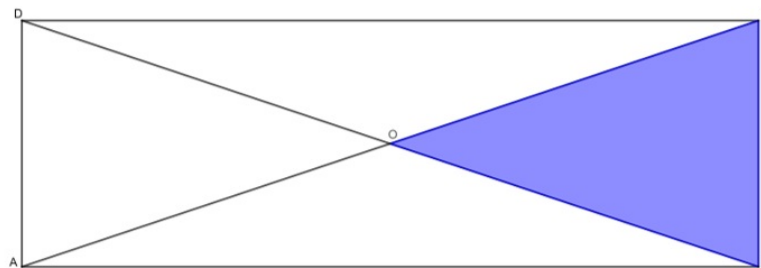
to explain why s/he agrees with him/her or why s/he doesn't, and so on. Collective discussions, with students justifying and explaining answers, provide insights into their thinking and learning needs.

Diagnostic question: The triangle within the rectangle

The learning focus of this question is to be able to calculate the area of plane figures by decomposing them into elementary figures, such as triangles, or using the most common formulas. In particular, students should be able to identify that the area of a triangle is equivalent to that of a rectangle with the same base and height equal to half of that of the triangle, or vice versa, that the area of a rectangle is equal to that of a triangle with the same base and height twice that of the rectangle.

The question proposed in this worksheet was administered to third-grade students in the national INVALSI test of 2012 [1]. The percentage of correct answers was quite low (24.1%). This suggests that the question brought out difficulties in various aspects: on one hand, difficulty in recognizing the equivalence of figures that are not congruent to each other, and on the other hand, difficulty in extracting information and understanding relationships between the elements of the depicted plane figures. The incorrect answers were distributed as follows: A (26.9%), B (30.8%) and D (16.4%) [1].

The figure shows a rectangle ABCD along with its diagonals. If the area of the rectangle is known, can you determine the area of the shaded triangle?



Tick (✓) the box next to the best answer.

- A** No, because the four triangles with vertex O are not all equal to each other ☐
- B** No, because I don't know the dimensions of the rectangle ☐
- C** Yes, because the four triangles are equivalent ☐
- D** Yes, because the four triangles with vertex O are isosceles ☐

It is possible that students who select option A may lack clarity on the distinction between equivalence and congruence, possibly assuming that if two triangles are not congruent, they cannot be equivalent. Option B might be chosen by students favoring an arithmetic approach to geometric problem-solving. When dimensions are not provided in the text or the figure, they may struggle to understand the relationship between the elements of the figures. The choice of option D may stem from a tendency to seek a pattern. In this case, if an understanding of equivalence of figures is missing, students may seek any known, evident property, even one that is not relevant to the calculation of the area of the rectangle.

The correct answer, C, assumes that students recognize that the four triangles, although not congruent, are all equivalent to a quarter of the initial rectangle. Each triangle has dimensions half those of the rectangle. Consequently, a triangle is always equivalent to a rectangle with one dimension equal to that of the rectangle, the other equal to half the corresponding dimension in the triangle.

Response activity: Transforming polygons

A prevalent misconception in mathematics is the belief that learning mathematics relies heavily on memory. Interviews with successful and enthusiastic students reveal that they prioritize understanding over memorization. Conversely, those who perceive difficulties in understanding tend to rely excessively on memorization [76]. The following hands-on activity provides students with the opportunity to grasp the meaning of common formulas for calculating areas and refine their understanding through discussion with classmates.

The worksheet offers semi-guided instructions, providing suggestions for working with specific figures (such as right-angled triangles and rhombuses) while leaving some room for unstructured exploration, to encourage students to explore alternative approaches.

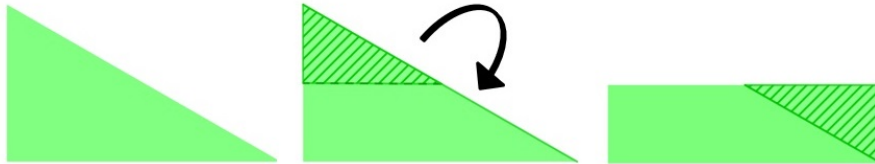
The activity should be conducted in pairs or small groups. For the teacher, actively listening to individual groups can help to identify difficulties that may be addressed through clarification or redirection to the whole class during the activity.

After completing the practical work, students are encouraged to share their results with the rest of the class. Following the response of a group, the teacher can model an improved response, prompting another group to add or clarify further.

The teacher supplies one colored cardboard figure of each type to each student/couple/group. The first activity is guided; students are provided with the cardboard cutout related to the right-angled triangle, and they must follow the instructions and observe the formula for calculating the area. Then, students are encouraged to apply a similar procedure with a scalene triangle.

A. Right-angled triangle

1. Fold the figure in half along one side: the triangle is divided into a smaller triangle and a trapezoid.
2. Cut along the fold.
3. Assemble the two resulting pieces into a rectangle.



The final rectangle has the same base as the initial triangle and its height is half the height of the initial triangle, so

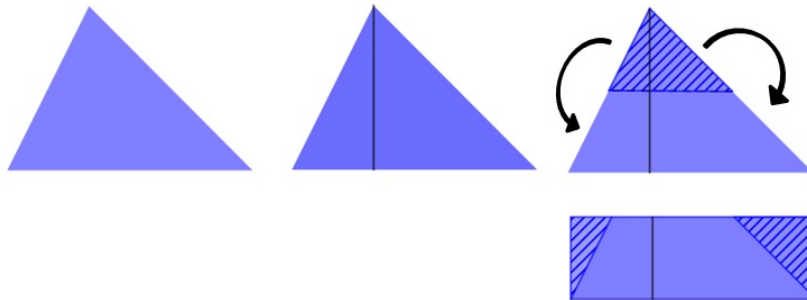
$$\text{Area of triangle: base} \times \text{half height} = \frac{b \times h}{2}$$

B. Scalene triangle

You can divide a scalene triangle into two right-angled triangles by drawing the height relative to one side.

4. Repeat the previous procedure on each of the right-angled triangles you obtain in this way. What do you get?

The transformation of a scalene triangle into a rectangle is illustrated.

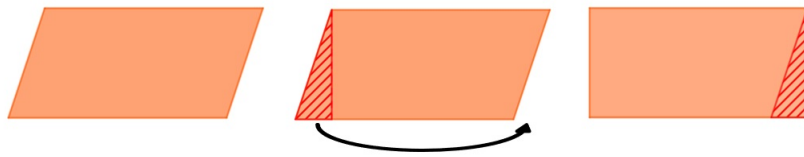


Then, the lesson moves on to the parallelogram; the transformation of a parallelogram into a rectangle is a procedure generally familiar to students in the first stage of secondary school, so no hints are given. However, students are encouraged to reflect through two questions.

C. Parallelogram

5. How can you cut the parallelogram to reassemble it into a rectangle?
6. How do the base and height of the rectangle compare to those of the parallelogram?

The transformation of a parallelogram into an equivalent rectangle is depicted.



At this point, the focus shifts to the rhombus: by remembering that a rhombus is a particular type of parallelogram, attention is drawn to the fact that it can be decomposed in the same way. Consequently, the same formula can be used to find the area. However, the presence of perpendicular diagonals in the rhombus leads to identifying another formula for calculating the area.

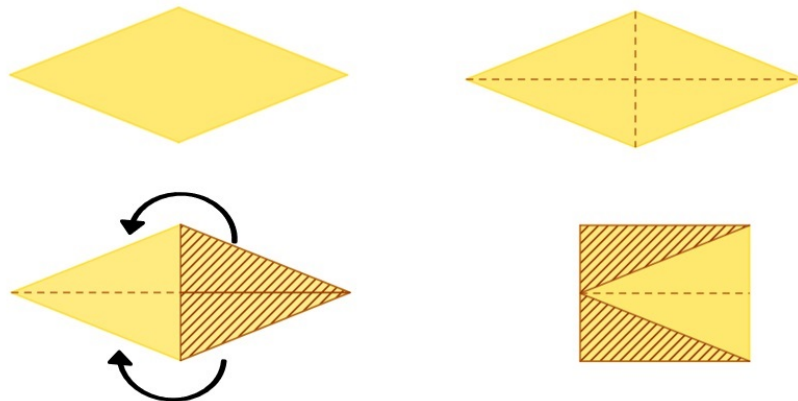
Through a guided activity, students are led to discover this formula by transforming the rhombus into a rectangle with dimensions equal to one of the diagonals and half the other.

D. Rhombus

The rhombus is a specific type of parallelogram. You can always find the area of a rhombus by using the same formula as for parallelograms.

In a rhombus, however, the diagonals are perpendicular. The area of a rhombus can also be obtained from the lengths of the diagonals.

7. Fold the rhombus along the diagonals: four congruent triangles are formed.
8. Cut along one diagonal, then cut one of the two pieces along the other diagonal.
9. Reassemble the pieces obtained to form a rectangle.



The final rectangle has sides equal to one diagonal of the rhombus and half the other diagonal.

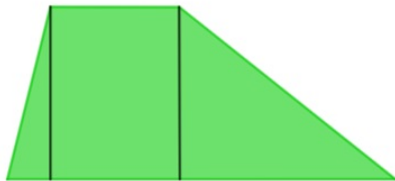
$$\text{Area of rhombus: } \text{diagonal}_1 \times \text{half diagonal}_2 = \frac{d_1 \times d_2}{2}$$

The sequence ends with the scalene trapezoid.

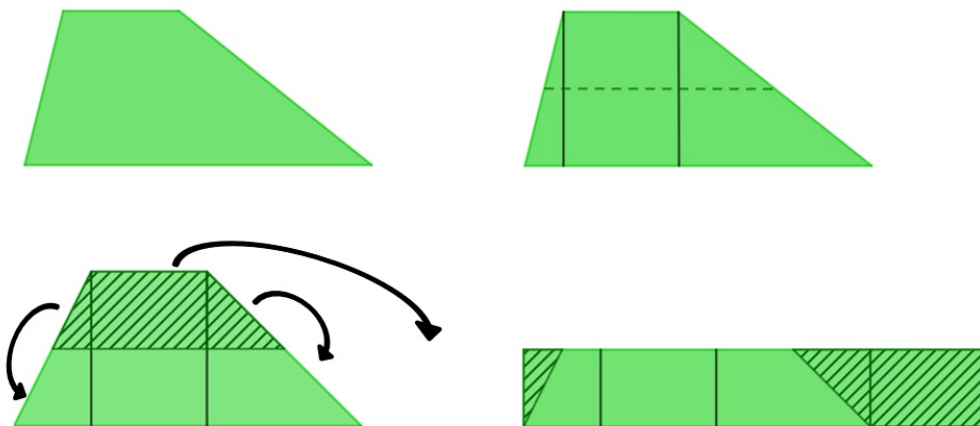
E. Trapezoid

If we draw the heights from the endpoints of the smallest parallel side to the longest parallel side, we can see the trapezoid is decomposed into a rectangle and two right-angled triangles.

10. What folds and cuts can you make to transform the trapezoid into a rectangle (similarly to how a right-angled triangle was transformed into a rectangle)?



The transformation of a trapezoid into an equivalent rectangle is demonstrated.



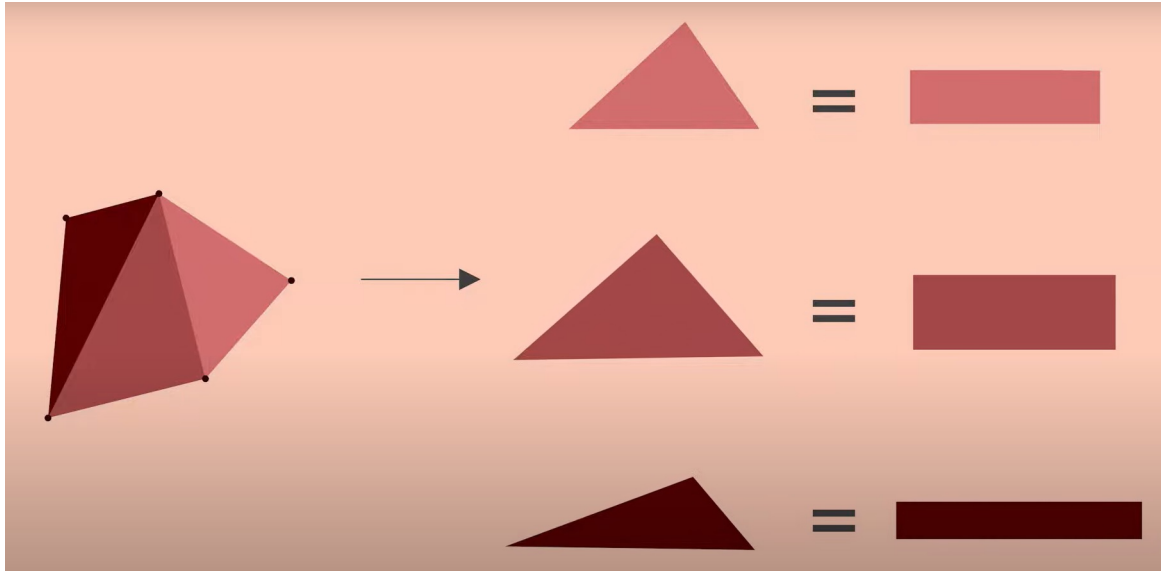
$$\text{Area of trapezoid} = \text{sum of bases} \times \text{half height} = \frac{(b_1 + b_2) \times h}{2}$$

While how to transform the trapezoid into the rectangle may not be immediately evident, it is hoped that activities with previous figures will facilitate the task. If no group can achieve the construction, the teacher can provide suggestions for the entire class.

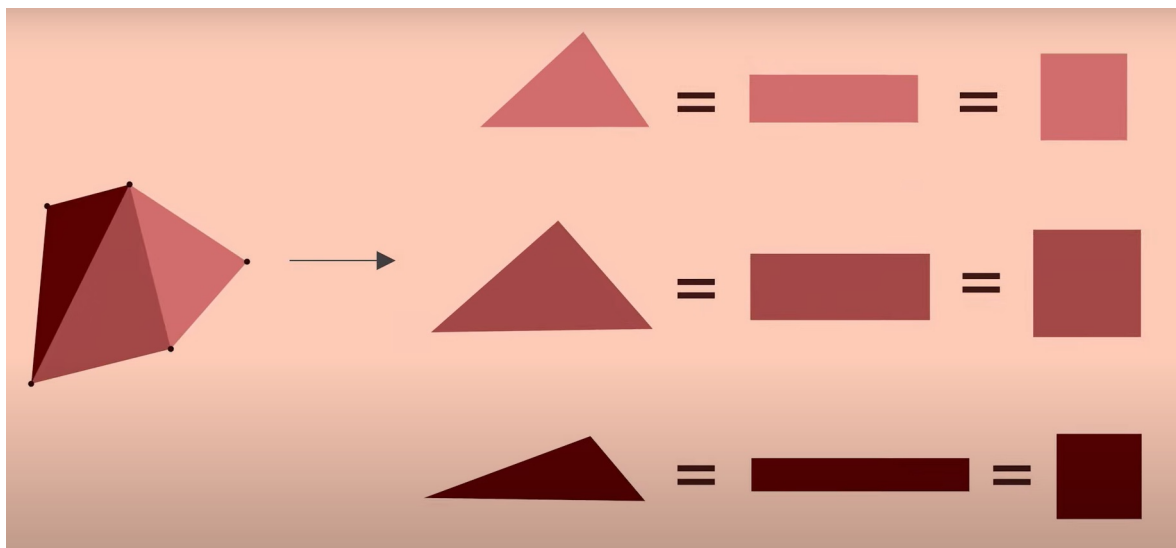
After completing the activity, if the class express an interest or curiosity, the teacher may recommend the video available at https://www.youtube.com/watch?v=9yoCbk_z_08.

This video demonstrates the transformation of any polygon into a square. While the content may involve concepts not yet encountered by the students, it can serve as a stimulating introduction to pique their curiosity.

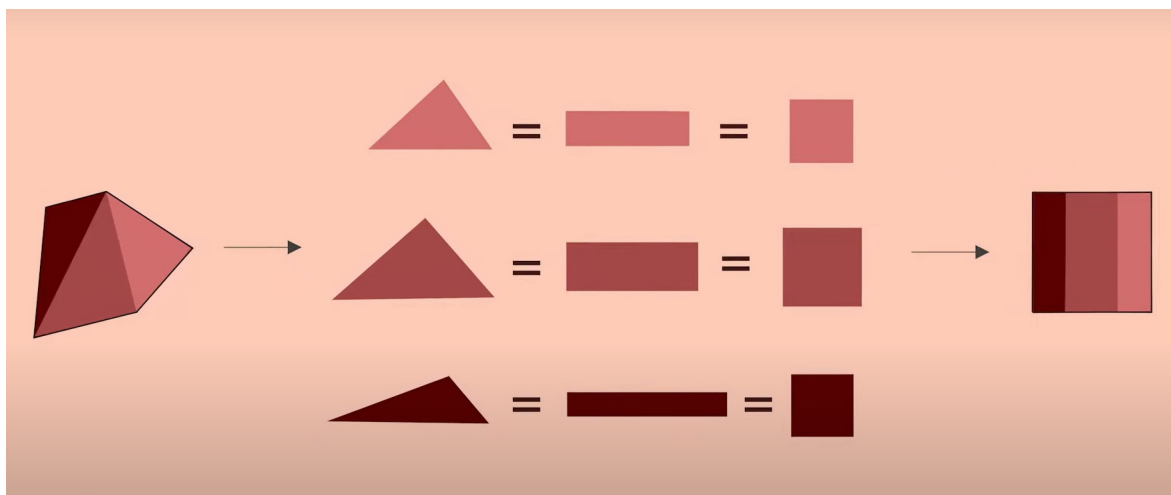
The polygon is divided into triangles, and each triangle is converted into a rectangle.



Euclid's second theorem is employed to transform rectangles into equivalent squares.



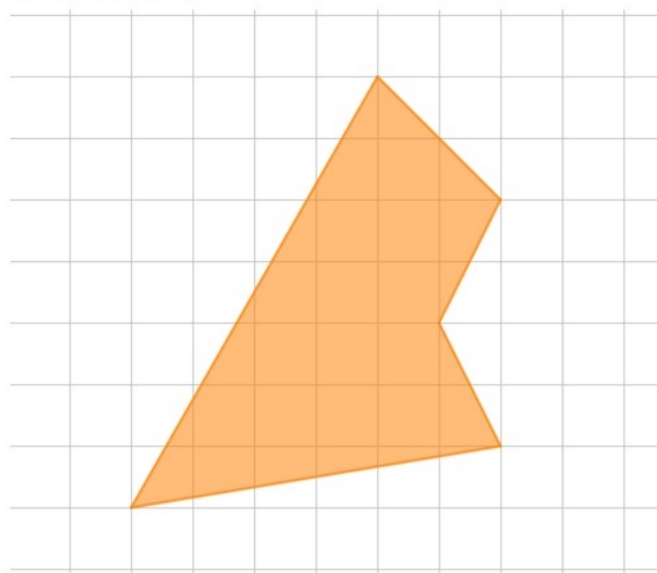
Finally, squares are added using the Pythagorean theorem.



Diagnostic question: Generic polygons

The purpose of this worksheet is to calculate the area of a generic polygon by summing or subtracting the areas of other polygons into which the figure can be decomposed or derived. The diagnostic question is taken from an INVALSI test administered in 2014 to students in the second year of secondary school [1]. The solution does not require the use of knowledge acquired after the end of the first cycle of education, so the question is also suitable for students in lower secondary school.

1 Observe the figure.



If the side of each grid square is 1 m, what is the area of the polygon?

- A 22 m²
- B 23 m²
- C 21 m²
- D 24 m²

☐
☐
☐
☐

In the INVALSI test, the percentage of correct responses was 30.7%, with a substantial number of incorrect answers (44.5%), and a significant percentage of missing answers (24.4%). This indicates that a considerable percentage of students struggle to employ effective problem-solving strategies [1].

Diagnostic activities such as this one can encourage students to contemplate different approaches to problem solving. Analyzing incorrect answers helps teachers understand students' reasoning. It is possible that students who correctly chose option C split the generic polygon into easily identifiable figures with clear base and height, as illustrated in Figure 1.

Alternatively, they might have determined the area by subtracting the area of the triangle from the outer rectangle, as depicted in Figure 2.

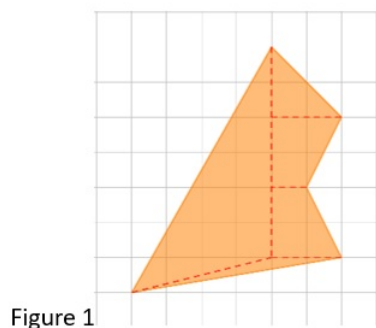


Figure 1

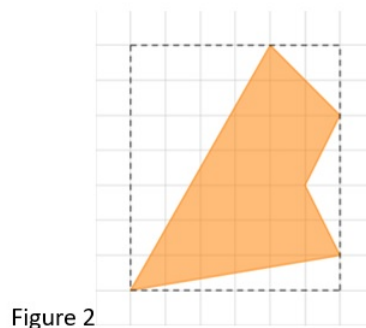


Figure 2

Encouraging discussion among students about the methods they used to obtain a correct answer can reveal diverse approaches. Incorrect responses may result from decomposing the generic polygon into figures where estimating the base and height is necessary, because the endpoints of these segments do not coincide with grid vertices, as illustrated in Figures 3 and 4. Another possibility is counting the complete squares within the figure and estimating the total sum of the remaining parts.

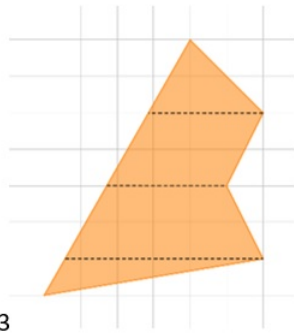


Figure 3

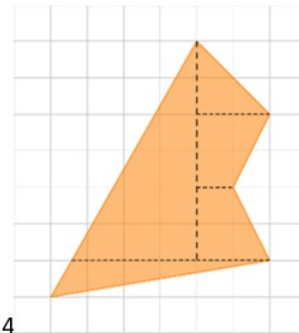


Figure 4

Diagnostic question: Units for measuring areas

The diagnostic question *Units for measuring areas* is a worksheet designed to make students reflect on the concept of the unit of measurement of a plane figure.

Calculations of areas of plane figures and the choice of the appropriate unit of measurement are generally introduced in the last two years of primary school. Both topics are resumed in the second year of lower secondary school and in the first two years of upper secondary school. Often what is reached at this level is superficial knowledge of the main formulas, and the notion of square meter as a unit of measurement. For this reason, it is useful to propose activities that review and consolidate notions already learned, as confirmed by various suggestions on how to teach these topics at different levels on the Scuolavalore-INDIRE website [12] and, previously, in *Matematica2001* [5] and *Matematica2003* [6].

Measuring a quantity means counting how many times a given unit of measurement is contained in the quantity to be determined.

For each figure in the table below, consider whether the polygon that has been singled out is a suitable unit to measure area.

A. 	B. 	C.
D. 	E. 	F.
G. 	H. 	I.
L. 	M. 	N.

For each of the following statements, choose the unit polygon or polygons, among A to N above, to which the statement applies.

- By arranging copies of the suggested unit polygon side by side, with no overlap, a plane can be covered without gaps.
- To determine the unit polygon, it is enough to know the length of a single side and a single angle.
- Which unit polygons satisfy both the conditions in statement 1 and statement 2?
- Among the unit polygons in Figure A to N, which would you choose as a unit of measurement to determine the area of a generic polygon?
Justify your answer.

The sequence of questions leads students to gradually identify the most suitable figure as a unit

of measurement for the area.

This unit must be an indefinitely reproducible model, capable of tiling the entire plane. However, it must also be easy to identify and to reproduce. Regular polygons (equilateral triangle, square, regular hexagon) are more suitable for this purpose: since each internal angle is known, it suffices to know the length of one side to identify each polygon uniquely and copy it. To reproduce a rectangle or a triangle or one of the other figures that satisfy condition 1, the lengths of at least two sides must be known. The choice of the square as a unit of measurement is the simplest, but also the most appropriate because all polygons can be transformed into rectangles, and the square is better suited to tiling these rectangles because all angles in squares and the rectangles are right angles.

4.1.3 Key concept: variations in areas and perimeters

Theoretical premises and references

In [34] D'Amore and Pinilla report that many students, regardless of their age, hold the misconception that there is a direct relational dependence between the perimeter and the area of a plane figure. A body of work started by Piaget in 1930s, and continued by his students, has highlighted the challenge students face in grasping the invariance of the surface measurement of a plane figure as the shape changes. Specifically, there is resistance in acknowledging that there are no inherent, predetermined relationships between the perimeter and area of plane figures. In other words, one cannot make a priori statements about the correlation between changes (increase, decrease, equality) in perimeter and changes (increase, decrease, equality) in the area of plane figures. These difficulties may manifest themselves at the primary school level, persist through university, and even be ingrained in the beliefs of certain teachers. The impediments to developing a satisfactory understanding of the relationships between perimeter and area are not solely of an epistemological nature but also have pedagogical implications.

Emma Castelfnuovo [26] points out that the confusion between area and perimeter in children often stems from the static nature of drawings, leading pupils to focus on the figure outline rather than its interior. The use of materials such as string, elastic bands, and sticks, which allow for movement of figures, aids in directing thoughts toward transformations and facilitates a clearer understanding of the concept of extension.

Conceptual progression

The conceptual progression begins with the comparison of different arguments related to squares and equilateral triangles with the same perimeter. Then the comparison is considered between the area of isoperimetric figures and the perimeters of equiextensive figures. In particular, drawing inspiration from the works of Emma Castelfnuovo, practical activities involving string and elastic band are proposed to promote a dynamic understanding of the concepts of perimeter and area.

Progression toolkit: variations in areas and perimeters

Learning focus	Capture the area variations in isoperimetric figures and the perimeter variations in equivalent figures.				
As students' conceptual understanding progresses they can:					
	Compare arguments regarding the perimeter of an equilateral triangle and that on a square. P	Compare areas of isoperimetric figures. A		Compare perimeters of equivalent figures. P	
Diagnostic question	Who is right?	Rectangles in sequence		Three triangles	
Response activity		Rectangles workshop 1	Triangles workshop 1	Rectangles workshop 2	Triangles workshop 2

Key:

P Prior understanding from earlier stages of learning

B Bridge to later stages of learning

Who is right?

Who is right?

The teacher asks the class:
Can a triangle and a square have the same perimeter?

Adam: Yes. To do it, the triangle has three sides, and the square has four.

Leah: Yes. To do it, a square is always bigger than a triangle.

Single: Yes. When it happens, the sides of the triangle are longer than those of the square.

Martha: Yes. When it happens, the sides of the triangle are equal to the sides of the square.

Who is right?

- A Adam
- B Leah
- C High
- D Martha

Rectangles in sequence

Rectangles in sequence

Observe the following sequence of figures:

4. Which of the following statements is true?
Tick (✓) the box next to the best answer.

- A The areas of the figures always remain the same.
- B The areas of the figures double with each step.
- C The perimeters of the figures always remain the same.
- D The perimeters of the figures increase with each step.

Three triangles

Three triangles

Observe the triangles in the figure:

5. Which of the following is true?
Tick (✓) the box next to the best answer.

- A The three triangles have the same area and perimeter.
- B The three triangles have the same area and different perimeter.
- C The three triangles have different areas and the same perimeter.
- D The three triangles have different areas and perimeters.

Talking heads

Two-tier multiple choice

Two-tier multiple choice

Rectangles workshop 1

Rectangles workshop 1

Observe the following sequence of figures:

1. Write down the area and perimeter for each rectangle in the sequence.

2. Write down the area and perimeter for the 10th rectangle in the sequence.

3. Write down the area and perimeter for the 100th rectangle in the sequence.

4. Write down the area and perimeter for the 1000th rectangle in the sequence.

5. Write down the area and perimeter for the 10000th rectangle in the sequence.

6. Write down the area and perimeter for the 100000th rectangle in the sequence.

7. Write down the area and perimeter for the 1000000th rectangle in the sequence.

8. Write down the area and perimeter for the 10000000th rectangle in the sequence.

9. Write down the area and perimeter for the 100000000th rectangle in the sequence.

10. Write down the area and perimeter for the 1000000000th rectangle in the sequence.

11. Write down the area and perimeter for the 10000000000th rectangle in the sequence.

12. Write down the area and perimeter for the 100000000000th rectangle in the sequence.

13. Write down the area and perimeter for the 1000000000000th rectangle in the sequence.

14. Write down the area and perimeter for the 10000000000000th rectangle in the sequence.

15. Write down the area and perimeter for the 100000000000000th rectangle in the sequence.

16. Write down the area and perimeter for the 1000000000000000th rectangle in the sequence.

17. Write down the area and perimeter for the 10000000000000000th rectangle in the sequence.

18. Write down the area and perimeter for the 100000000000000000th rectangle in the sequence.

19. Write down the area and perimeter for the 1000000000000000000th rectangle in the sequence.

20. Write down the area and perimeter for the 10000000000000000000th rectangle in the sequence.

21. Write down the area and perimeter for the 100000000000000000000th rectangle in the sequence.

22. Write down the area and perimeter for the 1000000000000000000000th rectangle in the sequence.

23. Write down the area and perimeter for the 10000000000000000000000th rectangle in the sequence.

24. Write down the area and perimeter for the 100000000000000000000000th rectangle in the sequence.

25. Write down the area and perimeter for the 1000000000000000000000000th rectangle in the sequence.

26. Write down the area and perimeter for the 10000000000000000000000000th rectangle in the sequence.

27. Write down the area and perimeter for the 100000000000000000000000000th rectangle in the sequence.

28. Write down the area and perimeter for the 1000000000000000000000000000th rectangle in the sequence.

29. Write down the area and perimeter for the 10000000000000000000000000000th rectangle in the sequence.

30. Write down the area and perimeter for the 100000000000000000000000000000th rectangle in the sequence.

31. Write down the area and perimeter for the 1000000000000000000000000000000th rectangle in the sequence.

32. Write down the area and perimeter for the 10000000000000000000000000000000th rectangle in the sequence.

33. Write down the area and perimeter for the 100000000000000000000000000000000th rectangle in the sequence.

34. Write down the area and perimeter for the 1000000000000000000000000000000000th rectangle in the sequence.

35. Write down the area and perimeter for the 10000000000000000000000000000000000th rectangle in the sequence.

36. Write down the area and perimeter for the 100000000000000000000000000000000000th rectangle in the sequence.

37. Write down the area and perimeter for the 1000000000000000000000000000000000000th rectangle in the sequence.

38. Write down the area and perimeter for the 10000000000000000000000000000000000000th rectangle in the sequence.

39. Write down the area and perimeter for the 100000000000000000000000000000000000000th rectangle in the sequence.

40. Write down the area and perimeter for the 1000000000000000000000000000000000000000th rectangle in the sequence.

41. Write down the area and perimeter for the 100th rectangle in the sequence.

42. Write down the area and perimeter for the 1000th rectangle in the sequence.

43. Write down the area and perimeter for the 100th rectangle in the sequence.

44. Write down the area and perimeter for the 1000th rectangle in the sequence.

45. Write down the area and perimeter for the 100th rectangle in the sequence.

46. Write down the area and perimeter for the 1000th rectangle in the sequence.

47. Write down the area and perimeter for the 100th rectangle in the sequence.

48. Write down the area and perimeter for the 1000th rectangle in the sequence.

49. Write down the area and perimeter for the 10000000000

Below we present the diagnostic questions and response activities related to the key concept *Variation in areas and perimeters*.

Diagnostic question: Who is right?

This diagnostic question replicates a problem from the INVALSI test given to 6th-grade students in 2012 [1]. The question format is “talking heads”: students must choose the best among four different answers and arguments.

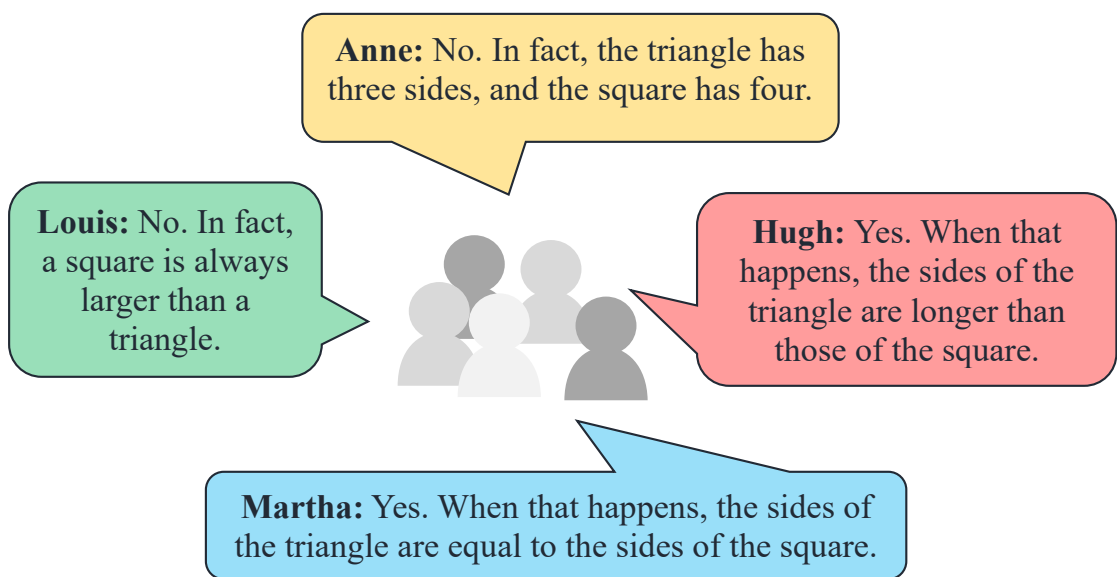
In this type of question, the focus should be on discussion, so it is useful for students to respond in pairs or small groups. In this way, the teacher can listen to students as they engage in

discussions. To encourage the participation of all students, including those with difficulties, roles may be assigned within groups. For instance, students who are well-versed in the topic can be tasked with writing down the answers of others. While they shouldn't respond themselves, they can lead other group members to the correct answer by posing suitable questions.

Who is right?

The teacher asks the class:

Can a triangle and a square have the same perimeter?



Who is right?

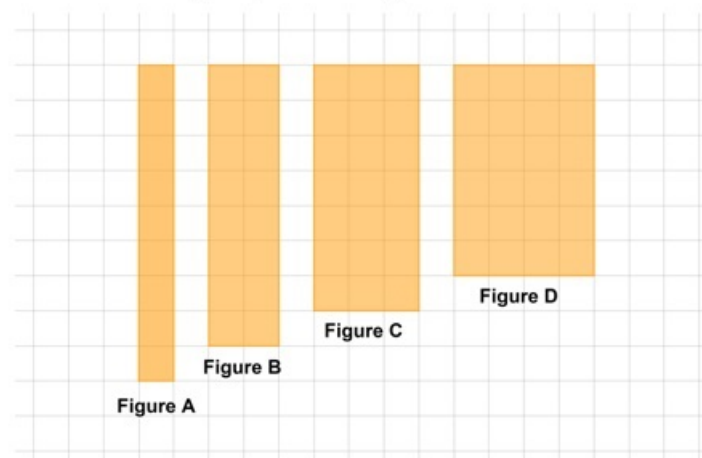
- A** Louis
- B** Anne
- C** Hugh
- D** Martha

Based on the responses provided by the students, the teacher can plan the subsequent activities, investigating variations in areas in isoperimetric polygons and variations in perimeters in equivalent polygons.

Diagnostic question: Rectangles in sequence

The question is a two-tier multiple choice question. The first part is taken from the INVALSI test for 6th-grade students in 2012 [1], while the second part has been added to encourage reflection among students who struggle with argumentation.

Observe the following sequence of figures:



1. Which of the following statements is true?

Tick (✓) the box next to the best answer.

- A** The areas of the rectangles are the same. ☐
- B** The areas of the rectangles double at each step. ☐
- C** The perimeters of the rectangles are the same. ☐
- D** The perimeters of the rectangles increase at each step. ☐

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

- A** When the base increases by one unit, the height decreases by one unit. ☐
- B** Every rectangle has a base and height different from those of the others. ☐
- C** As the base increases, the area also increases. ☐
- D** As the area increases, the perimeter also increases. ☐

We formulated the options for the second question by attempting to foresee the students' potential reasoning. In the national sample, only 35.7% of students provided the correct answer to the first question, that is, the invariance of the perimeter [1]. The correct answer to the first question should be associated with option A in the second question.

Observing the sequence of figures, it should immediately be evident that the areas of the figures are different. However, in the national sample, 4.4% of students selected option A for the first question [1]. This could be due to a genuine misunderstanding of the concept of area, or it could be caused by a terminology error: students might be thinking about perimeter but use the word "area". In the first scenario, the answer to the second question might be B, whereas in the second it could be A. In the national sample, 22.4% of students chose option B. In our view, may believe that the change in area is solely connected to the widening of the base. In such instances, we anticipate the answer to the second question to be C. If students opt for D for the first question, they might be assuming that an increase in the area of the figures is correlated to an increase in the perimeter. To validate this interpretation, they would need to select D as their answer to the second question. In the national survey, option D for the first question was chosen by 35.5% of students; this seems to support the considerations emerging from [34]. Of course, other combinations of responses to questions 1 and 2 are possible. It is the teacher's task to interpret the results and decide how to proceed.

The recommended response activities after this diagnostic question are:

- Rectangles workshop 1
- Triangles workshop 1.

Response activity: Rectangles workshop 1

In this response activity, students are required to compare the areas of isoperimetric rectangles, also highlighting the boundary cases. The activity is inspired by one due to Emma Castelnuovo [25]. Students can use their hands and a string to visualize the question.

Materials

- Twine
- Squared paper
- Stationery

Method

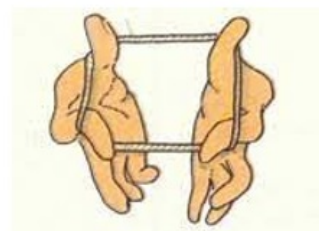
1. Cut the twine and tie the two ends to form a ring.
2. Take the twine between your fingers to form a rectangle.
3. Observe what happens when, keeping the string taut, you move your hands away or closer together.

What happens to your fingers when you move your hands away from each other or closer together?

What happens to the perimeter when you change one rectangle into another?

What happens to the area when you change one rectangle into another?

What happens when the rectangle collapses on the base or height?



The activity consists in a guided worksheet, divided into three parts.

In the first part (questions 1, 2 and 3) students reflect and discuss, making intuitive predictions on the proposed questions.

In second part (questions 4, 5, 6, 7 and 8) students experiment and calculate areas of rectangles with the usual formula. They then construct a table and a graph of the variation in the area as a function of a dimension (the base) of the rectangle.

4. Suppose that the total length of the twine between your fingers is 24 cm.
What is the sum of the lengths of the base and height?
5. Copy the table below into your notebook and fill it in: for each rectangle you can form with a twine of 24 cm, calculate the area. For simplicity, consider only integer values, starting from the limit case where the base is equal to 0.

Perimeter	Semiperimeter	Base	Height	Area
....		0		
		1		
				
				
				
			0	

Which rectangle has the greatest area?

Which rectangle has the least area?

6. Copy the columns of the table that include the lengths of the base and the corresponding areas.
7. Construct a graph on squared paper: the x-coordinates should be the lengths of the base; the y-coordinates should be the corresponding areas.
8. Describe the resulting graph.

Base	Area
0	
....	
....	
....	

In third part (questions 9, 10 and 11) students focus on the relationship between the dimensions of the isoperimetric rectangles and construct the corresponding graph.

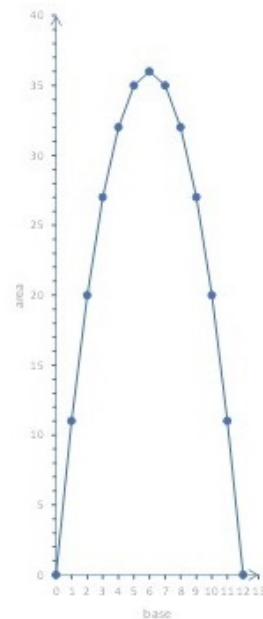
9. Now rewrite the table, keeping only the lengths of the base and the corresponding height.
10. Construct the corresponding graph on squared paper: the x-coordinates should be the lengths of the base; the y-coordinates should be the corresponding heights.
11. Describe the resulting graph.

Base	Height

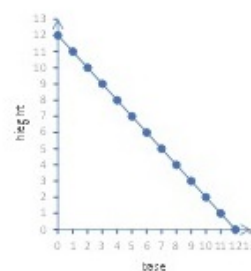
Students have to engage with the practical work and calculations; for the teacher, listening to individual groups as they work is crucial because it helps to highlight any difficulties that could be overcome through clarification or redirection aimed at the whole class during the activity.

This response activity also serves as a bridge to the topic of representations of relationships and functions in the Cartesian plane. At this level, it is not important to dwell on the equation of a parabola and a straight line. The important concept to highlight is the maximum point of the function, which can be read off the graph, and the linear relationship between the base and the height of the isoperimetric rectangles.

Base	Area
0	0
1	11
2	20
3	27
4	32
5	35
6	36
7	35
8	32
9	27
10	20
11	11
12	0



Base	Height
0	12
1	11
2	10
3	9
4	8
5	7
6	6
7	5
8	4
9	3
10	2
11	1
12	0



Response activity: Triangles workshop 1

The learning focus of this response activity is the comparison of isoperimetric triangles with the same base. Through observation and the use of artifact, students are invited to reflect on variations in area and the search for triangles with minimum and maximum area.

Triangles workshop 1

Emma is a mathematics teacher. Today she has brought an instructional tool to class for observing triangles. This tool consists of a cork board with two pins placed on it. She takes a piece of string, cuts it, and connects the two ends. She loops the string around the two pins and the tip of a pencil, keeping it taut. The two pins and the tip of the pencil form the vertices of a triangle. By adjusting the position of the pencil tip, various triangles can be created.

Predict

What will be the area and perimeter of the different triangles be?

Is there a triangle of maximum or minimum area?

Is there a triangle of maximum or minimum perimeter?

Explain

Why do you think you are right?

.....

Now carry out the investigation

Observe

Replicate a model of Emma's tool on your notebook and observe changes in the area and the perimeter as you transition between different triangles.

Consider limit cases as well.

Explain

Were your prediction and explanation correct?

If not, can you explain what you observed?

