UNIVERSITY OF CAMERINO

School of Advanced Studies

Doctor of Philosophy in Computer Science and Mathematics - XXXIV Cycle

Contextual Semantics Machinery

Supervisor

Prof. Emanuela Merelli

PhD Candidate

Sahil Imtiyaz

Academic Year 2022-2023

Declaration

I herewith declare that I have produced this thesis under the supervision of Prof. Emanuela Merelli at the University of Camerino, without the prohibited assistance of third parties and without making use of aids, other than those specified. Notions taken over directly or indirectly from other sources have been identified as such. This paper has not previously been presented in an identical or similar form to any other Italian or foreign examination board.

Sahil Imtiyaz
2023

* This dissertation is presented in partial fulfillment of the requirements for **Ph.D. degree** in the School of Advanced Studies of University of Camerino.

To, Ibraheem Mustafa.

Acknowledgements

I would like to thank my supervisor Emanuela Merelli for her guidance and support throughout these years in Italy. I specially thank Mario Rasetti, the former president of Institute of Scientific Interchange in Turin, for giving me an opportunity to work on his shared vision of the topological field theory of data. Thanks to Samson Abramsky along with his quantum group, in particular Rui Soares Barbosa and Giovanni Carù, for their valuable comments and discussions in University of Oxford. I also thank Giuseppe Longo at Ecole Normale Supèrieure CNRS Paris, for his helpful insights about my work and possible proposals to open questions in theoretical biology. I would also like to mention Serafim Rodrigues at Basque Center for Applied Mathematics, Bilbao Spain, for sharing collaborative possibilities of my work to the brain. Personal thanks to a dear friend Muhammad Saad at Paderborn University Germany for his compassion. I express my gratitude to the community of Camerino and the School of Advanced Studies, University of Camerino, Italia.

Abstract

Topological field theory of data, a programme recently proposed by Rasetti and Merelli, is a theoretical framework to extract hidden relations, *the emerging correlation patterns*, existing among the data which are part of the global data space; as well as, provides potential *semantics*, for instance as a *variational machinery*, based on physical field theories to express these emerging patterns. The theory is based on three inter-connected layers that work in synergy; the topological data analysis followed by the topological field theoretic modelisation and finally its formal language description; invoking influxes between physical theories and computation using mathematical structures to act as a connecting bridge for the ever increasing dichotomy between the physical and the digital.

This work is an instance of the physical genesis of computation based on the topological field theory of data. The thesis explores a way to discover directly from the space of observations; collection of data from a physical experiment, a specific emerging high degree correlation patterns among the data known as quantum contextuality, as well as, formalises a mathematical model of computation which provides semantics to these emerging patterns; the *contextual semantics machinery*, based on field theoretic description. As a result, establishes a formal connection between quantum contextuality and interactive computation advocating a next step forward for the theory as a whole.

The phenomenon of quantum contextuality mathematically corresponds to a class of novel patterns known to be *locally consistent and globally inconsistent* which has been recently shown by Abramsky and Branderburger using mathematical structure of sheaves; arising from the impossibility of an observer to visualise all contexts of a model simultaneously. The central feature of sheaves is the failure of local solutions to glue together to form a global solution

which advocates an *irreducible holistic doctrine*. We explore these patterns in the interactive computation by a mathematical model, whose expressiveness generalises the empirical models in the foundation of quantum physics, introducing the concept of *openness* unlike Turing-like interactive models. The computing synthesis of algebraic interpretation of quantum contextuality is that the computation doesn't depend only on the context but also on their collective structure. The results could encourage a deeper investigation into the foundations of computability.

.

Contents

List of Figures

Publication

- Sahil Imtiyaz and Emanuela Merelli, *Contextual Semantics Machinery*, under review.

Chapter 1

Introduction

1.1 General Introduction

Interaction as observation is the central concept in the foundations of quantum physics. It is the fundamental idea behind Bohr's complementarity and Heisenberg's principle of uncertainty: the value of a quantity cannot be known *a priori* but rather *discovered* by observation confirming their irreducibly holistic nature. A concept that places a fundamental restriction on the simultaneous assignment of values to variables of a system, which turns out to be the source of a strong non-classical correlation known as *quantum contextuality*.

Quantum contextuality imparts quantum computing with higher computational power than classical computing. The phenomenon of contextuality motivates the study of a mathematical structure independent of the formalism of quantum physics. In 2011, Abramsky and Branderburger proposed a new mathematical structure of quantum contextuality based on sheaf theory corresponding to *topological obstructions to the existence of a global section responsible for local consistency and global inconsistency* which came to be known as *contextual semantics* [4].

Contextual semantics has initiated a step towards a general theory of quantum contextuality and provided new insights to the theory of relational database [3], natural language semantics [5], robust constraint satisfaction [7] and logic [2] in computer science. These latest results motivate us to search for a mathematical structure for interactive computation. The interplay between computation and physics is not new. The theory of Petri nets for concurrency was influenced by the idea of relativistic space-time and provided a vision of the physical genesis of computation, even though non-locality was not taken into account [41]. The interactive computation paradigm further explains the qualitative description of non-locality arising from the fundamental incompleteness of the observer to visualise all contexts of the system simultaneously [45, 44]. Later, Milner pointed out bigraphs provide no constrains in linking arbitrarily any far apart context, which could allow counter-intuitive *action at a distance*, such as non-locality, due to global context dependency in interactive systems [43]. However, the Bell-Kochen-Specker theorem already proved the impossibility of local hidden variables to explain non-locality and contextuality [12, 33], together with, invoking inclusion of global variables because *contextuality is topological* [18, 38]. As a result, leads the way to geometric models of computation unlike Turing machine whose tape structure recalls local hidden variables.

The interplay between algebraic topology and concurrency invokes representation of the overall execution of the interaction among processes as simplicial complex, pioneering the geometric/topological models of computation [23]. The local partial orders among processes *always* admit a *consistent* global order. The consistency comes in by restricting processes to *local partially ordered spaces* which avoid non-trivial loops: a causality condition expressed as a loop free and fixed simplicial complex. Hence, the global structure comprising of contexts could constrain the computation but not vice-versa.

We allow computation to change the topology of the environment which in turn constrains the computation in a feedback loop. It allows non-trivial loops to emerge in a causal way and facilitates the definition of an *openness* responsible for local consistency and global inconsistency ($LC - GI$). The openness is expressed by associating a topological representation of the environment to a persistent Turing machine (PTM), to be known as topological interactive machine (TIM). The machine is based on topological field theory of data proposed by Rasetti and Merelli [42], followed by a qualitative conceptualisation [36]. It generalises computable function to partial function by defining the equivalence relation among functions which correspond to the set of paths feasible over the environment represented as a topological space. Typically, open environment can capture exactly the fundamental concept behind non- locality and contextuality, that is, not all computations are feasible in a context-sensitive non-linear environment. The machine expresses different degrees of correlation ranging from non-locality in Hardy model to strong contextuality in Kochen specker model. We restrict to the expressiveness of contextual semantics machinery which could invoke a deeper investigation into the foundations of computability based on interaction as observation.

1.2 Contextual Semantics Machinery

In order to justify the connection between quantum contextuality and interactive computation, we should be able to express contextual semantics, i.e. local consistent and global inconsistent computation in the interactive model. The model should also express its examples, e.g., Hardy model which signify non-locality and Kochen-Specker model which signify contextuality. The expressiveness of this class of computation in an interactive model is the main theme and the problem addressed in the thesis. In terms of TFTD, the thesis explore the potential semantics to *mine out* these emerging correlation patterns as a field theoretic machinery. The result of this expressivity in a model introduces the concept of openness unlike Turing-like interactive machines. The main question to address is, *What could be the underlying model of computation that can express the behaviour of the Bell-like models of quantum physics?* Precisely, *what could be computational generalisation of empirical models?*

The computational generalisation of empirical models in the foundations of quantum physics comprises broadly of five steps:

1. The first step is to formalise an interactive machine that can be used to express contextual semantics.

2. The second step is to show that this machine can emulate empirical model.

3. The third step is to formalise a condition that can quantify contextual semantics in the machine.

4. The fourth step is to allow expressiveness of contextual semantics in TIM.

5. The fifth step is to emulate examples of empirical models of quantum physics.

The computing synthesis of sheaf-based interpretation is that the computation doesn't depend only on the context but also on their collective structure. It implies, first, the collective representation of the environment could constrain the computation. Second, the constraint limits the accessibility of an unknown content in the environment due to dependency of the computation on the global contexts. It implies mathematical corollary of uncertainty postulate of quantum physics which is based on interaction as observation, a characteristic feature of the empirical models in quantum physics. Third, in this global schema, the context and the environment become distinct concepts. The interactive machine should take this synthesis into account.

Step 1: The first step is to formalise an interactive machine that can be used to express contextual semantics. The interactive machine should take the sheaf-theoretic synthesis into account. Classical Turing-like interactive models cannot express this type of computation due to the Bell-Kochen-Specker theorem, which recalls the linear tape structure of their environment as local hidden variables. Any inconsistency can be traced back due to *completeness*. Unlike above key sheaf concepts, first, the environment of the Turing-like interactive models is local whose structure is known (there exists a bijection between natural numbers and the tape structure), so there is no global structural constraint to the computation. Second, the concept of environment and contexts become interchangeable due to static nature of the environment. For instance, the labelled transition system and bigraphs compute by interacting with contexts in the static structure of the environment. Third, any unknown turns significantly known which can be accessed with exactness through an effective function during computation. It is expressed as explicit physical exchange of action among components of the model, for example, local variable declaration in process calculi and causal dependency relations in event structure models. These arguments could be explored in details based on the open debate/ philosophical discourses between Turing machines (formalists) and the interaction machines (empiricists), but it is out of scope of our thesis. However, 'what is the expressive power of interactive computation', is still in debate and quantifying paradoxes [1] can provide a promising way to understand the foundations of computability—bug as feature.

We choose the minimal extension of the Turing machine capturing the idea of interactive computation known as the persistent Turing machine (PTM) [31], and associate a topological representation of its environment to be known as the topological interactive machine [2] (TIM). Likewise key concepts, first, the contexts are distinguished from the global structure of the environment. The feature of openness which allows global structure to constrain the computation and vice-versa gives meaning to this distinction. Second, unlike linear tape, the content lies in the open neighbourhood of the topological environment. So the computation depends on the context and its corresponding class of feasible path over topological space. Third, the incapacity to localise even a known content exactly due to its dependence on global contexts and non-linear structure of environment turns it significantly unknown. Moreover, the interactive computation as a paradigm is based on interaction as observation [45].

The machine is constructed from PTM computation using topological data analysis. For brevity, consider each f has a probabilistic measure to transforms

[1] The paradox in a consistent and complete model is explainable at the level of observation without violating causality due to partial information about parameters of a model [39]. The *do-calculus* intervenes the observed reality to explore all desired parameters in order to avoid paradoxes but confines itself in localised simulation of the reality. The act of observing is reducible to act of doing under completeness and consistency. The limit to access its other unobserved part via doing or observing is quantified by *imagining* it using counterfactual reasoning.

On a somehow relatable footing the structure of hidden variables is not intervened directly rather its structure could be imagined from correlations in the observed empirical data. The act of observing is central in quantum physics expressing itself in various principles, for instance principle of uncertainty and Bohr's complementarity; as a result, act of observation is very different from act of doing, unlike equivalence between doing-observing dogma. It is also related to several philosophical discussions but the whole dogma is unreachable within the scope of our work.

Instead, as a very slight *gesture* we only focus one of possible corollary of observational postulate of quantum physics in sheaf based formalism; computation depends on global structure of the contexts, as well as, constrain the computation which in turn changes the global structure in a dynamic loop. The function is *discovered* rather than known *a priori* in a dynamic global contexts expressing a novel emerging pattern of local consistency and global inconsistency.

[2] Turing Machine has a monoidal structure. TIM would fall under braided monoidal category. The community of computational category theory addresses openness in Petri nets as symmetric monoidal category via (decorated) cospans [10]. Loosely, TIM is generated from symmetric monoidal functor except that it preserves the crossing over topological information decorated with simplicial complex unlike graphs.

i $\rightarrow$ o based on a context. The total PTM computation (behaviour) f : i $\rightarrow$ o produces empirical data in a given time t interval. Each $f_t : i \rightarrow o$ outputs data that is represented as an element in an arbitrary set Z. This set Z can be analysed by persistent homology, a procedure used in topological data analysis, to construct a topological space (environment) from PTM computations.

The TIM is a suitable model that satisfies the sheaf theory-based key concepts to express contextual semantics and emulate its examples. The machine has three levels: the topological environment (space) $\mathcal{E}$, (directed) simplicial complex $\mathcal{V}$ (contexts) and the process endowed with states (like tape with cells). In this hypothesis, classical Turing machine computes over a trivial topological space that allows computation to *always* run linearly.

Step 2: The second step is to show that this machine can emulate empirical model. Bell theorem can be represented as an empirical model with an associated topological space which exhibits its behaviour. Bell scenario consists of systems/agents N. At each site, agent makes a measurement from a measurement set M to observe output from output set O with a probability. Bell scenario is characterised by triplet (N,M,O). Statistics is calculated from the data obtained from any permutations between the triplet. The total probability space can be represented as a topological (behaviour) space. Each permutation corresponds to specific region in the space either classical, quantum or no-signalling region. The homotopy classes of its combinatorial structure encodes non-local and contextual behaviour. Whatever happens in the triplet equivalently affects it's topology.

Both triplet and topology can be entangled in the following way: to each input set of each agent, assign all values of the output set. Each element of input set can be represented as a vertices of simplicial complex embedded in topological space. The number of possible outcomes per input/vertex is represented as vector space. The structure is isomorphic to fiber bundle characterised as the gauge group. There exists an adjoint functor between the category of sheaves and the category of fiber bundles on a given topological space.

Step 3: The third step is to formalise a condition that can quantify contextual semantics in the machine. We compose a logical equality condition; $\mathbf{p} = e_1\mathcal{V}_1 + e_2\mathcal{V}_2 + e_3\mathcal{V}_3 + ... + e_n\mathcal{V}_n = \mathcal{I}$ where set $(e_1, e_2, ...e_n) \in \mathcal{E}$ are hidden environmental variables and $(\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n)$ as flexible variables that correspond to vertices of $\mathcal{K}$ and $\mathcal{I}$ is the model-specific topological invariant. The solution or possible states s of each $\mathcal{V}$ in the equality are contained in $\mathbf{S}$, possibly a fiber over each $\mathcal{V}$ through projection map π of TIM. The non-solvability of logical equality condition implies contextual semantics. Any state in a process can make transition to any other state(s) of other process given feasible path over the topological space. The computation between processes is seen as their permutations formalised as *symmetry group*. The environmental constraint to the computation as fundamental group which quantified as simplicial complex in the discrete setting using discrete Morse theory.

The formalisation can be visualised from Grothendieck's Galois theory connecting the algebraic variety (the process description) to the fundamental group of the topological space (topological environment). *Non-locality and contextuality in the computational sense of topological field theory is a symmetry breaking process.*

The classical Galois group applies to solution of polynomials. The polynomiality provides a combinatorial description of the topological space. The variables represent vertices of simplicial complex and the exponents represent in how many ways it is linked, quite similar to polynomial functor in dynamical system. The topological environment is combinatorially described as simplicial complex and given direction using discrete Morse theory. It introduces a notion of *strong collapse* which quantifies contextual semantics; non-existence of critical simplices and non-trivial closed paths in a simplicial complex. Strong collapse imply locality[3]. It categorises space as generic simplices representing feasible computation, critical simplices representing infeasible computation like deadlock and non-trivial closed loop representing LC-GI computation.

[3]i.e., not all measurements can be performed simultaneously which is core of quantum contextuality. The contractibility of topological space and collapsibility of its corresponding simplicial complex would mean that all measurement can be performed simultaneously and hence locality.

Step 4: The fourth step is to allow expressiveness of contextual semantics in TIM. The topological and geometric models of computation are local pospaces which avoid non-trivial loops. In this way, every local process *always* admits a global structure. Considering non-trivial loops seems to violate causality. The causality in the models come with two assumptions: first is underlying simplicial complex is acyclic (loop-free) and second is the simplicial complex is fixed[4]. It means that the process of computation doesn't alter the simplicial complex. But in TIM we allow computations to change the topology of environment and in turn environment constrains the feasibility of computation. It's like a feedback loop preserving causality. The feedback loop introduces the concept of openness in the machine. The expressivity of LC-GI computation is expressed as non-trivial loop in the topological environment characterising openness. Typically, an open environment can capture exactly the fundamental concept behind contextual semantics, that is, not all computations are feasible in a context sensitive topological environment. Topology is a bridge that connects contextual semantics to interactive computation.

Step 5: The fifth step is to emulate all of the examples of empirical models in the foundations of quantum physics. For example, Hardy model, Mermin Square and Kochen-Specker model, Popescu-Rohrlich Boxes and Greenberger-Horne-Zeilinger model in TIM.

Our Contributions

The paper explores and formally defines the mathematical structure which associates topological environment with PTM computation to encode contextual semantics. Empirical models of quantum physics emulate over TIM which is isomorphic to the structure of the fiber bundle characterised by the gauge group. The topological structure of the environment is further explored which encodes contextual semantics based on *strong collapse* of its combinatorial

[4]The dependence of *action* on underlying open and interactive topological environment constrains the scope of reductionism similar to resource sensitive (non-commutative) linear logic [29]. This interactivity was understood in terms of geometry [30] and further interpreted in computation [8]. The causal implications in this sense cannot be iterated after conditions are modified unlike classical logic which allows *explainable strange loops* in open dynamic environment expressing LC-GI.

structure using discrete Morse theory (DMT). We provide a generalisation based on interaction as observation paradigm in TIM to quantify examples of empirical models in the foundations of quantum physics. The examples include Hardy model, Peres-Mermin square, Kochen-Specker model, Popescu-Rohrlich Boxes and Greenberger-Horne-Zeilinger Model. The contextual semantics operationally arise due to non-trivial loops in the discrete topological environment which facilitates the concept of openness unlike Turing-type interactive models.