The model used for constructing this worksheet is the POE model (Predict - Observe - Explain) described by White and Gunston in [74] and is based on the classical model of scientific inquiry; a hypothesis is formulated and the reasons supporting it are provided, then relevant data are collected, and finally the results are discussed. In this case, students are shown a tool that allows them to reproduce isoperimetric triangles. Initially, they are asked to reflect on the nature of the different triangles that can be formed based on their own knowledge. In this first phase, students reflect individually and then share their reasoning in pairs or small groups. After an initial discussion, students, using the artifact, can observe and experiment, comparing their predictions with experimental evidence.

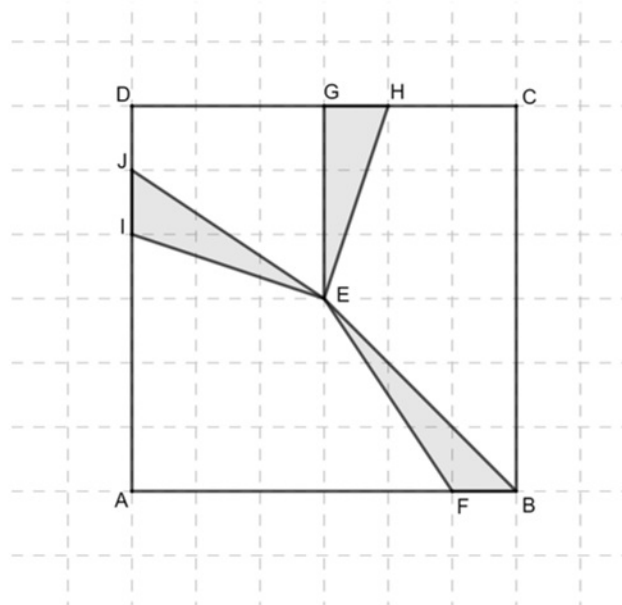
The worksheet provides guiding questions to facilitate reflection; it is helpful for each student to reproduce the model in their own notebook to develop and write a common response and explanation. At this point, after group discussion, students are encouraged to present their explanations to the class to make their reasoning visible.

Students who have completed *Rectangles workshop 1* should demonstrate familiarity with twine and an understanding of non-equivalent isoperimetric triangles. The limit cases, obtained when the tip of the pencil is on the same straight line as the base segment, should lead the students to identify the isosceles triangle as the one with the greatest area. Some students might argue for two triangles of maximum area, referring to symmetrical pencil positions; discussion can explore whether these are two distinct triangles or the same one.

Diagnostic question: Three triangles

Three triangles is a two-tier multiple-choice worksheet. The first tier is taken from the 2014 National INVALSI test for 8th grade. In the second tier, students must select the correct or best explanation for the answer they chose in the first tier.

Observe the triangles in the figure



1. Which of the following statements is true?

Tick (✓) the box next to the best answer.

- A** The three triangles have the same area and perimeter ☐
- B** The three triangles have the same area and different perimeter ☐
- C** The three triangles have different areas and the same perimeter ☐
- D** The three triangles have different areas and perimeters ☐

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

A The lengths of the bases and heights of the triangles are the same.

☐

B The lengths of the bases and heights of the triangles are different.

☐

C The triangles have a common vertex and bases of the same length.

☐

D The triangles are not congruent.

☐

Students are asked to recognize that the three triangles in the figure are equivalent but not isoperimetric. Depending on the responses and explanations provided, the teacher can decide how to proceed with the lesson; whether to review concepts of height or equivalence of plane figures, or to continue with the proposed response activities.

Response activity: Rectangles workshop 2

The response activity *Rectangles workshop 2* is designed similarly to *Rectangles workshop 1*. It is a guided worksheet where the perimeters of equivalent rectangles are compared, considering cases where the perimeter is the greatest or the least. For convenience, only integer values are considered for the dimensions of the rectangles. It will be the teacher's task, at the end of the activity, to guide the students towards abstraction. Drawing is used, but to make the work more dynamic, students can cut out the models they have constructed on squared paper.

For each pair/group, teacher provides 2 or 3 sheets of squared paper (1 cm grid). Each student works on their own notebook for drawing tables and graphs. However, they discuss and compare their work with others. In the first part (questions 1, 2, and 3), students construct equivalent rectangles either by drawing or by cutting out the figures.

Materials

- Sheet of squared paper (side 1 cm)
- Stationery

Method

1. Build a square of side 6 squares (6 cm)
What is the area of the square?
2. Construct as many square-equivalent rectangles as possible, considering only integer values for the lengths of the base and the height. Start with the base of length 1.
3. What happens to the perimeter as you change from one rectangle to another?

In the second part (questions 4, 5, 6, and 7), students calculate perimeters of rectangles and then construct the table and graph of variation in the perimeter as a function of one dimension (the base) of the rectangle.

4. Copy the table below in your notebook and fill it in by calculating the perimeter of each of the rectangles you have constructed.

Area	Base	Height	Perimeter
....	1		
	2		
			
			
			
		1	

Which rectangle has the greatest perimeter?

Which rectangle has the least perimeter?

5. Rewrite the table to include the length of the base and corresponding perimeter.
6. Construct a graph on squared paper: the x-coordinates should be the lengths of the base; the y-coordinates should be the corresponding perimeters.
7. Describe the graph that you obtain in this way.

Base	Perimeter
....	
....	
....	
....	

In the third part (questions 8, 9 and 10) students focus on the relationship between the dimensions of the equivalent rectangles and construct the corresponding graph.

8. Now rewrite the table, keeping only the lengths of the base and the corresponding height.
9. Construct the corresponding graph on squared paper: the x-coordinates should be the lengths of the base; the y-coordinates should be the corresponding heights.
10. Describe the resulting graph.

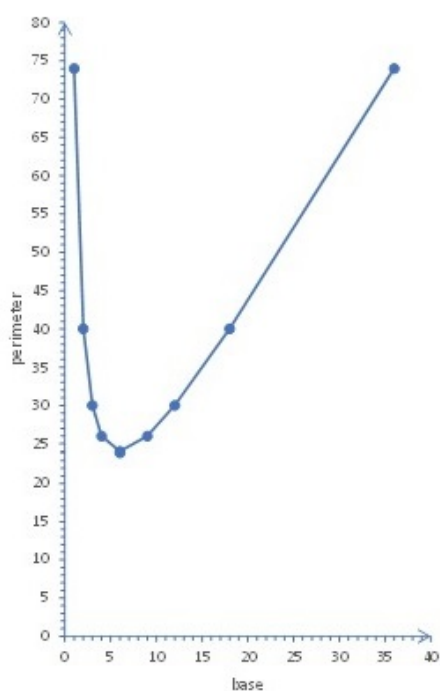
Base	Height
....	
....	
....	
....	

This activity can also be used to explore the topic of graphs of functions in the Cartesian plane. The key consideration is not necessarily the equation of the function linking the perimeter to the base, but rather an emphasis on the minimum point of the function, which can be read off the graph.

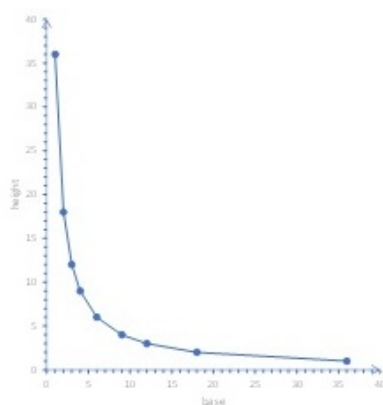
Regarding the relationship between base and height, it is important to stress that in the case of equivalent rectangles, the two quantities are inversely proportional, while the same cannot be said for isoperimetric rectangles. Pointing out the difference in the trend of the graphs can be useful in overcoming the misconception by those students who claim that two quantities are

inversely proportional when, as one increases, the other decreases.

Base	Perimeter
1	74
2	40
3	30
4	26
6	24
9	26
12	30
18	40
36	74



Base	Height
1	36
2	18
3	12
4	9
6	6
9	4
12	3
18	2
36	1

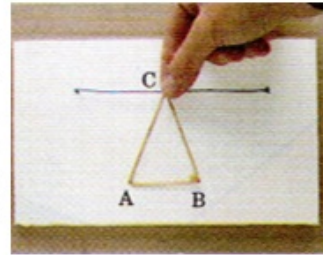


Response activity: Triangles workshop 2

The response activity *Triangles workshop 2* uses the POE model just like *Triangles workshop 1*. The artifact used is also very similar, but a guide is used to keep the distance of the third vertex from the base constant, and an elastic band is used to obtain different perimeters.

Triangles workshop 2

In today's class, Emma, a mathematics teacher, has introduced a tool for observing triangles. This tool consists of a cork board with two pins representing the ends of a segment. Parallel to the segment, she has placed two additional pins around which a wire was threaded. A small ring runs along the wire. An elastic band, tied to the initial points, passes through the ring. The vertices of the triangle are formed by the two initial points and the ring. By moving the ring, different triangles can be obtained.



Predict

What will the area and perimeter of the different triangles be?

Is there a triangle of maximum or minimum area?

Is there a triangle of maximum or minimum perimeter?

Explain

Why do you think you are right?

.....

Now carry out the investigation

Observe

Reproduce a model of the teacher's tool on your notebook and observe changes in the area and the perimeter as you transition between different triangles.
Consider limit cases as well.

Explain

Were your prediction and explanation correct?

If not, can you explain what you observed?

The learning focus is to explore variations in the perimeter of equivalent triangles. The model, along with the elastic's tendency to position itself in a state of minimal tension, should help understand that the triangle with the smallest perimeter is the isosceles one. Due to the limitations of the model, the mobile vertex cannot move indefinitely. Consequently, it becomes imperative for the teacher to guide pupils, encouraging abstraction to envision the limit positions.

Chapter 5

Stages of testing in class

5.1 Research strategy and sample selection

The resources developed in our project are designed to help teachers and their students understand where they are in the teaching-learning process. They are conceived as a formative assessment tool capable of making teaching effective and students' learning visible. At the moment, they represent an innovation in the Italian landscape of mathematics education, as there are no studies in the literature documenting the use of this type of worksheets as a formative assessment tool for teachers and students.

To evaluate whether this innovation can have an impact on teachers and students, we decided to initiate an initial pilot study in a very small context, involving a limited number of teachers and students.

We considered teaching through the worksheets as the experimental factor to be compared with traditional teaching (the control factor).

5.1.1 Participants

The study involved a total of 4 teachers and 11 7th-grade classes from the Istituto Comprensivo “F.lli Trillini” in Osimo (AN). These were divided into two groups: an experimental group and a control group.

We chose to use a recurring institutional sample: we applied the ordinary factor to the six second-year classes of the 2022/2023 school year (control group) and the experimental factor to the five second-year classes of the 2023/2024 school year (experimental group). The teachers involved are all mathematics teachers in the school to reduce selection bias. Environmental and historical conditions are similar for all classes in the comprehensive institute.

We chose this approach to minimize the effects that Trinchero in [71] believes could influence empirical research in education. When research, like ours, aims to evaluate whether the use of worksheets can be considered a tool to help teachers and students understand where they are in the teaching-learning process, it is necessary to isolate the use of this tool from other factors that may interfere with this relationship.

Assessing whether there is actual causality to evaluate the impact of a methodological innovation is not simple at all, since experimentation in educational sciences is not conducted in a laboratory but in the field, i.e., in classrooms.

Trinchero in [71] states:

Experimentation differs from standard research, interpretative research, and case study because the success of the research itself depends on the researcher's ability to manipulate and control factors, while in other research strategies, excluding action research, success depends on the researcher's ability to collect data on the objects of study while modifying the reality under examination as little as possible and considering as many factors as possible, rather than excluding them.

To isolate and study the effects of the experimental factor, it is necessary to construct an experimental design. We chose to adopt a two-group experimental design: experimental group and control group.

The success of this design depends on the two groups being as equivalent as possible and also on other factors that may affect the dependent variable being equivalent.

Trinchero in [71] discusses various effects that can interfere:

- Selection effect: when only teachers interested in the project are asked to participate in the experimentation. In this case, the comparison with the results of the control group can be influenced by the motivation of the teachers.
- Environment effect: occurs when the two groups are not in similar environmental conditions, such as classroom characteristics or scheduling differences.
- History effect: if there are events like strikes or issues in assigning classes to teachers that affect only one of the two groups.
- Drop-out effect: when there are cases of school dropouts among students during the experimentation or teachers dropping out of the project.
- Hawthorne effect: when the students subject to the experimentation feel they are the center of attention and, therefore, tend to adopt an attitude more in line with the teachers' expectations.
- Novelty effect: when the initial results of the new methodology yield better outcomes not because of the method itself but due to its novelty.
- Disruption effect: contrary to the previous effect, initially disappointing effects can occur due to teachers' unfamiliarity with the new method.
- Testing effect: the presence of an initial test is essential to see if there has been actual improvement, but from the content of the initial test, subjects might understand what the researcher expects from them, thus affecting the results.

It is necessary to try to minimize as much as possible other factors that can influence the research. By using a recurrent institutional cyclical sample (second-year classes from the 2022/2023 and 2023/2024 school years), we aimed to minimize environmental, historical, and dropout effects.

5.1.2 Research design

Phase 1, March 2023

The 7th-grade classes of the 2022/2023 school year have already covered topics related to the area and perimeter of plane figures with their teachers. Students are administered a test to assess their learning.

To introduce teachers to the worksheets and their use in the classroom, we conducted an intervention in the classes focusing on the key concept of height. The lesson was led by an external expert in co-presence with the regular teacher. The interventions serve a dual purpose:

- to train teachers on how to use worksheets in the classroom as a means of formative assessment, promoting dialogue and collaboration among students;
- to assess the impact of this methodology on students.

At the end of the activity, which serves as reinforcement of what the students have already done, we did not evaluate whether performance improved compared to the previous test. Instead, we administered a questionnaire to assess the effect of this teaching methodology on the students.

Phase 2, September 2023

A meeting was held to explain and hand over the worksheets to the teachers to initiate the experimentation in the 7th-grade classes of the 2023/2024 school year. It was decided not to include the presence of an external observer during the experimentation to give teachers the autonomy to decide how and when to use the worksheets. This was also done to prevent teachers from feeling observed and thereby possibly adopting attitudes more in line with the researcher's expectations (Hawthorne effect).

Phase 3, March 2024

At the end of the experimentation in the classes, students are administered the test to assess their learning and the satisfaction questionnaire used for the control classes.

The teachers had a discussion with the researcher and responded to a questionnaire to describe the just-concluded teaching experience.

5.2 Data collection tools

5.2.1 Teacher questionnaire setup

The purpose of the questionnaire for teachers is to evaluate the effect of using worksheets on the teaching process. The aspects on which they were asked to reflect are:

1. How the worksheets were used
2. The effect of the materials on their own teaching methods
3. The observed effect of the activities on the students

We chose to primarily use closed-ended questions, leaving an open-ended response option in case of negative feedback regarding the use of *teacher notes*, to access any reasons that were not considered in the research. To indicate the effect on their teaching and on the students, we opted for the semantic differential type of question to better specify the varying degrees of intensity of the intermediate responses.

Indicator 1: Use of the material. Below are the questions related to the current indicator:

1. Before presenting the activity in class, did you consult the general worksheet with the progression of topics?
☐Yes ☐No
1a. If you answered no, please explain the reasons for your choice:
.....
.....
1b. If you answered yes:
 - Did you strictly follow the suggested order in the progression? ☐Yes ☐No
 - Did you modify the order of the worksheets and topics? ☐Yes ☐No
 - Did you present the worksheets consecutively in a single learning unit? ☐Yes ☐No
 - Did you intersperse the worksheets with other activities related to the same topic? ☐Yes ☐No
2. What method did you use to present the worksheets? (multiple answers possible)
 - ☐ Projection on IWB and response by raising hands or using colored cards
 - ☐ Printed worksheets for individual use
 - ☐ Printed worksheets and work in pairs or small groups
 - ☐ Other... (please specify).....

Indicator 2: Effect on teaching. To indicate the effect on their teaching and on the students, we chose to use the semantic differential type of question to minimize the use of the response *quite*, which in different subjects might express different meanings or varying degrees of intensity.

3. Did you consult the teacher notes before presenting the activity in class? ☐Yes ☐No
3a. If you answered no, please explain the reasons for your choice:
.....
.....
If you answered yes...

4. Do you think reading the teacher notes influenced your behavior during the lesson?

☐Yes ☐No

4a. If you answered no, please explain the reasons for your choice:

.....
.....

5. If yes, in relation to which behaviors and how much?

5a. Identification of student misconceptions

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

5b. Attention to the language you used during teaching

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

5c. Attention to student responses to verify learning progression

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

5d. Attention to student responses to adapt teaching in the next phase

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

Indicator 3: Effect on students. Below are the questions related to the last indicator:

6. From your observations, do you think students have improved their skills in terms of...

6a. participation?

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

6b. collaboration?

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

6c. argumentation?

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

6d. metacognition?

not at all ☐1 ☐2 ☐3 ☐4 ☐5 very much

5.2.2 Student questionnaire setup

The purpose of the student questionnaire is to assess the impact of the worksheets on the students. Three indicators have been considered:

1. Quality of the worksheets
2. Student participation
3. Metacognition and self-assessment

A paper questionnaire was prepared to be administered in the presence of teachers to explain its purposes and avoid adding an introductory preamble. Regarding reliability and validity checks, given the small number of recipients, it is not possible to conduct a pretest, i.e., a pilot administration.

Indicator 1. Quality of the worksheets. To indicate the quality of the worksheets, we chose to use a semantic differential type of question to minimize the use of the response *quite*, which in different subjects might express different meanings or varying degrees of intensity.

1. The proposed activities seemed to me:

boring ☐1 ☐2 ☐3 ☐4 ☐5 exciting

2. The questions/requests of the sheet seemed to you:

difficult to understand ☐1 ☐2 ☐3 ☐4 ☐5 easy to understand

Indicator 2. Student participation. For this indicator, we highlighted certain attitudes and behaviors of the students and requested a frequency scale.

3. I answered all the questions there:

☐never ☐rarely ☐sometimes ☐often ☐always

4. I participated in class discussion:

☐never ☐rarely ☐sometimes ☐often ☐always

5. I collaborated with my classmates during the practical activity:

☐never ☐rarely ☐sometimes ☐often ☐always

6. I asked the teacher for explanations when I didn't understand:

☐never ☐rarely ☐sometimes ☐often ☐always

7. I asked classmates for explanations when I didn't understand:

☐never ☐rarely ☐sometimes ☐often ☐always

Indicator 3. Metacognition and self-assessment. For this indicator as well, we chose to use questions with a semantic differential scale.

8. During these activities I felt:

unable ☐1 ☐2 ☐3 ☐4 ☐5 able

excluded ☐1 ☐2 ☐3 ☐4 ☐5 involved

tense ☐1 ☐2 ☐3 ☐4 ☐5 relaxed

9. My knowledge before starting the class was:

nothing ☐1 ☐2 ☐3 ☐4 ☐5 very good

10. My knowledge at the end of the activities is:

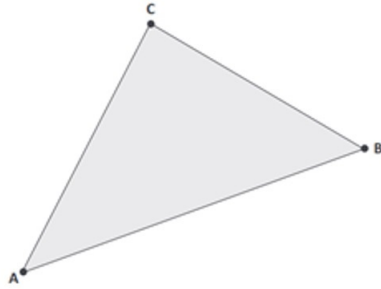
nothing ☐1 ☐2 ☐3 ☐4 ☐5 very good

5.2.3 Learning assessment test setup

For the construction of the learning assessment test, seven questions were considered.

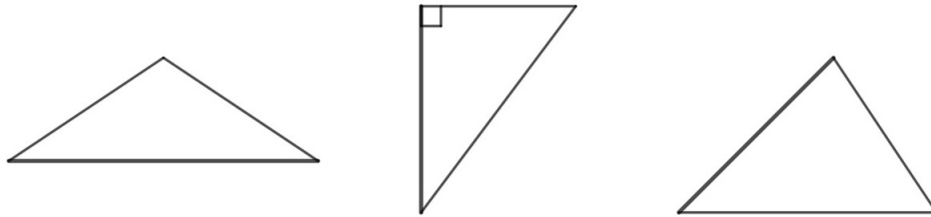
The first two questions explicitly ask to draw the height of the triangles represented in the figure.

1. Observe the figure.



Draw in triangle ABC the height CH relative to side AB.

2. Trace the height of the triangle relative to the highlighted side.

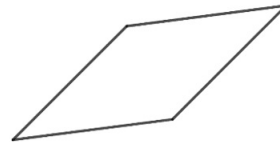


In question 3, students are asked to find the area of the figures represented by extracting the necessary data from the drawing. This question was chosen both to test knowledge of the usual formulas for calculating the area and to see if students could provide the correct basic representation and corresponding height.

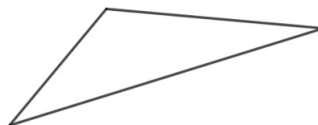
3. Find the area of the following plane figures, taking the necessary measurements with a ruler and marking on the figure what you have measured.



.....



.....



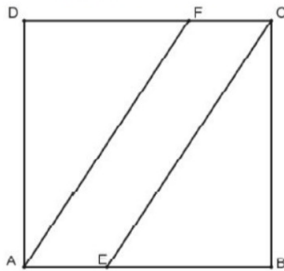
.....

In questions 4 and 5, students are asked to recognize the relationship that exists between the areas of two figures.

Question 4 is an open-ended question where one figure is contained within another.

Question 5 is a multiple-choice question where students must identify the equivalence of two triangles by inferring they have the same base and height.

4. Consider square ABCD with a side length of 6 cm. AE and FC each measure 2 cm. What is the area of quadrilateral AECF? Justify your answer.



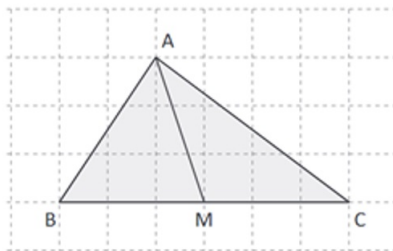
.....

.....

.....

.....

5. In the triangle shown, segment AM connects vertex A with the midpoint of side BC. The triangle is thus divided into two triangles.



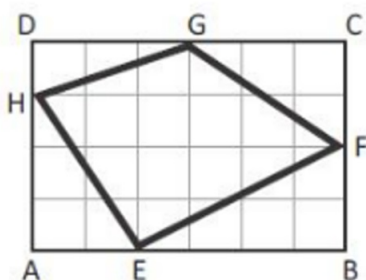
Are the two triangles ABM and AMC equivalent?

- ☐ Yes, because triangles ABM and AMC have a base and the corresponding height of equal dimensions.
- ☐ Yes, because side AM is common to triangles ABM and AMC.
- ☐ No, because triangles ABM and AMC are not congruent.
- ☐ No, because segment AM is the median relative to side BC of triangle ABC.

Questions 6 and 7 are open-ended questions asking to calculate the area of the represented figure.

In question 6, the quadrilateral is contained within a squared rectangle, while in question 7, there is a polygon with excess parts equal to the missing ones.

6. The figure shows quadrilateral EFGH whose vertices are on the sides of rectangle ABCD. The dimensions of the rectangle are 4 m and 6 m. Calculate the area of quadrilateral EFGH and explain how you found it.



.....

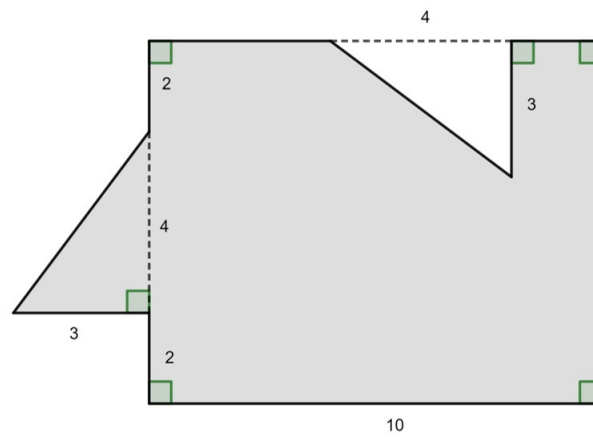
.....

.....

.....

.....

7. Calculate the area of the colored figure and explain how you found it.



The questions were taken from [1], [63], and [18].

Part IV

Data collection, analysis and conclusions

Chapter 6

Data collections and analysis

6.1 Description of the sample and the control

The following table shows the distribution of teachers in classes for the school years 2022/2023 and 2023/2024. The teachers involved have been labeled with numbers, while the classes have been labeled with letters.

school year 2022/2023	Teacher 1	Teacher 2	Teacher 3	Teacher 4	Teacher 5	Teacher 6
	Class A	Class B	Class C	Class D	Class E	Class F
school year 2023/2024	Teacher 1	Teacher 2	Teacher 3	Teacher 4	Teacher 4	
	Class A	Class B	Class C	Class D	Class E	

In the 2023/2024 school year, there are only five second-year classes. Teacher 5 is no longer at the school, and Teacher 6 does not teach second-year classes this year. As a result, there are only 4 teachers participating in the experiment compared to the 5 classes involved.

6.2 Analysis of teacher questionnaire

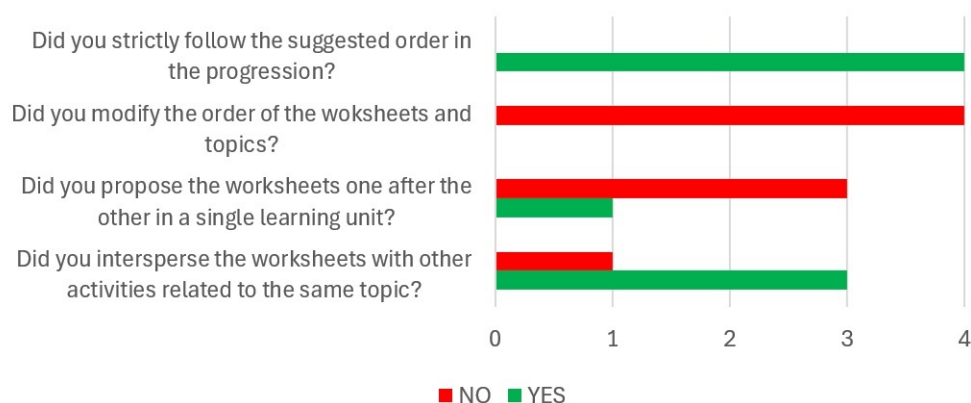
The analysis of the teacher questionnaires is based on the indicators highlighted during the preparation phase.

6.2.1 Indicator 1: use of the material

All the teachers involved consulted the learning progression toolkit. The presentation order of the worksheets was followed by all teachers except teacher 3. Teacher 2 conducted the activity

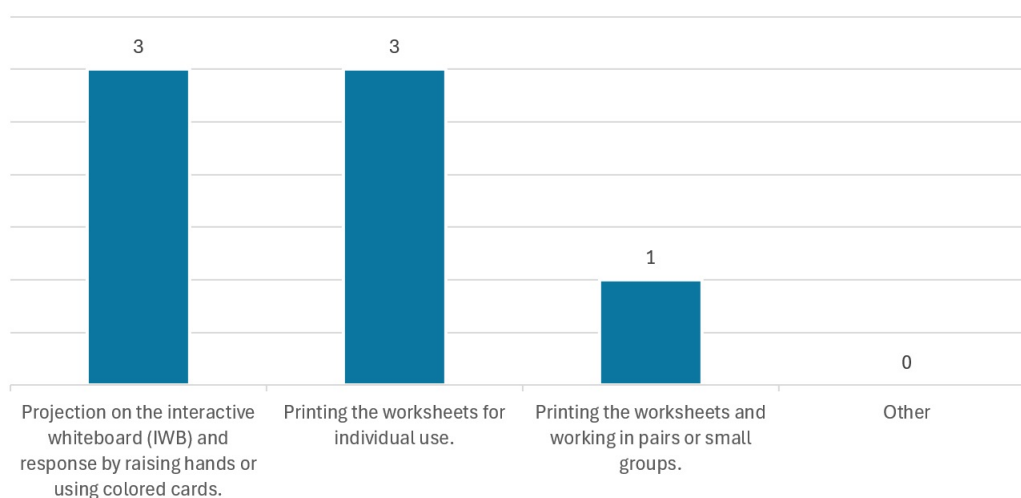
within a single learning unit, while for the others, the activity was distributed across several units. All of the teachers interspersed the worksheets with other activities.

How did you carry out the activity in class?



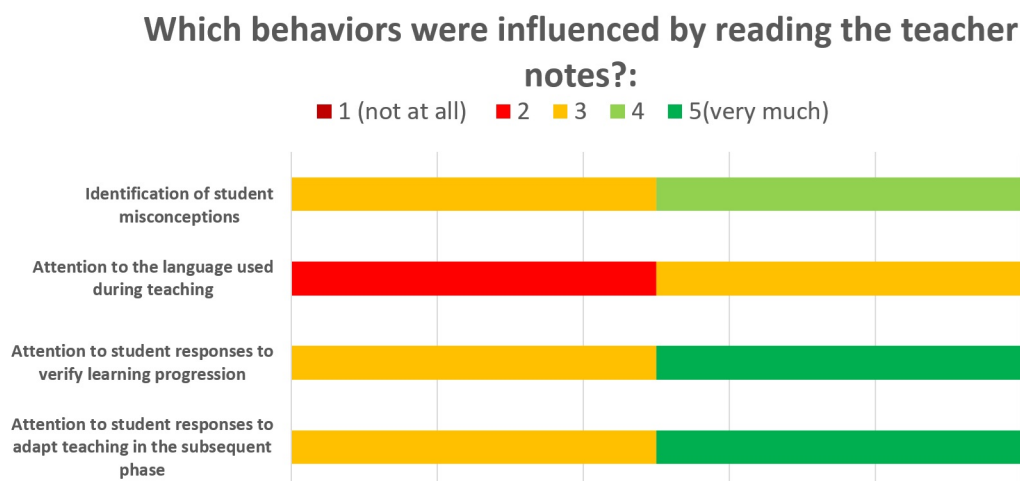
As for the presentation method, teacher 1 used the worksheets in various ways: projection on the Interactive Whiteboard (IWB), printing for individual use, and printing for pair or small group work. Teacher 2 projected the worksheets on the IWB, teacher 3 used both the IWB and printed the worksheets for individual use, while teacher 4 only printed the worksheets for individual use.

What method did you use to propose the worksheets? (Multiple answers are possible)



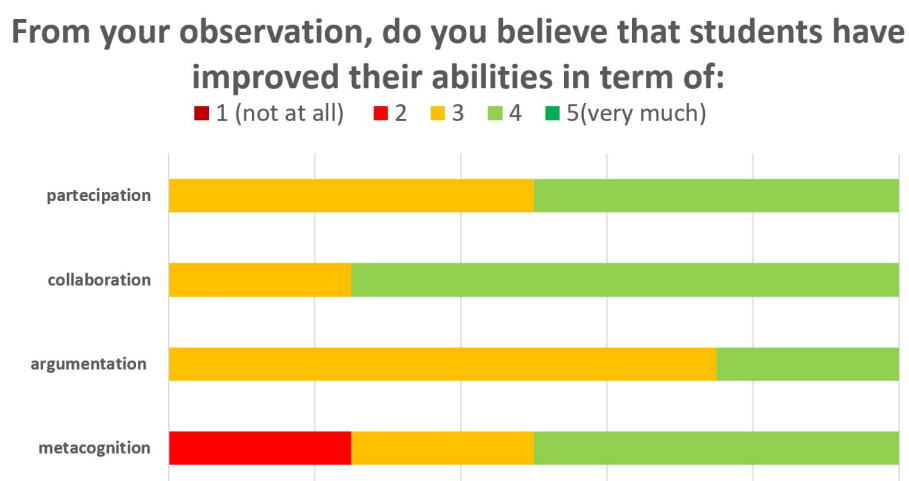
6.2.2 Indicator 2: effect on teaching

Only Teachers 1 and 2 stated that they consulted the teacher notes, and that this influenced their behavior in class during the activities. The specific behaviors influenced and the extent to which they were reported are as follows:



6.2.3 Indicator 3: effect on students

The teachers were asked to use their observations to assess any improvements in students in terms of participation, collaboration, argumentation, and metacognition. The bar chart summarizes the responses provided by the teachers.



6.3 Analysis of student questionnaire

The table shows the number of students who were administered the questionnaire in the experimental and control classes.

School year 2023/2024 experimental group	n. of students
Class A	21
Class B	21
Class C	17
Class D	22
Class E	20
Tot	101

School year 2023/2024 control group	n. of students
Class A	14
Class B	16
Class C	21
Class D	25
Class E	19
Class F	16
Tot	111

Since the activity with the worksheets in the control classes was conducted by the external researcher in the presence of the regular teachers, we decided to collect all the data from the 111 students and refer to it generically as the control group. We then compared the results of each experimental class with those of the control group.

For each question, the following are provided:

- a comparison graph of the overall responses from the experimental classes with those of the control group;
- a level-by-level comparison graph;
- test 1: a χ^2 test for levels 1 to 5 compared to the control group;
- test 2: a χ^2 test for 2 grouped levels (sum of levels 1, 2, and 3 vs. sum of levels 4 and 5) compared to the control group.

Test 1: χ^2 test for levels 1, 2, 3, 4, and 5. The 5 results corresponding to the values 1, 2, 3, 4, and 5 on the Likert scale are denoted by E_i .

Degrees of freedom: 4

$\alpha = 0.05$

Critical value: $\chi_c^2 = 9.49$

ϕ_{i0} is the probability of result E_i in the control group, composed of n_0 students.

ϕ_{ix} is the probability of result E_i in the sample class X, composed of n_x students.

$$H_0 : \phi_{ix} = \phi_{i0}$$

$$H_1 : \phi_{ix} \neq \phi_{i0}$$

- $F_{o,i,0}$ ($i = 1, 2, 3, 4, 5$) is the observed absolute frequency of result E_i in the control group and $F_{o,i,x}$ be the observed absolute frequency of result E_i in the experimental class X ($X = A, B, C, D, E$);
- ϕ_{i0} is the relative frequency in the control group, i.e. $F_{o,i,0}/n_0$;
- $f_{a,i,x}$ with ($X = A, B, C, D, E$) is the expected absolute frequency in the considered experimental class, i.e. $f_{a,i,x} = \phi_{i0} n_x$;
- n_0 is the size of the control group;
- n_x is the size of class X.

Class X:

	Level 1	Level 2	Level 3	Level 4	Level 5
$F_{o,i,x}$	$F_{o,1,x}$	$F_{o,2,x}$	$F_{o,3,x}$	$F_{o,4,x}$	$F_{o,5,x}$
$f_{a,i,x}$	$\frac{F_{o,1,0}}{n_0} n_x$	$\frac{F_{o,2,0}}{n_0} n_x$	$\frac{F_{o,3,0}}{n_0} n_x$	$\frac{F_{o,4,0}}{n_0} n_x$	$\frac{F_{o,5,0}}{n_0} n_x$

$$\chi^2 = \sum_{i=1}^5 \frac{(F_{o,i,0} - f_{a,i,x})^2}{f_{a,i,x}}$$

Test 2: χ^2 test for the sum of levels 1, 2, 3 and the sum of levels 4 and 5. We have considered the following events for the χ^2 test comparing proportions between the control group and class X:

E_P : The student responded with a 4 or 5.

E_N : The student responded with a 1, 2, or 3 (opposite event to E_P).

Degrees of freedom: 1

$\alpha = 0.05$

Critical value: $\chi_c^2 = 3.84$

ϕ_0 is the probability of the result E_P in the control group, composed of n_0 students.

ϕ_x is the probability of result E_P in the sample class X, composed of n_x students.

$$H_0 : \phi_x = \phi_0$$

$$H_1 : \phi_x \neq \phi_0$$

- $F_{o,0}$ is the observed absolute frequency of the event E_P in the control group;
- $F_{o,x}$ is the observed absolute frequency of the event E_P in the experimental class X ($X = A, B, C, D, E$);
- $f_{a,0}$ is the expected absolute frequency of the event E_P in the control group;
- $f_{a,x}$ is the expected absolute frequency of the event E_P in the in the experimental class X;
- n_0 is the size of the control group;
- n_x is the size of class X.

The expected frequencies are calculated as follows:

$$f_{a,0} = n_0 p$$

$$f_{a,x} = n_x p$$

where:

$$p = \frac{(F_{o,0} - F_{o,x})}{n_0 + n_x}$$

To simplify the calculation, we used the following table format:

Observed frequencies	E_P (level 1, 2 o 3)	E_N (level 4 o 5)	Total
control	$F_{o,0}$	$n_0 - F_{o,0}$	n_0
class X	$F_{o,x}$	$n_x - F_{o,x}$	n_x
Totals	$F_{o,0} + F_{o,x}$	$n_0 - F_{o,0} + n_x - F_{o,x}$	$n_0 + n_x$
Expected frequencies	E_P (level 4 o 5)	E_N (level 1, 2 o 3)	Total
control	$f_{a,0}$	$n_0 - f_{a,0}$	n_0
class X	$f_{a,x}$	$n_x - f_{a,x}$	n_x
Totals	$f_{a,0} + f_{a,x}$	$n_0 - f_{a,0} + n_x - f_{a,x}$	$n_0 + n_x$

$$\chi^2 = \frac{(F_{o,0} - f_{a,0})^2}{f_{a,0}} + \frac{(n_0 - F_{o,0} - (n_0 - f_{a,0}))^2}{n_0 - f_{a,0}} + \frac{(F_{o,x} - f_{a,x})^2}{f_{a,x}} + \frac{(n_x - F_{o,x} - (n_x - f_{a,x}))^2}{n_x - f_{a,x}}$$

For the purposes of the research, having the probability of the event E_P (levels 4 or 5) in the experimental class equal to or significantly higher than that in the control group is considered a positive result.

Therefore, if the null hypothesis (H_0) is rejected, we check whether the sum of the values corresponding to levels 4 and 5 is higher or lower than in the control group.

The analysis of the responses is divided based on the indicators.

The interpretation of the results emerging from this analysis is presented in the Chapter 7.

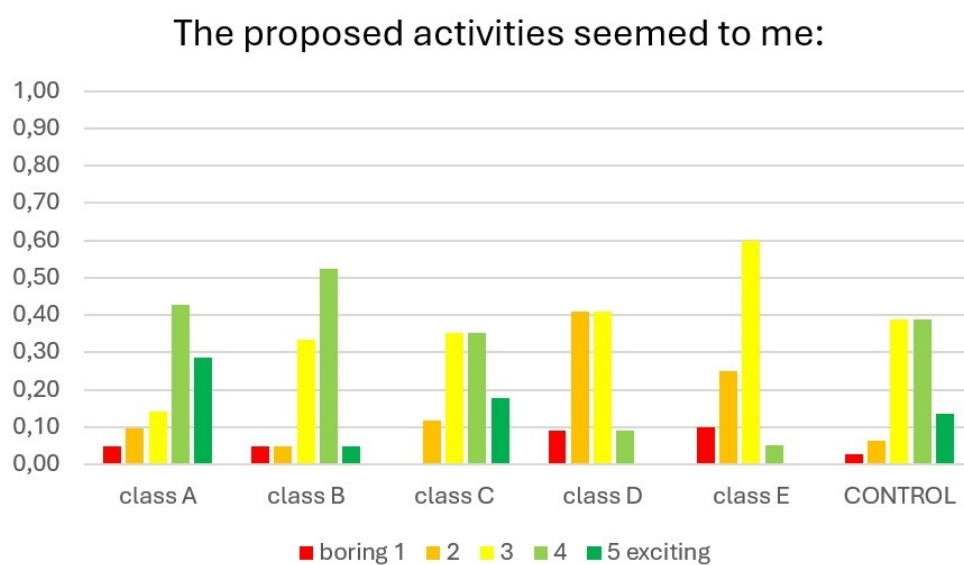
6.3.1 Indicator 1: quality of the worksheets

Question 1

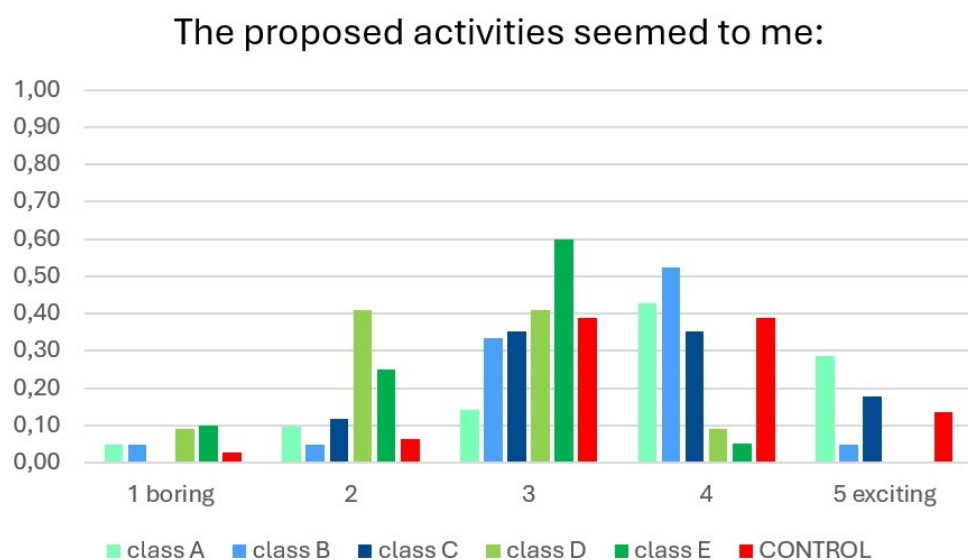
The proposed activities seemed to me:

boring 1 2 3 4 5 exciting

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	7,54	
Class B	2,77	
Class C	1,58	
Class D	53,08	
Class E	25,94	

The hypothesis H_0 can be rejected for classes D and E.

	Control	Class D	Class E
Level 1	3%	9%	10%
Level 2	6%	41%	25%
Level 3	39%	41%	60%
Level 4	39%	9%	5%
Level 5	14%	0%	0%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	2,63	
Class B	0,17	
Class C	0,00	
Class D	13,81	
Class E	15,29	

The hypothesis H_0 can be rejected for classes D and E. The sum of the frequencies of levels 4 and 5 in the considered classes is much lower compared to the control.

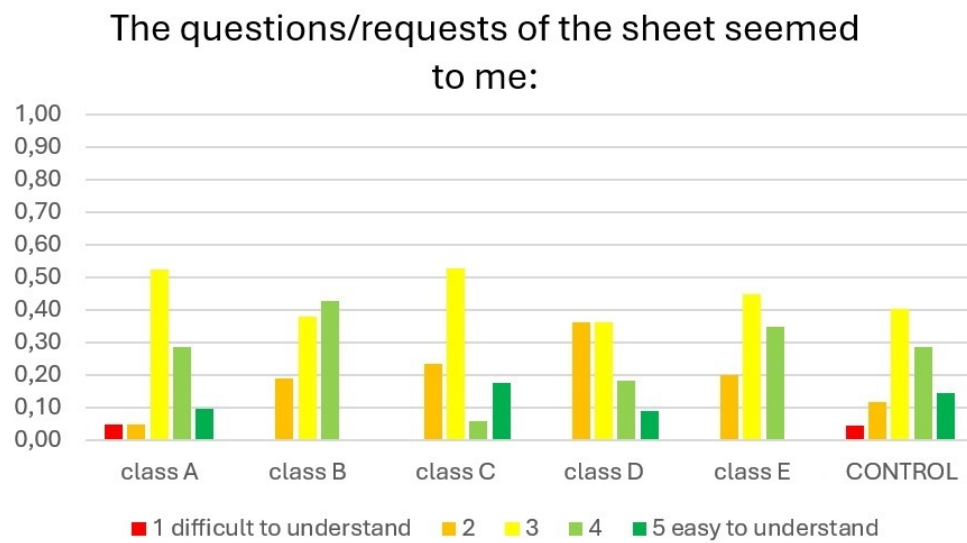
	Control	Class D	Class E
Sum of the frequencies of levels 4 and 5	52%	9%	5%

Question 2

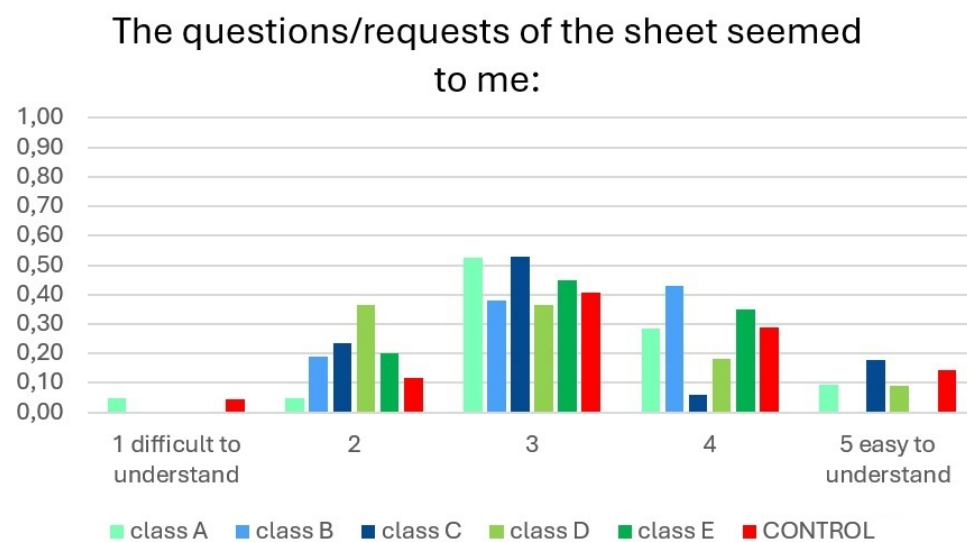
The questions/requests of the sheet seemed to me:

difficult to understand 1 2 3 4 5 easy to understand

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	1,94	
Class B	6,40	
Class C	6,67	
Class D	13,80	
Class E	5,32	

The hypothesis H_0 can be rejected for class D.

	Control	Class D
Level 1	5%	0%
Level 2	12%	36%
Level 3	41%	36%
Level 4	29%	18%
Level 5	14%	9%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	0,19	
Class B	0,00	
Class C	2,38	
Class D	1,94	
Class E	0,47	

The hypothesis H_0 can't be rejected.

6.3.2 Indicator 2: student participation

For this indicator, certain attitudes and behaviors of the students were highlighted, and a frequency scale was requested. In the first three questions in this group, students were asked how frequently they answered the questions, how frequently they participated in discussions, and how frequently they collaborated with their classmates. The analysis of the responses provides us with insights into how the lessons were conducted in the classroom, reflecting what the teachers themselves stated in their respective questionnaires.

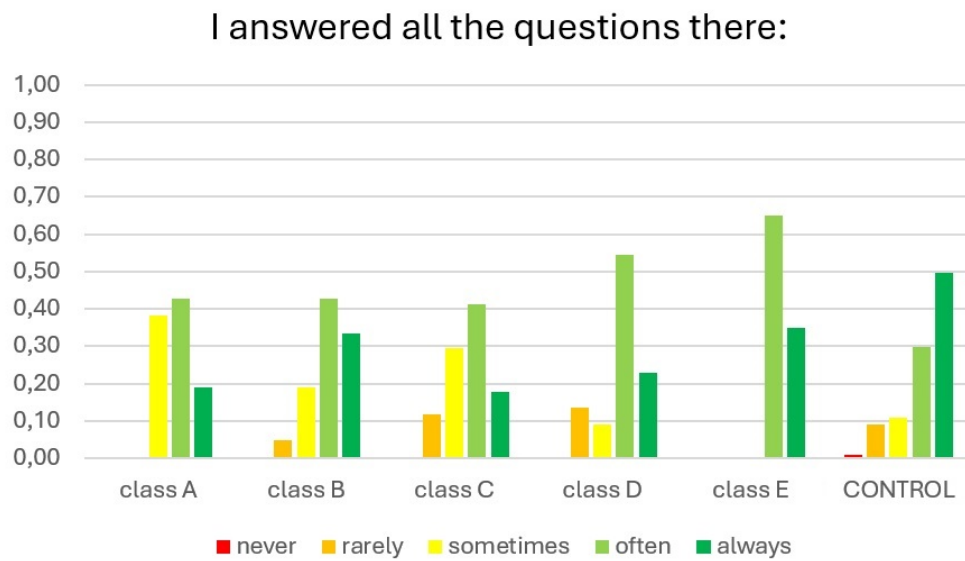
To analyze the students' responses and allow a quantitative comparison with the control data, we assigned the values 1, 2, 3, 4, 5 to *never*, *rarely*, *sometimes*, *often*, *always* respectively, and applied the same criteria used for the other questions.

Question 3

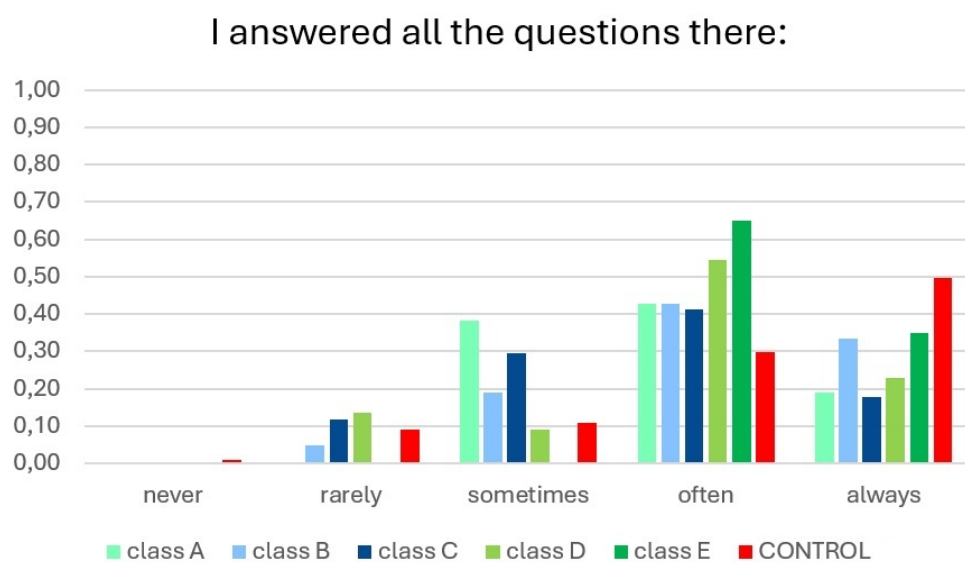
I answered all the questions there:

never rarely sometimes often always

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	21,70	
Class B	4,26	
Class C	9,98	
Class D	8,53	
Class E	13,37	

The hypothesis H_0 can be rejected for classes A, C, and E.

	Control	Class A	Class C	Class E
Level 1	1%	0%	0%	0%
Level 2	9%	0%	12%	0%
Level 3	11%	38%	29%	0%
Level 4	30%	43%	41%	65%
Level 5	50%	19%	18%	35%

Test 2

The χ^2 test provided the following values.

	χ^2	$\chi_c^2 = 3.84$
Class A	2,97	
Class B	0,10	
Class C	3,44	
Class D	0,04	
Class E	5,03	

The hypothesis H_0 can be rejected for class E, where all students reported always or almost always answering all the questions.

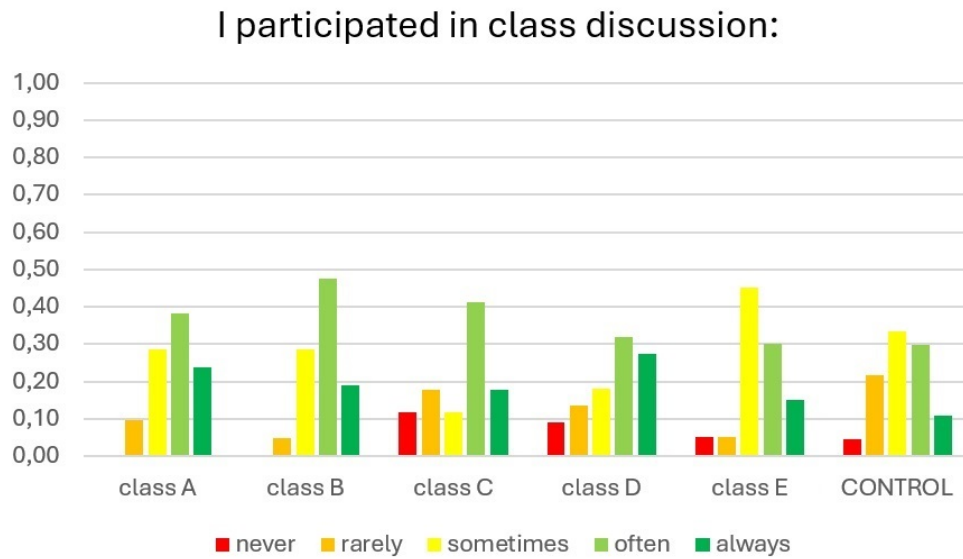
	Control	Class E
Sum of the frequencies of levels 4 and 5	79%	100%

Question 4

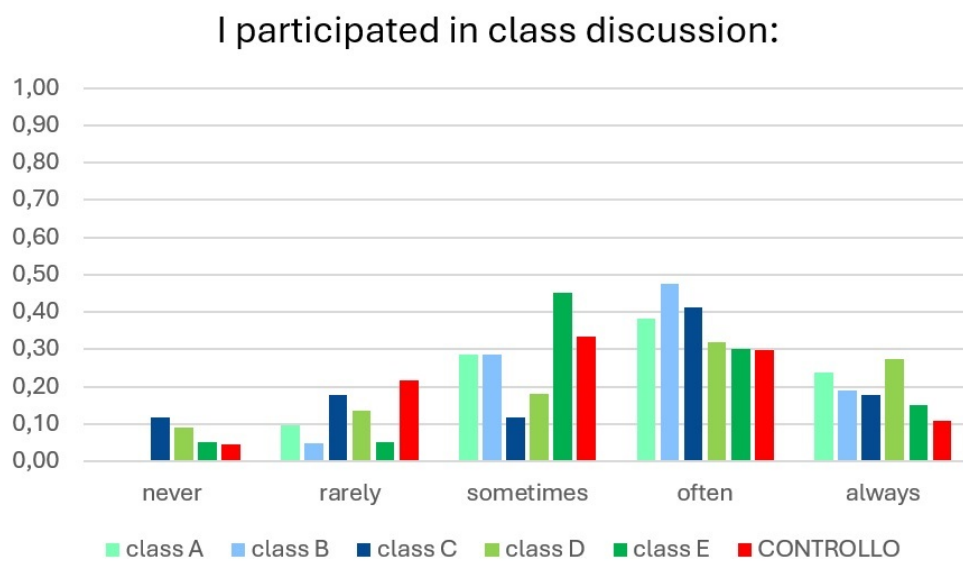
I participated in class discussion:

never rarely sometimes often always

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	6,29	
Class B	7,43	
Class C	5,97	
Class D	8,74	
Class E	3,71	

In this case, the hypothesis H_0 can be accepted for all classes.

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	3,27	
Class B	4,88	
Class C	2,01	
Class D	2,57	
Class E	0,14	

The hypothesis H_0 can be rejected for class B.

	Control	Class B
Sum of the frequencies of levels 4 and 5	41%	67%

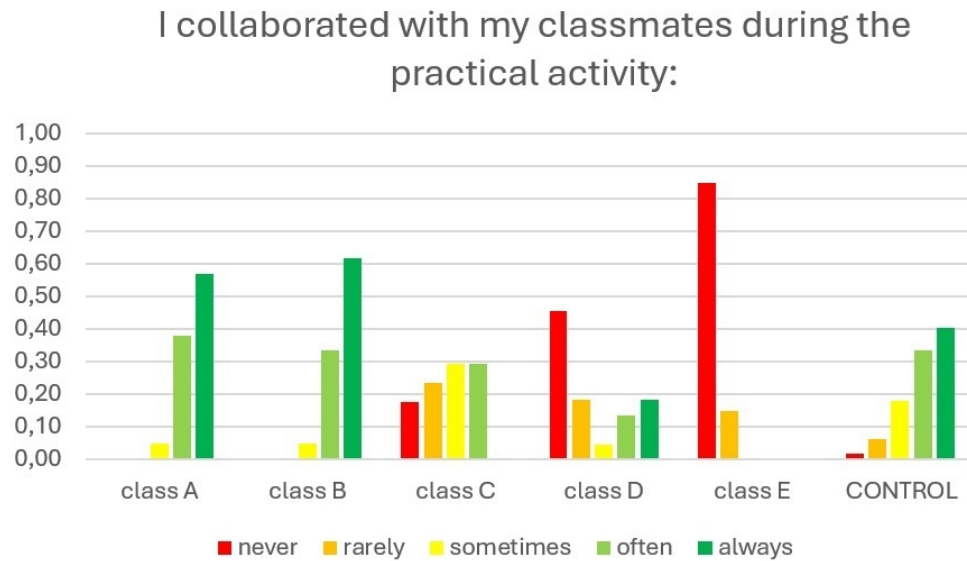
The frequencies of levels 4 and 5 in class B are higher compared to the control.

Question 5

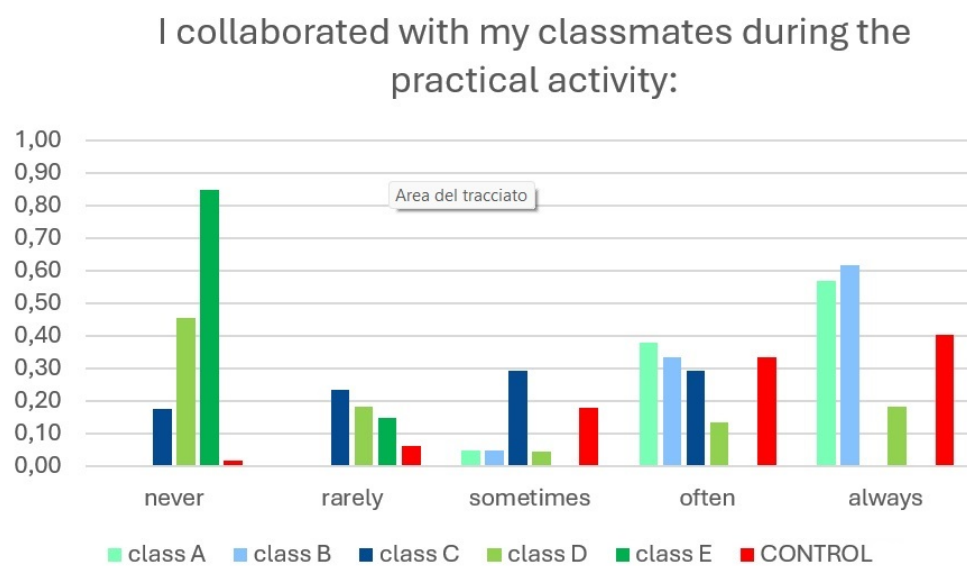
I collaborated with my classmates during the practical activity:

never rarely sometimes often always

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	5,32	
Class B	6,12	
Class C	39,88	
Class D	245,08	
Class E	789,11	

The hypothesis H_0 can be rejected for classes C, D, and E.

	Control	Class C	Class D	Class E
Level 1	2%	18%	45%	85%
Level 2	6%	24%	18%	15%
Level 3	18%	29%	5%	0%
Level 4	33%	29%	14%	0%
Level 5	41%	0%	18%	0%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	4,59	
Class B	4,59	
Class C	13,39	
Class D	14,67	
Class E	39,50	

For all classes, the critical value is exceeded, leading to the rejection of the hypothesis H_0 . However, in classes A and B, there is an improvement compared to the control, while in classes C, D, and especially E, there is a decline.

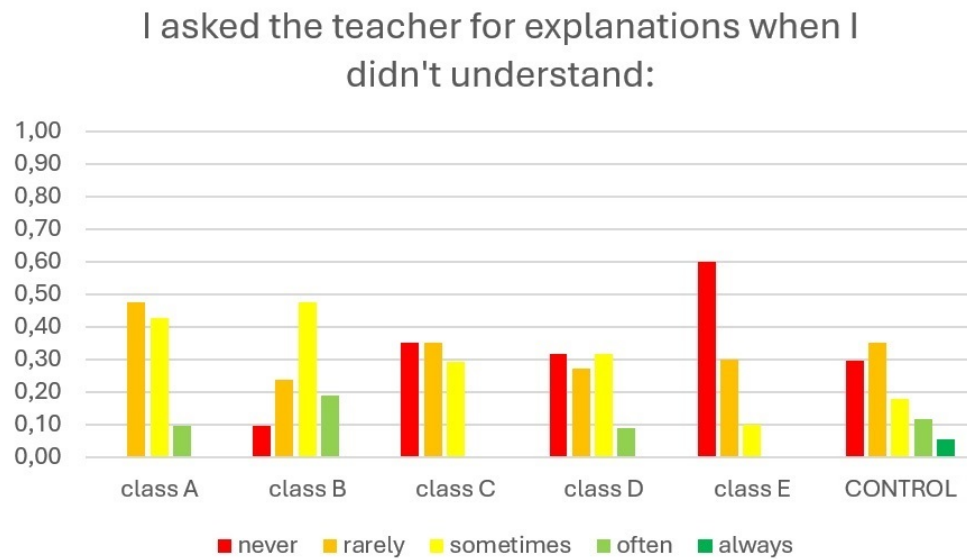
	Control	Class A	Class B	Class C	Class D	Class E
Sum of the frequencies of levels 4 and 5	74%	95%	95%	29%	32%	0%

Question 6

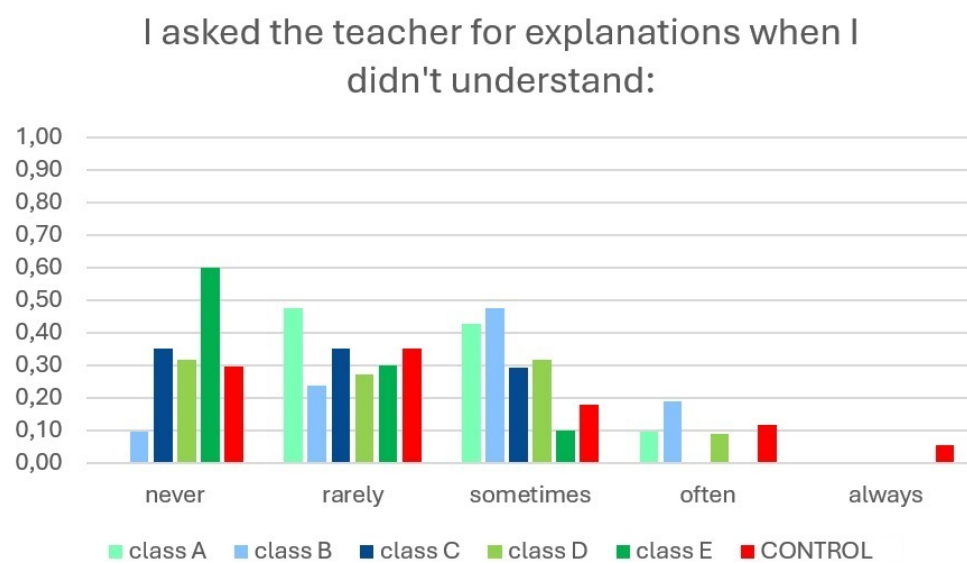
I asked the teacher for explanations when I didn't understand:

never rarely sometimes often always

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	15,59	
Class B	15,96	
Class C	4,31	
Class D	4,06	
Class E	10,45	

The hypothesis H_0 can be rejected for classes A, B, and E.

	Control	Class A	Class B	Class E
Level 1	30%	0%	10%	60%
Level 2	35%	48%	24%	30%
Level 3	18%	43%	48%	10%
Level 4	12%	10%	19%	0%
Level 5	5%	0%	0%	0%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	0,76	
Class B	0,05	
Class C	3,42	
Class D	18,82	
Class E	4,00	

The hypothesis H_0 can be rejected for classes D and E.

	Control	Class D	Class E
Sum of the frequencies of levels 4 and 5	17%	9%	0%

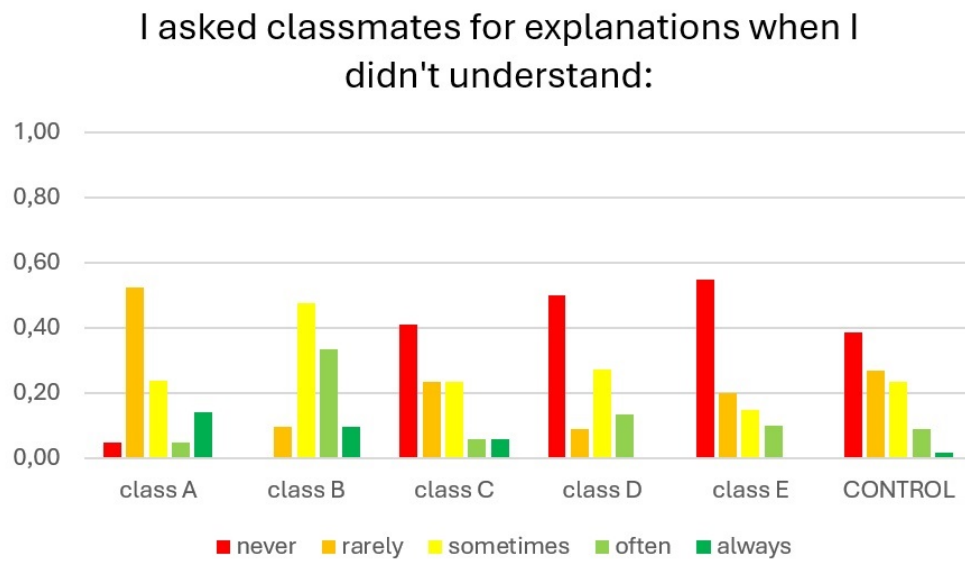
In both cases, the frequencies of levels 4 and 5 are lower than those of the control.

Question 7

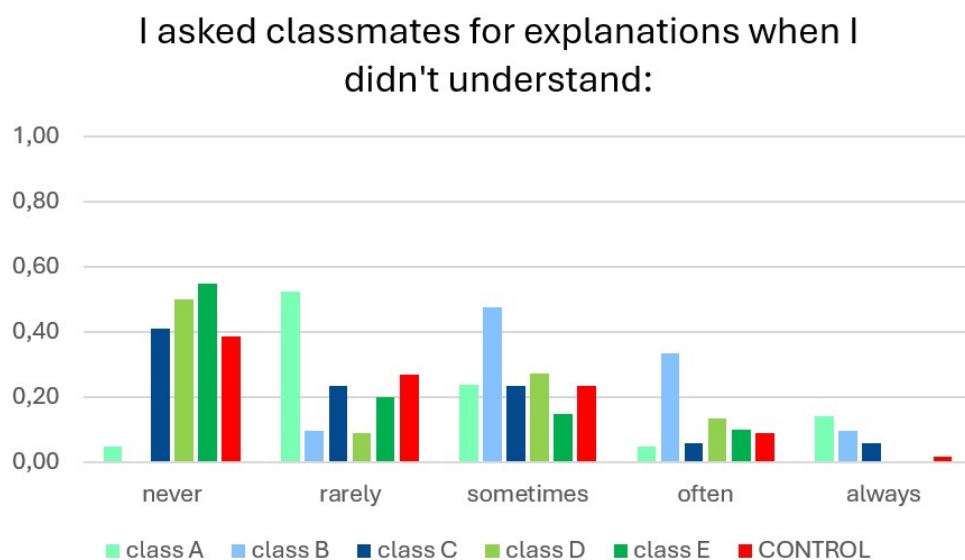
I asked classmates for explanations when I didn't understand:

never rarely sometimes often always

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	29,84	
Class B	36,51	
Class C	1,86	
Class D	4,40	
Class E	2,72	

The distribution of responses 1, 2, 3, 4, and 5 is significantly different from the control for classes A and B.

	Control	Class A	Class B
Level 1	39%	5%	0%
Level 2	27%	52%	10%
Level 3	23%	24%	48%
Level 4	9%	5%	33%
Level 5	2%	14%	10%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	1,12	
Class B	13,56	
Class C	0,01	
Class D	0,06	
Class E	0,01	

The frequency of levels 4 and 5 is higher than that of the control.

	Control	Class B
Sum of the frequencies of levels 4 and 5	11%	43%

6.3.3 Indicator 3: metacognition and self-assessment

For this indicator, questions with a semantic differential were also used.

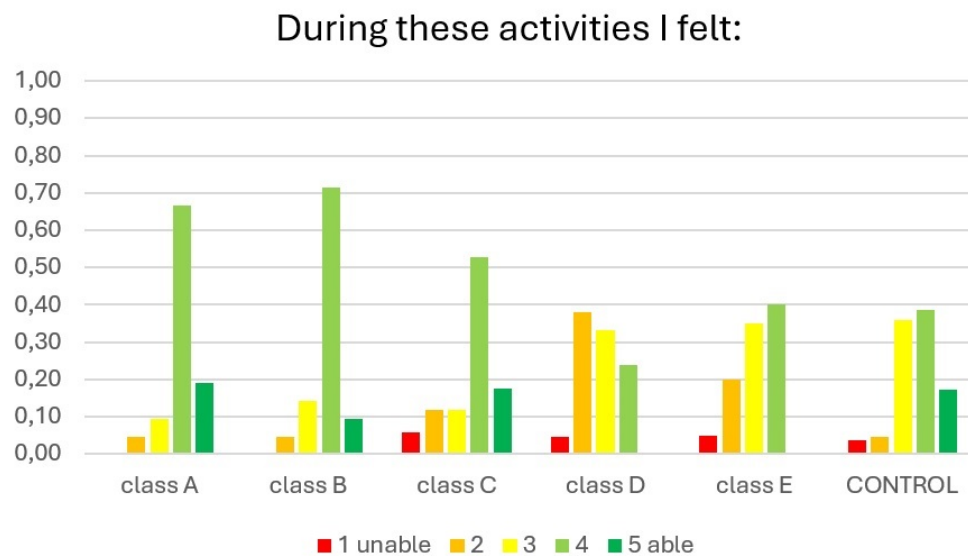
Question 8 In the first question in this group, students were asked to describe how they felt during the activities by choosing between three pairs of contrasting adjectives: *unable/able*, *excluded/involved*, and *tense/relaxed*.

For the first *unable/able* dichotomy:

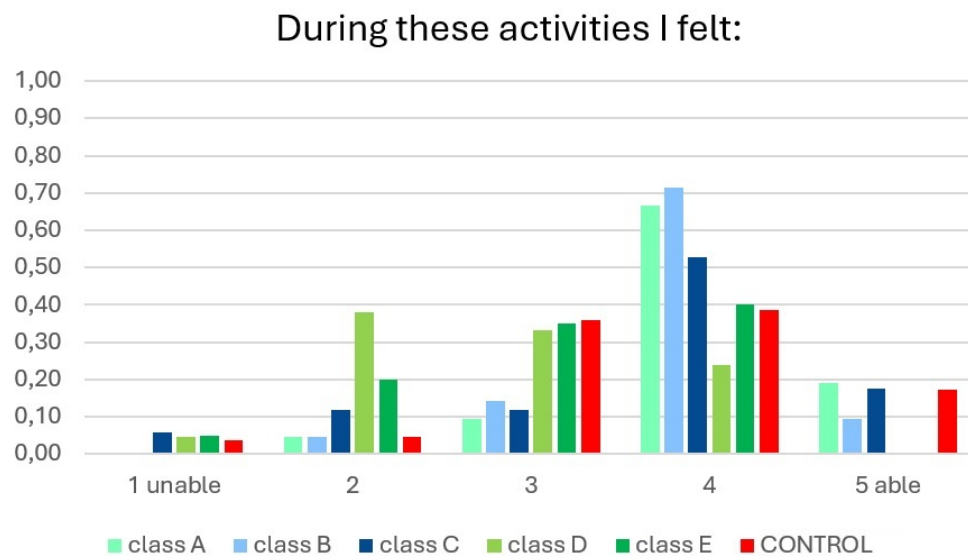
During these activities I felt:

unable 1 2 3 4 5 able

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	9,13	
Class B	10,02	
Class C	5,90	
Class D	57,53	
Class E	14,21	

The hypothesis H_0 can be rejected for classes B, D, and E.

	Control	Class B	Class D	Class E
Level 1	4%	0%	5%	5%
Level 2	5%	5%	38%	20%
Level 3	36%	14%	33%	35%
Level 4	39%	71%	24%	40%
Level 5	17%	10%	0%	0%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	6,59	
Class B	4,63	
Class C	1,31	
Class D	7,26	
Class E	1,71	

In this case, the hypothesis H_0 can be rejected for classes A, B, and D.

	Control	Class A	Class B	Class D
Sum of the frequencies of levels 4 and 5	56%	86%	81%	24%

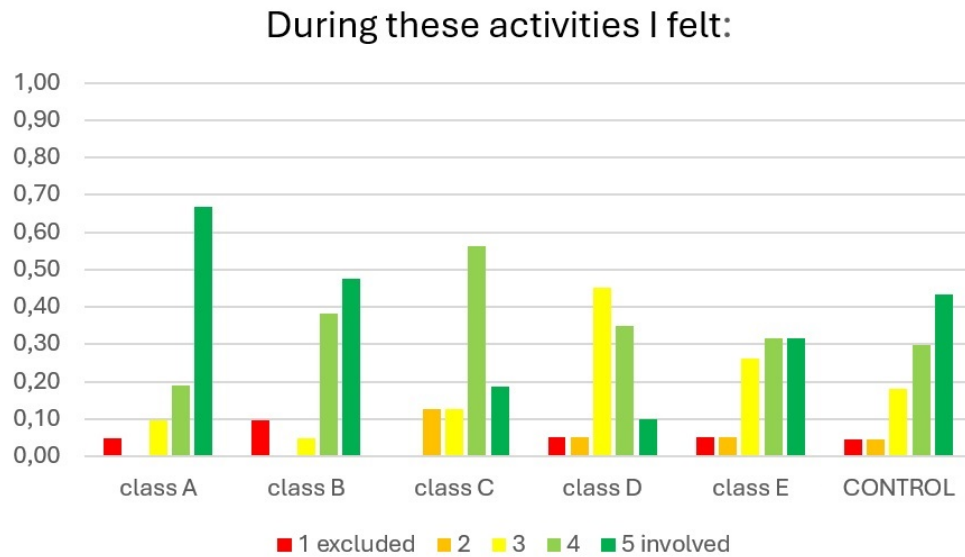
In classes A and B, the students felt more capable compared to the control group, unlike those in class D.

For the second dichotomy *excluded/involved* the data collected are as follow.

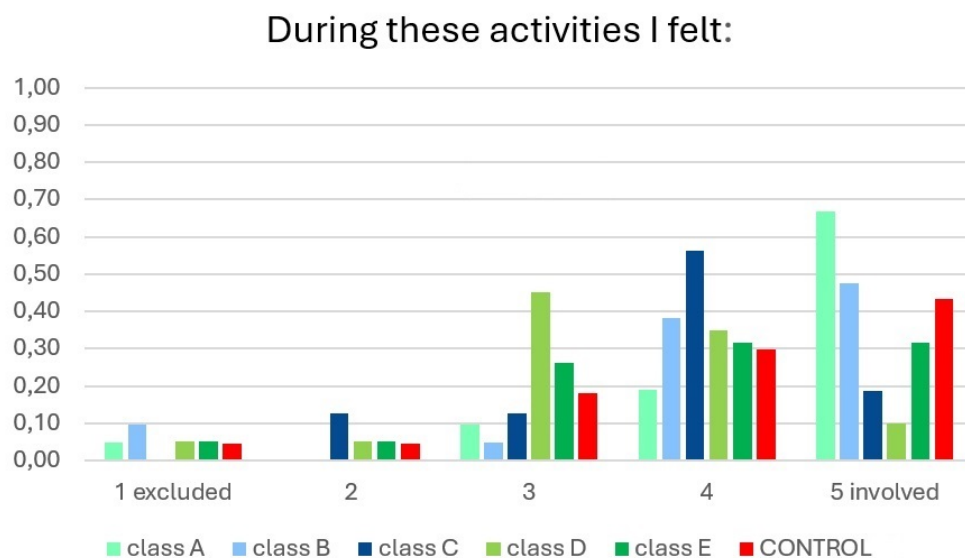
During these activities I felt:

excluded 1 2 3 4 5 involved

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	5,26	
Class B	4,76	
Class C	9,27	
Class D	13,40	
Class E	1,39	

The hypothesis H_0 can be rejected for class D.

	Control	Class D
Level 1	5%	5%
Level 2	5%	5%
Level 3	18%	45%
Level 4	30%	35%
Level 5	43%	10%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	1,53	
Class B	1,53	
Class C	0,03	
Class D	6,17	
Class E	0,77	

In this case as well, the hypothesis H_0 can be rejected for class D, where the number of students who felt involved is significantly lower compared to the control.

	Control	Class D
Sum of the frequencies of levels 4 and 5	73%	45%

Finally, the data of third dichotomy *tense/relaxed* are the following.

During these activities I felt:

tense 1 2 3 4 5 relaxed

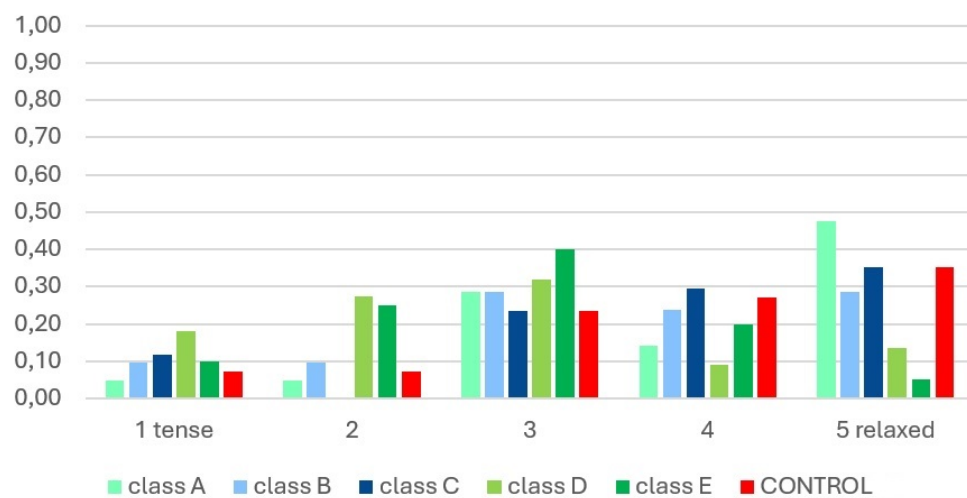
Comparison between classes

During these activities I felt:



Comparison between levels

During these activities I felt:



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	2,78	
Class B	0,89	
Class C	1,75	
Class D	22,14	
Class E	16,89	

The hypothesis H_0 can be rejected for classes D and E.

	Control	Class D	Class E
Level 1	7%	18%	10%
Level 2	7%	27%	25%
Level 3	23%	32%	40%
Level 4	27%	9%	20%
Level 5	35%	14%	5%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	0,00	
Class B	0,71	
Class C	0,04	
Class D	11,57	
Class E	9,52	

In this case as well, the hypothesis H_0 can be rejected for classes D and E.

	Control	Class D	Class E
Sum of the frequencies of levels 4 and 5	62%	23%	25%

The final questions, 9 and 10, concern students' perceptions of their knowledge level before and after the activity with the worksheets.

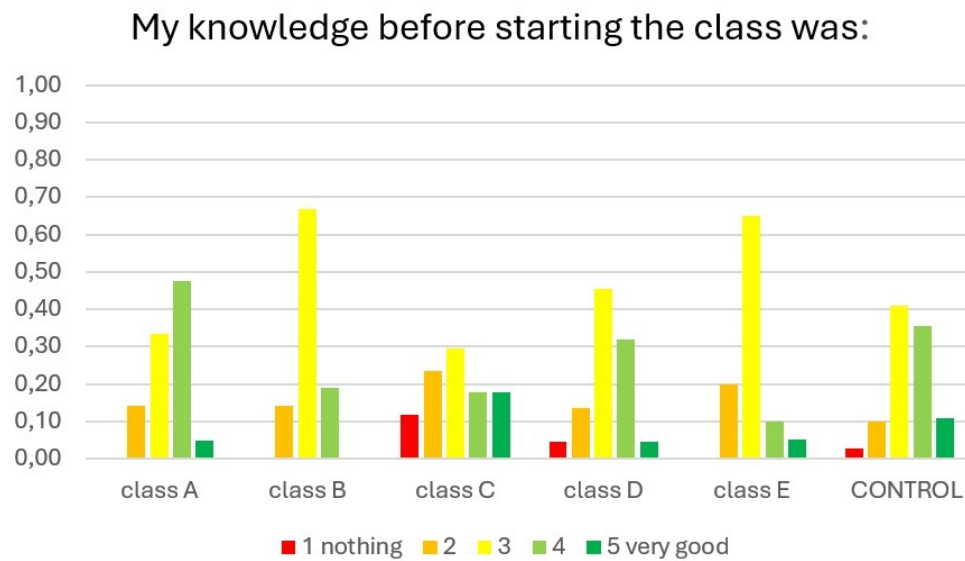
In the control group, an increase in self-assessment levels can be observed: the percentages for the lower levels (1, 2, and 3) decrease, while those for levels 4 and 5 increase.

Question 9

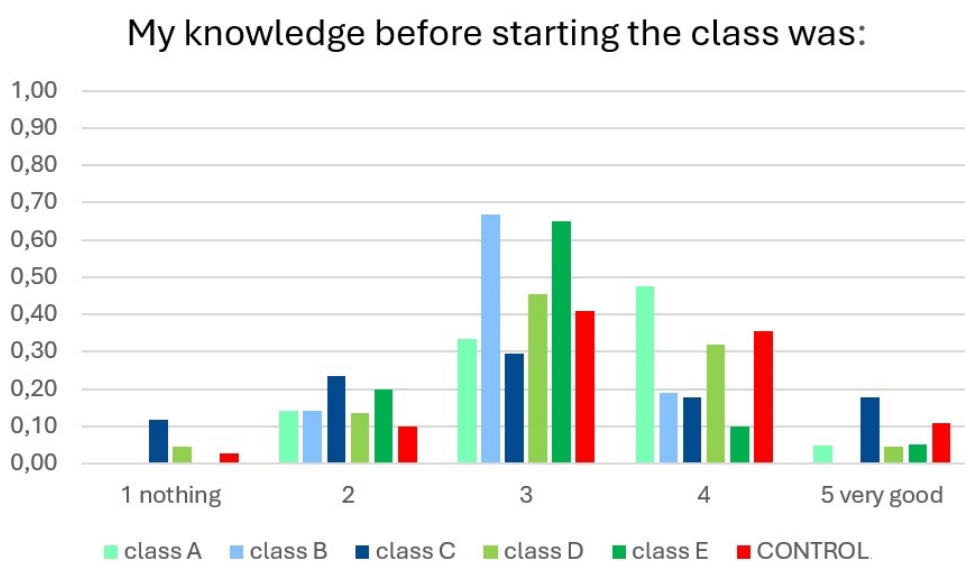
My knowledge before starting the class was:

nothing 1 2 3 4 5 very good

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	2,86	
Class B	8,25	
Class C	10,98	
Class D	1,57	
Class E	9,68	

The hypothesis H_0 can be rejected for classes C and E.

	Control	Class C	Class E
Level 1	3%	12%	0%
Level 2	10%	24%	20%
Level 3	41%	29%	65%
Level 4	35%	18%	10%
Level 5	11%	18%	5%

Test 2

	χ^2	$\chi_c^2 = 3.84$
Class A	0,26	
Class B	5,40	
Class C	0,73	
Class D	0,74	
Class E	6,86	

In classes B and E, the students perceive themselves as generally less competent compared to the control group.