Moreover, the foundation and the mathematical structure of the established connection is based on the topological field theory of data. The theory in itself still requires a strong effort on a number of formal aspects. The current work has addressed some aspects which include:

1. The extraction of emerging correlation patterns (the quantum contextuality) and providing potential semantics in field theoretic description (fiber bundles) which expresses the emerging pattern as a mathematical model of computation.

2. The work goes through two main layers of the theory: the construction of a topological space realised as a simplicial complex from data of physical experiment followed by the field theoretic modelisation of the space with two components; the state space representing the function and the topological space representing the structure isomorphic to fiber bundle characterised as a gauge group (semi-direct product of the symmetry group and the fundamental group); as well as, a specific classification of the irreducible representation of the gauge group of the theory in the current context of the study.

3. The extension from homological to homotopical methods which increases reach of the theory.

4. The *computational aspects of the theory* as a mathematical model of computation isomorphic to the structure of the fiber bundle.

1.3 Outline of thesis

The reader is first exposed to the relevant mathematical background in Chapter 2 to access the material of the thesis. Chapter 3 introduces the setting for understanding the background of contextual semantics and topological model of computation along with their interplay and few key observations. Chapter 4 and Chapter 5 formalises main results of the thesis. Chapter 6 gives examples of contextual semantics machinery. Chapter 7 provides discussion on promising future work.

Chapter 2

Background Theory

The chapter gives a compact mathematical background for the readers in order to access the material of the thesis. We start with a basic understanding of homology and cohomology along with some basic definitions and examples, followed by its relation to linear algebra via representation theory. The main aim is to give a clear conceptual understanding of local consistency and global inconsistency that arise due to cohomological obstructions based on interaction as observation. Moreover, the specific relevant mathematical ideas are introduced which are later used in the main content of the thesis.

2.1 The Idea of Cohomology

Cohomology plays a central role in algebraic topology and geometry. It is dual to the homological machinery. Homology characterises shape of a topological space by detecting *holes* in it. The concept looks for objects without boundary that are not the boundary of anything. It associates abelian groups with topological spaces in a mathematical sense. The interplay between groups and topological spaces is pervasive in mathematics, for instance, the fundamental group associated with the topological space.

The sequence of abelian groups and their corresponding sequence of homomorphisms is captured by an algebraic structure known as the chain complex. The image of each homomorphism or linear transformation is included in the kernel of the next. The homology describes the way images are included in the kernels associating simplicial complex associated to a topological space X. The chain complex of X is constructed using continuous maps from a simplex to X with homomorphisms capture how these maps restrict to the boundary of the simplex.

A chain complex is a sequence of maps $\cdots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots$, where C_i may be abelian groups, e.g., vector spaces, with maps satisfying $\partial_{i-1} \cdot \partial_i = 0$. Of main interest are the elements of C_i which are mapped to zero by ∂_i known as the i-dimensional cycles along with those that are in the image of ∂_{i+1} which are known as i-dimensional boundaries. $\partial_{i-1} \cdot \partial_i = 0$ asserts that every boundary is a cycle, so we may define a quotient vector space, the i-dimensional homology group H as, $H_i(C_*, \partial_*) = \frac{kernel(\partial_i)}{image(\partial_{i+1})}$. For example, suppose g and h be matrices whose product is zero. If $h \cdot v = 0$ for some matrix v, we cannot say $v = g \cdot u$. The failure measures a defect which quantifies holes in X.

Definition 1. A chain complex C of an abelian group (generally modules) is a family $\{C_n\}_{n \in \mathbb{Z}}$ of abelian groups together with maps $\partial = \partial_n : C_n \to C_{n-1}$ such that each composite $\partial \cdot \partial : C_n \to C_{n-2}$ is zero. The maps ∂_n are called the differentials of C. The kernel of ∂_n is the group of n-cycle of C denoted as $Z_n(C)$. The image of $\partial_{n+1} : C_{n+1} \to C_n$ is the group of n- boundaries of C denoted as $B_n(C)$. The n^{th} homology group of C is the quotient $H_n(C) = Z_n/B_n$ of C_n

The chain complex is exact if all its cycles are boundaries. It means the topological space associated with the chain complex is simply connected without any obstructions realised as holes in it, i.e., $H_n(C) = 0$. Homology thus measures to what extent the sequence of chain complex fails to be exact.

Remark 1. *C is exact, that is, exact at every C_n is equivalent to C is acyclic, that is, $H_n(C) = 0 \ \forall \ n$. It means there is no holes and the associated X is simply connected with no obstructions.*

Lets take an example to further understand the concept of homology. Suppose a simplicial complex $\mathcal{K}$ on $\{1, 2, 3, 4, 5\}$ which consists of all subsets of the set $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ and $\{5\}$ as shown in Figure 2.1.

For each integer i, let $F_i(\mathcal{K})$ be set of i-dimensional faces of $\mathcal{K}$. Let $\Bbbk^{F_i(\mathcal{K})}$ be a vector space over $\Bbbk$ whose basis elements e_σ corresponds to i-faces $\sigma \in F_i(\mathcal{K})$.

Example 1. *$F_2(\mathcal{K}) = \{1, 2, 3\}$, $F_1(\mathcal{K}) = \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$, $F_0(\mathcal{K}) = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ and $F_{-1}(\mathcal{K}) = \{\phi\}$. The corresponding field*

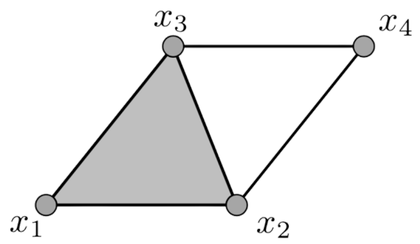


Figure 2.1 Simplicial complex

becomes $\Bbbk, \Bbbk^5, \Bbbk^5$ *and* $\Bbbk$. *The chain complex is shown in Figure* **??**. *The homomorphisms* ∂_2 *and* ∂_0 *both have rank 1. Since the remaining matrix* ∂_1 *has rank* 3, *we conclude that* $H_0(\mathcal{K}, \Bbbk)$ *and* $H_1(\mathcal{K}, \Bbbk) \cong \Bbbk$. *Topologically,* $H_0(\mathcal{K}, \Bbbk)$ *is non-trivial because* $\mathcal{K}$ *is disconnected, and* $H_1(\mathcal{K}, \Bbbk)$ *is non-trivial because* $\mathcal{K}$ *contains a triangle which doesn't bound a face of* $\mathcal{K}$.

The relation between rank of matrices and homology is given by rank-nullity theorem. The Betti numbers β of K can be determined entirely by the ranks of boundary operators which appear in the simplicial chain complex.

Proposition 1. *A simplicial complex* $\mathcal{K}$ *with* K_i *as set of all i-dimensional simplices in* $\mathcal{K}$. *Let* r_i *be the rank of the boundary map* $\partial_i : C_i(\mathcal{K}) \to C_{i-1}$, *for each dimension* $i \geq 0$ *we have*

$$\beta_i(\mathcal{K}) = \mathcal{K}_i - (r_i + r_{i+1})$$

The example can be understood in the light of Stanley-Reisner theory. The theory gives a correspondence between simplicial complex and the squarefree monomial ideals. A monomial is an element of a set which factors uniquely as a product of the variables in the set. It is squarefree if no variable appears more than once in this factorisation. Let $S = \Bbbk[x]$ the polynomial ring over $\Bbbk$ in n indeterminates of x. The Stanley-Reisner ideal $I_{\mathcal{K}}$ of a simplicial complex $\mathcal{K}$ is a squarefree monomial ideal generated by monomials corresponding to non-faces of $\mathcal{K}$. The Stanley-Reisner ring of $\mathcal{K}$ is the quotient ring $S/I_{\mathcal{K}}$. The rough idea is: Given a polynomial over $\Bbbk$, compute $I_{\mathcal{K}}$, a set not containing faces of $\mathcal{K}$. So, the ideal gives a measure of non-connected simplices of $\mathcal{K}$ so that the remaining is a connected simplicial complex. But for a simplicial complex which is triangulation of a topological space with holes in it, the ideal has a set which are not faces of $\mathcal{K}$ locally but globally it contains those faces due to identification of boundary. Here is an example to illustrate the idea 2.

Example 2. *Figure 2.2 is the standard triangulation of the real projective plane. The faces are abd, abf, acd,ace, aef, bce, bcf, bde, cdf and def. The Ideal is generated by ten minimal non-faces as, abc,abe, acf, ade, adf, bcd, bdf, bef, cde, cef. The projective plane is globally strange but locally well-behaved.*

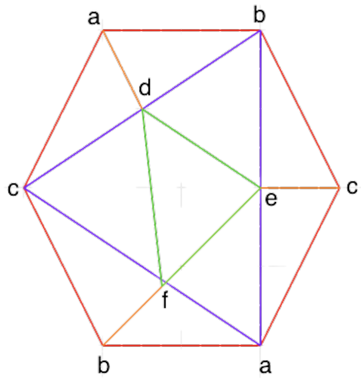


Figure 2.2 The standard minimal triangulation of the projective plane.

The concept of cohomology is exactly dual to the homological machinery where the direction of map is reversed. Cohomology doesn't look for subspaces for detecting holes, instead assigns a value to each sub-object in the space. It is a richer algebraic variant than homology and gives access to different parts of mathematics. Loosely, cohomology measures the failure of locally consistency of an algebraic object to be globally consistent. It pullback the space from target to the source unlike homology. Cohomology theories are related to the fiber bundles in physics and mathematics, for instance (cohomological) topological quantum field theory and theory of characteristic classes respectively.

Cohomology doesn't look for subspaces detecting holes, instead assigns a value to each sub-object in the space. Here, we provide a very concise idea about de Rham cohomology in order to understand the obstructions in a simpler way. It also gives a way to understand local consistency-global inconsistency that arise from these cohomological obstructions.

Terence Tao says about the de Rham cohomology as:

> The integration on forms concept is of fundamental importance in differential topology, geometry, and physics, and also yields one of the most important examples of cohomology, namely de Rham cohomology, which (roughly speaking) measures precisely the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds.

The fundamental theorem of calculus links integrating and differentiating a function as inverse operations. The volume of a sphere can be evaluated by gluing together its small surface areas. The main element of interest are the closed curves over a space. We first provide few example as:

Example 3. *Lets take an example of a curve in $\mathbb{R}^2$. The curve is oriented with initial and final point. We assign to such curve the value of the horizontal projection. If the curve flows to the right, this value increases and in the left this value would decrease. In such situations the dynamics of curve can be known from initial and the final point and the trajectory of curve in itself is not significant, i.e., it is state function. In mathematical sense, it means every closed curve is assigned a value of zero as shown in Figure 2.3. The value on the horizontal projection shown in orange colour, decreases as the curve moves over it until it becomes zero.*

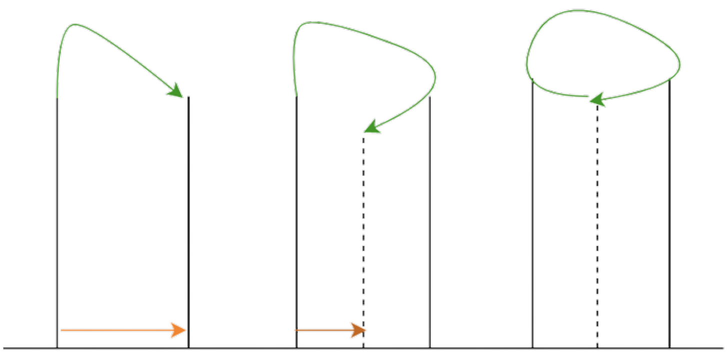


Figure 2.3 A vanishing loop.

Example 4. *Lets take another example as a vector field as shown in Figure 2.4. The length of the arrow represents the magnitude of the vector field. In upper portion, it is towards right direction and in lower portion its reverse direction.*

In this example, the value doesn't only depend on the initial and final position but also on the trajectory of the curve. The curve that first goes upwards and then goes downwards have a positive integral, while curves going vice-versa have a negative integral. Moreover, in this case, closed curves doesn't always have zero integral. (In this example the vector field is $f(x, y) = (y, 0)$, and the assignment is the line integral of curves.)

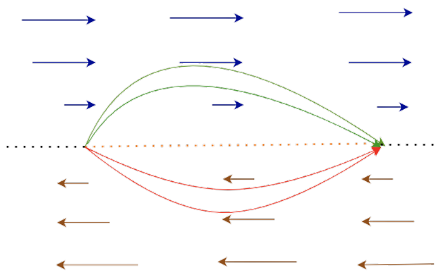


Figure 2.4 A non-vanishing loop.

Example 5. *For example angle swept in $\mathbb{R}^2 - (0, 0)$ shown in Figure 2.5. The angle swept by any curves (like one represented in green) except around origin would be zero. A curve that surrounds the origin sweeps an angle of 2π as shown by blue arrow around the origin , that is not what we though for closed curves. It can only happen when space has holes; zero homology for non-origin curves and 2π for curve around the origin.*

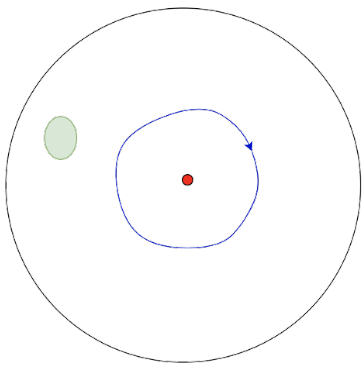


Figure 2.5 A vanishing loop.

In the example 2.4 the value of integral over curve is non-zero even if it doesn't have any hole. It doesn't break the fundamental theorem of calculus. The reason for non-zero value of closed curve is that we can't compute antiderivative of the function.

The relation between closed loops and the cohomology can be understood by introducing few concepts. The way little values of a quantity changes over space is measured by differential forms which are integrands over curves, solids or surfaces as in above example of Figure 2.4. The change in differential forms is called the exterior derivatives. The derivative is a measure of how fast a quantity is changing as it is moved around in various directions of the manifold. One of the important differential form to our theme is known as closed differential form. The integral over any small curve is zero. A differential form is a closed form if the exterior derivative is zero. In the sense of homological machinery, the abelian groups are replaced with k-differential forms and the boundary maps replaced with exterior derivatives ∂^{ext} satisfying all the properties like $\partial^2 = 0$.

We will take an analogous example of the vector calculus because it has similar notion of closed forms. If one associates each point in a space a vector, we get a vector field with magnitude and the direction. In vector calculus we have various derivatives for vector field like curl, divergence etc. One of the concept is the closed forms which are irrotational. If the curl of a vector is zero then the vector field is irrotational. The irrotational vector field is a closed form. The example 3 is a closed form evident from its Figure 2.3. In the example 5 in Figure 2.5, the field is also irrotational so is closed from. The idea is that even if the function (angle swept) is not globally defined continuously because of hole but is defined upon small patches yielding it as closed form. Unlike, in example 4 the curl is not zero as seen from its figure 2.4 there is a net circular moment. The exterior derivatives is non-zero, ($f(x, y) = (y, 0)$ gives $df = -dx \wedge dy \neq 0$).

The example 3 and example 5 both have closed form but they are different objects. They both have closed forms but how can we separate them further. The concept that distinguishes between them is the exact forms. Exact forms are differential forms which are derivates of another form. In vector field analogy it is the conservative vector field, i.e its gradient field. In example 3 the closed form is exact and in example 5 the closed form is not exact. The cohomology $H^1_{deRham} = kernel\ \partial^{ext}/image\ \partial^{ext} = closed\ 1forms/exact\ 1forms$. The existence of holes in a space is reason for non-vanishing curves. The non-vanishing trajectories is the measure of cohomological obstructions. In de Rham cohomology search for holes we find closed forms that are not exact which is similar to homological machinery as already discussed. Once we define the space of forms and the exterior derivative, we get chain of maps known as de Rham complex. Here, cohomology means looking for closed forms that are not exact.

Remark 2. *A conservative vector field is irrotational i.e., $curl(grad(f) = 0)$. The inverse is not true, i.e., if curl of a vector field is zero it doesn't mean the vector is a gradient say of some potential.*

The different categories of differential form gives different flow over manifold as shown in the Figure 2.6 using hodge decomposition theorem. The de Rham cohomology of closed forms along with the hodge decomposition theorem gives a more practical way to understand the meaning of obstruction and the LC-GI.

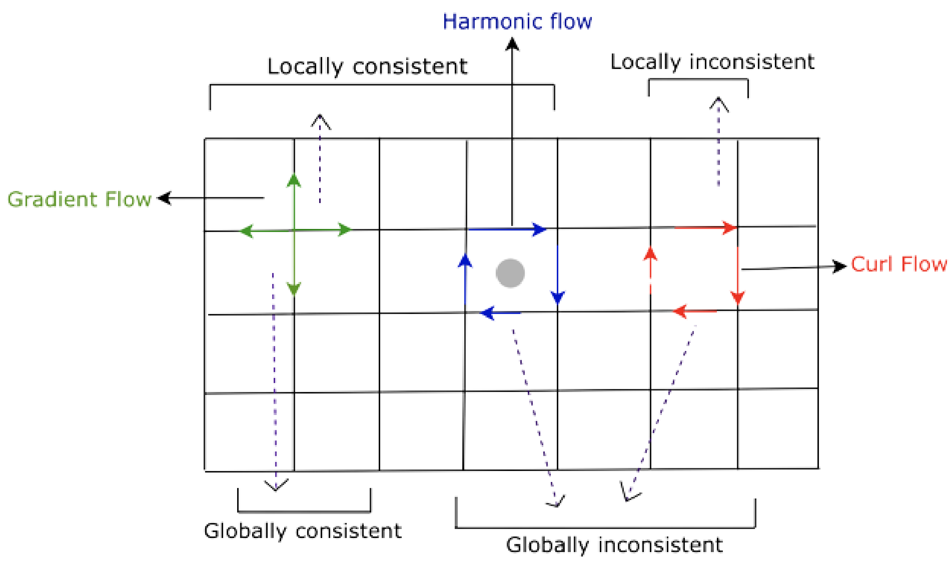


Figure 2.6 Flow categories over a space based on de Rham cohomology via Hodge decomposition. The gradient flow corresponds to local consistency-global consistency, curl flow corresponds to local inconsistency-global inconsistency and the harmonic flow (with a hole shown as grey circle based on closed forms) corresponds to local consistency-global inconsistency.