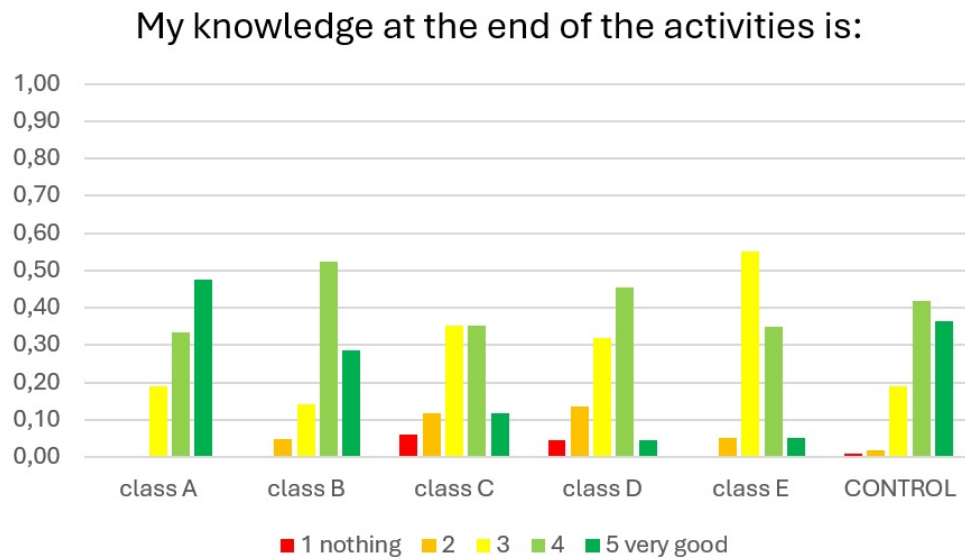
	Control	Class B	Class E
Sum of the frequencies of levels 4 and 5	46%	19%	15%

Question 10

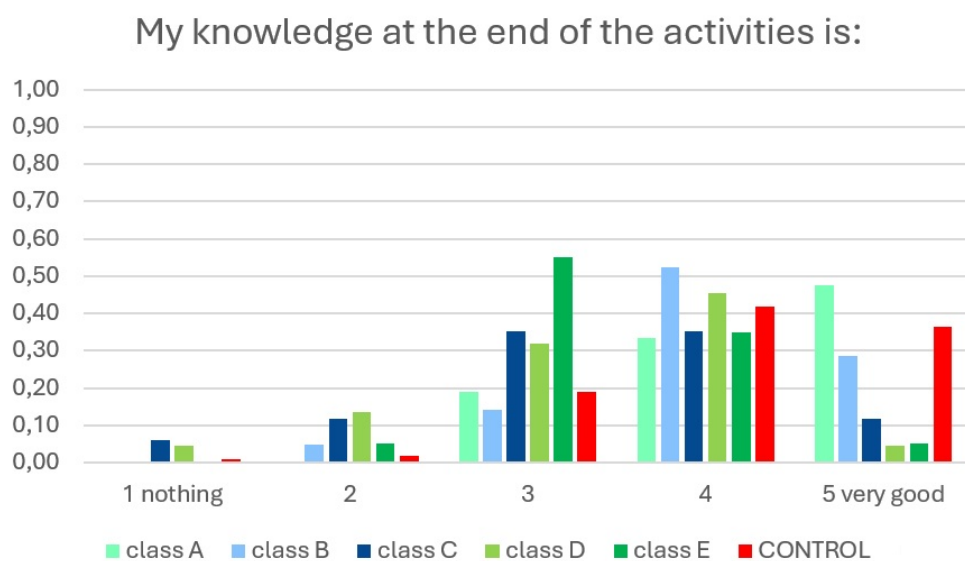
My knowledge at the end of the activities is:

nothing 1 2 3 4 5 very good

Comparison between classes



Comparison between levels



Test 1

	χ^2	$\chi_c^2 = 9.49$
Class A	1,67	
Class B	2,36	
Class C	19,22	
Class D	28,16	
Class E	20,44	

The hypothesis H_0 can be rejected for classes C, D, and E.

	Control	Class C	Class D	Class E
Level 1	1%	6%	5%	0%
Level 2	2%	12%	14%	5%
Level 3	19%	35%	32%	55%
Level 4	42%	35%	45%	35%
Level 5	36%	12%	5%	5%

Test 2

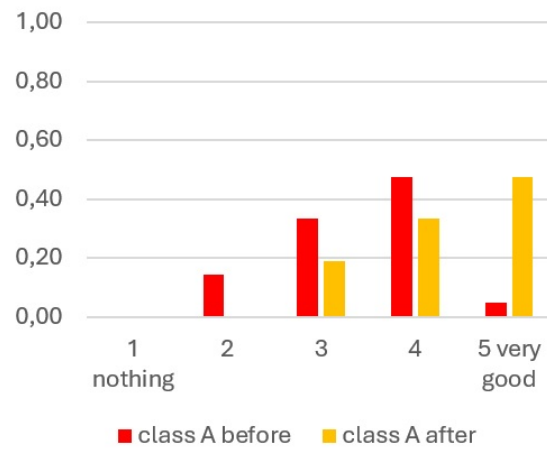
	χ^2	$\chi_c^2 = 3.84$
Class A	0,08	
Class B	0,08	
Class C	7,42	
Class D	7,47	
Class E	12,32	

The hypothesis H_0 can be rejected for classes C, D, and E, where students perceive themselves as less competent at the end of the course compared to the control group.

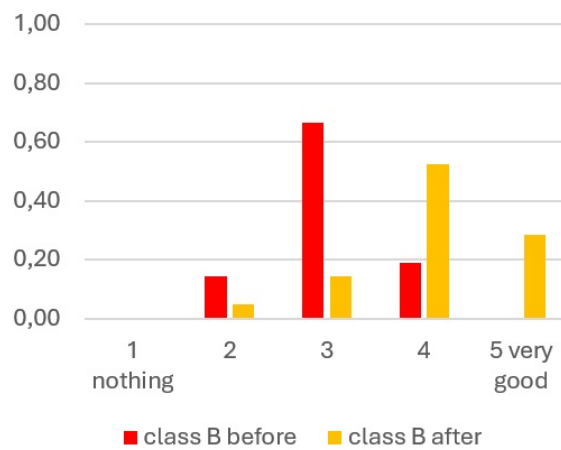
	Control	Class C	Class D	Class E
Sum of the frequencies of levels 4 and 5	78%	47%	50%	40%

In classes C, D, and E, the percentage of students who feel competent has increased compared to before, but the percentages of the highest values (4 and 5) remain low compared to those observed in the control group. It may be useful to compare the frequencies of levels 1 to 5 for each class, before and after the educational intervention.

Class A: Knowledge before
and after



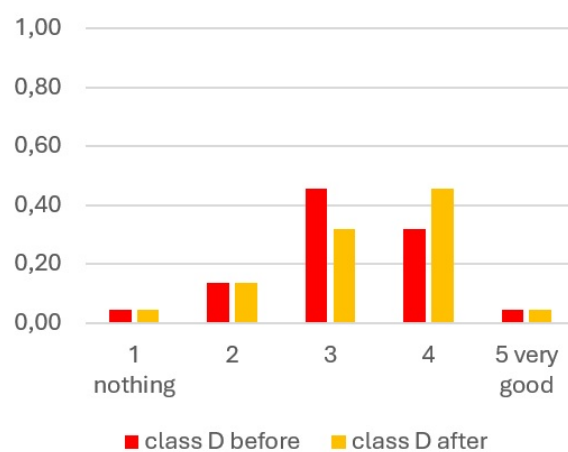
Class B: Knowledge before
and after



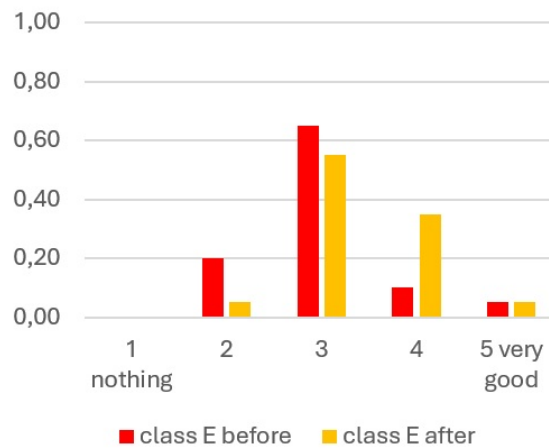
Class C: Knowledge before
and after



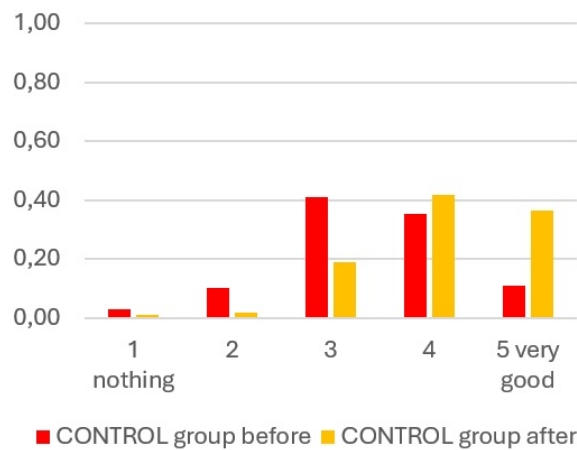
Class D: Knowledge before
and after



Class E: Knowledge before and after



Control group: knowledge before and after



6.4 Analysis of the learning assessment test

The learning assessment test was designed to answer the third research question:

Do the interventions with the worksheets improve the learning of the studied concepts?

To evaluate this, we decided to compare experimental classes of the school year 2023/2024 with

the corresponding classes taught by the same teacher during the 2022/2023 school year.

School year 2023/2024 experimental classes	n. of students
Classe A	22
Classe B	22
Classe C	--
Classe D + Class E	38

School year 2022/2023 control classes	n. of students
Classe A	13
Classe B	16
Classe C	22
Classe D	26

Since the teacher for Class C did not administer the learning assessment test, the comparison can be conducted only for three teachers:

Teacher 1: Class A 2022/2023(control group) vs. Class A 2023/2024(experimental group)

Teacher 2: Class B 2022/2023(control group) vs. Class B 2023/2024(experimental group)

Teacher 4: Class D 2022/2023(control group) vs. Class D 2023/2024 + Class E 2023/2024(experimental group)

Correction of learning assessment test

Each correct answer was assigned a score of 1, each incorrect answer a score of 0, and each missing answer a score of -1. Consequently, the theoretical minimum score is -7, and the maximum score is +7. In the classes, the minimum score recorded was -3, and the maximum was +7.

χ^2 test for scores from -3 to 7

The eleven outcomes corresponding to scores from -3 to +7 are denoted as E_i (i-th event).

Degrees of freedom: 10

$\alpha = 0.05$

Critical value: $\chi_c^2 = 18.31$

ϕ_{i0} is the probability of result E_i in the control group, composed of n_0 students.

ϕ_{ix} is the probability of result E_i in the sample class X, composed of n_x students.

$$H_0 : \phi_{ix} = \phi_{i0}$$

$$H_1 : \phi_{ix} \neq \phi_{i0}$$

- $F_{o,i,0,x}$ ($i = -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$) is the observed absolute frequency of result E_i in the control group and $F_{o,i,x}$ be the observed absolute frequency of result E_i in the experimental class X ($X = A, B, C, D + E$);
- ϕ_{i0} is the relative frequency in the control group, i.e. $F_{o,i,0,x}/n_{0,x}$;
- $f_{a,i,x}$ with ($X = A, B, C, D + E$) is the expected absolute frequency in the considered experimental class, i.e. $f_{a,i,x} = \phi_{i0} n_x$;
- n_{0x} is the size of the control group ($X = A, B, D$);
- n_x is the size of class X ($X = A, B, C, D + E$).

$$\chi^2 = \sum_{i=-3}^7 \frac{(F_{o,i,0,x} - f_{a,i,x})^2}{f_{a,i,x}}$$

Teacher 1: comparison of Class A 2022/23 - Class A 2023/24

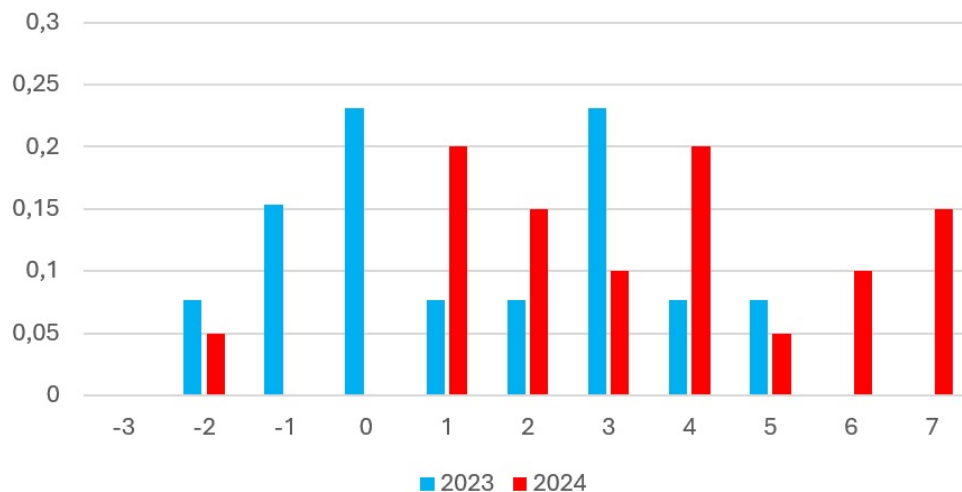
Table of Absolute Frequencies for Scores.

Teacher 1	-3	-2	-1	0	1	2	3	4	5	6	7
School year 2022/23	0	1	2	3	1	1	3	1	1	0	0
School year 2023/24	0	1	0	0	4	3	2	4	1	2	3

Table of Relative Frequencies for Scores.

Teacher1	-3	-2	-1	0	1	2	3	4	5	6	7
School year 2022/23	0	0,08	0,15	0,23	0,08	0,08	0,23	0,08	0,08	0	0
School year 2023/24	0	0,05	0	0	0,2	0,15	0,1	0,2	0,05	0,1	0,15

Learning test: teacher 1



	-3	-2	-1	0	1	2	3	4	5	6	7	Total
$F_{a,iA}$	0	1	0	0	4	3	2	4	1	2	3	20
$f_{a,iA}$	0,00	1,54	3,08	4,62	1,54	1,54	4,62	1,54	1,54	0,00	0,00	20,00

$$\chi^2 = 18.82$$

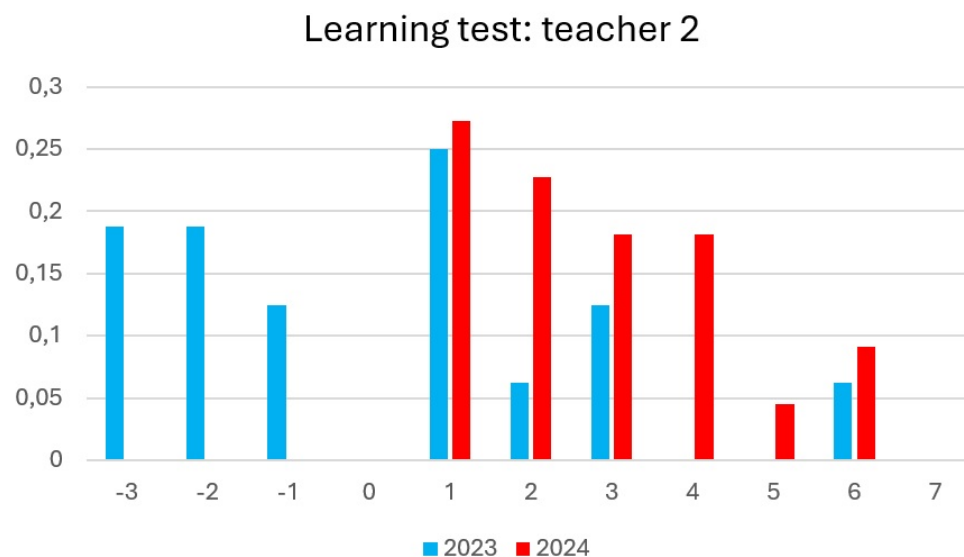
Teacher 2: comparison of Class B 2022/23 - Class B 2023/24

Table of Absolute Frequencies for Scores.

Teacher 2	-3	-2	-1	0	1	2	3	4	5	6	7
School year 2022/23	3	3	2	0	4	1	2	0	0	1	0
School year 2023/24	0	0	0	0	6	5	4	4	1	2	0

Table of Relative Frequencies for Scores.

Teacher 2	-3	-2	-1	0	1	2	3	4	5	6	7
School year 2022/23	0,19	0,19	0,13	0	0,25	0,06	0,13	0	0	0,06	0
School year 2023/24	0	0	0	0	0,27	0,23	0,18	0,18	0,05	0,09	0



	-3	-2	-1	0	1	2	3	4	5	6	7	Total
$F_{a,iB}$	0	0	0	0	6	5	4	4	1	2	0	22
$f_{a,iB}$	4,13	4,13	2,75	0,00	5,50	1,38	2,75	0,00	0,00	1,38	0,00	22,00

$$\chi^2 = 21.45$$

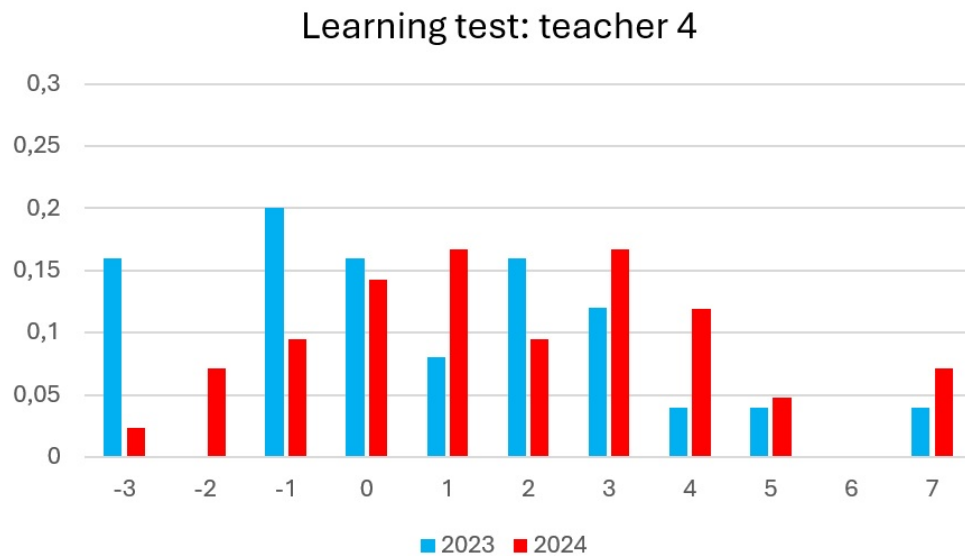
Teacher 4: comparison of Class D 2022/23 - Class D+E 2023/24

Table of Absolute Frequencies for Scores.

Teacher 4	-3	-2	-1	0	1	2	3	4	5	6	7
School year 2022/23	4	0	5	4	2	4	3	1	1	0	1
School year 2023/24	1	3	4	6	7	4	7	5	2	0	3

Table of Relative Frequencies for Scores.

Teacher 4	-3	-2	-1	0	1	2	3	4	5	6	7
School year 2022/23	0,16	0	0,2	0,16	0,08	0,16	0,12	0,04	0,04	0	0,04
School year 2023/24	0,02	0,07	0,10	0,14	0,17	0,10	0,17	0,12	0,05	0	0,07



	-3	-2	-1	0	1	2	3	4	5	6	7	Total
$F_{a,i,D+E}$	1	3	4	6	7	4	7	5	2	0	3	42
$f_{a,i,D+E}$	6,72	0,00	8,40	6,72	3,36	6,72	5,04	1,68	1,68	0,00	1,68	42,00

$$\chi^2 = 20.72$$

In all three cases, the null hypothesis (H_0) can be rejected.

By comparing the relative frequencies of the lowest scores (-3, -2, -1, or 0) and the highest scores (4, 5, 6, or 7) in the classes for the school years 2022/2023 and 2023/2024, further insights can be obtained.

The following tables show these comparisons.

Comparison of the lowest scores (-3, -2, -1, or 0) in the two school years considered for each teacher

Score -3, -2, -1 or 0	School year 2022/2023	School year 2023/2024
Teacher 1	46%	5%
Teacher 2	50%	0%
Teacher 4	52%	33%

Comparison of the highest scores (4, 5, 6, or 7) in the two school years considered for each teacher

Score 4, 5, 6 or 7	School year 2022/2023	School year 2023/2024
Teacher 1	15%	50%
Teacher 2	6%	32%
Teacher 4	12%	24%

The decrease in the lowest scores and the increase in the highest scores in the classes of Teachers 1 and 2 are more pronounced than those observed in the classes of Teacher 4.

Chapter 7

Interpretation of data and conclusions

7.1 Interpretation of teacher questionnaire data

The objectives of the teacher questionnaire were to determine how the worksheets were used, whether the proposed activities had an impact on the teaching method, and whether, according to the teachers, the activities had a positive impact on the students.

The first noticeable finding is that there is a clear difference between the attitude of Teachers 1 and 2 towards the proposed activities compared to that of Teachers 3 and 4. The former were more open to discussion and more willing to experiment than the latter.

Teachers 1 and 2 used the worksheets, taking into account both the progression of learning outlined in the progression toolkit and the guidelines provided in the teacher notes. They stated that they were mainly influenced by this in terms of paying attention to the students' responses, both to verify how much and how well they had learned, and to make decisions about how to structure the teaching in the following phases. Both teachers projected the worksheets on the interactive whiteboard (IWB) and worked with the whole class, but Teacher 1 also used printed worksheets to have students work in small groups and, at times, individually.

Teachers 3 and 4 consulted the progression toolkit, but in the case of Teacher 3, the proposed order of activities was not followed. Regarding the method of presentation, Teacher 4 only used printed materials for individual use, while Teacher 3, in some cases, used the projection on the IWB and in others, printed materials for individual use. Neither consulted the teacher notes, so they were not influenced by them and independently interpreted the students' responses.

As for the observed effects on the students, in general, all four teachers noted an improvement in students' participation and collaboration skills. However, improvements in argumentative skills and metacognition were less evident.

7.2 Interpretation of student questionnaire data

Students were asked to answer to questions regarding the quality of the worksheets, their participation in the activities, and their sense of self-efficacy regarding the tasks.

For the first indicator, in question 1, the hypothesis H_0 can be rejected for Classes D and E, both in test 1 and in test 2. The hypothesis H_0 can be rejected for question 2 in Class D during test 1, while in test 2, the null hypothesis cannot be rejected. A possible interpretation of these data, when comparing the students' responses with those provided by the teachers, is that, while in the control group the worksheets were considered mainly engaging and easy to understand, there is a significant difference in the experimental classes between those where the worksheets were projected on the interactive whiteboard (IWB) using voting systems, and those where the worksheets were printed and students worked mostly individually (Classes D and E). In these classes, the frequency of Likert scale values 4 and 5 for the first question was significantly lower compared to the control group (52% for the control, 9% for Class D, and 5% for Class E). For Class D, the response trends also differed from the control group for the second question, with students indicating that they found the worksheets rather difficult. In contrast, in the other classes, the response trends were very similar to those in the control group.

The second indicator concerns at student behavior in class during the activities. The worksheets were designed to stimulate dialogue and participation from all students. In this context, it is important that students ask for explanations from both teachers and peers.

In the first three questions, students are asked to indicate how frequently they answered questions, participated in class discussions, and collaborated with their peers. For question 3 (how frequently they answered questions), the null hypothesis H_0 can be rejected in test 1 for Classes A, C, and E, which show a distribution of Likert scale levels significantly different from the control group; however, in test 2, a significant difference is observed only for Class E, for which H_0 can be rejected. In the second question (participation in class discussions), there are no significant variations in the distribution of levels (test 1). The hypothesis H_0 can be rejected in test 2 only for Class B. In the third question (collaboration with peers), the hypothesis H_0 in test 1 can be rejected for Classes C, D, and E. In test 2, H_0 can be rejected for all classes, but for Classes A and B, there is an increase in the percentage of 4 and 5 values compared to the control (95% for Classes A and B, and 74% for the control). Meanwhile, for Classes C, D, and E, there is a noticeable decrease (29% in class C, 32% in class D, and 0% in class E).

Overall, the most significant differences from the control group concern Class E. 100% of the students in Class E reported that they answered all the questions often or always, but there were no significant differences compared to the control group in terms of participation in class discussions. Another striking aspect is the significant difference from the control group in the question regarding peer collaboration. 100% of the students stated that they never or rarely collaborated with their classmates to answer the questions. There are also differences for Classes C and D, where only 29% of students in Class C and 32% in Class D reported often or always collaborating with their peers. These results suggest that the students in these classes may have perceived the proposed activities primarily as individual work.

On the other hand, regarding students' participation in class discussions, there is a significant positive difference in Class B compared to the control group: the percentage of students who reported always or often participating in class discussions was high (67%), and there were no students who stated that they never participated in discussions. In Class A, the percentage of students who reported actively participating in discussions was also high (62%), though not significantly different from the control group. As for the question on peer collaboration, Classes A and B showed an improvement compared to the control group. In these classes, the teach-

ers reported following the guidelines in the teacher notes and using voting systems through hand-raising or colored cards.

The last two questions in this section concern how often students asked for help from the teacher or their peers when they were struggling. Here, too, there are differences compared to the control group.

In question 6, students in Classes A and B were less independent than those in the control group: the combined frequency of levels 2 and 3 in the control group was 53%, whereas in Class A it rose to 91% and in Class B to 72%. In fact, in these two classes, the null hypothesis H_0 for test 1 was rejected. The request for clarification from the teacher can have a dual interpretation: on the one hand, it could suggest that students frequently asked for explanations because the instructions were unfamiliar and required further clarification, while on the other hand, it could indicate that the questions were difficult to understand. However, this second hypothesis contrasts with the results of question 2, where no significant differences were found compared to the control group regarding the difficulty of the worksheets in these classes.

In Classes D and E, the percentage of students asking for clarification from the teacher was very low: the combined frequency of levels 4 and 5 was 17% in the control group, 9% in Class D, and 0% in Class E. For test 2, the hypothesis H_0 was rejected in these two classes. Comparing these results with those from question 3, especially regarding Class E, and considering the mode of presentation (printed worksheets for individual use), the interpretation we can make is that students may have perceived the worksheets as tests. Consequently, in this context, students might have thought it was not appropriate to ask for further explanations and did not expect any assistance.

Regarding question 7, which concerned asking for help from peers, when considering the levels separately (test 1), there are significant differences from the control group in Classes A and B. In test 2, a significant difference was found only in Class B, where 43% of the students reported asking their peers for help. When associating this result with the response to question 5 regarding peer collaboration, it seems that the students consulted and discussed with each other to formulate their answers.

Referring to the third indicator (Metacognition and self-assessment), for Class D, in questions 8a, 8b, and 8c, the null hypothesis H_0 can be rejected in both test 1 and test 2; the Likert scale values of 4 and 5 are lower compared to the control group. This suggests that students in Class D feel less capable, less involved, and less relaxed compared to the control group. Greater tension was also perceived by students in Class E (the null hypothesis H_0 was rejected for question 8c in both test 1 and test 2). Conversely, students in Classes A and B felt more capable compared to the control group. In Class B, the null hypothesis H_0 can be rejected for question 8a in both test 1 and test 2, while in Class A, it is rejected only in test 2. However, the percentage of high-level responses (4 and 5) is higher than that of the control group.

For questions 9 and 10, there are differences from the control group in Classes C, D, and E, while in Class B, the differences are only related to test 2 of question 9. However, I believe it is more meaningful to compare questions 9 and 10 class by class to see if students felt more competent at the end of the process compared to the beginning.

In Class A, the percentage of students who felt competent at the beginning of the process (sum of levels 4 and 5) was 52%; this percentage rose to 81% by the end of the process. In Class B, the increase was even more significant, rising from 19% to 81%.

In Class C, the sum of levels 4 and 5 at the beginning of the process was 35%, rising to 47% at the end. In Class D, it increased from 36% to 50%, and in Class E, from 15% to 40%. Although students in Classes C, D, and E also felt more competent at the end of the process, the increase was more modest compared to what was observed in Classes A and B.

7.3 Interpretation of learning assessment test data

When comparing the classes taught by the same teacher in the academic years 2022/2023 and 2023/2024, we observe that the distribution of scores is similar, but the null hypothesis H_0 can be rejected for all the cases considered. In fact, overall, there is an increase in the frequency of higher scores in the experimental classes compared to the control group. This increase is more pronounced in the classes of Teachers 1 and 2 compared to those observed in the classes of Teacher 4. It is important to note that the control classes took the learning assessment test after covering topics related to the area and perimeter of plane figures using traditional teaching methods, whereas in the experimental classes, the test was administered after working with the worksheets. These results may suggest that the use of worksheets enhances the learning of the concepts. However, due to the limited number of students who took the test and, more importantly, the fact that only three teachers were involved, caution is advised. Further evaluations, including long-term assessments, would be needed to more precisely evaluate any potential improvements.

7.4 Conclusions

Our experiment can be described as an exploratory investigation in which we aimed to place particular emphasis on qualitative analysis, rather than quantitative. Given the specific nature of the samples considered and the limited number of students and teachers involved, this experiment represents a first attempt to assess the impact of the worksheets on both teachers and students.

None of the teachers involved had previously been trained on the potential and use of formative assessment strategies or peer instruction. Some of them read the reflection prompts in the Teacher Notes, but the ability to grasp these prompts depended on the sensitivity and competence of the individual teachers.

Despite this, the results are encouraging and, in my opinion, demonstrate once again that it is the teacher who makes the difference.

Any educational tool by itself always has some impact on students. If we think of the use of worksheets as a reinforcement activity, they certainly had a positive impact on students, both in terms of concept comprehension and on the students' sense of self-efficacy. This is what almost all students pointed out, albeit at different levels.

But, if we focus on what works best, following Hattie's approach in [46], even though we do not have the opportunity to quantify the effect size (ES), we need to consider how the worksheets were introduced.

What emerges from the interpretation of the responses in both teacher and student questionnaires is that when the activities were designed to encourage student thinking, invite collaboration, and foster metacognition, the students enjoyed the work, felt more capable, and felt they had learned more.

If a teacher's goal at the end of the first cycle of education is to foster a positive attitude towards mathematics, then using this tool with a formative assessment approach can be a step towards achieving that objective.

Appendix

Appendix A

Proposals for a new approach to mathematics education

The examples provided in this section are some of the teaching proposals that I analyzed during my first year of doctoral studies. Reading these proposals can further help clarify what is meant by mathematical workshops, teaching via games, problem solving, and argumentation. For convenience, and following the structure established in Chapter 1, the proposals have been divided into these four groups. However, in practice, the groups are not mutually exclusive, as most proposals share features with more than one group. Education based on problems, workshops, and games encourage exploration, discovery, and, at the same time, practicality, argumentation, and reflection on one's own work.

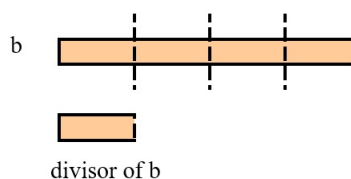
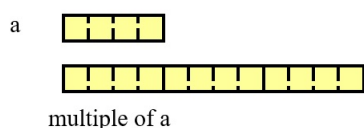
A.1 Mathematics workshop proposals

A.1.1 Drawing numbers: an example for finding the Least Common Multiple and the Greatest Common Divisor

The starting point of this work proposed by Tovenà [70] from University of Rome *Tor Vergata* is the geometric representation of numbers in various ways.

To represent multiples and divisors, segments or strips of paper, or sticks of different lengths are useful.

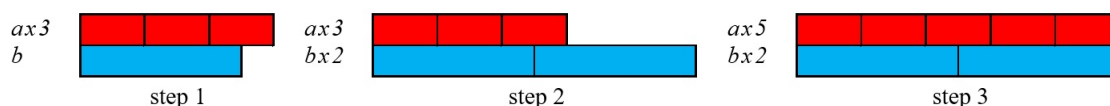
Given a number a , represented by a segment or a strip of paper, one of its multiples can be shown by placing copies of the segment or strip consecutively, while the divisor can be indicated by folding the strip into equal parts.



Taking two numbers, a and b (in the example shown below $a = 4$ and $b = 10$), we can visualize the least common multiple by arranging two strips of paper on two parallel lines with an initial alignment.



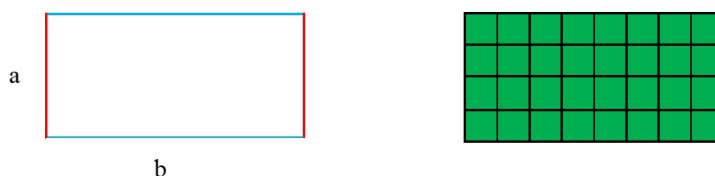
The aim is to construct two parallel lines of equal length. Copies of the smaller object are added to see if the length of the larger one is reached. If this does not occur, adding another piece will cause the line to be crossed (step 1). Next, the larger object must "catch up" with the smaller one (step 2). This alternating pursuit continues until the final alignment is achieved (step 3).



In the example, the number obtained by multiplying $a \times 5$, or by multiplying $b \times 2$ is the same (the number 20) and is a multiple of both, thus it is a common multiple. However, it is not the only one: students can continue the activity to find other alignments and reach the conclusion that between two numbers a and b there exist infinite common multiples, but only one will be the smallest. Now the teacher can stimulate the curiosity of the students by asking: does the least common multiple (lcm) always exist for two numbers a and b ? The answer, for students before secondary school, is not always as obvious as it seems.

To understand common divisors, it is helpful instead to represent the product of two numbers as a rectangle.

Considering two non-zero numbers a and b , the product can be visualized in this way:



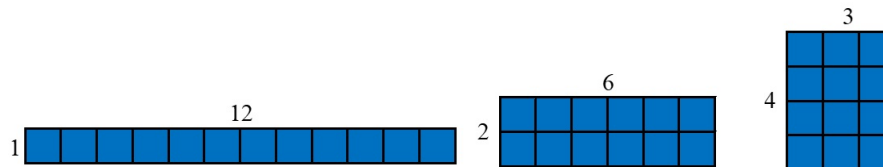
By separating the rows and columns, we we rediscover the concept of product as repeated addition.



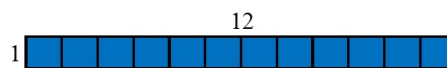
For multiplication, the commutative property holds, so it is equivalent to represent the rectangle with the base equal to b and the height equal to a or vice versa.

Any number can be represented by at least one rectangle, because every number can be obtained by multiplying the number itself by the unit. For example, let's take the number $b = 12$.

Possible representations are:



To show the division of b by a number a , we can imagine moving from the representation of the dividend b as a rectangle with dimensions 1 and b , to one where it will be represented by a rectangle with dimensions equal to the divisor a and the quotient q : all the squares of the initial representation of b are distributed to form a rectangle with a dimension equal to a . returning to the example with $b = 12$, the initial representation could be:

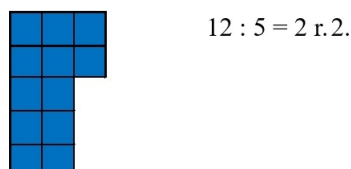


or vice versa with one column and twelve rows.

If a is equal to 4, for example, we can imagine moving the squares to form 4 rows; the number of columns then provides the quotient.



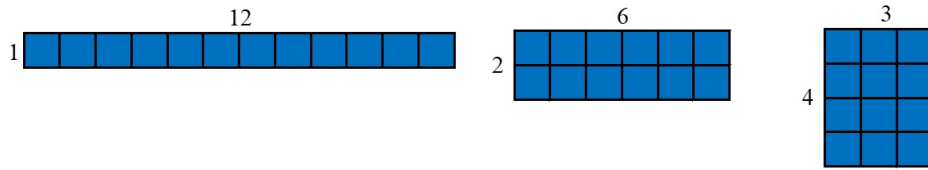
Sometimes this is not possible: if a were 5, not all the squares would form the rectangle and there would be a remainder.



The number 5 is not a divisor of 12.

The divisors of a number will be the dimensions of all possible rectangles that represent it.

For the number 12, the possible representations are:

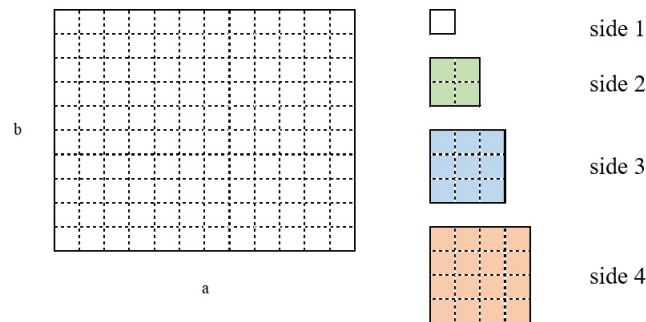


The divisors will therefore be: 1, 2, 3, 4, 6 and 12.

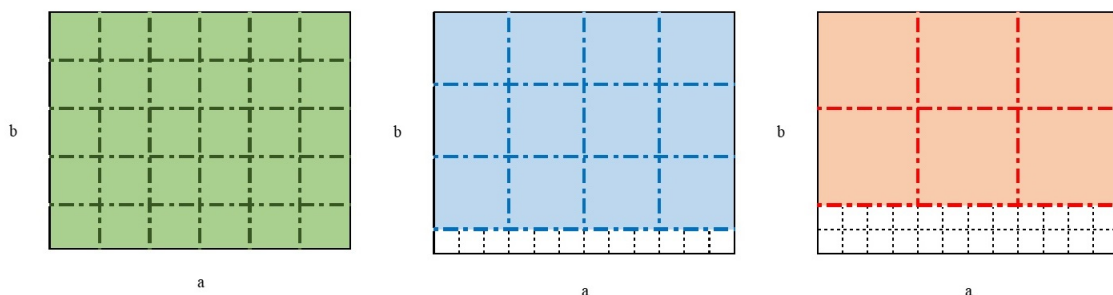
To represent the greatest common divisor between two numbers a and b , we can think of constructing a rectangle with dimensions a and b and try to cover it entirely, that is, tile it, with squares of different sizes.

It is evident that it is always possible to tile with squares of side length 1, for rectangles of any dimensions a and b .

With squares of side length 2, it will be possible to tile only if numbers a and b are even, with squares of side length 3 only if they are both multiples of 3, and so on.



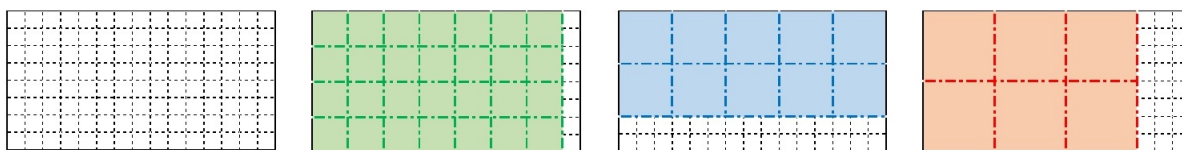
In the depicted case, it is possible to tile the rectangle with squares of side length 2, whereas with squares of side length 3 and 4, it is not possible to cover the entire figure, as shown below.



The measure of the maximum possible side length is 2, which represents the greatest common divisor (GCD).

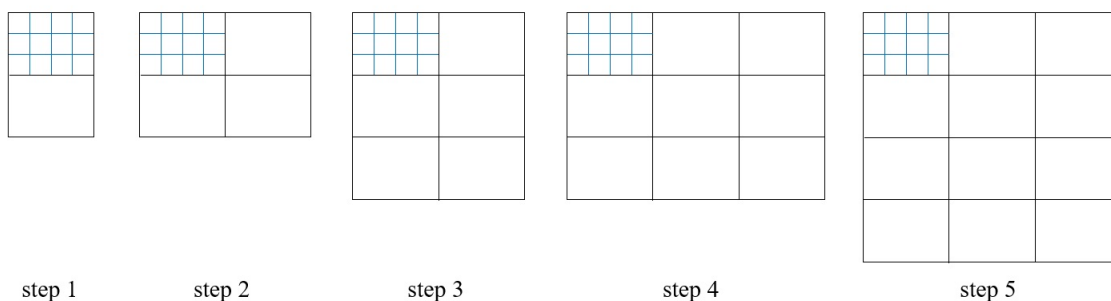
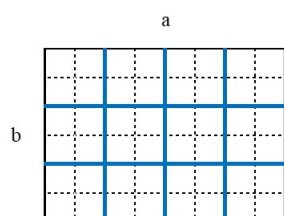
The rectangular representation allows us to simultaneously visualize both sides, and it can be seen that the side of the square divides both a and b .

If the numbers considered are coprime, meaning they have no common divisors other than the number 1, then it is only possible to tile with squares of side length 1, which will therefore be the greatest common divisor (GCD). The following figure illustrates the example of 8 and 15.



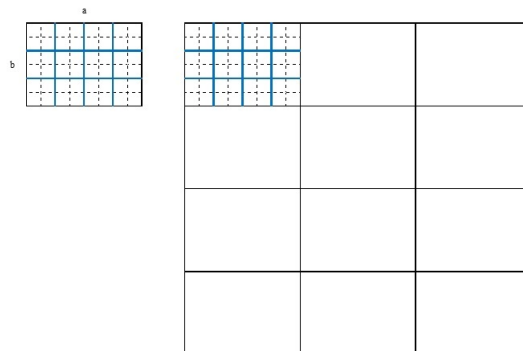
The rectangular representation can also be used to visualize the least common multiple (lcm); this also highlights the relationships between lcm and GCD.

Starting from the rectangle $a \times b$ (which has an internal subdivision linked to the GCD), it is used as a tile to build a square; the procedure recalls what has already been seen in the linear representation. You start from the smaller side and add a rectangle until it fills or exceeds the measurement of the larger side (step 1), then repeat the procedure with the larger side (step 2), bouncing from one side to the other until a square is built (steps 3, 4, and 5).



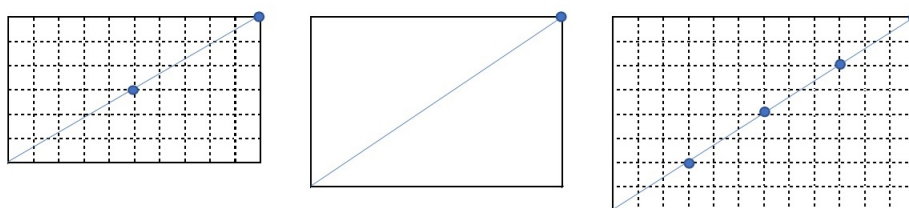
We realize that the minimum side of the square that can be obtained with these rectangular tiles is the least common multiple (lcm) between a and b .

In this way, students have in the same image the vision of the GCD as the side of the inner square and that of the lcm as a side of the outer square.

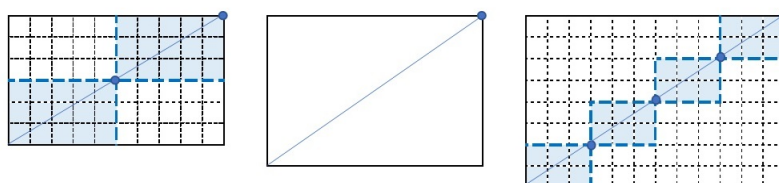


Let us consider a numerical example. Let's imagine tiling with a rectangle with sides 8 and 6: to obtain the square, the longer side must be repeated 3 times ($8 \times 3 = 24$) and the shorter one must be repeated 4 times ($6 \times 4 = 24$); $\text{lcm}(8; 6) = 24$. To tile the interior of the rectangle, we can use squares of side length at most 2; $\text{GCD}(8; 6) = 2$.

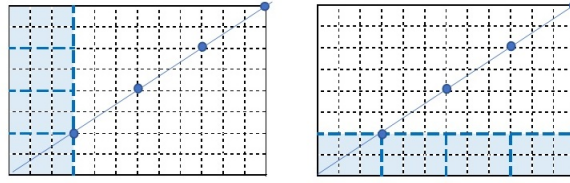
As shown in the next figure, to further highlight the relationship between lcm and GCD, another representation can be used: the idea is to start from rectangular grids and use the diagonals. A grid of size $a \times b$ is constructed, and the diagonal is drawn, asking the students to identify the number of times the diagonal touches a vertex of a square, without counting the starting point from which they drew the diagonal. Why do these intersections occur? And how many are there?



Students can make various examples, choose which attempts to make, and how to organize the research. It is observed that the vertices appear at regular intervals and that the presence of internal vertices on the diagonal generally does not depend on the individual side but on the relationship between the two. Furthermore, between one vertex and the next, rectangular structures that repeat can be highlighted.

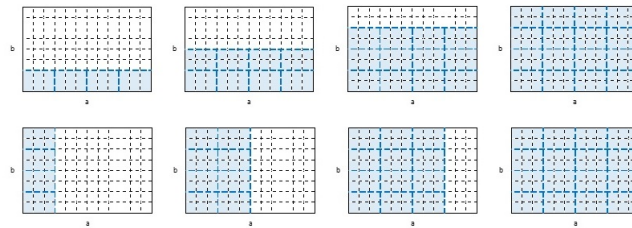


By imagining moving the rectangles to the sides, it can be observed that they can entirely cover both the base and the height, meaning the number of rectangles is a common divisor of both a and b , and the largest possible. The number of small rectangles therefore represents the GCD.



Since both the base and height are entirely covered by the small rectangles, the sum of their areas is simultaneously a multiple of a and a multiple of b . Therefore, the sum of the areas of the small rectangles is a common multiple of a and b (and it is the least, otherwise the diagonal would encounter another vertex). Geometrically, the sum of the areas of the small rectangles is thus the lcm.

At this point, it can be observed that to tile the entire initial rectangle, the sum of the areas of the small rectangles (lcm) must be repeated a number of times equal to the GCD.



The area of the rectangle ($a \times b$) is therefore the product of the lcm and the GCD.

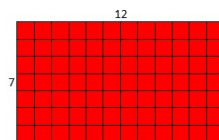
$$a \times b = lcm(a; b) \times GCD(a; b).$$

At the end of the path, therefore, an algebraic reflection is reached. The starting point is an accessible activity, with rectangles and tiling, but gradually the work becomes more complex. Students can justify the properties of numbers without resorting to literal calculation. It is an engaging yet naive approach that can be a stimulus for further exploration when students have developed adequate abstract reasoning skills.

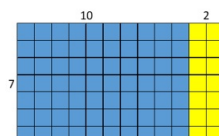
A.1.2 Seeing and touching algebra

Another activity proposed by Castellini uses the representation of the product of two numbers as a rectangle. In this work, the concepts of “low floor, high ceiling, and wide walls” are also highlighted [23].

The activity starts from a multiplication (for example 7×12) graphically represented by a rectangle of dimensions 7 and 12.



By dividing the side of 12 units into two parts (for example, $10 + 2$) the rectangle is divided into two rectangles: one of dimensions 7 and 10, and the other of dimensions 7 and 2.



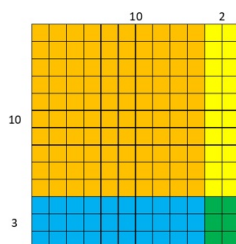
The multiplication 12×7 can therefore be written as:

$$12 \times 7 = (10 + 2) \times 7 = 10 \times 7 + 2 \times 7 = 70 + 14.$$

The numbers 14 and 70 also appear in the usual multiplication algorithm.

$$\begin{array}{r} 7 \\ \times \\ 12 \\ \hline 14 \\ + \\ 70 \\ \hline 84 \end{array}$$

The same procedure is repeated for a multiplication where both factors have two digits: for example 12×13 , considering 12 as $10 + 2$ and 13 as $10 + 3$; now the 12×13 rectangle is divided into four rectangles.



So

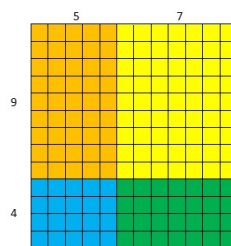
$$12 \times 13 = (10 + 2) \times (10 + 3) = 10 \times 10 + 10 \times 2 + 10 \times 3 + 2 \times 3 = 100 + 20 + 30 + 6.$$

By observing the previous expression, we get 1 hundred (100), 5 tens (20+30), and 6 units, which gives the number 156.

$$\begin{array}{r}
 12 \quad \times \\
 13 \quad \hline
 36 \quad + \\
 120 \quad \hline
 156
 \end{array}$$

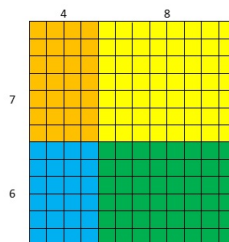
But at this point we can also think of other terms into which to decompose the factors.
For example

$$12 \times 13 = (5 + 7) \times (9 + 4) = 5 \times 9 + 7 \times 9 + 5 \times 4 + 7 \times 4,$$

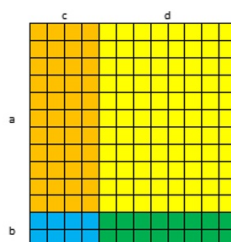


or

$$12 \times 13 = (4 + 8) \times (7 + 6) = 4 \times 7 + 8 \times 7 + 4 \times 6 + 8 \times 6,$$



to gradually arrive at the general rule



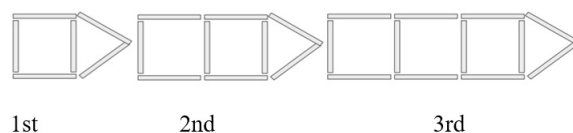
$$(a + b) \times (c + d) = ac + bc + ad + cd.$$

So, starting from a “low floor” (multiplication and distributive property) we reach a “high ceiling” (the product of binomials), touching “wide walls”, because the activity can also be used to make sense of the multiplication algorithm.

Using the same approach, other operations with polynomials can be introduced (such as squaring a binomial, cubing a binomial, etc.), paving the way for a view of algebra as a generalization of arithmetic.

Algebra, in this way, is literally seen and touched.

Another significant example of how a mathematics workshop allows for the realization of algebra, often perceived by students as overly abstract, is the following. We start from the request to construct a sequence of three figures with wooden sticks, which are provided by the teacher:



and then continue with the following figures. The question is: how many sticks are needed to construct the fiftieth figure?

Students easily realize that at each step, three sticks must be added, but the use of a table can help perceive the regularities better.

figure	triangles	square	sticks		
1	1	1	6	3+3	3x2
2	1	2	9	3+3x2	3x3
3	1	3	12	3+3x3	3x4
.....					
n	1	n	3 +nx3	3+3+3 n+1 times	3(n+1)

To obtain the fiftieth figure ($n = 50$) you need to use $3(n + 1)$ sticks, which equals 153.

At this point, the activity can continue with other stimulating questions, such as:

1. Can there be a figure made from 59 sticks? Why?
2. If it exists, at which step would a figure made of 84 sticks be found? Explain how you figured it out.

The answer of question 1 is negative since 59 is not a multiple of 3. In question 2, students need to work backward, dividing by 3 and subtracting 1 to find the correct step ($84 \div 3 = 28, n = 28 - 1 = 27$).

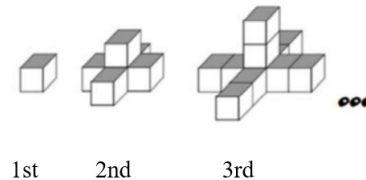
In this way, by moving back and forth on the representation with wooden sticks, students begin to see the questions from multiple perspectives and perhaps start to grasp the theory of remainders in division.

The workshop using a sequence of figures built with sticks is an example of how algebra can be

presented as a tool for creating models. Once students are familiar with this type of approach, it is useful to continue with similar activities but with some variations.

In this way, students are encouraged to reflect on the analogies and differences with the previous experience, deriving new insights.

For example, more complex figures involving three dimensions can be considered, such as those represented in the following image:



Observing the figures, students can use different reasoning strategies the following table. One approach, for instance, involves imagining adding 5 cubes at each step, corresponding to the directions indicated by the five visible faces of the initial cube.

figure	number of cubes			
1	1			
2	6	1+5		1+5x1
3	11	6+5	1+5+5	1+5x2
4	16	11+5	1+5+5+5	1+5x3
.....				
n				1+5x(n-1)

Alternatively, for each figure, one could count the cubes placed in the central column and those arranged laterally.

figure	number of cubes			
1	1			
2	6	2+4	1+1+4	1x2+4x1
3	11	3+8	1+1+1+4+4	1x3+4x2
4	16	4+12	1+1+1+1+4+4+4+	1x4+4x3
.....				
n				1xn+4x(n-1) = n+4x(n-1)

Depending on their reasoning, students may arrive at seemingly different algebraic expressions that, although formally different, have the same meaning.

$$\begin{aligned}
 1 + 5 \times (n - 1) &= 1 + 5n - 5 = 5n - 4, \\
 n + 4 \times (n - 1) &= n + 4n - 4 = 5n - 4,
 \end{aligned}$$

Writing the algebraic expression is important because it trivially manages to give meaning even to the many procedural exercises of simplifying algebraic expressions that are so abundant in textbooks.

The National Guidelines [57] emphasize the importance for students to build:

an adequate vision of mathematics, not reduced to a set of rules to be memorized and applied, but recognized and appreciated as a context for addressing and posing significant problems (...).

Workshop activities like these help students not to lose their sense of mathematics, moving within a virtuous circle that promotes the development of mathematical thinking by connecting informal and formal reasoning.

Students start from observation and informal reasoning about a real situation (the series of figures built with sticks or cubes) and arrive at a formal representation (the series of operations and the corresponding general formula); at this point, transformations and manipulations can be made within the formal system (simplification of the obtained expressions), and the results can be reinterpreted in the real situation.

It is a thinking pattern that replicates what we use, even unconsciously, in solving everyday problems (we observe, make hypotheses, reason, and apply theoretical reasoning to solve the real problem), thus generating a deeper understanding compared to what is achieved by working only on the formal level.

A.2 Teaching-via-game proposals

The examples presented in this section offer an informal and playful approach to studying particular aspects of geometry, with the intention of also evoking positive and lasting emotions.

A.2.1 Leonardo da Vinci's legacy in mathematical learning

These characteristics are also evident in the proposal by Daniele Pasquazi from the University of Rome *Tor Vergata*, author of the game *I ludi geometrici di Leonardo da Vinci*. The game has been inspired by Leonardo's studies on geometric figures, particularly related to the "squaring" of curved figures, that is, the transformation of irregular figures into equivalent ones of which it is easy to determine the area. In addition to reproducing Leonardo's drawings with the figures to be "squared" and the preliminary explanation of some equivalences, students have at their disposal lunules, triangles, portions of a circle made of plexiglass to construct and modify these figures concretely. The students, divided into groups, must reproduce and square the various figures using the available materials.

The proposal for this game arose to address a specific need: recent surveys [63] conducted on students aged between 11 and 14 years revealed a tendency to confuse the concepts of equality, equivalence, and isoperimetry of plane figures. Often, students also tend to use an arithmetic approach, corresponding to the application of formulas, in solving geometric problems. Therefore, it is necessary to develop the capacity for geometric perception, which allows one to recognize, understand, and use the characteristics that define a figure.

Moreover, contextualizing within a geometric framework often provides an excellent starting point for developing problem-solving strategies for arithmetic problems as well. The geometric object that can be drawn allows abstract mathematical objects to be concretized. However, the drawing is static: once drawn, the figure can be observed from a single point of view. Sometimes, however, to understand the deeper properties of a figure, it is useful to observe its movement. The use of a manipulable material allows the object to be observed from different perspectives,

to make observations and conjectures. Hands-on activities, as observed in Montessori studies and confirmed by recent research in neuroscience, has enormous educational value; tactile experience activates neurological processes that allow the imagining of movements performed with educational tools, even when facing drawn figures. If a mathematical concept has been associated with physical interaction, then that concept can be recalled simply by remembering the manipulation with the artifact.

Manipulative activity, which predominantly stimulates the right cerebral hemisphere, subsequently facilitates the processing of abstract thought, which pertains to the left hemisphere. In students aged between 11 and 14 years, the processes of rationalization are not yet fully mature, resulting in some difficulty in keeping track of all the interconnected steps of reasoning. The correct mental models that students must gradually build can be facilitated by manipulating appropriate artifacts. The “figurative demonstration” through material that simulates the mathematical object allows one to proceed with a certain rigor, even if it is not yet a formal demonstration.

The triangles, lunules, and portions of a circle used to square the figures present in Leonardo da Vinci’s Codex Atlanticus serve as stimuli to convey the concept of equivalence between seemingly very different plane figures.

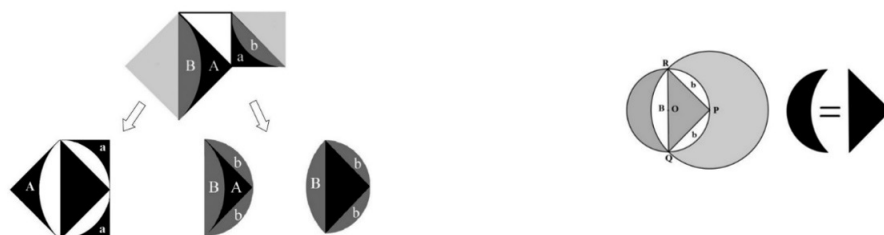
We start from the duplication of the square, using the Pythagorean theorem, to arrive at the duplication of the circles inscribed in the squares constructed on the sides of a right triangle.



By observing the mixed figures in the figure below labelled with A, a, B, and b, we obtain the following equivalences:



Then, by examining in detail the surfaces A and B, we can derive other equalities, which ultimately lead us to the equivalence between the lunule and the isosceles triangle by the difference of congruent parts.



Perhaps Leonardo's goal was to construct a square with an area exactly equal to that of a given circle, and the difficulty of achieving this led him to make thousands of attempts.

Sharing this experience with students can be a stimulus to convey the idea that often, more than solving a problem, the process of exploration and inquiry is important. The attempts to solve this unsolvable problem, using only an unmarked ruler and a retractable compass, have provided a wealth of educational and developmental stimuli.

Students feel involved; they know that each of their attempts is important. The game intrigues and stimulates them; proposing the activity in small groups also encourages interaction and communication among students who must collaborate to reach a solution.

Another aspect not to be underestimated in this experience is the aesthetic aspect. The beauty of Leonardo's figures, which are often found in the architectural details of historic buildings, can have a strong appeal for those students who are not usually interested in mathematics. The opportunity to recreate the drawings, perhaps enriching them artistically, allows all students to participate and overcome prejudices that sometimes hinder finding the right spirit to tackle mathematics.

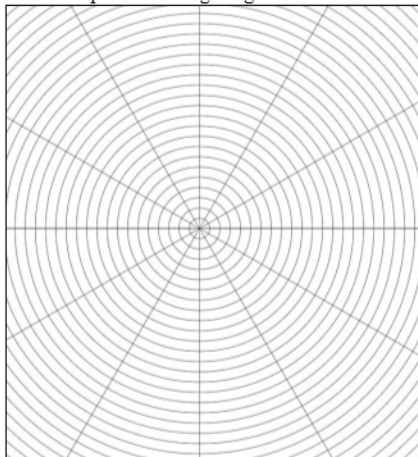
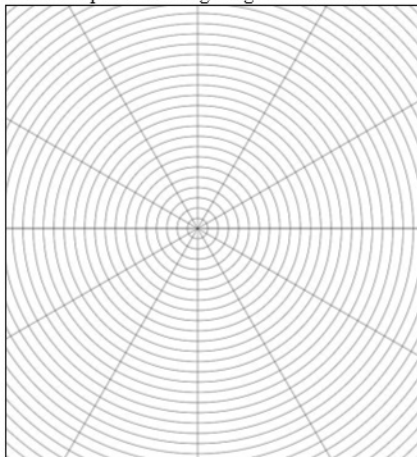
A.2.2 Exploring spirals through immersive learning

The importance of interaction with artifacts and the presence of a “non-mathematical” approach bring the work of Pasquazi closer to that proposed by the group from Turin led by Sabena. In this case, the proposal is not a game but an immersive experience in a context different from the mathematical one to identify geometric figures [21]. It starts with the observation of a historic building, Palazzo Madama in Turin, in search of decorations, stairs, and other architectural elements in the form of a spiral. From observation, it then moves on to manipulation through the use of a mathematical device, Archimedes' Spirograph. Through the artifact, students have the opportunity to mechanically construct the spiral, directly observing that it is determined by the combination of rectilinear motion (translation) and circular motion (rotation). On these bases, it is then possible to inspire in students mathematical questions, identifying analogies and differences between various types of spirals, the presence of regularities, changes related to the variation of certain parameters, the presence of boundary cases, and so on. Gradually, students move away from manipulative action and begin to reflect and generalize, aided by collective discussion mediated by the teacher.

Only after this initial immersion can structured activities be introduced, through which students will be able to geometrically construct various types of spirals and thus gradually arrive at mathematical knowledge of the spiral. Below are two examples [22].

Activity 1

Based on definitions, try drawing the two spirals.

The point that draws the SPIRAL OF ARCHIMEDES moves away from the pole at a constant speed, while the ray vector rotates at a constant speed. For this reason, the points located in the intersections of the polar grid are distant from the pole by a number of "squares" which follows an arithmetic progression (e.g. 0, 1, 2, 3, 4, 5, ...). Draw the spiral following the given definition. 	The point that draws the LOGARITHMIC SPIRAL moves away from the pole with a uniformly accelerated motion, while the ray vector rotates at a constant speed. For this reason, the points located in the intersections of the polar grid are distant from the pole by a number of "squares" which follows a geometric progression (e.g. 0, 1, 2, 4, 8, 16, ...). Draw the spiral following the given definition. 
--	--

On the same sheet, with different colors, try to trace other Archimedean and logarithmic spirals using different arithmetic and geometric progressions. What do you observe?

Activity 2

Open GeoGebra and try to follow the steps of the procedures for kinematic construction of spirals.

Procedure:

SPIRAL OF ARCHIMEDES	
- Construct a <i>slider</i> t (found in the toolbar). Indicate the name t, select <i>number</i>, the range of variation (from 0 to 300) and increment 0.0001. - Build a <i>slider</i> a. Indicate name a, dial <i>number</i>, the range of variation (from -5 to 5) and the increment 0.01. - Build a <i>slider</i> b. Enter name b, dial <i>number</i>, the range of variation (from -5 to 5) and the increment 0.01. - Define in the <i>Algebra View</i> the point $P = (t, 0)$, which will move along the positive abscissa axis as time t varies. - Rotate point P by an <i>Angle of given size</i>: select the point P, then the origin O and indicate as rotation angle: $a \cdot \text{Distance}(O, P) + b$ - Rename the rotated point as P'. - Operate the time slider by clicking on the Play symbol. - Move sliders a and b with the mouse to see how the spiral changes.	

In the first activity, students are required to reproduce on paper, using a polar coordinate plane, both an Archimedean spiral and a logarithmic spiral, which, if introduced cinematically, differ in terms of translational velocity. In the Archimedean spiral, both rectilinear and curvilinear motion occur at a constant velocity: equal rotation angles correspond to equal distances in rectilinear motion, so with each rotation, one moves away from the pole following an arithmetic progression. In the logarithmic spiral, however, the translational velocity is not constant but

progressively increases, and one moves away from the pole following a geometric progression. The use of the polar plane simplifies the construction because the rotation angles are already highlighted.

The second activity considers another artifact, in this case, digital. Students are asked to construct the Archimedean spiral using the dynamic geometry software Geogebra. This activity is not properly manipulative, but it still involves a laboratory and constructive activity during which students must activate different skills.

In this experience the students are not engaged in a game, but the informal context and the workshop, linked to direct observation and manual contact with the spiralograph, make learning creative, non-linear and therefore unpredictable. The teacher plays the important role of bringing out the mathematical meanings by promoting collective discussion and focusing attention, when necessary, on the most relevant interventions, which can be used to introduce the mathematical perspective and correct terminology.

A.3 Problem-solving: teaching through problems

In this section, we deepen our understanding of the *Problemi per matematici in erba* [11] website, which we mentioned in Chapter 1 and which we consider particularly useful and accessible to teachers.

This website targets teachers of lower secondary education and draws on experiences from the *MathUp*[10] training courses and the research of the *Centro Matematica* [7].

To encourage teachers to promote a mathematical education that sparks a taste for research and discovery, problems already tested in various classes within MathUp courses are presented on the site. These problems are believed to possess the right qualities to achieve this goal. The general characteristics of a good problem are clarified and explained, but the most interesting aspect, in my opinion, is the contribution of experimentation. These characteristics are reviewed and corrected based on feedback from students and teachers who have tried working with the problems.

Another interesting aspect of the website is the section concerning the link between mathematical problems and the aspects teachers must continually confront in educational planning. This includes the *Indicazioni Nazionali per il Curricolo della Scuola dell'Infanzia e del Primo Ciclo di Istruzione* [57], the key competences for lifelong learning described in the *Reference Framework attached to the Council of Europe's Recommendation* of May 22, 2018, [3] and *Matematica per il Cittadino* (Matematica 2001) [5]. Additionally, there is a section about the role of students' mistakes in the teaching-learning process of mathematics.

The core of the website is a list of problems, which are categorized based on various criteria:

- the mathematical domain or domains to which the problem refers, namely the thematic cores of the National Guidelines (numbers, space and figures, relationships and functions, data and forecasts);
- the process to be implemented (analyzing a text, observing, describing, classifying, representing, counting, measuring, conjecturing, etc.);
- the curricular core (algebra, angles, functions, 2D geometry, 3D geometry, etc...), meaning the themes, topics, and content to which the problem relates;
- the grade level at which it can be used;

- the presentation mode, i.e., the presence of certain text characteristics that allow the problems to be divided into various categories.

The presentation mode of the problem is an aspect worthy of attention. While acknowledging that a problem may have various characteristics and that it is often challenging to fit it into a defined schema, several categories are considered.

What makes the website *Problemi per matematici in erba* unique and what differentiates it from other websites that feature lists of problems (such as the website of the *Transalpine Mathematical Rally Association*[13], the *Kangourou Games*[4], the *Bocconi Mathematical Games*[8], etc.) is the extensive space given to solutions and comments. The reflections arise from both theoretical considerations and classroom experiments.

The comments revolve around the properties attributed to a *good problem*: mathematical significance, multiple solutions, openness to new perspectives, memorability.

Below, some significant examples of problems according to the presentation method are presented.

Narrative problems and sphinx problems

These problems are characterized by a lengthy text and are contextualized within a story. Narrative problems are rich in references to usual situations that facilitate understanding: students reproduce the story in their minds and base the solving process on this representation.

In Sphinx problems, the story is merely a motivational pretext, while the actual problem is solving an enigma.

An example of a narrative problem is *Snow White and the 77 mushrooms*: students must imagine distributing 77 mushrooms among the 7 dwarfs, starting with the smallest, Dopey, and giving each subsequent dwarf one more mushroom than the previous one. In the end, they need to find out how many mushrooms Dopey received.

In this problem, the context is imaginary, but the proposed situation, distributing a certain number of objects among different individuals, is a familiar situation for children. Even though initially most students start working through trial and error, gradually becoming more reasoned, many manage to visualize the distribution action using graphical representations and expressions.

This is one of the cases where the synergy between narrative thinking and the logical thinking necessary for the solving process occurs, as discussed by Rosetta Zan in [77]. Narrative thinking supports logical thinking because

the information necessary for solving is consistent from a narrative perspective, and the information necessary for representation is consistent from a logical perspective, especially concerning the question asked.

An example of a Sphinx problem is *Regular Polygons and Fractions*. In this case, the narrative context, a student getting bored during class and discussing with their desk mate, is merely an engaging pretext to introduce the actual problem. The problem consists of finding other regular polygons, besides the square, that can be drawn around a point completing a full rotation.

The narrative context is plausible and refers to a peer discussion that is an integral part of all workshop activities; however, it is not directly connected to the logical thinking required to solve the problem.

Snow White the 77 Mushrooms

Snow White divides her harvest of 77 mushrooms among the seven dwarfs. She starts serving the youngest, Puppy, and then proceeds to serve all the other: Bashful, Grumpy, Sneezy, Doc, Happy and Sleepy. Each dwarf receives one more mushroom than the one immediately before him, and Snow White manages to distribute all the collected mushrooms. How many mushrooms did Dopey receive?

Regular polygons and fractions

Antonio, a seventh grade student, was not in the mood to listen to the lesson a few days ago. Bored, he was staring at an empty page in his notebook when suddenly, almost without realizing it, he exclaimed: "Look! Around every point, there are exactly four squares! So, the angle of the square is $1/4$ of the full rotation angle.

Want to bet that the angle of the regular pentagon is $1/5$ of the full rotation angle, the one of the regular hexagon is $1/6$, and so on?"

Filippo, his desk mate, disagrees: "No, that can't be right! Can't you see that the angle of the regular pentagon is larger than that of the square? And yet, $1/5$ is smaller than $1/4$! And with the hexagon, the angle becomes even larger, while $1/6$ is smaller than both $1/4$ and $1/5$."

Filippo is right, but Antonio's observation is also accurate: with four squares having a common vertex, a full rotation angle is "completed", so the angle of a square is exactly $1/4$ of the full rotation angle.

Do you know other regular polygons, besides the square, with which you can complete the entire rotation angle around a point? Which ones are they? Do you think you've found them all?

And do you know other regular polygons, besides the square, that have an angle representing a fraction of the full rotation angle like $1/n$ (for example, $1/3$, $1/4$, $1/5$, $1/6$, ...)? Which ones are they? Do you think you've found them all?

Abstract Context Problems

These are problems where the protagonists are numbers or figures without a story.

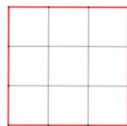
Contextualizing a mathematical concept within a real and familiar framework can make it easier to understand. However, a primary obstacle in solving a problem can be transforming the narrative text into a mathematical model. Abstract context problems can be used to overcome this type of difficulty when the teacher's goal is not to work on the mathematization of real problems but, for example, to work on recognizing a pattern that repeats regularly and therefore transitioning from an arithmetic to an algebraic solution.

In the problem *Slicing a square*, for example, you have to cut each side of a square into equal parts and count the little squares obtained, separating them based on how many sides they share with the original square's boundary (0, 1, or 2 sides). Increasing the number of equal parts the side is divided into each time, you need to predict the result obtained by performing n cuts.

An abstract problem can be used also as a consequence of solving a problem that has opened a new scenario without the need to create a new contextualization. This is the case, for example, with the problem *Regular Polygons and Fractions: A Bit More Variety*, which connects back to the Sphinx problem presented earlier but, at this point, no longer needs a motivating story. The problem is very similar to the first one, with the variation of considering different regular polygons.

Slicing a square *Let's start with a square, highlighting its sides (colored red in the figure); cut it into little squares by dividing each side into three equal parts. As we can see from the figure,*

out of the 9 little squares obtained, one has no red sides, 4 have one red side, 4 have two red sides, and none have more than two.

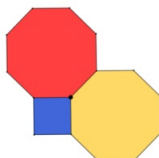


Let's continue and cut the square into little squares by dividing each side into 4 equal parts: how many little squares do we get? How many without red sides? How many with one red side? How many with two red sides? How many with more than two? And what if we start by dividing each side into 5 equal parts? And into 10? And into 57? In this last case, we're not interested in the exact number you calculate; however, we'd like you to indicate which operations are needed to calculate it.

If you've answered convincingly to the last questions, you won't have difficulty saying what happens when you slice a square into little squares starting from a division of the side into n equal parts: n is a number, an integer, positive, but... we don't know which number it is; so, the operations you indicate on the number n (to get the total number of little squares and the number of those with none, or 1, or 2 red sides) must "work" for any number. Are you done? And now, how could you check the accuracy of your answer?

Regular polygons and fractions : a little more variety

In the figure below, you can see three regular polygons: two octagons and one square. They share a vertex and perfectly fill the full rotation angle around that vertex without overlaps or gaps.



Riccardo, who is currently studying fractions, sees the figure in a book and exclaims, "look! In the figure, I can see that $3/8 + 3/8 + 1/4 = 1$ ".

Riccardo is right! In the figure, you can "read" that the full rotation angle is the sum of three angles: a right angle (the angle of the square) and two angles equal to the internal angle of a regular octagon. The angle of the square is $1/4$ of the full rotation angle, while that of the regular octagon is 135° , which is $3/8$ of the full rotation angle. So, it's indeed true that, $3/8 + 3/8 + 1/4 = 1$.

Can you find at least three other combinations of regular polygons (not all identical!) that fill the full rotation angle around a point?

And can you "read" in your figures sums of fractions that result in 1?

A hint: it might help to know that the internal angle of a regular polygon with n sides, expressed as a fraction of the full rotation angle, is $(n-2)/2n$.

Flash problems

These are problems with short texts that may or may not have a reference context, like in these two examples:

The two cars

In car A, there are two males; in car B, there is one male and one female.

I see one of the two cars speeding by, but I can only see one crew member: it's a male.

What is the probability that the passing car is car A?

The product is 3024

The product of four consecutive natural numbers is 3024. What are these four numbers?

Both of these problems don't require a deep knowledge base and have a quick solution. However, they are not trivial because the correct solution is easily and correctly obtained only if the right strategy is used.

In the first problem, using the symbols M_{A1} and M_{A2} for the occupants of car A and M_B and F_B for the occupants of car B, you quickly arrive at the solution: knowing that you've seen a male, there are three possible cases (M_{A1}, M_{A2}, M_B) , but two of these favor car A. Thus, the probability that car A passed by is $2/3$.

In the second problem, the key point is that 3024 is a relatively small number and is not divisible by 5: by trial and error one can start with $1 \times 2 \times 3 \times 4$ (too small), then with $6 \times 7 \times 8 \times 9$ (correct solution).

Game problems

Game problems are generally divided into two parts. The first part involves explaining the rules of the game, while the second part, which constitutes the actual problem, is a series of questions that lead to reflection on what happened during the game.

On the website, only one problem of this type is proposed, *A Game with Dice*, but there are various activities available online that can be presented in the form of a game. Indeed, games can generally be seen not only as a motivator but also as a means to simultaneously build a positive relationship with mathematics and meaningful learning. Students, to follow the game rules and "win", must curb impulsivity and exercise control over decisions, training in metacognition.

A game with dice

We propose a game to be played in pairs

You need two dice and two tables (one for each player) with numbers from 1 to 12, each repeated twice.

Each turn consists of rolling the two dice simultaneously by one player.

On his turn, the player can erase from his table, alternatively, either the two numbers rolled or the sum of the two numbers rolled.

Table example:

1	2	3	4	5	6	7	8	9	10	11	12
12	11	10	3	8	7	6	5	4	3	2	1

Let's provide an example for clarification; with a roll of 1 - 2, you could erase one (only one!) of the following combinations:

- the two numbers 1 and 2;
- the single number 3.

It's up to the player to decide which combination is more advantageous to use.

If players obtain this combination at a point in the game where they have already erased all the number 1s, then they can only erase one number 2 or one number 3; if they obtain it at a point where they have already erased numbers 1 and 3, then they can only erase one number 2. And if they have already erased all numbers 1, 2, and 3, they cannot erase anything and must pass the dice to their opponent.

The winner is the first to erase all 24 numbers from their table.

In class, play at least a couple of rounds in pairs, keeping all the tables, especially those of players who did not win.

Observe the numbers that remain to be erased in the tables of players who did not win: is there a particularly frequent number?

Could you have predicted this phenomenon?

Can you quantify the probability that, by rolling two dice as in this game, the player finds themselves in conditions to erase the number 1? And the number 2? And all the other numbers up to 12?

The proposed game highlights the existence of events that (although all random, casual, unpredictable) do not all have the same probability of occurring; they are not all *equiprobable*. To erase the number 12, number 6 must appear on both dice, while the number 6 can be erased either when it appears on one of the two dice or as the sum of other numbers (1 + 5, 5 + 1, 2 + 4, 4 + 2, 3 + 3). To win, one needs a rational behavior facing uncertain events: recognizing that, even if we cannot say *anything for sure* about what will happen, we can still say *something* and act accordingly.

Video problems

This category of problems is introduced by a video presentation from the teacher. Many of these problems were introduced during the COVID-19 pandemic emergency to continue working on problems even in remote learning, but their use in class under normal conditions is also beneficial. They involve problems constructed around concrete objects and the questions that can arise from observing and manipulating them. Since there is no written text to decode, the initial obstacle to solving the problem, which is reading the mathematical text, can be bypassed.

In [68] Silvia Sbaragli and Silvia Demartini highlights how a deep understanding of the mathematical text is fundamental. This allows for the identification and separation of relevant information from irrelevant information. Reading for understanding is a complex and multi-component process, involving the use of a series of cognitive resources.

When transforming the problem text into an abstract mathematical model, students encounter difficulties related to both the linguistic component (words, phrases, conjunctions, etc.) and the presence of different forms and sources of information (words, numbers, graphs, figures, etc.). Additionally, there is a habituation to certain recurring and stereotypical textual characteristics

that shift students' attention to the selective search for numbers and keywords, overshadowing the understanding of the situation.

Reading to understand is a complex and multi-component process, that is, it involves the use of a series of cognitive resources. To transform the text of the problem into an abstract mathematical model, the pupils encounter difficulties linked both to the linguistic component (words, sentences, conjunctions, etc...) and to the presence of different forms and sources of information (words, numbers, graphs, figures, etc.) and to the habit of certain recurring and stereotyped textual characteristics which shift students' attention to the selective search for numbers and keywords, making the understanding of the situation fade into the background.

A video presentation can also be a useful alternative for students with learning difficulties related to reading text.

A.4 Argumentation in lower secondary school

To demonstrate how argumentation can be promoted in students aged 11-14, we can analyze the activities proposed by the *Language and Argumentation* working group, which has already been mentioned in Section 1.4. The Mathematics Department of the University of Genoa, within the framework of the *Progetto Nazionale Lauree Scientifiche*, has established a collaboration, now decades old, with some *Istituti Comprensivi* in the Liguria region to initiate workshops on argumentation. The working group involves about thirty teachers and is coordinated by Francesca Morselli.

The peculiar characteristics of the project are:

- a) close collaboration between school and university in the planning, experimentation (with the presence of observers in the classroom who assist the teacher and handle documentation), and subsequent analysis phases;
- b) the willingness to involve not individual teachers but entire institutes, benefiting from the exchange, parallel experiments, and the work of already established teams. The reference pole is the *Istituto Comprensivo* of Carcare (Savona), which has been part of the project since 2008-09 and has made argumentation the cornerstone of its school educational project (POF - *Piano dell'offerta formativa*). Over the years, other comprehensive institutes have participated in the project;
- c) the choice to develop medium-long-term paths, tested over several years (thus progressively refined) and gradually integrated into the educational offer of the involved schools;
- d) the willingness to involve, where possible, also literature teachers, creating interdisciplinary paths.

The objective is, therefore, to design and then experiment with extensive activities around the theme of argumentation.

These activities are not teaching units but long-term paths that fit into many different activities and disciplinary fields. Indeed, the transversal nature of argumentation is recognized, aiming to work also with a perspective of continuity between school levels.

Each of the paths developed targets the entire class and involves several phases:

1st phase: individual work;

2nd phase: sharing in small groups;

3rd phase: mathematical discussion (plenary, orchestrated by the teacher);

4th phase: individual report on what has been learned.

As an example, below is a path designed for 6th grade [9].

Activity 1. **Think of a number..**

Individual work.

Think of a number, multiply it by two, add five, subtract the number you thought of, add eight, subtract two, subtract the number you thought of, subtract one.

Do you think it is possible for the teacher, even without knowing the number you thought of, to guess your result? If so, how?

After this initial phase of individual reflection, there is a comparison in small groups and then a collective discussion in class.

Through an exploration with numerical examples, students should observe that the result does not change, no matter what number is thought of, so the teacher can indeed guess the result beforehand. At this point, the teacher stimulates collective discussion to reach the crucial point, which is the search for the reasons why the result does not change. To encourage this search, the problem can be represented in the form of an expression, which allows “seeing” that the number thought cancel each other out and therefore do not affect the final result.

Activity 2.

Individual work.

Write down in the form of an expression the sequence of calculations in the game, using a different color for the number you thought of.

In small groups:

Try to write an expression that works for any number you may have thought of.

Class discussion:

Transcribe the expressions proposed by the class: in your opinion, which one would a mathematician choose? Why?

In this part of the path, students make hypotheses, justify their answers, compare their solving strategies, and get used to moving from observing a regularity to searching for the reasons behind that regularity.

Here are some examples of answers provided by the students:

Student 1:
 $100 \times 2 = 200$
 $200 + 5 = 205$
 $205 - 100 = 105$
 $105 + 8 = 113$
 $113 - 2 = 111$
 $111 - 100 = 11$
 $11 - 1 = 10$

Student 2:
 $N \times 2 = A$
 $A + 5 = B$
 $B - N = C$
 $C + 8 = D$
 $D - 2 = E$
 $E - N = F$
 $F - 1 = G$

Student 3:
 $100 \times 2 + 5 - 100 + 8 - 2 - 100 - 1 = 10$
 $nx2 + 5 - n + 8 - 2 - n - 1 = 10$

Student 4
 $4 \times 2 = 8 + 5 = 13 - 4 = 9 + 8 = 17 - 2 = 15 - 4 = 11 - 1 = 10$
 $nx2 = 8 + 5 = 13 - n = 9 + 8 = 17 - 2 = 15 - n = 11 - 1 = 10$

Even if not all answers are correct, they form the basis for discussion and comparison. Writing an expression allows students to begin using algebra as a demonstrative tool. For 6th-grade students, this is one of their first experiences of writing expressions that also include letters. The conclusion of this first part is a reflection on the different ways to represent the game (relational representation, in the form of an expression; procedural representation, in the form of a sequence of operations). Students are called to discuss the correctness of the two representations, but also and above all, the usefulness of the two expressions in relation to the proposed problem; a

student's comment on the expression $n \times 2 + 5 - n + 8 - 2 - n - 1 = 10$ is, in fact, "you can understand that the thought number is irrelevant".

Activity 3.

Individual work.

The path continues with the request to individually write a report on the collective discussion held in class and with a series of exercises to consolidate what has been learned (transition from natural language to mathematical language and vice versa) and to trigger new reflections. For example:

Write down as an expression the sequence of the following game: Think of an even number, triple it, add four, subtract the number you thought of, add six, subtract three, subtract twice the number you thought of, subtract seven.' Without choosing the number, 'guess' the result. Check by retracing the game with two different even numbers to verify the correctness of your result. If you were to think of an odd number instead of an even one, what would change?

It can be noted that argumentation, at this stage, has been used as a means to convey some mathematical content. The game is very similar to the previous one, but the students are invited to reflect on the properties of even and odd numbers.

Given that this is a long-term path, the work continues with other prompts. To learn to argue, students are encouraged, first, to express their opinions on the truth or falsity of statements coming from different contexts, even non-mathematical ones, and then to try to convince a classmate who has a different opinion from their own. Some examples:

If the 17th of the month falls on a Friday, you have to be careful that day!

The sum of two odd numbers is an even number.

A statement is true if it is true for the majority of people.

This exercise encourages students to reflect on the validity, usefulness, and clarity of the proposed arguments. If the first statement is considered true, what theoretical basis would support it? Furthermore, the activity invites reflection on the functions and limitations of examples: having a large amount of evidence supporting a thesis (a statement that is true in most cases) does not guarantee its absolute truth. For instance, to prove that the sum of two odd numbers is always even, it is not enough to provide numerous examples; a general expression independent of the choice of individual numbers is required.

Another question in which students are further invited to reflect on the need for a general argument and on the role of counterexamples is the following:

Imagine being an ancient sage who wants to quickly identify prime numbers and conjectures that if an integer ends in 7 and is not divisible by 3, then it is a prime number. First of all, you need to understand the statement at hand: according to this statement, what are the conditions for the number to be prime?

It is quite easy for the students to find examples supporting the statement, but also counterexamples (such as 77), making use of the work done on distinguishing between empirical verification and the search for a general rule, on the use of appropriate examples and counterexamples, and on the use of algebra as a demonstrative tool.

Appendix B

The worksheets

Here are all the worksheets along with their respective toolkits, organized based on the corresponding Key Concept.

B.1 Height

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons

Key concept

AP1.1: Height

What is the subject?

An area of interest in the thematic focal point SPACE AND FIGURES is the concept of area and perimeter of a plan figure.

How does this key concept develop understanding of the subject?

To measure the areas of plane figures, a square is used as the unit of measurement. In particular, every polygon can be transformed into an equivalent rectangle. Knowing how to identify the height relative to one side of a polygon is the first step in making this transformation.

The conceptual progression starts from recognizing that in a polygon there are as many heights as there are sides and that the height can be represented by a segment of length equal to the maximum distance of a point of the polygon from the side designated as the base.

Using the progression toolkit to support student learning

Diagnostic questions can be used to quickly identify where students are in their conceptual progression. Once the starting points have been established, it is possible to decide how to continue the teaching, using further diagnostic questions or activating recovery actions with different means and methods (workshop activities, problem solving, traditional lessons, etc.). Response activities can be used as additional means of advancing student understanding.

Progression toolkit:		What I am teaching	
Learning focus	CONCEPTUAL PROGRESSION		
As students' conceptual understanding progresses they can:		Observable learning outcomes to guide my teaching focus	
Diagnostic questions		Questions to find out what my students know and understand	
Response activities		Activities to move my students' understanding forwards	

Progression toolkit: height

Learning focus		The height of a polygon relative to a side is a segment whose length is the maximum distance between a point on the polygon and the straight line on which the side lies.			
As students' conceptual understanding progresses they can:		CONCEPTUAL PROGRESSION			
Recognize that there are as many heights as there are sides in a triangle. P	Understand that heights can also be represented by segments not internal to the figure.	Understand that height can be represented by segments perpendicular to the sides or their extensions in the same direction.	Recognize that in a polygon there are as many heights as there are sides and that the heights can be represented by segments perpendicular to the sides or their extensions in the same direction.	Use and distinguish the concepts of perpendicularity, parallelism, horizontality, and verticality.	Know the definitions and properties of the main plane figures.
	Let's talk about heights!	Triangles and heights	Quadrilaterals and heights		
			Let's draw heights		Is everything clear?

Key:

P Prior understanding from earlier stages of learning

B Bridge to later stages of learning

Let's talk about heights!

Let's talk about heights!

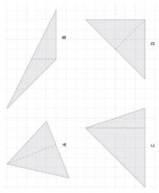
What do you think about the height of a triangle?

Look at these statements and tick (✓) the option that comes closest to your thinking.