Before going further, we review elementary definitions for general readers.

Given a finite topological space X , we define for every point $x \in X$ the minimal open set $\mathcal{U}_i$ as the intersection of all the open sets which contain x. The arbitrary intersection of all the open sets in finite topological spaces are open. The minimal open sets constitute a basis for the topology of X. The open sets correspond to the measurement context of the empirical model. X may be given a poset structure by defining $y \leq x$ if and only if $\mathcal{U}_y \subseteq \mathcal{U}_x$. The finite posets and finite topological spaces are in bijective correspondence. It can be represented by Hasse diagram. The Hasse diagram is a digraph whose vertices are the points of X and whose edges are the ordered pairs (x, y) such that $x < y$ and there exists no $z \in X$ such that $x < z < y$. The Hasse diagram has a corresponding simplicial complex $\mathcal{K}$.

Definition 2. A topology T on a set X consists of subsets of X satisfying the following properties:

1. The empty set ϕ and the space X are both sets in the topology.
2. The union of any collection of sets in T is contained in T.
3. The intersection of any finitely many sets in T is also contained in T

Definition 3. A topological space is a pair (X, T) where X is a set and T is a set of subsets of X satisfying above axioms. T is called a topology.

A *finite topological space* is a topological space with finitely many points.

Definition 4. A simplicial complex is a finite collection of simplices $\mathcal{K}$ such that $\sigma \in \mathcal{K}$ and $\tau \leq \sigma$ implies $\tau \in \mathcal{K}$, and $\sigma, \sigma_0 \in \mathcal{K}$ implies $\sigma \cap \sigma_0$ is either empty or a face of both.

Definition 5. A directed or an ordered abstract simplicial complex is a collection $\mathcal{K}$ of finite ordered sets called ordered simplices, with the property that if $\sigma \in \mathcal{K}$ is an ordered simplex and $\tau \subseteq \sigma$ is a subset ordered by the induced ordering, then τ is also an ordered simplex.

In an ordered simplicial complex, unlike ordinary simplicial complex the set of $n + 1$ vertices could support as many as $(n + 1)!$ distinct n-simplices that correspond to all possible orderings of the set of vertices. On the level of homology nothing goes wrong but for homotopy equivalence the way two vertices are attached is significant. We explore ordinary simplicial complexes and use Forman's discrete Morse theory for ordering it. This notion of ordering is significant for computing homology and homotopy of simplicial complexes.

2.2 An Idea of Representation Theory

Homology is also computed using linear algebra using concept of signed incidence matrices. Given a graph with edges and vertices and an orientation. A signed incidence matrix D is a matrix such that $D_{ij} = 1$ if edge say V_j leaves vertex V_i, -1 if edge V_j enters V_i, and 0 otherwise. An simple example is given in Figure 2.8, the corresponding incidence matrix is:

$$D = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The rank-nullity theorem for boundary matrix, say ∂_k states that the dimension of the domain of ∂_k is the sum of the dimension of its image and the dimension of its kernel (nullity). The pictorial realisation of linear algebra and the abelian group is depicted in Figure 2.9. Above the chain complex and the homomorphism is a vector space which can be described in matrix formulation. The arrows in the cone shaped space of the Figure 2.9 describe nonzero entries in matrix representation of the boundary operator: 1-simplex has two nonzero entries under its column in ∂_1; these lie in the rows corresponding to the two 0-complex which form its faces and so on. The cone shaped space over the chain complex shows how matrix representation encodes information about simplices of $\mathcal{K}$. For instance, C_3 simplices consist of 3-simplices of $\mathcal{K}$, C_2 consists of 2-simplices, C_1 consists of 1-simplices and C_0 consists of 0-simplices. One of the general results to compute homology from matrices is Smith decomposition.

The interplay between groups and matrices is a subject matter of a comprehensive theory known as representation theory of the group. The theory studies abstract algebraic structures, for instance, the groups by representing its elements as linear transformation of vector spaces. It makes groups more concrete by describing its elements as matrices and their algebraic operation like linear transformation using matrix operations. Before proceeding, we provide a brief definition of groups.

Groups: A binary operation on a set is a rule to combine two elements of the set. A non-empty set A is a group under some operation, say $\bullet$, if it satisfies the following four conditions.

1. $\bullet$ is associative, i.e., $(a_1 \bullet a_2) \bullet a_3 = a_1 \bullet (a_2 \bullet a_3)$ for $a_1, a_2, a_3 \in A$
2. There exists an element e called identity element such that $a_1 \bullet e = a_1$
3. To each element a_i of A there corresponds an element a_j of A, called inverse of a_i, such that $a_i \bullet a_y = e$
4. Closure property, $a_1 \in A$ and $a_2 \in A$, implies $a_1 \bullet a_2 \in A$.

The pair $(A, \bullet)$ with first property is the semigroup. A semigroup with third property is a monoid. The pair $(A, \bullet)$ with closure property is a groupoid. Examples of groups are group of matrices, group of linear transformations and group of permutations.

Conjugacy class: Two elements a_1 and a_2 in a group are conjugate if there is an element a such that $a_2 = aa_1a^{-1}$. This is equivalence relation whose equivalence classes are called conjugacy classes. The conjugacy class for abelian group is a set consisting of one element.

Example 6. *Take an example of a symmetry group S_4, It consists of* 24 *permutations of four elements, having five conjugacy classes. The conjugacy class correspond to the following permutations; no change, interchanging two elements and other two elements unchanged, cyclic permutation of three elements, cyclic permutation of all four elements and interchanging two elements and also other two elements.*

Let G be a finite group, $\mathbb{F}$ a field and V a vector space over $\mathbb{F}$. a homomorphism ρ from G to $GL(V)$ is the group of all nonsingular linear transformations of V which is called representation of G over $\mathbb{F}$. The main idea is to associate to each group element say $g \in G$, an $n \times n$ matrix $\rho(g)$. So every group element is associated with a corresponding matrix which makes ρ a function. Further, the matrices should multiply in the same way as the group elements, i.e., $g \cdot h = \rho(g) \times \rho(h)$, where g, h are group elements and $\rho(g)$ and $\rho(h)$ are their corresponding matrices. We used different sign to distinguish between matrix and group operations; $\times$ for matrix operation and $\cdot$ for group operation which makes this function a homomorphism. Lets take an example of additive group with $< 1 >$ as its generator and associate with generator a square matrix as shown below.

$$\mathbb{Z}_4 = \{0, 1, 2, 3\} \qquad \rho(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The square matrix is rotation matrix. Take a standard basis vector $(1, 0)$ which lie in first quadrant and apply the matrix to it. The image of $(1, 0)$ under this matrix multiplication is $(0, 1)$. The vector has been rotated counter-clockwise by 90°. Applying the matrix to vector $(0, 1)$ rotates it further to 180° changing vector to $(-1, 0)$. It means the matrix rotates $\mathbb{R}^2$ by 90° counter-clockwise. Since there is a homomorphism between group $\mathbb{Z}_4$ and the rotation matrix, so it allows us to determine associated matrices for other elements of this group. In $\mathbb{Z}_4$, $1 + 1 = 2$ (1 is generator of the group), that mean we can associate to the group element 2, the matrix which we get by multiplying the matrix for 1 $\rho(1)$ to itself. Following the same procedure, the matrices associated with group elements $2, 3, 0$ of $\mathbb{Z}_4$ are $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ respectively. The matrices has rotates the vector in $\mathbb{R}^2$ from 0° to 360° which comes back to the initial position, that's why the matrix associated with group element 3 is the identity matrix. It is quite reasonable in group operation of $\mathbb{Z}_4$ because if one adds it four times we get zero. Every element of $\mathbb{Z}_4$ has a corresponding matrix representation, and the addition operation of the group is a matrix multiplication as shown in Figure 2.7.

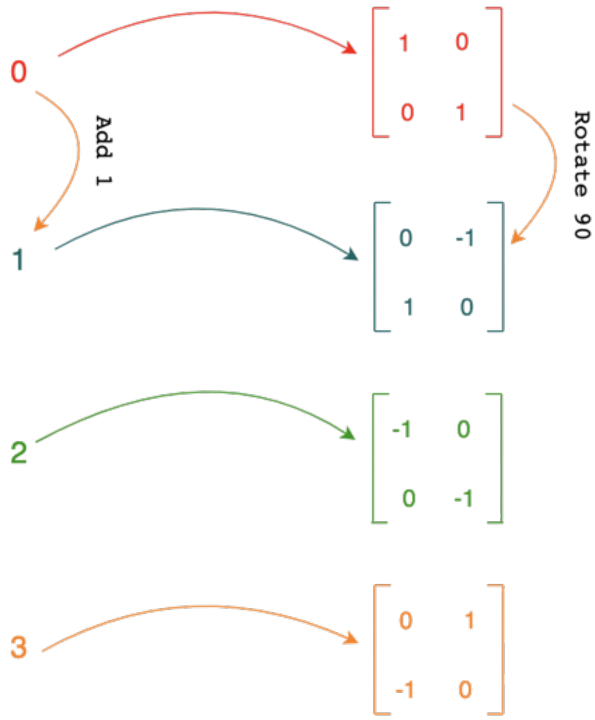


Figure 2.7 Group elements of $\mathbb{Z}_4$ and their corresponding matrices.

Definition 6. A representation of a finite group G on a finite dimensional vector space V is a group homomorphism $\rho : G \to GL(V)$ of G to the group of automorphisms of V.

Definition 7. The trivial representation of G is the representation $\rho : G \to GL(V)$ such that $\rho(g) = 1 \ \forall \ g \in G$

Definition 8. A sub-representation of a representation V is a vector sub-space W of V which is invariant under G.

If V and W are representations, then the direct sum and the tensor product of both vectors are also representations.

Definition 9. A representation V is called irreducible if it has no proper non-zero invariant subspaces.

Definition 10. Any representation is a direct sum of irreducible representations

Definition 11. (Schur) If V and W are irreducible representations of G and $\rho : V \to W$ is a homomorphism, then:

1. Either ρ is an isomorphism or $\rho = 0$.

2. If $V = W$, then $\rho = \lambda$ I for some $\lambda \in \mathbb{F}$, where I is the identity.

An effective tool for understanding the representation of a finite group G is called character theory.

Definition 12. If V is a representation of G, its character χ_V is the function on the group defined by $\chi_V(g) = Tr(g|V)$, the trace of g on V

The character of a representation of a group G is a function on the set of conjugacy classes in G. It gives information about the irreducible representations of a group G in the form of a character table.

The table lists the conjugacy classes $[g]$ of G across the top, usually with the number of elements in each conjugacy class over it, and the irreducible representations V of G listed on the left of the table. The entries corresponding to the rows and columns is the character χ_V. The character table of any finite group is constructed using Schur's orthogonality relations of irreducible representations.

Remark 3. *The number of irreducible representations of a finite group is always equal to the number of conjugacy classes.*

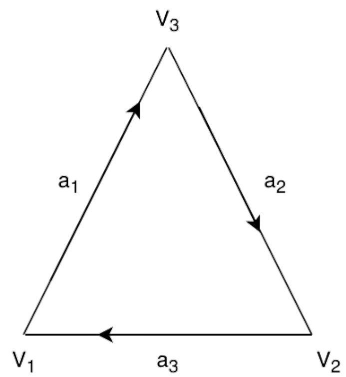


Figure 2.8 A simple triangle example

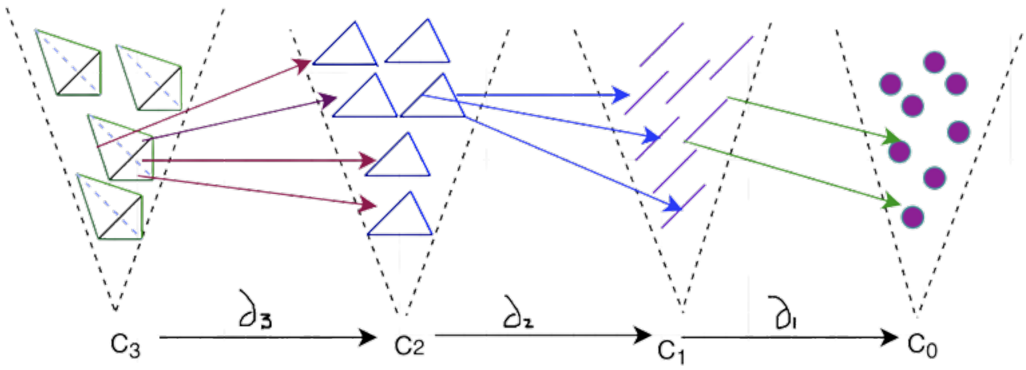


Figure 2.9 The chain complex

2.3 Interaction as Observation

The logical reason for LC-GI is based on incapacity of an observer to visualise the overall contexts of a model simultaneously. There is a close relation between impossible figures and cohomology based on interaction as observation.

Take an example of *Penrose tribar* to illustrate the connection between observation and cohomology [40]. Suppose Q is some non-simply connected region of the plane over which we shall draw the tribar. Let G be a (abelian) group of the tribar which measures distance of observer (eye) from the tribar. The tribar is shown in Figure 2.11. We may divide the region Q as being glued together from three smaller regions Q_1, Q_2 and Q_3 as shown in Figure 2.12. There are overlapping pair denoted as points over each Q_i which are to be glued together. The drawing is consistent on each Q_i except one main ambiguity that one doesn't know distance away from the observer's eye as shown in Figure 2.13. Fix a point A_{12} on region Q_1 where it overlaps with Q_2, and a point A_{13} where it overlaps with Q_3. Similarly A_{21} on Q_2 to be overlapped with A_{12} on Q_1, A_{31} on Q_3 matched with A_{13} and A_{23} on Q_2 matched with Q_3 which corresponds to A_{32} on Q_3. The observer sees object O_1 and object O_2 which the drawing of Q_1 and Q_1 depicts respectively. The point A_{12} on O_1 may not be the same distance from the observer's eye E as the corresponding point A_{21} on O_2. Let the ratio of these distances be d_{12}, thus we have,

$$d_{ij} = \frac{distance\ from\ E\ to\ point\ on\ O_i\ depicted\ by\ A_{ij}}{distance\ from\ E\ to\ point\ on\ O_j\ depicted\ by\ A_{ji}}$$

The ratio doesn't depend on the particular matched pair of A_{ij} and A_{ji} chosen on the overlap between Q_1 and Q_2. This d_{ij} represents the (rescaling) factor that one must move out by when we pass from O_j to O_i at the region of overlap. If one changes its chosen distance from the observer's eye then the pain say (d_{ij}, d_{ik}) is replaced according to $(d_{ij}, d_{ik}) \rightarrow (\lambda d_{ij}, \lambda d_{ik})$ for some positive number λ. Its rescaling the observed objects O_1, O_2 and O_3 in and out until they all come together as one consistent structure. It means, the rescaling reduces the three ratios d_{12}, d_{23} and d_{31} simultaneously to one. It follows: Let λ_1, λ_2 and λ_3 be rescaling factors to be equal to one means, $\frac{\lambda_1}{\lambda_2}(d_{12}) = 1$, $\frac{\lambda_1}{\lambda_3}(d_{13}) = 1$ and $\frac{\lambda_2}{\lambda_3}(d_{23}) = 1$. Which implies, $d_{12} = \frac{\lambda_2}{\lambda_1}$, $d_{13} = \frac{\lambda_3}{\lambda_1}$ and $d_{23} = \frac{\lambda_3}{\lambda_2}$. And if there are no such λ_1, λ_2 and λ_3, then the object is impossible.

It is exactly the basic concept in cohomology theory, the collection $\{d_{ij}\}$ is a cocycle. If the ratio equals to one, then the cocycle is called a coboundary. The rescaling map $(d_{ij}, d_{ik}) \rightarrow (\lambda d_{ij}, \lambda d_{ik})$ provides coboundary freedom which are equivalent if they can be converted to one another under this freedom. Under the equivalence, the whole observational doctrine is expressed in cohomology group $H^1(Q, R^+)$, where Q is the space possibly non-connected and R^+ forms a group of positive numbers λ_i under scaling operation.

Roger Penrose was inspired by M.C. Escher's artistic work 2.10 while attending 1954 International Congress of Mathematicians in Amsterdam. The sheaf theoretic interpretation of quantum contextuality was also inspired by Escher's ascending and descending lithograph. The image is locally consistent when seen as pieces but gluing together as a whole turns to be globally inconsistency. We put another Escher art, the print gallery which gives a deep sense of observation as interaction, forming strange loops similar to fractals 2.10.

Figure 2.10 Print Gallery (original title: Prententententoonstelling —1956), a famous lithograph by M.C. Escher. Notice a hole in the middle. Indistinguishability between experienced object and experiential subject.