Statements	Always true	Mostly true	Sometimes true	Mostly false	Always false
A. In a triangle there are three heights					
B. All the heights of a triangle meet the same point					
C. A height of a triangle is a line segment					
D. The height of a triangle is the same for all triangles					
E. The height of a triangle is the same for all triangles					
F. The height of a triangle is the same for all triangles					

Triangles and heights

Observe the triangles.



A. In one of the triangles the dotted segment is NOT a height. In which?

Think (✓) the line next to the correct answer.

☐ A. In Triangle A

☐ B. In Triangle B

☐ C. In Triangle C

☐ D. In Triangle D

Quadrilaterals and heights

Observe the figures

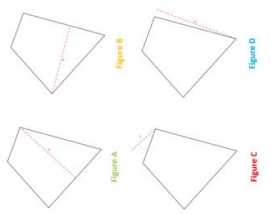


Figure A

Figure B

Figure C

Figure D

Confidence grid

Two-tier multiple choice

Two-tier multiple choice

Let's draw heights

Let's draw heights



Material

- Quadrilaterals drawn on large sheets (30cm x 70cm)
- Ruler (at least 40 cm long) and set square
- Stationery (pencil, eraser, markers, etc.)

Method

Each pupil chooses a quadrilateral and marks it in color, and then they draw the height of the quadrilateral.

1st step

Each pupil marks height relative to the side which has chosen as a dashed line, in the same color.

2nd step

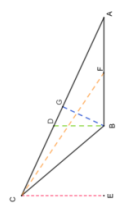
Each group compares their drawing with those by other groups.

Answer

- Is the result what you expected?
- What can be said in general about the heights of quadrilaterals?
- How many heights do you expect to find in a polygon with more than four sides?

Is everything clear?

Observe the figure



A. Observe the figure

10. Which of the following pairs of segments represents two of the heights of triangle ABC?

Think (✓) the line next to correct answer.

☐ A. CE and CF

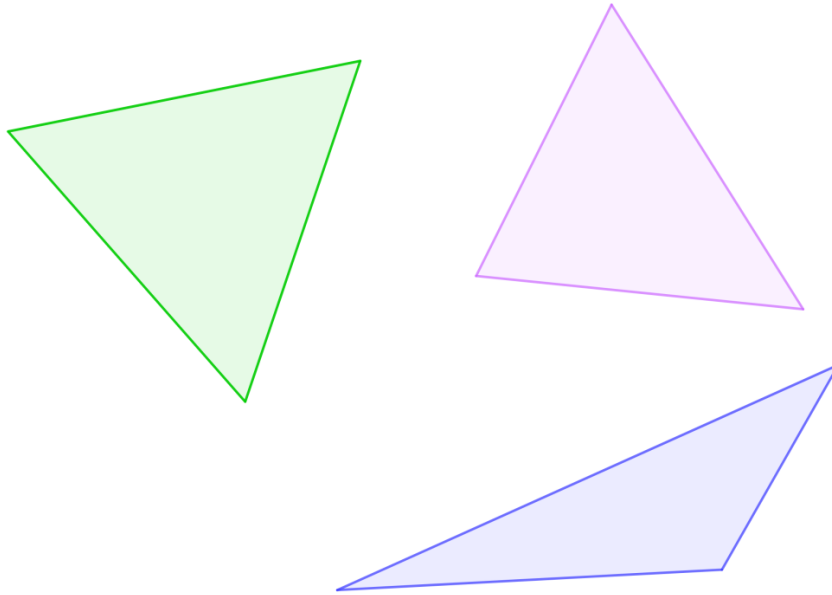
☐ B. BD and BE

☐ C. CE and BE

☐ D. CF and BD

Multiple choice

Let's talk about heights!



What do you think about the height of a triangle?

Look at these statements.

For each statement, tick (✓) **the** option that comes closest to your thinking.

Statements	I'm sure this is correct	I think this is correct	I think this is wrong	I'm sure this is wrong
A In a triangle there are three heights				
B All the heights of a triangle have the same length				
C A height of a triangle is always internal to the triangle				
D Every height of an acute angled triangle is always internal to the triangle				
E Every height of an obtuse angled triangle is always external to the triangle				

Space and figures > Subject AP: area and perimeter > topic AP1: areas and perimeters of polygons > Key concept AP1.1: Height

Diagnostic question

Let's talk about heights!

Overview

Learning focus:	The height of a polygon relative to a side is a segment whose length is the maximum distance between a point of the polygon and the straight line on which the side lies.
Observable learning outcome:	Recognize that in a triangle there are as many heights as there are sides and that the heights can also be represented by segments that are external to the figure.
Question type:	Confidence grid
Key words:	Height, triangles

P

PRIOR UNDERSTANDING

This diagnostic question probes understanding of ideas that are usually taught at age 5-11, to aid transition from earlier stages of learning.

What does the research say?

The concept of height is often considered a simple one but, in reality, it is the basis of many misconceptions, which sometimes derive from the teachers' choices. We often read in textbooks that a height is a segment that *starts* from a vertex and *falls* perpendicularly on the opposite side. We should ask ourselves, however, whether the height is really a segment or a quantity, whether it can *start* from a vertex and *fall* on one side, whether it concerns only some figures and not others, whether there are multiple heights in a polygon or only one. The teacher often makes it clear that the segment that identifies the height has a vertex of the polygon as its end point, leading the students to exclude those segments which, although representing the distance between a vertex and a side, do not have points in common with the vertices of the figure or which are totally or partially outside the polygon. Even the practice of using a plumb line to identify height can lead to confusing verticality with perpendicularity, so students do not recognize a representation of height if the segment is not vertical from the observer's point of view. (Sbaragli S., 2010. Qui cade sua... altezza. La Vita Scolastica. 18, 25-27).

Ways to use this question

Students should complete the grid individually, with paper and pencil, or the worksheet can be projected onto the IWB and students can use colored cards to "vote".

If there is a variety of answers, a structured discussion can be initiated in class. A student can be asked to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; ask another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and really understand their learning needs.

Differentiation

For the inclusion of pupils with additional requirements, you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

R: Statements A e D are correct; the other claims are incorrect.

How to respond – what next?

Statement B is correct only if the triangle is equilateral.

Statements C, D and E invite reflection on the segments representing the height relative to a specific side, which can also be external to the triangle. The expressions "every" and "always" draw attention to the presence of the three heights and their characteristics.

Students who mark statement E as correct probably forget to consider the height relative to the side opposite to the obtuse angle.

To consolidate the concept of height of a polygon you can continue with diagnostic questions:

- Triangles and height
- Quadrilaterals and height

References

Sbaragli S. (2010) *Qui cade sua... altezza*. *La Vita Scolastica*. 18, 25- 27.

Sbaragli S. (2016) *L'importanza dei saperi fondanti. Il caso dell'altezza dei poligoni*. In: D'Amore, B. & Sbaragli S. (2016) (Eds). *La matematica e la sua didattica*. Convegno del trentennale. Bologna: Pitagora.

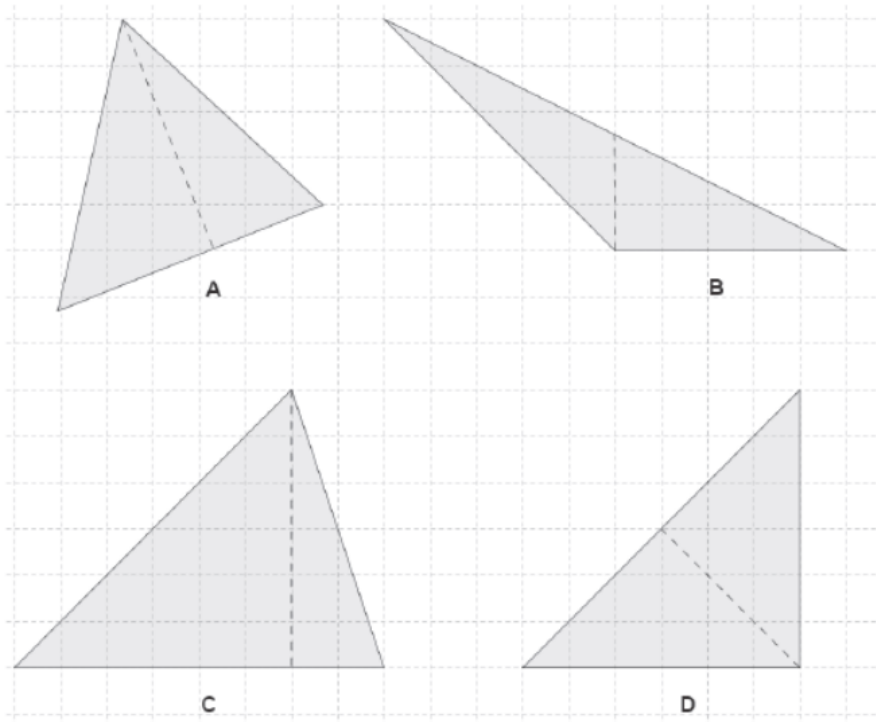
Clara Bertinetto, Arja Metiäinen, Johannes Paasonen, Eija Voutilainen *Contaci! Numeri, relazioni, dati vol. 1* Zanichelli

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Triangles and heights

Observe the triangles.



1. In one of the triangles the dotted segment is NOT a height. In which?

Tick (✓) the box next to the correct answer.

A In triangle A

B In triangle B

C In triangle C

D In triangle D

☐☐☐☐

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

A A height does not divide the base in half

☐

B A height cannot be oblique

☐

C A height is not vertical

☐

D A height is not perpendicular to base

☐

Space and figures > Subject AP: area and perimeter > topic AP1: areas and perimeters of polygons > Key concept AP1.1: Height

Diagnostic question

Triangles and heights

Overview

Learning focus:	The height of a polygon relative to a side is a segment whose length is the maximum distance between a point of the polygon and the straight line on which the side lies.
Observable learning outcome:	The height can be represented by segments perpendicular to the sides or their extensions in the same direction.
Question type:	Two-tier multiple choice
Key words:	Height, triangles

What does the research say?

The concept of height is often considered a simple one but, in reality, it is the basis of many misconceptions, which sometimes derive from the teachers' choices. We often read in textbooks that a height is a segment that *starts* from a vertex and *falls* perpendicularly on the opposite side. We should ask ourselves, however, whether the height is really a segment or a quantity, whether it can *start* from a vertex and *fall* on one side, whether it concerns only some figures and not others, whether there are multiple heights in a polygon or only one. The teacher often makes it clear that the segment that identifies the height has a vertex of the polygon as its end point, leading the students to exclude those segments which, although representing the distance between a vertex and a side, do not have points in common with the vertices of the figure or which are totally or partially outside the polygon. Even the practice of using a plumb line to identify height can lead to confusing verticality with perpendicularity, so students do not recognize a representation of height if the segment is not vertical from the observer's point of view. (Sbaragli S., 2010. Qui cade sua... altezza. La Vita Scolastica. 18, 25-27).

From the results of the INVALSI 2017 national test, it emerges that the most frequent incorrect answer in Q1 is D (39.1%). Almost everyone recognizes representation C as correct (only 6.2% of incorrect answers) and representation A (13.5% consider it incorrect), while there is equality of percentages between answers D (39.1%) and B (39.4%).

Ways to use this question

Students must complete the grid individually, with paper and pencil, or the worksheet can be projected onto the IWB and students can use colored cards to "vote".

If there is a variety of answers, a structured discussion can be initiated in class. Ask one student to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; ask another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and really understand their learning needs.

Differentiation

For the inclusion of pupils with additional requirements, you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

1. B
2. D

How to respond – what next?

The high percentages of responses indicating B and D as misrepresentations suggest that the concept of height “falling from the top and reaching the base” is probably deep-rooted. Answer B to question 2 seems to confirm this interpretation. Indicating A or D as incorrect could instead suggest the persistence of the misconception of height as vertical. In this case we would expect the answer C to question 2. The answer A to question 2 could be given by those students who confuse height and median or who, when they think of a triangle, instinctively imagine an isosceles triangle.

If misunderstandings persist, teachers should activate remedial actions with the means and methods most suited to them (workshop activities, problem solving, traditional lesson etc...).

The response activity could be used in follow-up:

- Let's draw heights

References

Sbaragli S. (2010) *Qui cade sua... altezza*. *La Vita Scolastica*. 18, 25- 27.

Sbaragli S. (2016) *L'importanza dei saperi fondanti. Il caso dell'altezza dei poligoni*. In: D'Amore, B. & Sbaragli S. (2016) (Eds). *La matematica e la sua didattica*. Convegno del trentennale. Bologna: Pitagora.

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<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Quadrilaterals and heights

Observe the figures

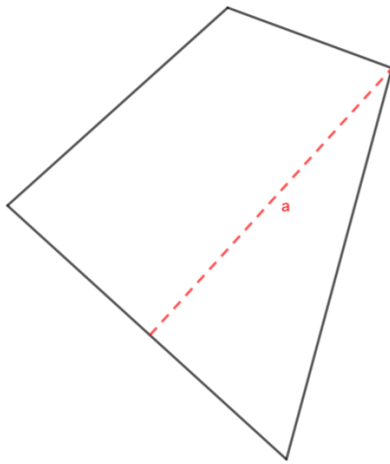


Figure A

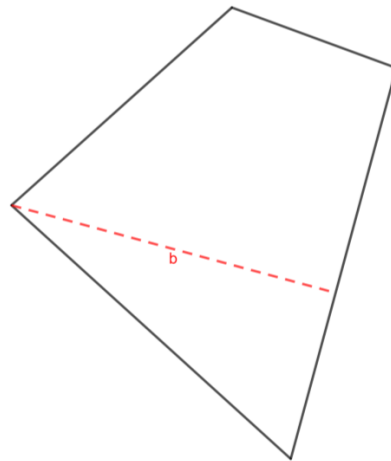


Figure B

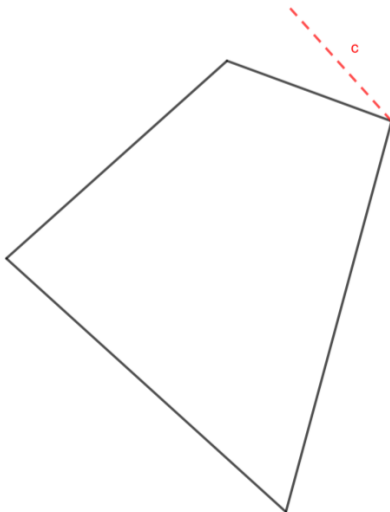


Figure C

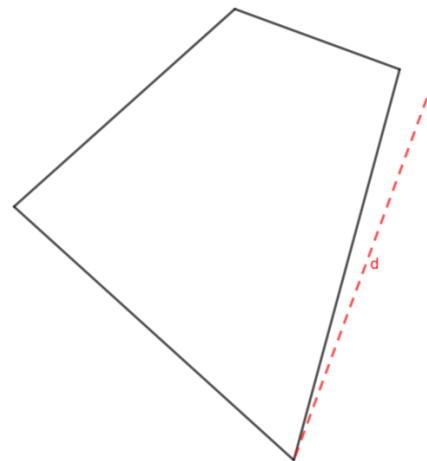


Figure D

1. In which figure is the segment labelled by a letter NOT a height?

Tick (✓) the box next to correct answer.

A Figure A

☐

B Figure B

☐

C Figure C

☐

D Figure D

☐

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

A The length of the labelled segment is not the maximum distance

☐

B The labelled segment is not perpendicular to the side

☐

C The labelled segment is not inside the polygon

☐

Space and figures > Subject AP: area and perimeter > topic AP1: areas and perimeters of polygons > Key concept AP1.1: Height

Diagnostic question

Quadrilaterals and heights

Overview

Learning focus:	The height of a polygon relative to a side is a segment whose length is the maximum distance between a point of the polygon and the straight line on which the side lies.
Observable learning outcome:	Recognize that in a polygon there are as many heights as there are sides and that they can be represented by segments perpendicular to the sides or their extensions in the same direction.
Question type:	Two-tier multiple choice
Key words:	Height, quadrilaterals

What does the research say?

The concept of height is often considered a simple concept but, in reality, it is the basis of many misconceptions, which sometimes derive from teachers' didactic choices. We often read in textbooks that height is a segment that *starts* from a vertex and *falls* perpendicularly on the opposite side. We should ask ourselves, however, whether the height is really a segment or a quantity, whether it can *start* from a vertex and *fall* on one side, whether it concerns only some figures and not others, whether there are multiple heights in a polygon or only one. The teacher often makes it clear that the segment that identifies the height has a vertex of the polygon as its end, leading the students to exclude those segments which, although representing the distance between a vertex and a side, do not have points in common with the vertices of the figure or which are totally or partially outside the polygon. Even the practice of using a plumb line to identify height can lead to confusing verticality with perpendicularity, so students do not recognize a representation of height if the segment is not vertical from the observer's point of view. (Sbaragli S., 2010. Qui cade sua... altezza. La Vita Scolastica. 18, 25-27).

Ways to use this question

Students must complete the grid individually, with paper and pencil, or the worksheet can be projected onto the IWB and students can use colored cards to "vote".

If there is a variety of answers, a structured discussion can be initiated in class. Ask one student to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; ask another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and really understand their learning needs.

Differentiation

For the inclusion of pupils with additional requirements, you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

1. C
2. A

How to respond – what next?

This diagnostic question focuses on the definition of height as a segment whose length is equal to the maximum distance of the points of the polygon from the straight line containing a side. Students who consider the segments represented in figures A and B incorrect may not recognize that the segments are perpendicular to the side they refer to; therefore, one would expect B as the answer to question 2. Instead, those who choose option D might think that the height must be inside the quadrilateral. The same motivation could also be given by students who identify figure C as the one in which the segment does not represent the height: in this case, however, the associated motivation will not be the correct one (answer A) but option C.

If misunderstandings persist, teachers should activate remedial actions with the means and methods most suited to them (workshop activities, problem solving, traditional lesson etc...).

The response activity may also be used for this purpose:

- Let's draw heights

The response activity could be used in follow-up:

- Is everything clear?

References

Sbaragli S. (2010) *Qui cade sua... altezza. La Vita Scolastica*. 18, 25- 27.

Sbaragli S. (2016) *L'importanza dei saperi fondanti. Il caso dell'altezza dei poligoni*. In: D'Amore, B. & Sbaragli S. (2016) (Eds). *La matematica e la sua didattica*. Convegno del trentennale. Bologna: Pitagora.

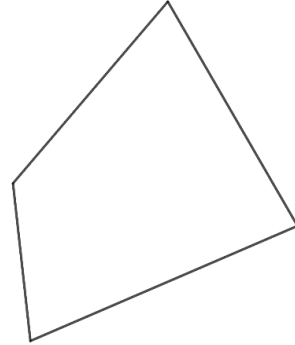
Botta E. & Sbaragli S. (2016). *Il caso dell'altezza. Un sapere fondante*. *Nuova secondaria*, XXXIV, 1, 112-116

Campolucci L. & Sbaragli S. (2022). *Altezze dei poligoni. Seconda parte*. Gaia Edizioni.

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Let's draw heights



Material

- Quadrilaterals drawn on large sheets (50cm x 70cm)
- Ruler (at least 40 cm long) and set square
- Stationery (pencils, erasers, markers, etc.)

Method

Form groups of 4 students and place the sheet with the drawing at the center of the group.

1st step

Each pupil chooses a side of the quadrilateral and marks it in color, and may also marks the line containing that side.

2nd step

Each pupil marks height relative to the side s/he has chosen as a dashes line, in the same color.

Observe

Each group compares their drawing with those by other groups.

Answer

- 1 Is the result what you expected?
 - 2 What can be said in general about the heights of quadrilaterals?
 - 3 How many heights do you expect to find in a polygon with more than four sides?
-

Space and figures > Subject AP: area and perimeter > topic AP1: areas and perimeters of polygons > Key concept AP1.1: Height

Response activity

Let's draw heights

Overview

Learning focus:	The height of a polygon relative to a side is a segment whose length is the maximum distance between a point on the polygon and the straight line on which the side lies.
Observable learning outcome:	Recognize that in a polygon there are as many heights as there are sides and know how to represent heights through segments perpendicular to the sides or to their extensions in the same direction.
Activity type:	Application and practice
Key words:	Height, quadrilaterals

This activity can help develop students' understanding by addressing critical points that may emerge from the following diagnostic question:

- Quadrilaterals and heights

What does the research say?

The concept of height is often considered a simple one but, in reality, it is the basis of many misconceptions, which sometimes derive from the teachers' choices. We often read in textbooks that a height is a segment that *starts* from a vertex and *falls* perpendicularly on the opposite side. We should ask ourselves, however, whether the height is really a segment or a quantity, whether it can *start* from a vertex and *fall* on one side, whether it concerns only some figures and not others, whether there are multiple heights in a polygon or only one. The teacher often makes it clear that the segment that identifies the height has a vertex of the polygon as its end point, leading the students to exclude those segments which, although representing the distance between a vertex and a side, do not have points in common with the vertices of the figure or which are totally or partially outside the polygon. Even the practice of using a plumb line to identify height can lead to confusing verticality with perpendicularity, so students do not recognize a representation of height if the segment is not vertical from the observer's point of view. (Sbaragli S. , 2010. Qui cade sua... altezza. La Vita Scolastica. 18, 25-27).

Ways to use this activity

This activity offers students the opportunity to practice and clarify the concept also through comparison with other groups and collective discussion.

To draw heights, students must be careful to identify the point (or points) of the figure furthest from the line containing the chosen side, from which the height will be drawn.

Group work and the final comparison allows students to self-correct and reflect once again on the meaning of height. The final discussion, prompted by questions, allows us to generalize the definition for all polygons.

Differentiation

For the inclusion of pupils with additional requirements, provision may be made, where necessary, for a support teacher.

Expected answers

Practical application with the construction of heights with ruler and set square is expected to strengthen awareness of the concept of height. By comparing their drawing with those by other groups, the children should highlight any errors and arrive at a generalisation.

References

Campolucci L. & Sbaragli S. (2022). Altezze dei poligoni. Seconda parte. Gaia Edizioni.

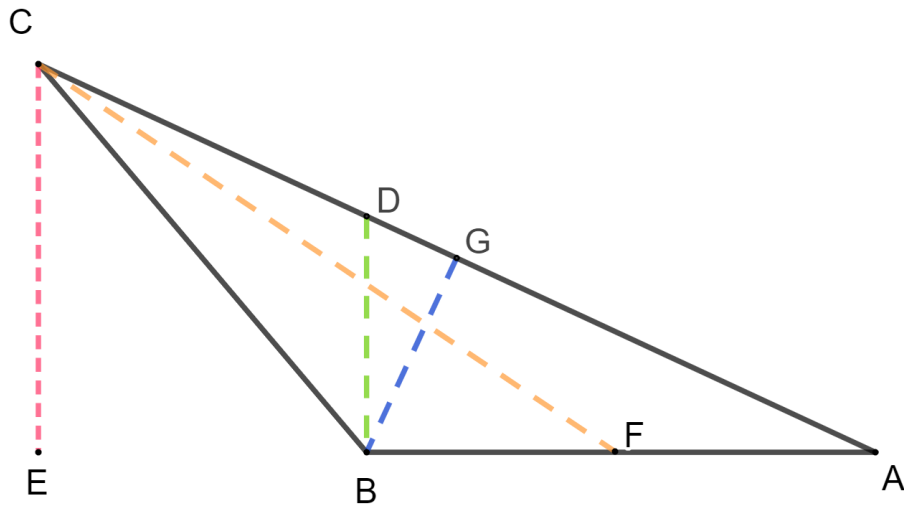
<https://missioneinsegnante.it/2022/12/12/altezze-dei-poligoni-seconda-parte/>

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Is everything clear?

A. Observe the figure



1A. Which of the following pairs of segments represents two of the heights of triangle ABC?

Tik (✓) the box next to correct answer.

A CE and CF

☐

B BD and BG

☐

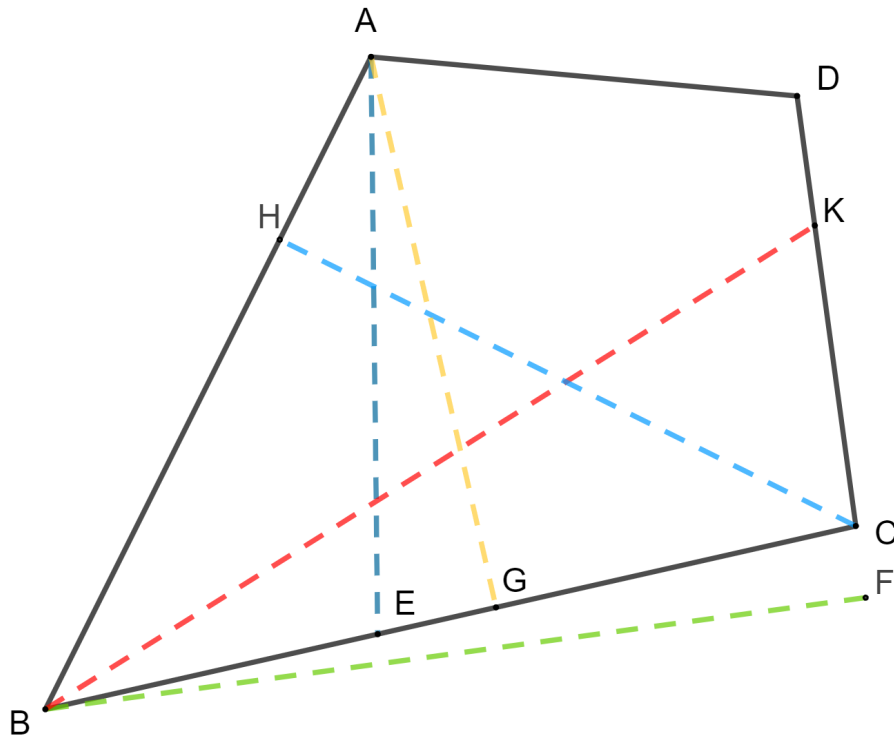
C CE and BG

☐

D CF and BD

☐

B. Observe the figure



1B. Which of the following triples of segments represents three of the heights of the quadrilateral ABCD?

Tik (✓) the box next to correct answer.

A AE, BK, and CH

B AE, BF, and CH

C BK, BF, and AG

D CH, BF, and AG

☐☐☐☐

Space and figures > Subject AP: area and perimeter > topic AP1: areas and perimeters of polygons > Key concept AP1.1: Height

Response activity

Is everything clear?

Overview

Learning focus:	The height of a polygon relative to a side is a segment whose length is the maximum distance between a point on the polygon and the straight line on which the side lies.
Observable learning outcome:	Use and distinguish the concepts of perpendicularity, parallelism, horizontality, and verticality. Know the definitions and properties of the main plane figures.
Activity type:	Multiple choice
Key words:	Height

What does the research say?

The concept of height is often considered a simple one but, in reality, it is the basis of many misconceptions, which sometimes derive from the teachers' choices. We often read in textbooks that a height is a segment that *starts* from a vertex and *falls* perpendicularly on the opposite side. We should ask ourselves, however, whether the height is really a segment or a quantity, whether it can *start* from a vertex and *fall* on one side, whether it concerns only some figures and not others, whether there are multiple heights in a polygon or only one. The teacher often makes it clear that the segment that identifies the height has a vertex of the polygon as its end point, leading the students to exclude those segments which, although representing the distance between a vertex and a side, do not have points in common with the vertices of the figure or which are totally or partially outside the polygon. Even the practice of using a plumb line to identify height can lead to confusing verticality with perpendicularity, so students do not recognize a representation of height if the segment is not vertical from the observer's point of view. (Sbaragli S., 2010. Qui cade sua... altezza. La Vita Scolastica. 18, 25-27).

Question A appeared on the INVALSI 2013 national test for first-year secondary school students: the percentage of correct answers (C) was 34.8. The most common incorrect option was A (28%).

Ways to use this activity

Students must complete the grid individually, with paper and pencil, or the worksheet can be projected onto the IWB and students can use colored cards to "vote".

If there is a variety of answers, a structured discussion can be initiated in class. Ask one student to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; ask another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and really understand their learning needs.

Differentiation

For the inclusion of pupils with learning additional requirements, you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

1A. C

1B. D

References

Sbaragli S. (2010) *Qui cade sua... altezza. La Vita Scolastica*. 18, 25- 27.

Sbaragli S. (2016) *L'importanza dei saperi fondanti. Il caso dell'altezza dei poligoni*. In: D'Amore, B. & Sbaragli S. (2016) (Eds). *La matematica e la sua didattica*. Convegno del trentennale. Bologna: Pitagora.

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B.2 Areas of polygons

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons

Key concept

AP1.2: Areas of polygons

What is the subject?

An area of interest of the thematic focal point SPACE AND FIGURES is the concept of area and perimeter of a plan figure.

How does this key concept develop understanding of the subject?

To measure the areas of flat figures, a square is used as a unit of measurement. Every polygon can be transformed into an equivalent rectangle. Knowing how to transform figures into the equivalent rectangle allows pupils to internalize the formulas for calculating the areas of polygons without having to rely only on memory.

The conceptual progression starts with recognizing equivalent figures. Then we consider the transformation of triangles and quadrilaterals into equivalent rectangles to arrive at identifying the square as a unit of measure.

Using the progression toolkit to support student learning

Diagnostic questions can be used to quickly identify where students are in their conceptual progression. Once the starting points have been established, it is possible to decide how to continue the teaching, using further diagnostic questions or activating recovery actions with different means and methods (workshop activities, problem solving, traditional lessons, etc.). Response activities can be used as additional means of advancing student understanding.

Progression toolkit:	
Learning focus	What I am teaching
As students' conceptual understanding progresses they can:	Observable learning outcomes to guide my teaching focus
Diagnostic questions	Questions to find out what my students know and understand
Response activities	Activities to move my students' understanding forwards

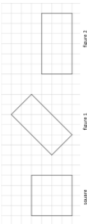
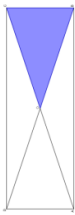
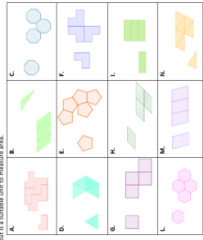
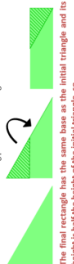
Progression toolkit: areas of polygons

Learning focus	Determine areas of plane figures by decomposing them into elementary figures or using common formulas.			
As students' conceptual understanding progresses they can:	CONCEPTUAL PROGRESSION			
	Identify equidecomposable figures. P	Identify equivalent figures obtained by adding or deleting matching congruent parts. B	Transform triangles and quadrilaterals into equivalent rectangles.	Calculate the area of any polygon.
Diagnostic questions	Odd one out	Equivalent figures	The triangle within the rectangle	Units of measuring areas
Response activities		Are they equivalent?	Transforming polygons	

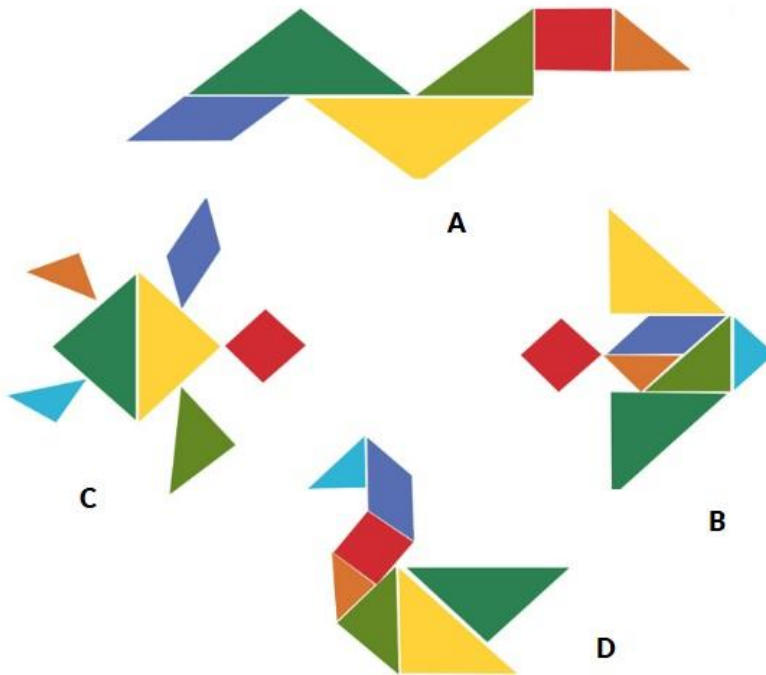
Key:

P Prior understanding from earlier stages of learning

B Bridge to later stages of learning

GENERIC POLYGONSET Odd one out	GENERIC POLYGONSET Equivalent figures 1. Observe the figure below. Figure 1 is equivalent to the square. Figure 2 is not.  Read the following statements carefully. <table><tr><td>The square can be decomposed into 16 squares equivalent to each other.</td><td>☐</td></tr><tr><td>The square can be decomposed into 16 triangles equivalent to each other.</td><td>☐</td></tr><tr><td>The square can be decomposed into 16 rectangles equivalent to each other.</td><td>☐</td></tr><tr><td>The square can be decomposed into 16 trapezoids equivalent to each other.</td><td>☐</td></tr><tr><td>The square can be decomposed into 16 circles equivalent to each other.</td><td>☐</td></tr></table> Questions - Which statements help explain why Figure 1 is equivalent to the square? - Which statements help explain why Figure 2 is not equivalent to the square?	The square can be decomposed into 16 squares equivalent to each other.	☐	The square can be decomposed into 16 triangles equivalent to each other.	☐	The square can be decomposed into 16 rectangles equivalent to each other.	☐	The square can be decomposed into 16 trapezoids equivalent to each other.	☐	The square can be decomposed into 16 circles equivalent to each other.	☐	GENERIC POLYGONSET The triangle within the rectangle The figure shows a rectangle ABCD along with its diagonals. If the area of the rectangle is known, can you determine the area of the shaded triangle?  Tick (✓) the box next to the best answer. <table><tr><td>☐ A. Yes, because the four triangles with vertex O are not all equal in size.</td><td>☐</td></tr><tr><td>☐ B. No, because I don't know the dimensions of the rectangle.</td><td>☐</td></tr><tr><td>☐ C. Yes, because the four triangles are equivalent.</td><td>☐</td></tr><tr><td>☐ D. Yes, because the triangles four with vertex O are identical.</td><td>☐</td></tr></table>	☐ A. Yes, because the four triangles with vertex O are not all equal in size.	☐	☐ B. No, because I don't know the dimensions of the rectangle.	☐	☐ C. Yes, because the four triangles are equivalent.	☐	☐ D. Yes, because the triangles four with vertex O are identical.	☐	GENERIC POLYGONSET Generic polygons 1. Observe the figure.  If the side of each grid square is 1 m, what is the area of the polygon? <table><tr><td>☐ A. 22 m²</td><td>☐</td></tr><tr><td>☐ B. 23 m²</td><td>☐</td></tr><tr><td>☐ C. 21 m²</td><td>☐</td></tr><tr><td>☐ D. 24 m²</td><td>☐</td></tr></table>	☐ A. 22 m ²	☐	☐ B. 23 m ²	☐	☐ C. 21 m ²	☐	☐ D. 24 m ²	☐	GENERIC POLYGONSET Units for measuring areas Measuring a quantity means counting how many times a given unit of measurement is contained in the quantity to be determined. For example, to measure the area of a polygon, we need to know how many times a unit of area is contained in the polygon that has been assigned to a suitable unit to measure area.  For each of the following statements, choose the unit polygons or polygons, among A to N above, to which the statement applies.	Units for measuring areas
The square can be decomposed into 16 squares equivalent to each other.	☐																														
The square can be decomposed into 16 triangles equivalent to each other.	☐																														
The square can be decomposed into 16 rectangles equivalent to each other.	☐																														
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☐ D. 24 m ²	☐																														
Odd one out	Odd one out	Multiple choice	Multiple choice	Observe and explain																											
Are they equivalent?	Transforming polygons GENERIC POLYGONSET Transforming polygons Materials • A set of geometric shapes: right-angled triangles, scalene triangles, isosceles triangles, equilateral triangles, squares, rectangles, parallelograms, rhombus, scalene trapezoid. • Stationery (pencil, eraser, markers, ruler, square, etc.), scissors, adhesive tape. Method Students work in pairs or groups of 3. Each pair/group is provided with a figure per type. A. Right-angled triangle - Observe the figure below. The triangle is divided into a smaller triangle and a trapezoid. - Cut along the fold. - Assemble the two resulting pieces into a rectangle.  B. Scalene triangle The figure rectangle has the same base as the initial triangle and its height is half the height of the initial triangle, so $\frac{b \times h}{2}$. You can divide a scalene triangle into two right-angled triangles by drawing the height relative to one side. - Observe the figure below. Consider one each of the right-angled triangles you obtain in this way. What do you get?	Multiple choice	Multiple choice	Multiple choice	Multiple choice																										

Odd one out



Which of these images is the odd one out?

Explain the reasons for your answer.

.....

.....

.....

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Diagnostic question

Odd one out

Overview

Learning focus:	Equivalence between equidecomposability and equality of area in polygons.
Observable learning outcome:	Identify equidecomposable figures.
Question type:	Odd one out
Key words:	equi-extension, plane figures

P

PRIOR UNDERSTANDING

This diagnostic question probes understanding of ideas that are usually taught at age 5-11, to aid transition from earlier stages of learning.

What does the research say?

The tangram is a widely enjoyed puzzle that can be used to illustrate the concept of equidecomposability of polygons. The strength of the game lies in the fact that each piece can be dissected into parts congruent with the two smallest pieces. Consequently, one can assess whether two compound figures have the same area without measurement. The equivalence between equi-extension (the property of having the same area) and equidecomposability in polygons serves as the foundation for introducing calculations of areas, treated in subsequent sheets.

Way to use this question

The worksheet can be displayed on the IWB, and the teacher can instruct students to independently write in their notebooks the reasons that influenced their choice. At a later stage, students are encouraged to share their observations with the rest of the class, facilitating comparisons of answers and reasons. This type of discussion provides students with the opportunity to delve into their thinking processes and truly understand their learning needs.

Differentiation

To include students with learning difficulties, the teacher can read the questions aloud to the class. In certain situations, the assistance of a support teacher for one or more students may be more appropriate.

Expected answers

Figure A differs from other figures as it does not contain the small blue triangle.

How to respond – what next?

This serves as an introductory sheet with an activity acting as an "icebreaker", particularly accessible to students with difficulties. If understanding does not emerge spontaneously, it may be necessary to emphasize that equivalent figures are formed by possibly different arrangements of the same parts.

You can continue the activity with the diagnostic question:

- Equivalent figures

Reference

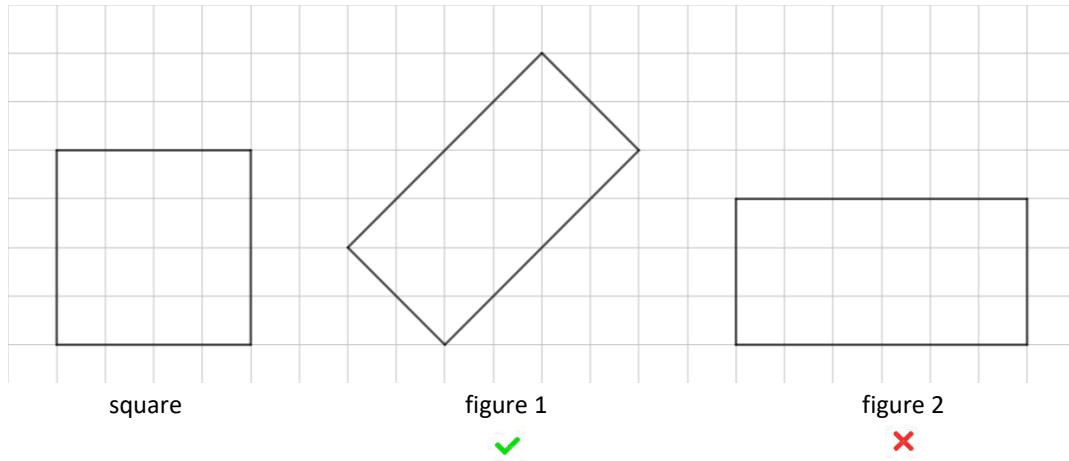
Ferri L., Matteo A., Sgobbi F., Bruno S., Maffia A. *Da zero a infinito: Corso di matematica per la scuola secondaria di primo grado. Guida per l'insegnante*, Fabbri Editori

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Equivalent figures

1 Observe the image below: Figure 1 is equivalent to the square, Figure 2 is not.



Read the following statements carefully:

The square can be decomposed into 16 squares equivalent to each other.	The square can be decomposed into 32 triangles equivalent to each other.
The square can be decomposed into 4 triangles equivalent to each other.	The square can be decomposed into 8 rectangles equivalent to each other.
The square can be decomposed into 8 triangles equivalent to each other.	The square can be decomposed into 4 rectangles equivalent to each other.
The square can be decomposed into 16 triangles equivalent to each other.	The square can be decomposed into 4 squares equivalent to each other.

Questions

- Which statements help explain why Figure 1 is equivalent to the square?
- Which statements help explain why Figure 2 is not equivalent to the square?

2 The figures below have been obtained by deleting pieces from a square. Figure 1 is equivalent to Figure 2 but not to Figure 3.

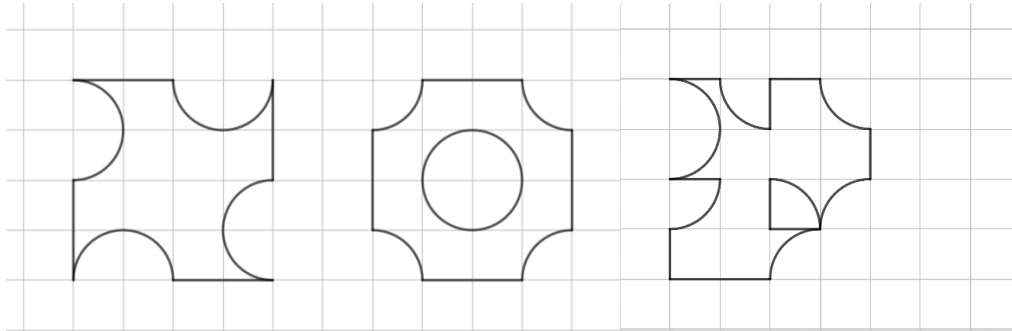


figure1

figure 2

figure 3



Read the following statements carefully:

In figure 1, we can count 8 whole squares.	With the missing parts of Figure 1, we can build 2 circles.
Eight quarter circles have been deleted from Figure 1.	Four semicircles have been deleted from Figure 1.

Questions

3. Which statements help explain why Figure 2 is equivalent to Figure 1?
4. Which statements help explain why Figure 3 is not equivalent to Figure 1?

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Diagnostic question

Equivalent figures

Overview

Learning focus:	Equivalent figures can be obtained by adding or deleting matching congruent parts.
Observable learning outcome:	Explain the reasons behind the conclusion that two or more figures are or are not equivalent to each other.
Question type:	Peer discussion
Key words:	equiextension, plane figures, circle

B

BRIDGING

This activity uses tiling to introducing the concept of units of measure for areas of plane figures.

What does the research say?

From the analysis of the results of the INVALSI tests first- and third-year pupils of lower secondary school and second-year pupils of upper secondary school (grade 6, 8 and 10 respectively), it appears that, among the questions on Space and Figures, where more than 60% of answers are incorrect, 11 out of 40 concern areas of plane figures. The mistakes arise when the children have to identify equivalent figures or recognize the base of a figure and its relative height. The learning objectives for the third year of lower secondary school, defined in the *Indicazioni nazionali per il curriculum della scuola dell'infanzia e del primo ciclo di istruzione* (2012) include:

"to determine the area of simple figures by breaking them down into elementary figures, for example triangles, or by using common formulas."

School textbooks often focus on exercises that involves applying formulas to data that are provided in the statement of the exercise. What is missing, however, are exercises on seeking and identifying elements that allow to calculate areas by observing plane figures.

Way to use this question

Students should complete this activity in pairs or small groups. The focus should be on discussion. It is through discussions that students can check their understanding and test their explanations.

Each group can be given a single worksheet to fill in as group work. This fosters discussion. However, each group member should be able to report back to the class. Listening to each group's conversations can often give the teacher insight into how their students think. At the end, each student writes the answer individually in their notebook. This is likely to help them consolidate their learning.

Differentiation

Reading the statements aloud to the whole class may help students with additional requirements. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

Question 1

1. It can be verified that rectangle 1 is equivalent to the square, by dividing the square into:
 - 4 right-angled isosceles triangles with hypotenuse of length 4 squares
 - 8 right-angled isosceles triangles with sides of length 2 square
 - 16 right-angled isosceles triangles with hypotenuse of length 2 squares
 - 32 right-angled isosceles triangles with sides of length 1 square
2. It can be verified that rectangle 2 is not equivalent to the square, by dividing the square into:
 - 16 squares with a side of length 1 square
 - 8 rectangles measuring 1 x 2 squares
 - 32 right-angle isosceles triangles with sides of length 1 square

Question 2

3. Three statements help explain why Figure 2 is equivalent to Figure 1:
 - 4 semicircles have been deleted from Figure 1
 - 8 quarter circles have been deleted
 - with the missing parts of Figure 1 you can build 2 circles.
4. Two statements help explain why Figure 3 is not equivalent to Figure 1:
 - 8 quarter circles have been deleted from Figure 1
 - in Figure 1 one can count 8 whole squares.

How to respond – what next?

In order to answer Question 1, students are invited to break down the square into equal parts (isosceles or scalene right-angled triangles, rectangles, squares) and to verify the equivalence of the figures by reassembling the parts to construct rectangles 1 or 2. This tiling activity acts as a bridge to introduce units of measurement for areas (diagnostic question: Unit for measuring areas).

For Question 2, the focus is on establishing the equivalence of plane figures by deleting congruent parts. The choice of curved sides is meant to make pupils realize that the parts into which the figures are broken down do not necessarily have to be polygons.

To consolidate the knowledge acquired with this diagnostic question, you can continue with the following response activity:

- Are they equivalent?

References

Indicazioni nazionali per il curricolo della scuola dell'infanzia e del primo ciclo di istruzione. 2012

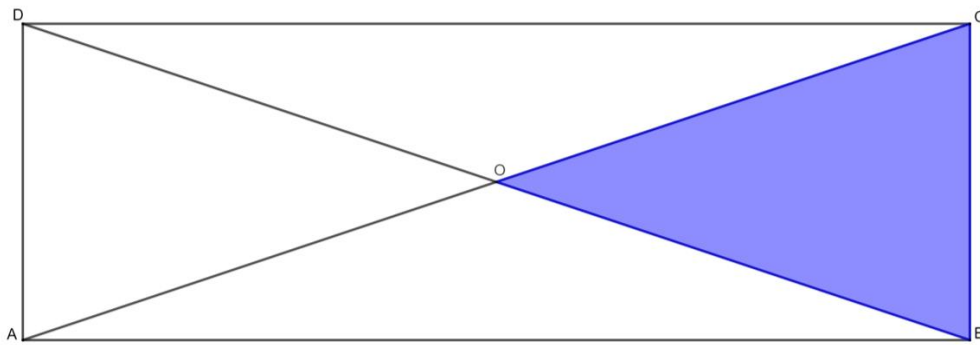
<https://www.gestinv.it/>

BEST. Best evidence science teaching. University of York. Department of education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

The triangle within the rectangle

The figure shows a rectangle ABCD along with its diagonals. If the area of the rectangle is known, can you determine the area of the shaded triangle?



Tick (✓) the box next to the best answer.

- A** No, because the four triangles with vertex O are not all equal to each other ☐
- B** No, because I don't know the dimensions of the rectangle ☐
- C** Yes, because the four triangles are equivalent ☐
- D** Yes, because the four triangles with vertex O are isosceles ☐

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Diagnostic question

The triangle within the rectangle

Overview

Learning focus:	Calculate the area of plane figures by decomposing them into elementary figures, such as triangles, or using basic formulas
Observable learning outcome:	Understand that the area of a triangle equals that of a rectangle with the same base, and height half the height of the triangle. Conversely, the area of a rectangle is equal to that of a triangle with the same base and height double the height of the rectangle
Question type:	Multiple choice
Key words:	Rectangle, isosceles triangle, equivalence of plane figures

What does the research say?

This question was presented eight-grade students in the 2012 INVALSI national test.

The correct response rate was notably low at 24.1%. This indicates difficulties in various aspects, including the challenge of recognizing non-congruent figures as equivalent and struggling to extract information and comprehend relationships among the elements of the depicted plane figures.

Incorrect answers are distributed as follows: A 26.9%, B 30.8% and D 16.4%.

Way to use this question

Students must fill in the sheet individually, with paper and pencil. Alternatively, the worksheet can be projected onto the IWB and students can “vote” using colored.

If there is a range of responses, you may choose to engage in a structured discussion in class. A student can be asked to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; ask another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and gain full awareness of their learning needs.

Differentiation

To include pupils with learning difficulties, you can choose to read the questions to the class. In some situations, the assistance of a support teacher for one or more students may be more appropriate.

Expected answer

C

How to respond – what next?

Students who select option A may lack clarity on the distinction between equivalence and congruence, possibly assuming that if two triangles are not congruent, they cannot be equivalent. Answer B might be chosen by students favoring an arithmetic approach in geometric problem-solving. Without dimensions provided in the text or the figure, they may struggle to understand the relationship between the elements of the figures. The choice of answer D may stem from a tendency to seek a pattern. In this case, if an understanding of equivalence of figures is missing, students may seek any known, evident property, even one that is not relevant to the calculation of the area of the rectangle.

The correct answer, C, assumes that students recognize that the four triangles, although not congruent, are all equivalent to a quarter of the initial rectangle. Each triangle has dimensions half those of the rectangle. Consequently, a triangle is always equivalent to a rectangle with one dimension equal to that of the rectangle, the other equal to half the corresponding dimension in the triangle.

If students still face challenges in recognizing this equivalence, they can engage with the practical activity outlined in the response activity:

- Transforming polygons

References

Prova INVALSI 2012 grado 08 domanda n.6

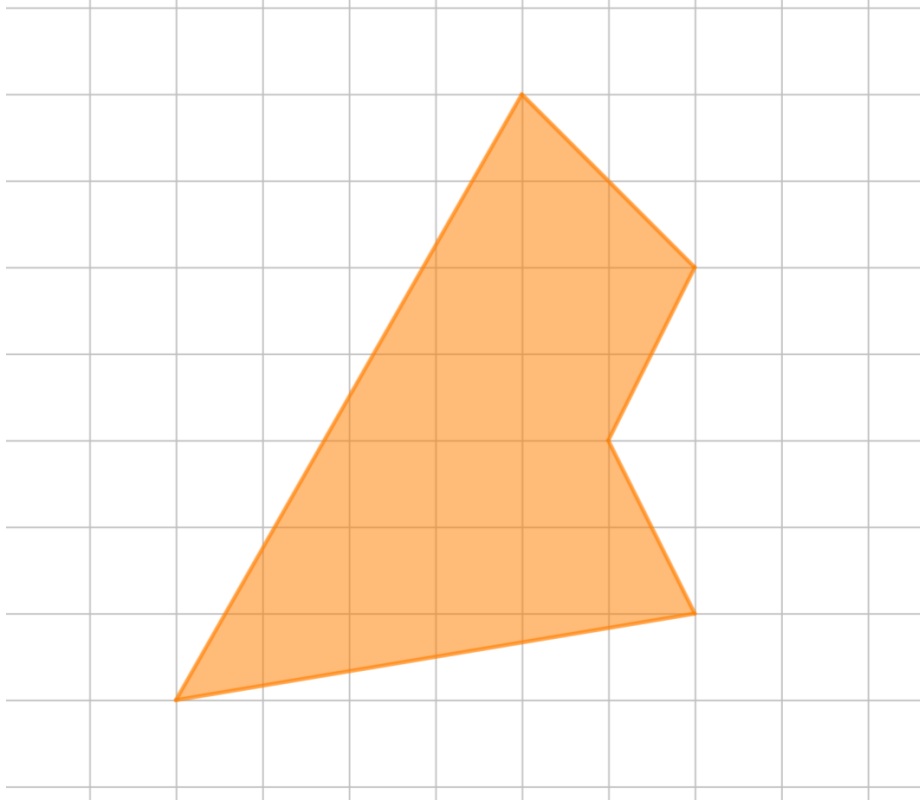
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<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Generic polygons

1 Observe the figure.



If the side of each grid square is 1 m, what is the area of the polygon?

A 22 m²

B 23 m²

C 21 m²

D 24 m²

☐☐☐☐

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Diagnostic question

Generic polygons

Overview

Learning focus:	Calculate the area of a generic polygon
Observable learning outcome:	Calculate areas of generic polygons by adding or deleting areas of other polygons, through decomposition or inference.
Question type:	Multiple choice
Key words:	Areas of plane figures, composition of plane figures

What does the research say?

The question was presented to second-year secondary school students in the 2014 INVALSI test. The solution does not require knowledge beyond primary school, making it suitable as a diagnostic question at the lower secondary school level.

In a survey of INVALSI results at national level, the percentage of correct responses was 30.7%, with a substantial number of incorrect answers (44.5%), and a significant percentage of missing answers (24.4%). This indicates that a considerable percentage of students struggle to employ effective problem-solving strategies.

Diagnostic activities such as this one can encourage students to contemplate different approaches to problem solving.

Ways to use this question

Students must fill in the sheet individually, with paper and pencil. Alternatively, the worksheet can be projected onto the IWB and students can answer in their notebook, and then share the answer with the class.

If there is a range of responses, you may choose to engage in a structured discussion in class. A student can be asked to justify the answer s/he gave; you can then ask another student to explain why s/he agrees with the first student; then you can ask a third student to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and gain full awareness of their learning needs.

Differentiation

For the inclusion of pupils with learning difficulties, you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

1 C

How to respond – what next?

Pupils who select option C are likely to have split the generic polygon into easily identifiable figures with clear base and height dimensions, as illustrated in Figure 1.

Alternatively, they might have determined the area by subtracting the area of the triangle from the outer rectangle, as depicted in Figure 2.

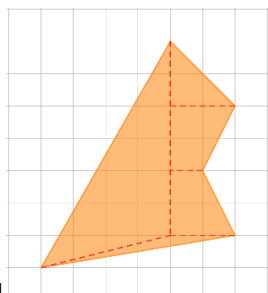


Figure 1

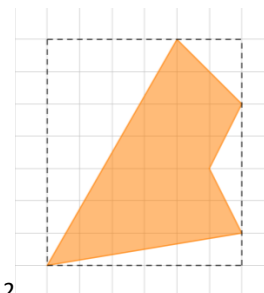


Figure 2

Encouraging discussion among students about the methods they used to obtain a correct answer can reveal diverse approaches.

Incorrect responses may result from decomposing the generic polygon into figures where estimating the base and height is necessary, because the endpoints of these segments do not coincide with grid vertices, as illustrated in Figures 3 and 4. Another possibility is counting the complete squares within the figure and estimating the total sum of the remaining parts.

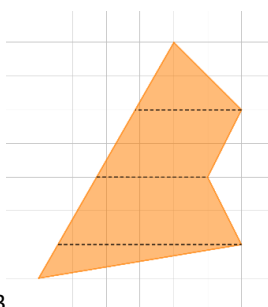


Figure 3

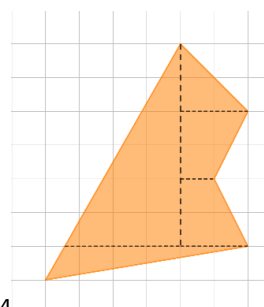


Figure 4

References

Prova INVALSI 2014 grado 10 domanda n.7

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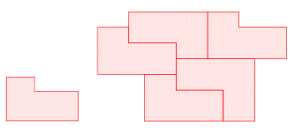
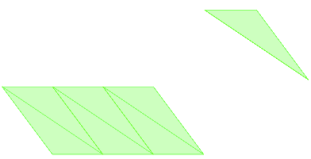
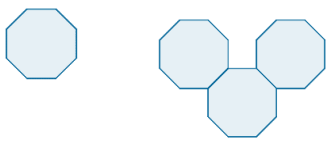
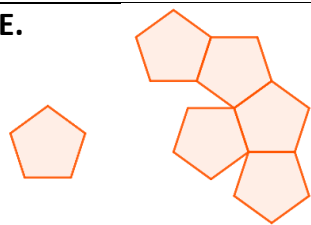
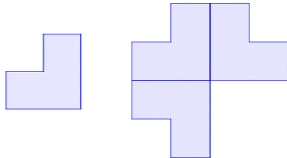
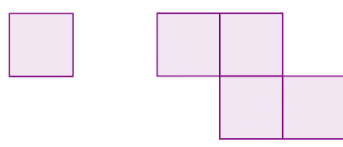
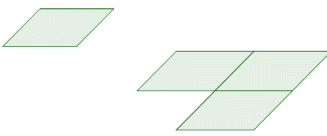
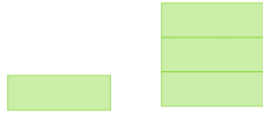
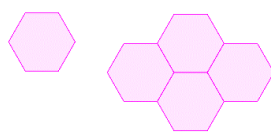
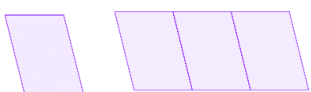
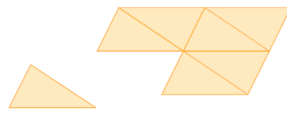
BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Units for measuring areas

Measuring a quantity means counting how many times a given unit of measurement is contained in the quantity to be determined.

For each figure in the table below, consider whether the polygon that has been singled out is a suitable unit to measure area.

A. 	B. 	C. 
D. 	E. 	F. 
G. 	H. 	I. 
L. 	M. 	N. 

For each of the following statements, choose the unit polygon or polygons, among A to N above, to which the statement applies.

- 1** By arranging copies of the suggested unit polygon side by side, with no overlap, a plane can be covered without gaps.
- 2** To determine the unit polygon, it is enough to know the length of a single side and a single angle.
- 3** Which unit polygons satisfy both the conditions in statement 1 and statement 2?
- 4** Among the unit polygons in Figure A to N, which would you choose as a unit of measurement to determine the area of a generic polygon?
Justify your answer.

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Diagnostic question

Unit of measuring areas

Overview

Learning focus:	Identify the appropriate unit of measurement for areas of plane figures.
Observable learning outcome:	Recognize the square meter and its multiples and submultiples as units of measurement for areas of plane figures.
Question type:	Observe and explain
Key words:	tessellations, equivalence of plane figures, units of measurement

What does the research say?

Calculations of areas of plane figures and the choice of the appropriate unit of measurement are generally introduced in the last two years of primary school. Both topics are resumed in the second year of lower secondary school and in the first two years of upper secondary school. Often what is reached at this level is superficial knowledge of the main formulas, and the notion of square meter as a unit of measurement. For this reason, it is useful to propose activities that review and consolidate notions already learned, as confirmed by various suggestions on how to teach these topics aimed at different school levels on the Scuolavalore-INDIRE website and, previously, in *Matematica2001* and *Matematica2003*.

Ways to use this question

Students must answer the questions individually, with paper and pencil. Alternatively, the worksheet can be projected onto the IWB and students can answer in their notebook, and then share the answer with the class.

If there is a range of responses, you may choose to engage in a structured discussion in class. A student can be asked to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; ask another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and gain full awareness of their learning needs.

Differentiation

For the inclusion of pupils with learning difficulties you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

1. A, B, D, F, G, H, I, L, M, N

Among the given figures, only the regular pentagon and the regular octagon cannot be used to tile the two-dimensional plane.

2. C, D, E, G, L

A regular polygon is determined by the number of sides and the length of the side. In all other cases, it is necessary to know at least the length of two sides in order to determine the unit polygon.

3. D, G, L

4. G, the square

All polygons can be transformed into rectangles, but not into equilateral triangles or the regular hexagons.

How to respond – what next?

Answering the sequence of questions 1 to 4 should help highlighting the most appropriate conditions for choosing the unit of measurement for areas. This unit must be an indefinitely reproducible model, capable of tiling the entire plane. However, it must also be easy to identify and to reproduce. Regular polygons (equilateral triangle, square, regular hexagon) are more suitable for this purpose: since the measure of each internal angle is known, it suffices to know the length of one side to identify each polygon uniquely and copy it. To reproduce a rectangle or a triangle or one of the other figures that satisfy condition 1, the lengths of at least two sides must be known. The choice of the square as a unit of measurement is the simplest, but also the most appropriate because all polygons can be transformed into rectangles, and the square is better suited to tiling these rectangles because all angles in squares and the rectangles are right angles.

References

Matematica 2001- Materiali per un nuovo curriculum di matematica con suggerimenti per attività e prove di verifica (scuola elementare e scuola media).- UMI-CIIM novembre 2001

Matematica 2003 - Attività didattiche e prove di verifica per un nuovo curriculum di Matematica.- UMI-CIIM maggio 2003

Andrea Gorini, *Matematica a sorpresa, Geometria 2*, Principato (2011)

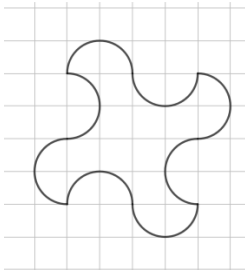
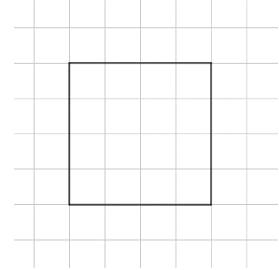
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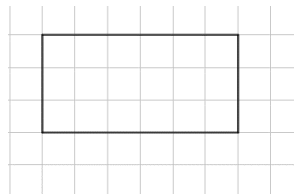
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Are they equivalent?

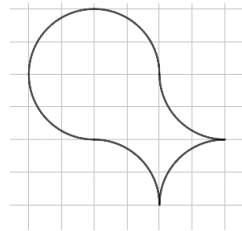
Among the figures below, identify those that are equivalent to the square to the right.



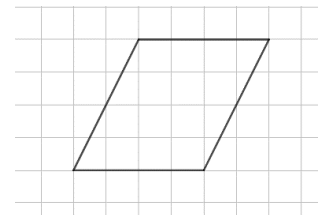
A



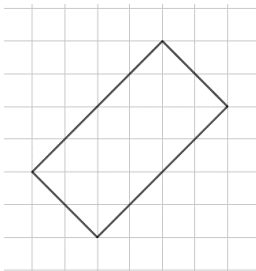
B



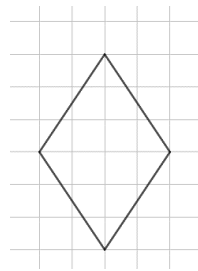
C



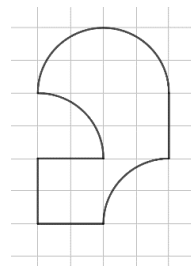
D



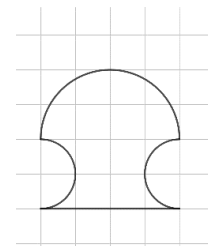
E



F



G



H

Tick (✓) the box next to the answer you think is correct.

1 D, E

2 A, C, D, E, G

3 C, F, G, H

4 D, G

5 B, D, E, F

☐
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☐

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Response activity

Are they equivalent?

Overview

Learning focus:	Equivalent figures can be obtained by adding or deleting matching congruent parts.
Observable learning outcome:	Recognize equidecomposable, and therefore equivalent, figures.
Activity type:	Multiple choice
Key words:	equiextension, plane figures

This activity can help develop students understanding by addressing the critical points revealed by the following diagnostic question:

- Equivalent figures

What does the research say?

From the analysis of the results of the INVALSI tests first- and third-year pupils of lower secondary school and second-year pupils of upper secondary school (grade 6, 8 and 10 respectively), it appears that, among the questions on Space and Figures, where more than 60% of answers are incorrect, 11 out of 40 concern areas of plane figures. The mistakes arise when the children have to identify equivalent figures or recognize the base of a figure and its relative height. The learning objectives for the third year of lower secondary school, defined in the *Indicazioni nazionali per il curricolo della scuola dell'infanzia e del primo ciclo di istruzione* (2012) include:

"to determine the area of simple figures by breaking them down into elementary figures, for example triangles, or by using common formulas."

School textbooks often focus on exercises that involves applying formulas to data that are provided in the statement of the exercise. What is missing, however, are exercises on seeking and identifying elements that allow to calculate areas by observing plane figures.

Ways to use this activity

Students must fill in the sheet individually, with paper and pencil, or they can "vote" using colored cards.

If there is a range of responses, you may choose to engage in a structured discussion in class. You may ask a student to justify the answer s/he gave; ask another student to explain why s/he agrees with him/her; again, ask to another to explain why s/he disagrees, and so on. This type of discussion gives students the opportunity to explore their thinking and gain full awareness of their learning needs.

Differentiation

For the inclusion of pupils with learning difficulties you can choose to read the questions to the class. In some situations, the intervention of a support teacher for one or more students may be more appropriate.

Expected answers

Option 2 is the correct answer; the other answers are wrong.

How to respond – what next?

Pupils who find it difficult to think that a square can be broken down into parts containing curved lines could answer 1 or 5. If they choose 1, they have likely to have recognized figure D and E as the only ones that are equidecomposable with the square among the figures with no curved sides. Instead, the choice of option 5 might indicate a difficulty in identifying matching congruent parts.

Figures D and G are the easiest to decompose and compose again into a square, so option 4 could be chosen by those pupils who respond impulsively, relying heavily on intuition.

Option 3 could be chosen by pupils who respond at a glance, or by counting an approximate number of squares.

If the students' answers show that misunderstandings persist, teachers should activate remedial actions through their preferred means and methods (workshop activities, problem solving, traditional lessons, etc.).

References

Indicazioni nazionali per il curricolo della scuola dell'infanzia e del primo ciclo di istruzione. 2012

<https://www.gestinv.it/>

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Transforming polygons

Materials

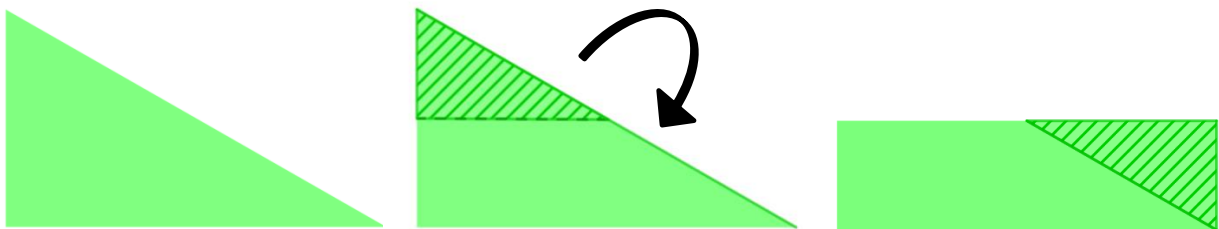
- Cardboard models of polygons: right-angled triangle, scalene triangle, parallelogram, rhombus, scalene trapezoid.
- Stationery (pencil, eraser, markers, ruler, square, etc.), scissors, adhesive tape

Method

Students work in pairs or groups of 3. Each pair/group is provided with a figure per type.

A. Right-angled triangle

1. Fold the figure in half along one side: the triangle is divided into a smaller triangle and a trapezoid.
2. Cut along the fold.
3. Assemble the two resulting pieces into a rectangle.



The final rectangle has the same base as the initial triangle and its height is half the height of the initial triangle, so

$$\text{Area of triangle: base} \times \text{half height} = \frac{b \times h}{2}$$

B. Scalene triangle

You can divide a scalene triangle into two right-angled triangles by drawing the height relative to one side.

4. Repeat the previous procedure on each of the right-angled triangles you obtain in this way. What do you get?

C. Parallelogram

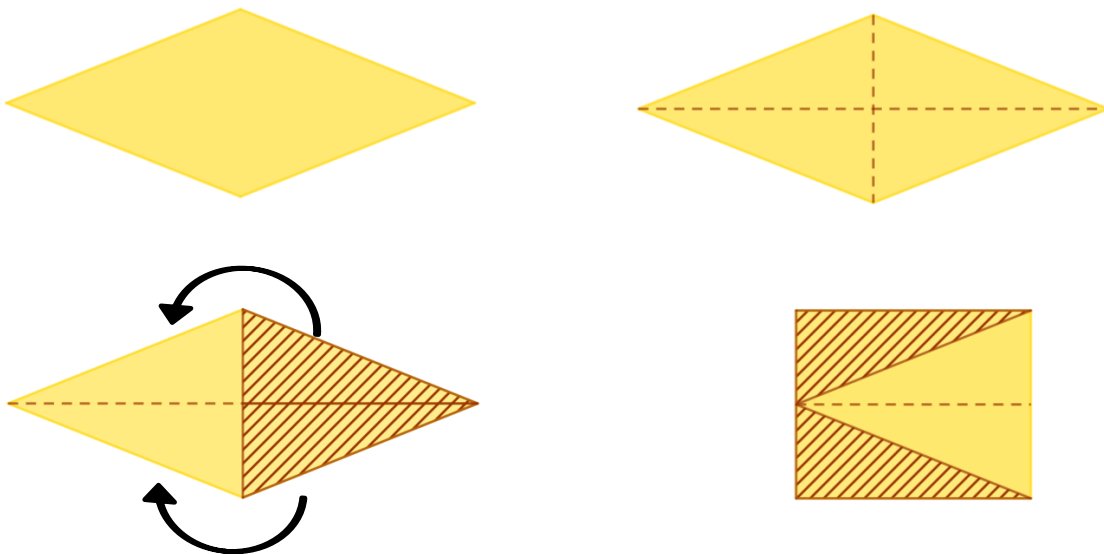
5. How can you cut the parallelogram to reassemble it into a rectangle?
6. How do the base and height of the rectangle compare to those of the parallelogram?

D. Rhombus

The rhombus is a specific type of parallelogram. You can always find the area of a rhombus by using the same formula as for parallelograms.

In a rhombus, however, the diagonals are perpendicular. The area of a rhombus can also be obtained from the lengths of the diagonals.

7. Fold the rhombus along the diagonals: four congruent triangles are formed.
8. Cut along one diagonal, then cut one of the two pieces along the other diagonal.
9. Reassemble the pieces obtained to form a rectangle.



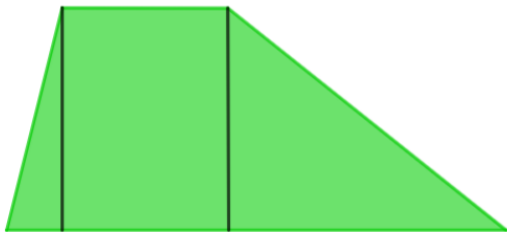
The final rectangle has sides equal to one diagonal of the rhombus and half the other diagonal.

Area of rhombus: $\text{diagonal}_1 \times \text{half diagonal}_2 = \frac{d_1 \times d_2}{2}$

E. Trapezoid

If we draw the heights from the endpoints of the smallest parallel side to the longest parallel side, we can see the trapezoid is decomposed into a rectangle and two right-angled triangles.

10. What folds and cuts can you make to transform the trapezoid into a rectangle (similarly to how a right-angled triangle was transformed into a rectangle)?



Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.2: areas of polygons

Response activity

Transforming polygons

Overview

Learning focus:	Determine the area of plane figures using common formulas
Observable learning outcome:	Understand and deduce the meaning of the most common formulas for calculating the area of triangles and quadrilaterals
Activity type:	Application and practice
Key words:	equidecomposable figures, areas of plane figures

This activity can help develop the student's understanding of basic formulas for calculating the areas of triangles and quadrilaterals.

It is designed to improve understanding of the diagnostic question

- The triangle while the rectangle

What does the research say?

A prevalent misconception in mathematics is the belief that learning mathematics relies heavily on memory. Interviews with successful and enthusiastic students reveal that they prioritize understanding over memorization. Conversely, those who perceive difficulties in understanding tend to rely excessively on memorization (Zan, 2007). This hands-on activity provides students with the opportunity to grasp the meaning of common formulas for calculating areas and refine their understanding through discussion with classmates.

Ways to use this activity

The activity should be conducted in pairs or small groups. The worksheet offers semi-guided instructions, providing suggestions for working with specific figures (such as right-angled triangles and rhombuses) while leaving some room for unstructured exploration, to encourage students to explore alternative approaches.

For the teacher, actively listening to individual groups can help to identify difficulties that may be addressed through clarification or redirection to the whole class during the activity.

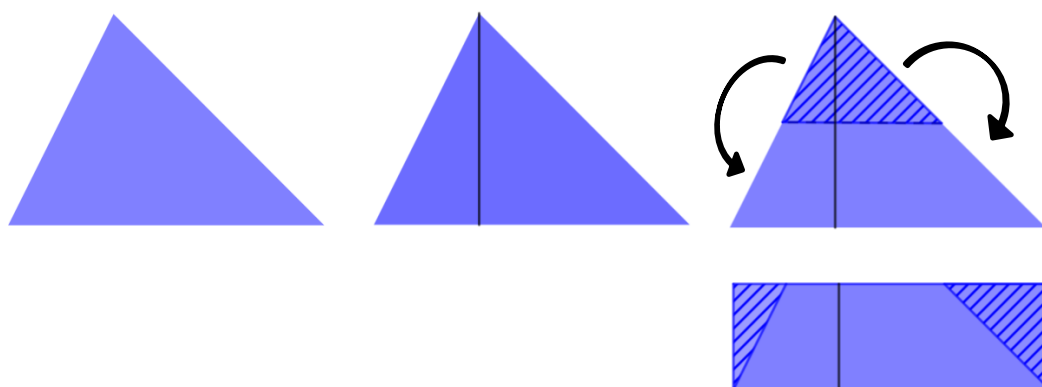
After completing the practical work, students are encouraged to share their results to the class. Following the response of a group, the teacher can model an improved response, prompting another group to add or clarify further.

Differentiation

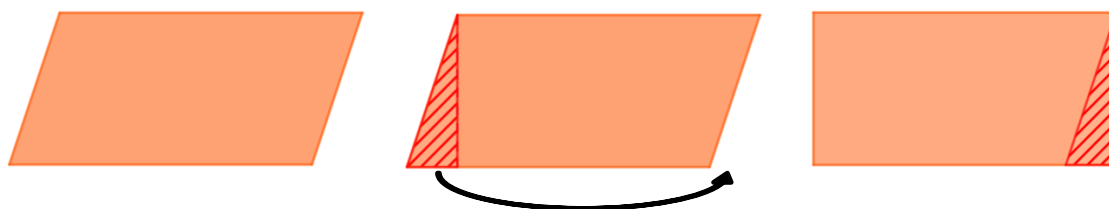
Students who find the work challenging may be allocated to a group of three. Assistance may be sought from an additional support teacher if appropriate.

Materials

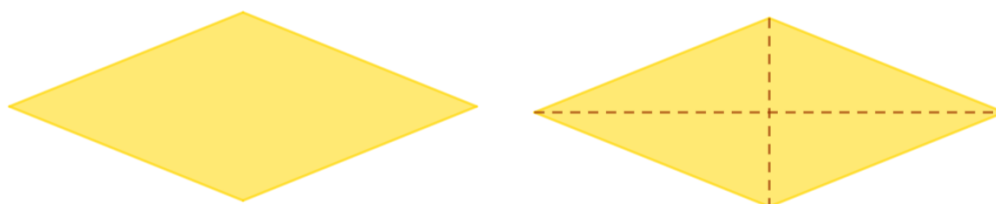
Prepare one colored cardboard figure of each type per each student/couple/group. It is advisable to have spare figures in case of accidental damage.

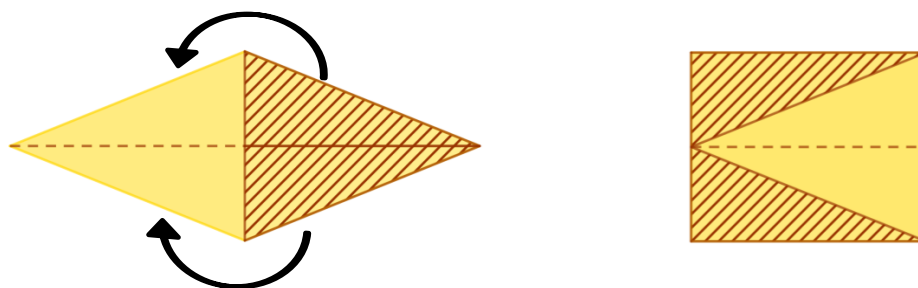
Expected answersScalene triangle

$$\text{Area of triangle} = \text{base} \times \text{half height} = \frac{b \times h}{2}$$

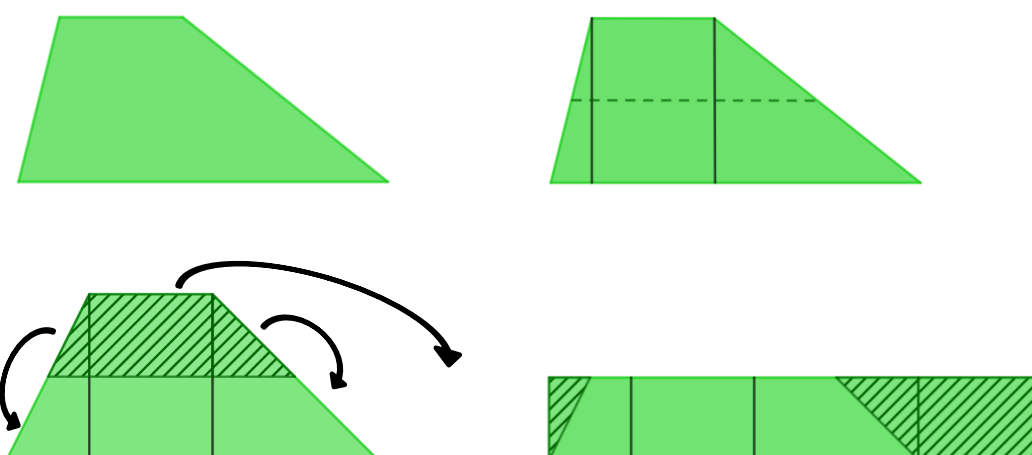
Parallelogram

$$\text{Area of parallelogram} = b \times h$$

Rhombus



Trapezoid



$$\text{Area of trapezoid} = \text{sum of bases} \times \text{half height} = \frac{(b_1 + b_2) \times h}{2}$$

The transformation of the trapezoid into a rectangle may require additional thought. However, experience with the previous figures should make the task easier. In case none of the groups find the expected construction, the teacher can step in and provide support to the entire class.

After completing the activity, if the class express an interest or curiosity, you may recommend the video available at https://www.youtube.be/watch?v=9yoCbk_z_08. This video demonstrates the transformation of any polygon into a square. While the content in the video may involve concepts not yet encountered by the students, it can serve as a stimulating introduction to pique their curiosity.

References

Rosetta Zan (2007) *Difficolta in matematica-osservare, interpretare, intervenire*, Springer

Andrea Gorini, *Matematica a sorpresa, Geometria 2*, Principato

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

B.3 Variations in areas and perimeters

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons

Key concept

AP1.3: Variations in areas and perimeters

What is the subject?

An area of interest of the thematic core SPACE AND FIGURES is the variations in area and perimeter of plan figures.

How does this key concept develop understanding of the subject?

Frequently, students commonly make mistakes by confusing the area and perimeter of plane figures or by assuming that equivalent figures are always isoperimetric or congruent. To effectively differentiate and comprehend these two concepts, it's helpful to compare the perimeter of equivalent figures or the area of isoperimetric figures.

The conceptual progression starts with comparing various arguments concerning isoperimetric equilateral squares and triangles. Then we explore the comparison between the area of isoperimetric figures and the perimeters of equivalent figures.

Using the progression toolkit to support student learning

Diagnostic questions can be used to quickly identify where students are in their conceptual progression. Once the starting points have been established, it is possible to decide how to continue the teaching, using further diagnostic questions or activating recovery actions with different means and methods (workshop activities, problem solving, traditional lessons, etc.). Response activities can be used as additional means of advancing student understanding.

Progression toolkit:	
Learning focus	What I am teaching
As students' conceptual understanding progresses they can:	Observable learning outcomes to guide my teaching focus
Diagnostic questions	Questions to find out what my students know and understand
Response activities	Activities to move my students' understanding forwards

Progression toolkit: variations in areas and perimeters

Learning focus	Capture the area variations in isoperimetric figures and the perimeter variations in equivalent figures.		
As students' conceptual understanding progresses they can:	CONCEPTUAL PROGRESSION 		
	Compare arguments regarding the perimeter of an equilateral triangle and that on a square. P	Compare areas of isoperimetric figures. B	Compare perimeters of equivalent figures. B
Diagnostic question	Who is right?	Rectangles in sequence	Three triangles
Response activity		Rectangles workshop 1	Rectangles workshop 2 Triangles workshop 2

Key:

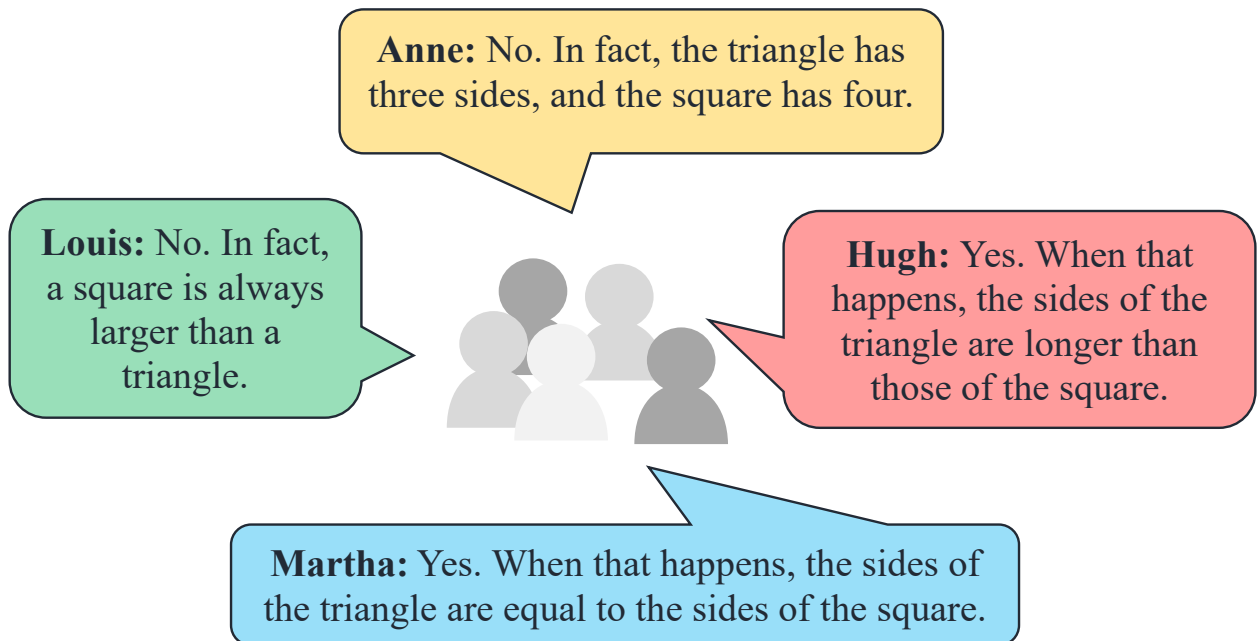
P Prior understanding from earlier stages of learning**B** Bridge to later stages of learning

Who is right?	Rectangles in sequence	Three triangles	Two-tier multiple choice
Talking heads	Two-tier multiple choice	Rectangles workshop 2	Triangles workshop 2
Rectangles workshop 1	Triangles workshop 1	Rectangles workshop 2	Triangles workshop 2
Application and practice	Predict-Observe-Explain	Application and practice	Predict-Observe-Explain

Who is right?

The teacher asks the class:

Can a triangle and a square have the same perimeter?



Who is right?

- A** Louis
- B** Anne
- C** Hugh
- D** Martha

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Diagnostic question

Who is right?

Overview

Learning focus:	Recognize and understand the relationships between the elements of the plane figures
Observable learning outcome:	Compare arguments regarding the perimeter of an equilateral triangle and that on a square.
Question type:	Talking heads
Key words:	Isoperimetric figures

P

PRIOR UNDERSTANDING

This diagnostic question probes understanding of ideas that are usually taught at age 5-11, to aid transition from earlier stages of learning.

What does the research say?

A paper by D'Amore and Pinilla reports that many students, regardless of their age, hold the misconception that there is a direct relational dependence between the perimeter and the area of a plane figure. According to this mistaken belief, when comparing two plane figures A and B, if the perimeter of A is greater than the perimeter of B, then the area of A is also greater than the area of B, and vice versa. This misconception extends to equality, suggesting that isoperimetric figures must have equal extension. The paper identified that these misunderstandings stem not only from conceptual challenges but also from educational obstacles. The predominant focus is on teaching convex and standard figures, and this leads to neglecting concave figures. Additionally, students are rarely shown the relationships between the perimeter and area of the same figure, nor are they exposed to transformations that preserve or alter the area or perimeter.

This question was presented to sixth grade students during the 2012 INVALSI national test. Only 38.3% correctly chose answer C. The most common incorrect answer was A, selected by 31%, while options B and D had similar percentages at 14.1% and 13%, respectively.

Ways to use this question

To facilitate discussion, students should work in pairs or small groups, and the teacher should focus on guiding these conversations.