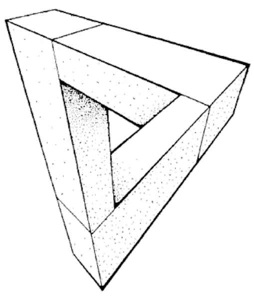


Figure 2.11 The Penrose Tribar. Courtesy [40]

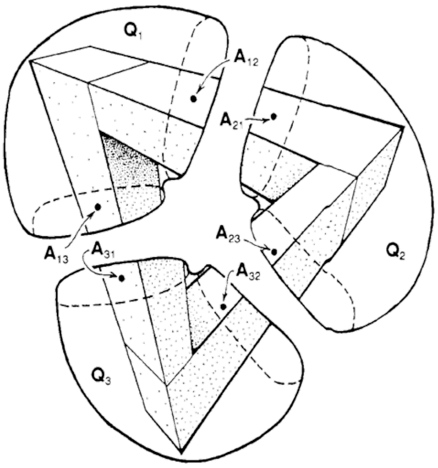


Figure 2.12 The tribar shown pieced together out of overlapping smaller drawings, each of which depicts a possible structure. Courtesy [40]

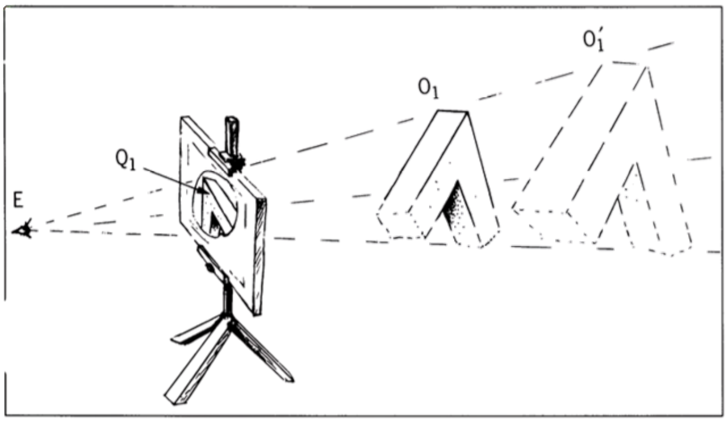


Figure 2.13 There is a local $\mathbb{R}^+$ ambiguity in any plane drawing as to the distance from the observer's eye to the object depicted. Courtesy [40]

2.4 Sheaf Theory

Sheaves take two types of input spaces. One is the space they are on which is the horizontal component of topological type. The other input is the vertical component of the algebraic type. Examples are sheaves of rings/fields on manifold/any type of topological space. The literal meaning of sheaf is, *a* *bundle of grain stalks laid lengthways and tied together after reaping* as shown in Figure 2.14.

Figure 2.14 A Sheaf of Wheat. Courtesy Internet Image

It is exactly the intuitive mathematical picture of the sheaves with horizontal knot as the topological space and each vertical stalks as sheaves. The individual sheaf is called a *stalk*. If one chooses a spot in a stalk which is next to another spot on other stalk, the whole selection of all spots throughout the bundle of sheaf is called the cross-section of sheaf.

Lets take an example of sheaves *of* vector spaces *on* abstract simplicial complexes. It consists of simplicial complex as base space/ horizontal component and the vector spaces as the vertical component. Here is a simple example of a simplicial complex in Figure 2.15. The subset simplices are AB,BC,CD,AD, BD and ABD in addition to individual simplices. The partial diagram giving the subset relations is shown in left side of Figure 2.16 which comprises of the base horizontal space. One can go from one simplices to another using homological machinery. There is some data on each vertices of the simplicial complex which is the vertical component of the sheaf. It means, we have vector space on each vertices which is shown separately on right side of Figure 2.16. One can move between vector spaces using linear maps such that diagram commutes. Each vector space is called the stalk of the sheaf. The object on the right hand side of Figure 2.16 is actually a presheaf. But every presheaf can be sheafied. So, we don't distinguish between sheaf and a presheaf which is also not subject matter of our discourse.

Theorem 1. *Every presheaf can be uniquely realised as a sheaf.*

Loosely, a cross-section is to pick a vector from every vector space in Figure 2.16 so that when we walk around via linear map that vector is already there. A section of a sheaf is an element $s \in \prod_{\sigma \text{ is a simplex}} (O(\sigma))$ such that $O(\sigma \to \sigma')s(\sigma) = s(\sigma')\ \forall\ \sigma \subseteq \sigma'$, i.e., σ is the face of σ'. $s(\sigma)$ and $s(\sigma')$ are elements of O.

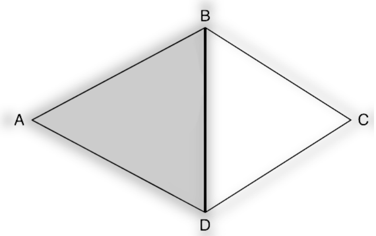


Figure 2.15 An example of simplicial complex to illustrate sheaf structure.

The corresponding sheaf structure is given as:

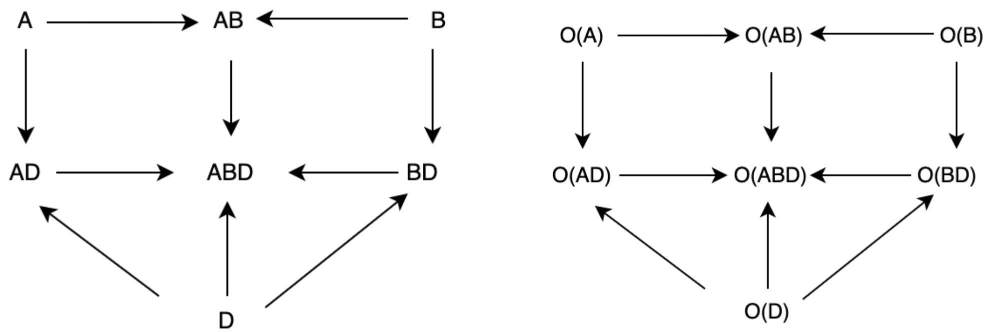


Figure 2.16 On the left is the horizontal component of Sheaf which comprises of the power set of the simplicial complex in 2.15, and on the right is the vertical component of sheaf which comprises of the vector spaces associated to each vertex of the simplicial complex.

Basic idea of sheaf theory: Sheaves comprise of two components that operate together: the horizontal topological component over which its vertical algebraic type resides. A basic example is sheaf *of* vector space *on* abstract simplicial complex. The individual sheaf is called a stalk. The idea is to move between vectors in each vectors space (stalk) via linear maps in the vertical component together moving from higher simplices to 0-simplex on its horizontal topological component.

Suppose X is a topological space with open set $\mathcal{U}$ *covering* X. Let $\mathcal{U}_i$ be open subsets of $\mathcal{U}$, where i denotes the number of open subsets of $\mathcal{U}$ in order to cover it. Sheaf theory operates on a function r defined on $\mathcal{U}$ of X and restricts to functions $r|_{\mathcal{U}_i}$ defined on $\mathcal{U}_i \subset \mathcal{U}$. $\mathcal{U}$ is recovered by gluing together the restrictions to $\mathcal{U}_i$. The *restriction-gluing* is main theme of sheaf theory. A (pre)sheaf $\mathcal{O}$ on X can be described as a rule/function which assigns to each point, a set O consisting of the elements of the functions at that point defined by its neighbourhood, i.e., $\mathcal{O}(\mathcal{U}) := O^{\mathcal{U}}$. The set O turns out to be output set in sheaf generalisation of empirical model. The sets $O^{\mathcal{U}_i}$ for all $\mathcal{U}_i$ can then be glued together by a suitable topology so as to form bundle projected onto X. The individual *nice* function for this sheaf is then a cross-section of the projection of this bundle. The combinatorial representation of X is a finite simplicial complex $\mathcal{K}$. When X is discrete, the presheaf $\mathcal{O}$ is trivially a sheaf. In fact, every presheaf can be uniquely realised as a sheaf. A section of a sheaf is an element $\kappa \in \prod_{\sigma,\tau \in \mathcal{K}} \mathcal{O}(\sigma)$ such that $\mathcal{O}(\sigma \to \tau)\kappa(\sigma) = \kappa(\tau)\ \forall\ \sigma \subseteq \tau$, i.e., σ is the face of τ and $\kappa(\sigma)$, $\kappa(\tau)$ are elements of O. The map $\mathcal{O}(\sigma \to \tau)$ is a linear map that transforms matrix $\kappa(\sigma)$ to $\kappa(\tau)$ over vertical component of sheaf given its corresponding collapsible or acyclic $\mathcal{K}$, i.e., the cochain complex is exact.

The *cohomological obstructions* which arise in sheaf theory is the main mathematical concept to express phenomena of non-locality and contextuality. The obstructions arise due to non-simply connected regions of the topological space which has non-zero or non-vanishing cohomology groups. There could locally exist a section κ via linear map but globally non-extendible due to non-simply connected X.

2.5 Sheaf Generalisation of the Empirical Model

Generalisation of empirical models through sheaf structure: Quantum contextuality restricts the simultaneous assignment of values to all measurements. This feature is captured by introducing the measurement cover $\mathcal{U}$ which covers the space X. The subsets of $\mathcal{U}$ denoted by $\mathcal{U}_i$ represents the set of compatible families of measurement contexts, i.e., those which can be measured simultaneously with, $\bigcup_{\mathcal{U}_i \in \mathcal{U}} \mathcal{U}_i = X$. The combinatorial description of X is represented to be a simplicial complex $\mathcal{K}$ which is homeomorphic to X. It comprises of finite measurement set representing all the available measurements in a general experiment. Each measurement $\mathcal{U} \in X$ produces an outcome in a set $O_{\mathcal{U}}$. The measurement set, the measurement contexts and the outcome set constitute the measurement scenario. X is equipped with a discrete topology and defines sheaf of events which relates open subsets $\mathcal{U}$ of X to the space of sets O.