Providing a worksheet for each group encourage discussion, with each member prepared to report back to the class. Monitoring group conversations can offer insights into students' thought processes.

Encourage students to explain their chosen answer, fostering a deeper understanding of their thought processes. Discussions can involve students agreeing or disagreeing with one other, providing opportunities for students to explore their thinking and comprehend their learning needs.

Differentiation

For differentiation, enhancing discussion quality can involve carefully selecting groups or assigning specific roles to students. For instance, designating students with strong prior knowledge as scribes, prohibiting them from contributing answers but encouraging them to question and record group discussion, promotes engagement from multiple group members. In case of learning difficulties, the support teacher's contribution may be valuable in reading or explaining statements.

Expected answers

The correct statement is C.

How to respond – what next?

Both negative options, A and B, can be chosen for different reasons.

The choice of option A may rest on an implicit assumption that squares are always larger than triangles.

Those who choose option B may attribute the difference in perimeter to the different number of sides. This idea might stem from a misreading or a deeper misconception. In the first case, discussion in pairs may solve the problem.

The positive options focus on side length.

Option C correctly states that, in order for an equilateral triangle and a square to have the same perimeter, the sides of the triangle must be longer than those of the square.

Option D incorrectly assumes that if two regular polygons have the same perimeter, the lengths of the sides must be the same, overlooking the fact that the number of the sides is different.

By scrutinizing the justifications given by students, teachers can assess the clarity of the concept of isoperimetric polygons. This insight aids in planning subsequent activities, such as exploring area variations in isoperimetric polygons and perimeter variations in equivalent polygons.

References

D'Amore B., Fandiño Pinilla M.I. (2005). Relazioni tra area e perimetro: convinzioni di insegnanti e studenti. *La matematica e la sua didattica*. [Bologna, Italia]. 2, 165-190.

Prova INVALSI 2012 grado 06 domanda n.24

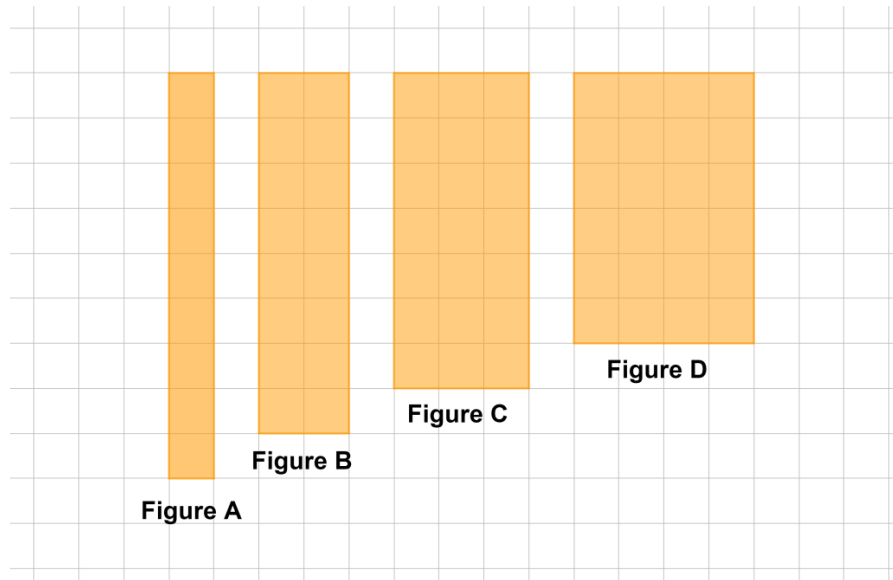
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BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Rectangles in sequence

Observe the following sequence of figures:



1. Which of the following statements is true?

Tick (✓) the box next to the best answer.

- A** The areas of the rectangles are the same.
- B** The areas of the rectangles double at each step.
- C** The perimeters of the rectangles are the same.
- D** The perimeters of the rectangles increase at each step.

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2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

- | | | |
|----------|---|--------------------------|
| A | When the base increases by one unit, the height decreases by one unit. | ☐ |
| B | Every rectangle has a base and height different from those of the others. | ☐ |
| C | As the base increases, the area also increases. | ☐ |
| D | As the area increases, the perimeter also increases. | ☐ |

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Diagnostic question

Rectangles in sequence

Overview

Learning focus:	Recognize and understand the relationships between the elements of the plane figures
Observable learning outcome:	Compare areas of isoperimetric figures
Question type:	Two-tier multiple choice
Key words:	Isoperimetric figures, areas of plane figures, perimeter

What does the research say?

A paper by D'Amore and Pinilla reports that many students, regardless of their age, hold the misconception that there is a direct relational dependence between the perimeter and the area of a plane figure. According to this mistaken belief, when comparing two plane figures A and B, if the perimeter of A is greater than the perimeter of B, then the area of A is also greater than the area of B, and vice versa. This misconception extends to equality, suggesting that isoperimetric figures must have equal extension. The paper identified that these misunderstandings stem not only from conceptual challenges but also from educational obstacles. The predominant focus is on teaching convex and standard figures, and this leads to neglecting concave figures. Additionally, students are rarely shown the relationships between the perimeter and area of the same figure, nor are they exposed to transformations that preserve or alter the area or perimeter.

The initial part of this diagnostic question is extracted from the 2012 INVALSI national test for grade 6. In 35.7% of cases, students answered C correctly. The percentages of incorrect answers were 4.4% for A, 22.4% for B, and 35.5% for D. The high percentage of response D appears to support the observations made by D'Amore and Pinilla.

Ways to use this question

Students must complete the grid individually, with paper and pencil, or the worksheet can be projected onto the IWB and students can use colored cards to show their choices.

If there is a range of answers, a structured discussion can be initiated in class. The teacher can ask a student to justify the answer s/he gave; then ask another student to explain why s/he agrees with him/her; again, ask another to explain why s/he disagrees, and so on. This sort of discussion gives students the opportunity to explore their thinking and really understand their learning needs.

Differentiation

For the inclusion of pupils with learning difficulties, the question can be read aloud to the class. In some cases, the support of a teacher may be necessary for certain students.

Expected answers

1. C
2. A

How to respond – what next?

When answering the question, students should recognize that the areas of the figures are different. The low percentage of students choosing option A in the national sample (4.4%) could be due to a real misunderstanding of the concept of area, or it could be a terminology error, where perimeter is considered instead of the area. In the first scenario, the answer to the second question might be B, whereas in the second it could be A. Students who choose answer B to the first question may believe that the change in area is solely connected to the widening of the base. In such instances, we anticipate the answer to the second question to be C. If students opt for D, they might be assuming that an increase in the area of the figures, is correlated to an increase in the perimeter. To validate this interpretation, they would need to select D as their answer to the second question.

Obviously other combinations between the answers to questions 1 and 2 are possible. The teacher's role is to interpret the results and decide how to proceed.

The recommended response activities after this diagnostic question are:

- Rectangles workshop 1
- Triangles workshop 1

References

D'Amore B., Fandiño Pinilla M.I. (2005). Relazioni tra area e perimetro: convinzioni di insegnanti e studenti. *La matematica e la sua didattica*. [Bologna, Italia]. 2, 165-190.

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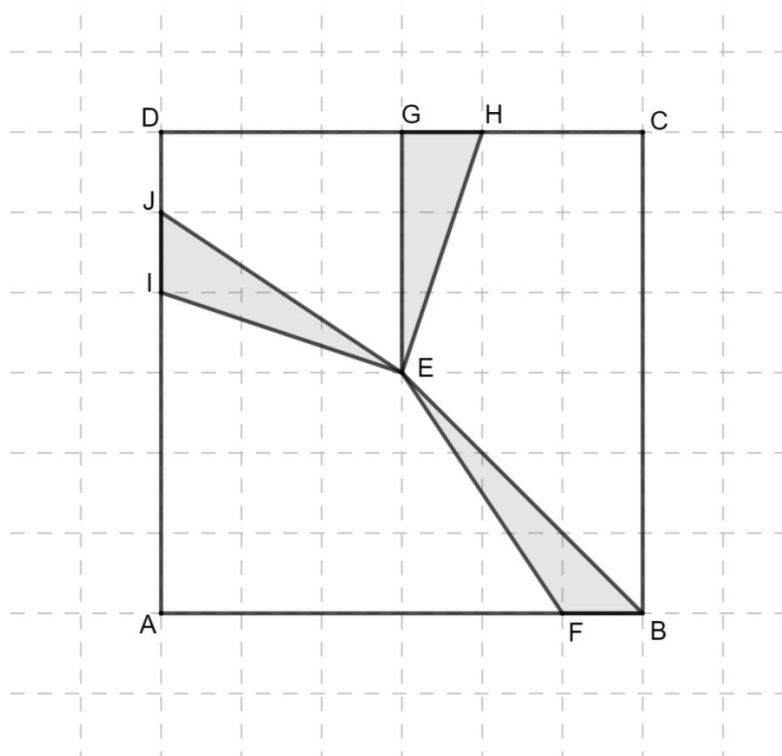
www.gestinv.it

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Three triangles

Observe the triangles in the figure



1. Which of the following statements is true?

Tick (✓) the box next to the best answer.

- A** The three triangles have the same area and perimeter ☐
- B** The three triangles have the same area and different perimeter ☐
- C** The three triangles have different areas and the same perimeter ☐
- D** The three triangles have different areas and perimeters ☐

2. Explain the reason for your choice

Tick (✓) the box next to the answer that comes closest to your thinking.

- A** The lengths of the bases and heights of the triangles are the same.
- B** The lengths of the bases and heights of the triangles are different.
- C** The triangles have a common vertex and bases of the same length.
- D** The triangles are not congruent.

☐☐☐☐

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Diagnostic question

Three triangles

Overview

Learning focus:	Recognize and understand the relationships between the elements of the plane figures
Observable learning outcome:	Compare perimeters of equivalent figures
Question type:	Two-tier multiple choice
Key words:	Isoperimetric figures, area of plane figures, perimeter

What does the research say?

A paper by D'Amore and Pinilla reports that many students, regardless of their age, hold the misconception that there is a direct relational dependence between the perimeter and the area of a plane figure. According to this mistaken belief, when comparing two plane figures A and B, if the perimeter of A is greater than the perimeter of B, then the area of A is also greater than the area of B, and vice versa. This misconception extends to equality, suggesting that isoperimetric figures must have equal extension. The paper identified that these misunderstandings stem not only from conceptual challenges but also from educational obstacles. The predominant focus is on teaching convex and standard figures, and this leads to neglecting concave figures. Additionally, students are rarely shown the relationships between the perimeter and area of the same figure, nor are they exposed to transformations that preserve or alter the area or perimeter.

The first question in this worksheet was presented to eighth-grade students in the 2014 INVALSI national test. The results indicate that less than half the students (44.5%) answer correctly. The majority of incorrect responses for Question 1 were option D (33.9%), while the percentages for option A and C were very similar (10.5% and 9.7% respectively).

Ways to use this question

Students must answer the questions individually, with paper and pencil, or the teacher can project the worksheet to the IWB and the pupils can use colored cards to show their choice.

If there is a range of answers, a structured discussion can be initiated in class. The teacher can ask a student to justify the answer s/he gave; then ask another student to explain why s/he agrees with him/her; again, ask another to explain why s/he disagrees, and so on. This sort of discussion gives students the opportunity to explore their thinking and really understand their learning needs.

Differentiation

For the inclusion of pupils with learning difficulties, the question can be read aloud to the class. In some cases, the support of a teacher may be necessary for certain students.

Expected answers

1. B

2. A

How to respond – what next?

Students who choose answer D for Question 1 are likely to have confused the concept of equivalence and isoperimetry with that of congruence. Consequently, they may believe that if two figures are not congruent, they cannot have the same perimeter or area. In this case, they might associate option D for the first question with option D for the second question.

Choosing answer A for the first question might result from the belief that the figures have base and height of the same length, and this alone is sufficient, as in the case of rectangles, to give the same perimeter. This misunderstanding could be reinforced by selecting answer 2A for the second question. Answer C to the first question could stem from a similar misunderstanding, where it's recognized that the bases and heights of the different triangles have the same lengths, but it is thought that they determine the perimeter rather than the area. In this case, students might associate option A with the second question. Reason 2B could be chosen by those who struggle to identify the base and height when they are not drawn horizontally and vertically, or have difficulty reading measurements off a drawing. Reason 2C, associated with a wrong answer to question 1, may be given by those who recognize that the bases are the same but face challenges identifying the height as the distance of vertex E from the base of the triangle. However, motivation 2C, when associated with the correct answer to the first question, may come from students who superficially do not consider that if the common vertex were drawn at a different point inside the square, the heights of the triangles would be different.

Depending on the reasons provided, the teacher can decide whether to continue with practical activities or revisit the concepts of height or equivalence of plane figures.

The recommended response activities after this diagnostic question are:

- Rectangles workshop 2
- Triangles workshop 2

References

D'Amore B., Fandiño Pinilla M.I. (2005). Relazioni tra area e perimetro: convinzioni di insegnanti e studenti. *La matematica e la sua didattica*. [Bologna, Italia]. 2, 165-190.

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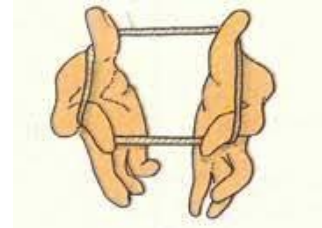
BEST. Best Evidence Science Teaching. University of York. Department of Education.

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Rectangles workshop 1

Materials

- Twine
- Squared paper
- Stationery



Method

1. Cut the twine and tie the two ends to form a ring.
2. Take the twine between your fingers to form a rectangle.
3. Observe what happens when, keeping the string taut, you move your hands away or closer together.

What happens to your fingers when you move your hands away from each other or closer together?

What happens to the perimeter when you change one rectangle into another?

What happens to the area when you change one rectangle into another?

What happens when the rectangle collapses on the base or height?

4. Suppose that the total length of the twine between your fingers is 24 cm.
What is the sum of the lengths of the base and height?
5. Copy the table below into your notebook and fill it in: for each rectangle you can form with a twine of 24 cm, calculate the area. For simplicity, consider only integer values, starting from the limit case where the base is equal to 0.

Perimeter	Semiperimeter	Base	Height	Area
....		0		
		1		
				
				
				
			0	

Which rectangle has the greatest area?

Which rectangle has the least area?

6. Copy the columns of the table that include the lengths of the base and the corresponding areas.
7. Construct a graph on squared paper: the x-coordinates should be the lengths of the base; the y-coordinates should be the corresponding areas.
8. Describe the resulting graph.

Base	Area
0	
....	
....	
....	

9. Now rewrite the table, keeping only the lengths of the base and the corresponding height.
10. Construct the corresponding graph on squared paper: the x -coordinates should be the lengths of the base; the y -coordinates should be the corresponding heights.
11. Describe the resulting graph.

Base	Height

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Response activity

Rectangles workshop 1

Overview

Learning focus:	Compare areas of isoperimetric rectangles. Recognize and describe regularity in a sequence of figures.
Observable learning outcome:	Verify that with the same perimeter, the rectangle with the maximum area is the square. Verify that the graph of the function that links the base and the height of isoperimetric rectangles is a straight line.
Activity type:	Application and practice - calculations
Key words:	Isoperimetric figures, perimeter of plane figures, area of plane figures, Cartesian plane, graphs of functions

This activity can help develop student understanding of the relationships between perimeter and area in rectangles.

It can be used after the diagnostic question:

- Rectangles in sequence

B

BRIDGING

This activity serves as a bridge with the study of the representations of relations and functions in the Cartesian plane.

What does the research say?

A body of work started by Piaget in 1930s, and continued by his students, has highlighted to the challenge students face in grasping the invariance of the surface measurement of a plane figure as the shape changes. Specifically, there is resistance to acknowledging that there are no inherent, predetermined relationships between the perimeter and area of plane figures. In other words, one cannot make a priori statements about the correlation between changes (increase, decrease, equality) in perimeter and changes (increase, decrease, equality) in the area of plane figures. These difficulties may manifest themselves at the primary school level, persist through university, and even be ingrained in the beliefs of certain teachers. The impediments to developing a satisfactory understanding of the relationships between perimeter and area are not solely of an epistemological nature but also have didactic implications (D'Amore, Fandiño Pinilla, 2005).

Emma Castelnuevo pointed out that the confusion between area and perimeter in children often stems from the static nature of drawings, leading pupils to focus on the figure outline rather than its interior. The use of materials such as string, elastic bands, and sticks, which allow for movement of figures, aids in directing thoughts toward transformations and facilitates a clearer understanding of the concept of extension. (Castelnuevo, 1963).

Ways to use this activity

This activity should be conducted in pairs or small groups. The activity consists in a guided worksheet, divided into three parts:

- First part (questions 1, 2 and 3): Students reflect and discuss, making intuitive predictions on the proposed questions.
- Second part (questions 4, 5, 6, 7 and 8): Students experiment and calculate areas of rectangles with the usual formula. They then construct a table and a graph of the variation in the area as a function of a dimension (the base) of the rectangle.
- Third part (questions 9, 10 and 11): Students focus on the relationship between the dimensions of the isoperimetric rectangles and construct the corresponding graph.

For the teacher, listening to individual groups as they work often helps to highlight any difficulties that could be overcome through clarification or redirection aimed at the whole class during the activity.

At the end of the practical work, it is useful to ask the students to report on their results to the class. After a group responds, it may be helpful to model a better response. In order to achieve this, the teacher can ask another group to add or clarify further, and so on.

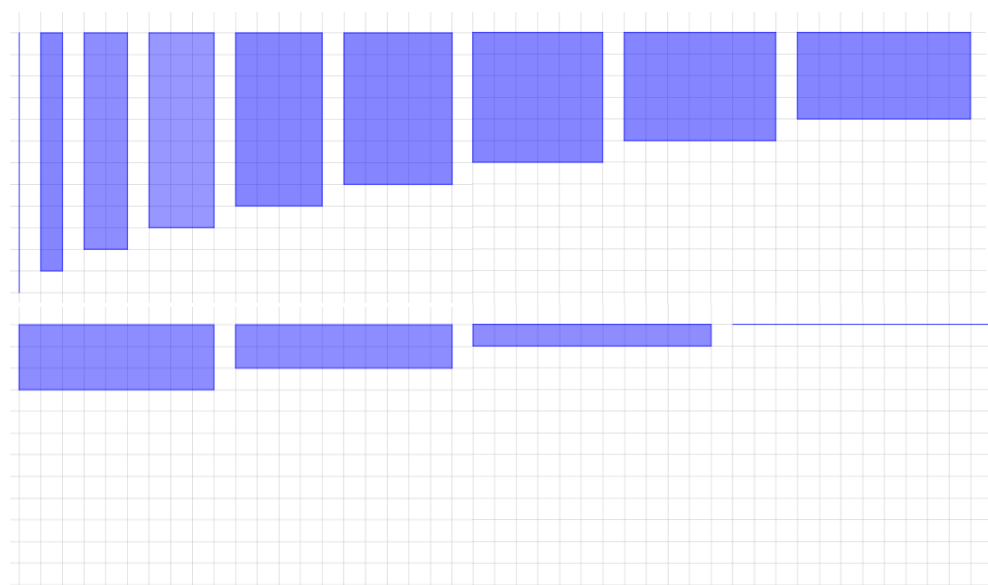
Differentiation

If necessary, it may be advisable that students with difficulties to work in groups of three or benefit from the assistance of a support teacher.

Materials

For each pair/group provide a piece of twine about 30 cm long. Each student works on their own squared notebook for drawing tables and graphs.

Expected answers



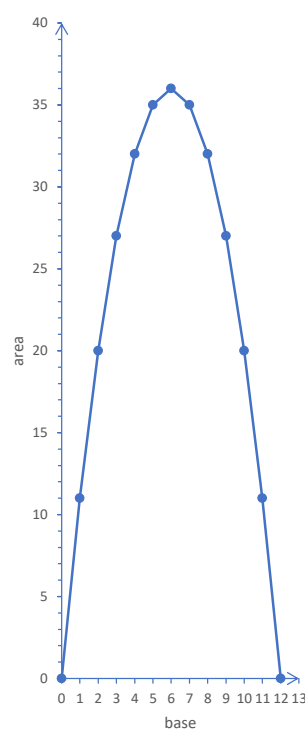
By considering only integer values and the two boundary cases, 13 rectangles are obtained. The use of string should help students understand that the length of the two segments constituting the boundary cases is equal to the semiperimeter.

Perimeter	Semi-perimeter	Base	Height	Area
24	12	0	12	0
24	12	1	11	11
24	12	2	10	20
....				
....				
24	12	6	6	36
....				
....				
24	12	11	1	11
24	12	12	0	0

Boundary rectangles have the minimum area, which is zero, while the square has the maximum area, equal to 36 cm^2 .

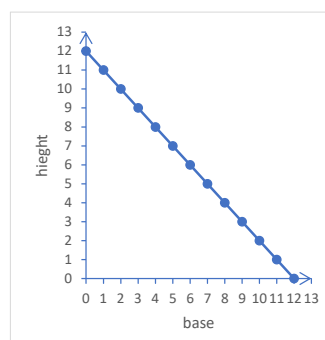
Tabulating areas as a function of the base and plotting the data on the Cartesian plane gives a parabola, where the maximum value is evident. The corresponding equation is $y = -x^2 + 12x$

Base	Area
0	0
1	11
2	20
3	27
4	32
5	35
6	36
7	35
8	32
9	27
10	20
11	11
12	0



Tabulating heights as a function of the base and plotting the data on a Cartesian plane results in the straight line with the equation $y = 12 - x$

Base	Height
0	12
1	11
2	10
3	9
4	8
5	7
6	6
7	5
8	4
9	3
10	2
11	1
12	0



How to respond – what next?

The practical activity described requires a significant amount of time. It can also be used to explore the topic of graphs of functions in the Cartesian plane. At this level, it is not important to focus on the equation of a parabola or a straight line. The important concept is the maximum point of the function, which can be read off the graph.

The line of inquiry can continue by exploring variations in the perimeter of equivalent rectangles, using the sheet:

- response activity: rectangles workshop 2

Alternatively, you can continue by analyzing variations in areas of isoperimetric triangles with the same base:

- response activity: triangles workshop 1

References

Emma Castelnuovo. *La matematica. Figure piane A*. La Nuova Italia. (2005)

Emma Castelnuovo. *Didattica della matematica*. Utet

D'Amore B., Fandiño Pinilla M.I. (2005). Relazioni tra area e perimetro: convinzioni di insegnanti e studenti. *La matematica e la sua didattica*. [Bologna, Italia]. 2, 165-190.

BEST. Best Evidence Science Teaching. University of York. Department of Education.

<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Triangles workshop 1

Emma is a mathematics teacher. Today she has brought an instructional tool to class for observing triangles. This tool consists of a cork board with two pins placed on it. She takes a piece of string, cuts it, and connects the two ends. She loops the string around the two pins and the tip of a pencil, keeping it taut. The two pins and the tip of the pencil form the vertices of a triangle. By adjusting the position of the pencil tip, various triangles can be created.

Predict

What will be the area and perimeter of the different triangles be?

Is there a triangle of maximum or minimum area?

Is there a triangle of maximum or minimum perimeter?

Explain

Why do you think you are right?

.....

Now carry out the investigation

Observe

Replicate a model of Emma's tool on your notebook and observe changes in the area and the perimeter as you transition between different triangles.
Consider limit cases as well.

Explain

Were your prediction and explanation correct?

If not, can you explain what you observed?

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Response activity

Triangles workshop 1

Overview

Learning focus:	Compare areas of isoperimetric triangles that have the same base. Recognize and describe regularity in a sequence of figures.
Observable learning outcome:	Verify that for the same perimeter and base, the triangle of maximum area is the isosceles triangle.
Activity type:	Predict, explain, observe, explain – practical/demonstration
Key words:	Isoperimetric figures, perimeter of plane figures, area of plane figures

This activity can help develop student understanding of the relationships between perimeter and area in triangles.

It can be used to improve understanding of diagnostic question:

- Rectangles in sequence

What does the research say?

A body of work started by Piaget in 1930s, and continued by his students, has highlighted to the challenge students face in grasping the invariance of the surface measurement of a plane figure as the shape changes. Specifically, there is resistance to acknowledging that there are no inherent, predetermined relationships between the perimeter and area of plane figures. In other words, one cannot make a priori statements about the correlation between changes (increase, decrease, equality) in perimeter and changes (increase, decrease, equality) in the area of plane figures. These difficulties may manifest themselves at the primary school level, persist through university, and even be ingrained in the beliefs of certain teachers. The impediments to developing a satisfactory understanding of the relationships between perimeter and area are not solely of an epistemological nature but also have didactic implications (D'Amore, Fandiño Pinilla, 2005).

Emma Castelnuovo pointed out that the confusion between area and perimeter in children often stems from the static nature of drawings, leading pupils to focus on the figure outline rather than its interior. The use of materials such as string, elastic bands, and sticks, which allow for movement of figures, aids in directing thoughts toward transformations and facilitates a clearer understanding of the concept of extension. (Castelnuovo, 1963).

Ways to use this activity

Students are encouraged to complete this activity in pairs or small groups, with a focus on engaging in discussions. The understanding and testing of concepts occurs through these discussions, allowing students to predict outcomes and articulate their reasoning.

To initiate the activity, each group should reflect and discuss the given questions, using their knowledge to predict *what* they think will happen, and then to explain *why* they think they will be right. If there is disagreement within a group, the teacher can step in to guide the students with thoughtful questions.

Following this, students should reproduce the model in their notebooks and, if needed, use the teacher's tool to observe different perspectives, particularly paying attention to boundary cases. Each group should have the opportunity to revise or improve their explanations. Within the group, students must develop a unified answer and explanation to document in their notebooks. After the group discussion, students are invited to present their explanations to the class, illustrating their reasoning visible. A guided class discussion further helps in enhancing students' argumentative skills and fostering awareness of their learning.

Differentiation

To promote participation and discussion, specific roles can be assigned, such as designating a student with strong prior knowledge as a scribe who can only write what the others say, encouraging contributions from students facing difficulties. This strategy allows stronger students to ask appropriate questions to elicit understanding from their classmates, without providing the answer.

Materials

For the whole class:

- a cork board
- pins
- twine

For each student:

- notebook
- stationery

Expected answers

Students who have completed *rectangles workshop 1* should demonstrate familiarity with twine and an understanding of non-equivalent isoperimetric triangles. The limit cases, obtained when the tip of the pencil is on the same straight line as the base segment, should lead the students to identify the isosceles triangle as the one with the maximum area.

Some students might argue for two triangles of maximum area, referring to symmetrical pencil positions; discussion can explore whether these are two distinct triangles or the same one.

If the pencil is not removed from the board while keeping the string taut, it describes an ellipse. The teacher can decide whether to introduce this curve, considering the class's level of understanding.

Unforeseen aspects may arise during class discussion, and the teacher's role is to evaluate interventions and decide how to incorporate them into the ongoing understanding of the topic.

References

Emma Castelnuovo. *La matematica. Figure piane A*. La Nuova Italia. (2005)

Emma Castelnuovo. *Didattica della matematica* UTET

D'Amore B., Fandiño Pinilla M.I. (2005). Relazioni tra area e perimetro: convinzioni di insegnanti e studenti. *La matematica e la sua didattica*. [Bologna, Italia]. 2, 165-190.

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Rectangles workshop 2

Materials

- Sheet of squared paper (side 1 cm)
- Stationery

Method

1. Build a square of side 6 squares (6 cm)

What is the area of the square?

2. Construct as many square-equivalent rectangles as possible, considering only integer values for the lengths of the base and the height. Start with the base of length 1.
3. What happens to the perimeter as you change from one rectangle to another?
4. Copy the table below in your notebook and fill it in by calculating the perimeter of each of the rectangles you have constructed.

Area	Base	Height	Perimeter
....	1		
	2		
			
			
			
		1	

Which rectangle has the greatest perimeter?

Which rectangle has the least perimeter?

5. Rewrite the table to include the length of the base and corresponding perimeter.
6. Construct a graph on squared paper: the x-coordinates should be the lengths of the base; the y-coordinates should be the corresponding perimeters.
7. Describe the graph that you obtain in this way.

Base	Perimeter
....	
....	
....	
....	

8. Now rewrite the table, keeping only the lengths of the base and the corresponding height.
9. Construct the corresponding graph on squared paper: the x -coordinates should be the lengths of the base; the y -coordinates should be the corresponding heights.
10. Describe the resulting graph.

Base	Height
....	
....	
....	
....	

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Response activity

Rectangles workshop 2

Overview

Learning focus:	Compare perimeters of equivalent rectangles. Recognize and describe regularities in a sequence of figures.
Observable learning outcome:	Verify that for the same area, the rectangle of minimum perimeter is a square. Verify on the Cartesian plane that the function that links the bases and heights of equivalent rectangles is a hyperbola.
Activity type:	Application and practice - calculations
Key words:	Isoperimetric figures, perimeter of plane figures, area of plane figures, Cartesian plane, graphs of functions

This activity can help develop understanding of student on the relationships between perimeter and area in rectangles.

It can be used to improve understanding of diagnostic question:

- Three triangles

B

BRIDGING

This activity serves as a bridge to the study of the representations of relations and functions in the Cartesian plane.

What does the research say?

A body of work started by Piaget in 1930s, and continued by his students, has highlighted to the challenge students face in grasping the invariance of the surface measurement of a plane figure as the shape changes. Specifically, there is resistance to acknowledging that there are no inherent, predetermined relationships between the perimeter and area of plane figures. In other words, one cannot make a priori statements about the correlation between changes (increase, decrease, equality) in perimeter and changes (increase, decrease, equality) in the area of plane figures. These difficulties may manifest themselves at the primary school level, persist through university, and even be ingrained in the beliefs of certain teachers. The impediments to developing a satisfactory understanding of the relationships between perimeter and area are not solely of an epistemological nature but also have didactic implications (D'Amore, Fandiño Pinilla, 2005).

Emma Castelnuovo pointed out that the confusion between area and perimeter in children often stems from the static nature of drawings, leading pupils to focus on the figure outline rather than its interior. The use of materials such as string, elastic bands, and sticks, which allow for movement of figures, aids in directing thoughts toward transformations and facilitates a clearer understanding of the concept of extension. (Castelnuovo, 1963).

Ways to use this activity

This activity should be conducted in pairs or small groups. The activity consists in a guided worksheet, divided into three parts:

- First part (questions 1, 2 and 3): Students construct the equivalent rectangles by drawing or cutting out the figures, reflect and discuss, making intuitive predictions on the proposed questions.
- Second part (questions 4, 5, 6 and 7): Students calculate perimeters of rectangles with the usual formula. They then construct a table and a graph of the variation in the perimeter as a function of a dimension (the base) of the rectangle.
- Third part (questions 8, 9 and 10): Students focus on the relationship between the dimensions of the equivalent rectangles and construct the corresponding graph.

For the teacher, listening to individual groups as they work often helps to highlight any difficulties that could be overcome through clarification or redirection aimed at the whole class during the activity.

At the end of the practical work, it is useful to ask the students to report their results to the class. After a group responds, it may be helpful to model an even better response. In order to achieve this, the teacher can ask another group to add or clarify further, and so on.

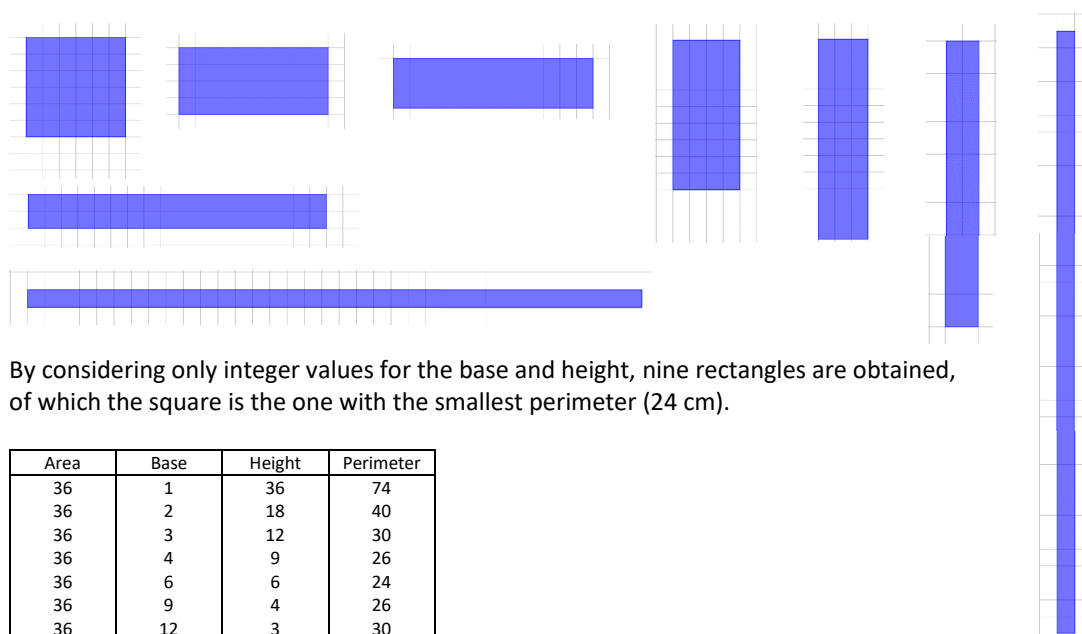
Differentiation

If necessary, it may be advisable that students with difficulties work in groups of three or benefit the assistance of a support teacher.

Materials

For each pair/group, provide 2 or 3 sheets of squared paper (1 cm grid). Each student works on their own notebook for drawing tables and graphs.

Expected answers

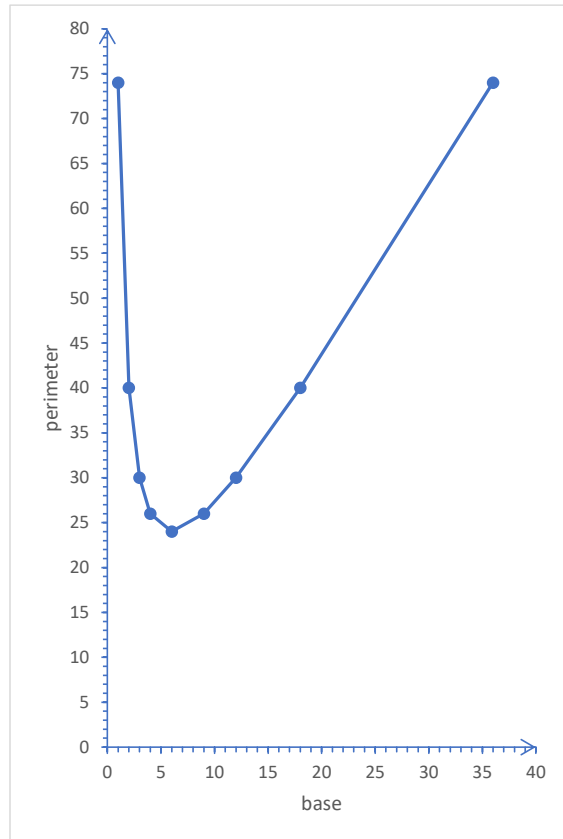


By considering only integer values for the base and height, nine rectangles are obtained, of which the square is the one with the smallest perimeter (24 cm).

Area	Base	Height	Perimeter
36	1	36	74
36	2	18	40
36	3	12	30
36	4	9	26
36	6	6	24
36	9	4	26
36	12	3	30
36	18	2	40
36	36	1	74

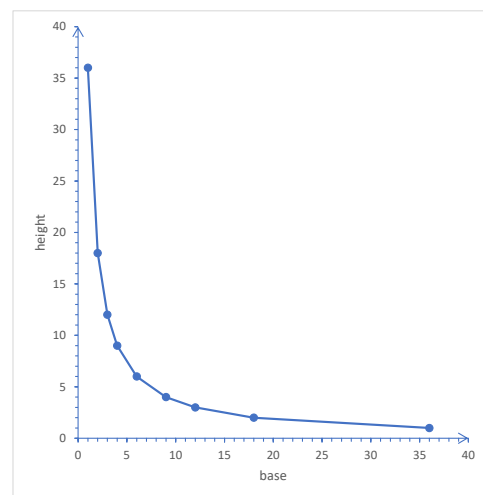
Tabulating the values of the perimeter as a function of the base and reporting the data on the Cartesian plane gives a function whose minimum value is evident. The corresponding equation is: $y = (2x^2 + 72)/x$

Base	Perimeter
1	74
2	40
3	30
4	26
6	24
9	26
12	30
18	40
36	74



Tabulating the height as a function of the base and reporting the data on the Cartesian plane gives a hyperbola with equation $y = 36/x$.

Base	Height
1	36
2	18
3	12
4	9
6	6
9	4
12	3
18	2
36	1



How to respond – what next?

The practical activity requires a significant amount of time. It can also be used to explore the topic of graphs of functions in the Cartesian plane. At this level, it is not important to focus on the equation of functions. The important concept is the minimum point of the function, which can be read off the graph.

Regarding the relationship between base and height, it is important to stress that in the case of equivalent rectangles, the two quantities are inversely proportional, while the same cannot be said for isoperimetric rectangles. Pointing out the difference in the trend of the graphs can be useful to overcome a misconception for many students who state that two quantities are inversely proportional when one increases and the other decreases.

This line of inquiry can continue by studying the variations in the perimeters of equivalent triangles with the same base, using the sheet:

- response activity: triangles workshop 2

Alternatively, you can continue by analyzing the variation of the area in isoperimetric rectangles:

- response activity: rectangles workshop 1

References

Emma Castelnuovo. *La matematica. Figure piane A*. La Nuova Italia. (2005)

Emma Castelnuovo. *Didattica della matematica*. Utet

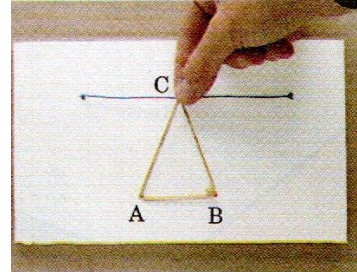
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<https://www.york.ac.uk/education/research/uyseg/research-projects/bestevidencescienceteaching/>

Triangles workshop 2

In today's class, Emma, a mathematics teacher, has introduced a tool for observing triangles. This tool consists of a cork board with two pins representing the ends of a segment. Parallel to the segment, she has placed two additional pins around which a wire was threaded. A small ring runs along the wire. An elastic band, tied to the initial points, passes through the ring. The vertices of the triangle are formed by the two initial points and the ring. By moving the ring, different triangles can be obtained.



Predict

What will the area and perimeter of the different triangles be?

Is there a triangle of maximum or minimum area?

Is there a triangle of maximum or minimum perimeter?

Explain

Why do you think you are right?

.....

Now carry out the investigation

Observe

Reproduce a model of the teacher's tool on your notebook and observe changes in the area and the perimeter as you transition between different triangles. Consider limit cases as well.

Explain

Were your prediction and explanation correct?

If not, can you explain what you observed?

Space and figures > Subject AP: area and perimeter > Topic AP1: areas and perimeters of polygons > Key concept AP1.3: variations in areas and perimeters

Response activity

Triangles workshop 2

Overview

Learning focus:	Compare perimeters of equivalent triangles that have the same base. Recognize and describe regularity in a sequence of figures.
Observable learning outcome:	Verify that for the same area and base, the triangle of minimum perimeter is an isosceles triangle.
Activity type:	Predict, explain, observe, explain – practical/demonstration
Key words:	Equivalent figures, perimeter of plane figures, area of plane figures

This activity can help develop student understanding by addressing the critical points revealed by the following diagnostic question:

- Three triangles

What does the research say?

A body of work started by Piaget in 1930s, and continued by his students, has highlighted the challenge students face in grasping the invariance of the surface measurement of a plane figure as the shape changes. Specifically, there is resistance to acknowledging that there are no inherent, predetermined relationships between the perimeter and area of plane figures. In other words, one cannot make a priori statements about the correlation between changes (increase, decrease, equality) in perimeter and changes (increase, decrease, equality) in the area of plane figures. These difficulties may manifest themselves at the primary school level, persist through university, and even be ingrained in the beliefs of certain teachers. The impediments to developing a satisfactory understanding of the relationships between perimeter and area are not solely of an epistemological nature but also have didactic implications (D'Amore, Fandiño Pinilla, 2005).

Emma Castelnovo pointed out that the confusion between area and perimeter in children often stems from the static nature of drawings, leading pupils to focus on the figure outline rather than its interior. The use of materials such as string, elastic bands, and sticks, which allow for movement of figures, aids in directing thoughts toward transformations and facilitates a clearer understanding of the concept of extension. (Castelnovo, 1963).

Ways to use this activity

Students are encouraged to complete this activity in pairs or small groups, with a focus on engaging in discussions. The understanding and testing of concepts occur through these discussions, allowing students to predict outcomes and articulate their reasoning.

To initiate the activity, each group should reflect and discuss posed questions, using their knowledge to predict *what* they think will happen, and then to explain *why* they think they will be right. If there is disagreement within a group, the teacher can step in to guide them with thoughtful questions.

Following this, students should reproduce the model in their notebooks and, if needed, use the teacher's tool to observe different perspectives, particularly paying attention to boundary cases. Each group should have the opportunity to revise or improve their explanations. Within the group, students must develop a unified answer and explanation to document in their notebooks. After the group discussion, students are invited to present their explanations to the class, making their reasoning visible. A guided class discussion further aids in enhancing students' argumentative skills and fostering awareness of their learning.

Differentiation

To promote participation and discussion, specific roles can be assigned, such as designating a student with strong prior knowledge as a scribe who can only write what others say, encouraging contributions from students facing difficulties. This strategy allows stronger students to ask appropriate questions to elicit understanding from their classmates, without providing the answer.

Materials

For the whole class:

- a cork board
- pins
- wire with a little ring
- elastic band

For each student:

- notebook
- stationery

Expected answers

The wire is at a constant distance from the segment identified by the pins: that should prompt students to infer that the triangles are all equivalent. The stretchability of the elastic band should, however, suggest that perimeters are not equal.

The model, along with the elastic's tendency to position itself in a state of minimal tension, can aid in comprehending that the triangle with the smallest perimeter is the isosceles one. Due to the limitations of the model, the mobile vertex cannot move infinitely. Consequently, it becomes imperative for the teacher to guide pupils, encouraging abstraction to envision the limit positions.

References

Emma Castelnuovo. *La matematica. Figure piane A*. La Nuova Italia. (2005)

Emma Castelnuovo. *Didattica della matematica* UTET

D'Amore B., Fandiño Pinilla M.I. (2005). Area e perimetro Relazioni tra area e perimetro: convinzioni di insegnanti e studenti. *La matematica e la sua didattica*. [Bologna, Italia]. 2, 165-190.

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