A *section over* $\mathcal{U}$ is a function $\kappa : \mathcal{U} \to O$ that describes the event in which the measurements in $\mathcal{U}$ were performed, and the outcome $\kappa(m)$ was *observed* for each $m \in \mathcal{U}$. The function κ is generalised mathematically in a *sheaf structure* that assigns set of sections over O to each *measurement context* $\mathcal{U}$. It is represented topologically as a bundle diagram as shown in Figure 3.3. There exists a unique section for a family of subset in $\mathcal{U}$ if its corresponding family of sections is *compatible* and has simultaneous existence i.e., $\kappa_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \kappa_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$. For brevity, the non-collapsibility of the simplicial complex constrains the extendability of contexts to global section due to cohomological obstructions [6]

Measurement scenarios and the sheaf of events define the experiment setting and are therefore independent of any physical system. The application of this sheaf scenario on an actual physical system is captured by the notion of empirical model. The empirical model $\mathbf{e}$ in this sense is a compatible family $\{\mathrm{p}_{\mathcal{U}_i}\}_{\mathcal{U}_i \in \mathcal{U}}$, where $\mathrm{p}_{\mathcal{U}_i}$ is a probability distribution on $\mathcal{O}(\mathcal{U})$. The compatibility of the family is independent of the choice of $\mathcal{U}$. The model $\mathbf{e}$ is possibilistic expandable if and only if every section is a member of compatible family. It is possibilistically non-extendable if for some section κ (cross-section), there is no such family. The model $\mathbf{e}$ is strongly contextual if for every κ there is no such family. A simple understanding of this mathematical setting is comprehensively expressed in the tabular representation of the empirical models, for instance, the Hardy table shown in Figure 6.1. The sheaf theoretic generalises the strong forms of non-classical behaviour exhibited by quantum physics.

Definition 13. Let Q be finite set of all measurements equipped with discrete topology X, producing outcome O_m for each $m \in X$. The set X, cover $\mathcal{M}$ and outcome O constitute measurement scenario denoted as $< X, \mathcal{M}, O >$ with $\mathcal{O}(\mathcal{U}) := \prod_{m \in \mathcal{U}} O_m$ is sheaf of events.

Definition 14. An empirical model is a compatible family $\{\mathbf{e}_{\mathcal{U}}\}_{\mathcal{U}\in\mathcal{M}}$ of probability distribution over the events at each context $\mathcal{O}(\mathcal{U})$. Such models are defined as (pre)sheaves $\mathcal{O}$ satisfying conditions:

1. $\mathcal{O} \neq \phi \; \forall \, \mathcal{U} \in \mathcal{M}$.
2. $\mathcal{O}$ is a *flasque beneath the cover* i.e. the restriction map $\rho^{\mathcal{U}_j}_{\mathcal{U}_i}$ is surjective whenever $\mathcal{U}_i \subset \mathcal{U}_j \subset \mathcal{C}$ for some context $\mathcal{C} \in \mathcal{M}$.
3. Every family $\{s_{\mathcal{C}} \in \mathcal{O}(\mathcal{C})\}_{\mathcal{C}\in\mathcal{M}}$ which is compatible ($\forall \, \mathcal{U}_i, \mathcal{U}_j \in \mathcal{M}$, we have $s_{\mathcal{U}_i}|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_{\mathcal{U}_j}|_{\mathcal{U}i \cap \mathcal{U}_j}$) induces a global section in $\mathcal{O}(X)$

Let $\mathbf{e}$ be an empirical model on a scenario $< X, \mathcal{M}, O >$. Given a context $\mathcal{C} \in \mathcal{M}$ and a section $\kappa \in \mathbf{e}(\mathcal{C})$, we say that $\mathbf{e}$ is possibilistic or logically contextual at κ if κ is not part of any compatible family.

Theorem 2. *(Abramsky & Brandenburger [4]) Let* $\mathbf{e}$ *be an empirical model. The following are equivalent:*

1. $\mathbf{e}$ *is extendable;*

2. $\mathbf{e}$ *admits a realisation by a deterministic hidden-variable model;*

3. $\mathbf{e}$ *admits a realisation by a factorisable hidden-variable model.*

Theorem 3. *(Rui Barbarosa [11]) Let* $\mathcal{K}$ *be a simplicial complex. Then any empirical model* $\mathbf{e}$ *defined on* $\mathcal{K}$ *is extendable if and only if* $\mathcal{K}$ *is acyclic.*

2.6 Topological Field Theory of Data

Topological field theory of data TFTD is a broad programme to provide a novel framework for complex systems. Complex systems are systems composed of many non-identical elements, entangled in loops of non-linear interaction. For example, neurons in the brain cortex and living cell. These systems are based on irreducibly holistic doctrine. The doctrine pushes for an overwhelming evidences for a novel paradigm of complex systems. A typical feature of complex systems is the emergence of non-trivial superstructures which cannot be reconstructed by a reductionistic approach. The systems don't have a blueprint but are perceived through large amount of data. Reconstruction of complex system is a search for a model that can be represented as a mathematical construct able to reproduce the observed data reasonably well.

The theory is based on three levels which need to operate together:

1. Topological Data Analysis: The homology methods explores the sufficient tools for the efficient reconstruction of the topological structures from the space of data encoding patterns. It allows topological data analysis of data coherent with combinatorial, algebraic and topological aspects of the data.

2. Topological field theory: The output of the first level is the combinatorial structure of the space of data. This step explore the dynamics over the topological space as a formal machine. It gives access to many mathematical theories that could be applied computationally machine theoretic way to emulate/ provide explanations of the data from complex systems. The physical example is Yang Mills field theory in which variables are the connection field over manifold and the gauge group is a Lie group under which action (Chern Simon Action) is invariant. One of the mathematical structure significant at this level is the fiber bundle quantified by the gauge group. The level provides the construct, emulating the physical (statistical) field theories for extracting the useful information to characterise the behaviour in presence of non-linearity and self-interactions.

3. Formal language theory: The main question addressed at this level is about the relevance of topological landscape with structure of formal language theory. It gives a way to study syntactical aspects of language through algebraic structures in order to understand their behaviour. It gives a functor from semantics of transformations of field theory into automated self-organising learnability.

Our work is an instance of the computational aspects of the topological field theory of data. The formal connection between quantum contextuality and interactive computation implicitly goes through all these levels. We construct a discrete topological space from the empirical model which is part of the first level of TFTD. The empirical models are based on observational data from a physical experiment. The formal description of a topological interactive machine emulating the empirical model which is isomorphic to the structure of a fiber bundle is the second level of TFTD. The computational aspects implicit in the machine-theoretic generalisation of the empirical models touch the algebraic part of the third level of TFTD.

2.7 The Fiber Bundle

Suppose X is a topological space. F-bundle with fiber(s) F over X is a total space E and a surjective map $\pi : E \to X$ with the property that X admits an open cover $\mathcal{U}_i$ such that for each i there is a homeomorphism $\psi : \pi^{-1}(\mathcal{U}_i) \to F \times \mathcal{U}_i$ with projection $\theta : F \times \mathcal{U}_i \to \mathcal{U}_i$ back to X such that all maps commute, i.e., $\theta \cdot \psi = \pi$. Moreover, for $\mathcal{U}_i, \mathcal{U}_j$ the transition functions belong to the structure group $\mathcal{G}$ of the fiber bundle, i.e., $\psi_j^{-1} \cdot \psi_i : F \times (\mathcal{U}_i \cap \mathcal{U}_j) \to F \times (\mathcal{U}_i \cap \mathcal{U}_j)$.

Transition from local to global section: The fiber bundle is locally a product space but globally can have a different topology. It gives a passage from local to global in the same sense as in sheaf theory. A section κ of a fiber bundle gives an element of the fiber over every point of X. Its a map $\kappa : X \to E$ such that $\pi \cdot \kappa$ is identity in X, i.e., for every $x \in X$, $\kappa(x)$ is an element of the fiber over x. For example, in a special case of a vector bundle, the zero section is defined when $\kappa(x) = 0$. The value of κ is sought in each fiber of the vector bundle which looks like a horizontal line – cross-section of a fiber bundle. A section of the fiber bundle means to specify the exact fiber state in each fibers of the bundle through local gauge transformation. These local sections can be extended to global sections only if the bundle is trivial. The local sections in non-trivial spaces cannot be globally extended: $LC - GI$.

The fiber bundle structure has also an equivalent description in terms of covering spaces. The covering spaces of X with fiber F are classified by the *action* of fundamental group $\pi(X)$ on F, i.e., $\pi(X) \to Aut(F)$. F consists of a set and $Aut(F)$ is the group of permutations of this set which constitutes E. The fiber bundle has two components: the group of permutations of states of F and its interplay with X. The interplay is the essence of Grothendieck's Galois theory. It gives a way to reconstruct the fundamental group of X as a group of transformations of the universal covering spaces which is same as the automorphisms of the fiber over X.

Definition 15. A fiber bundle is a triple (E, π, X) where $\pi : E \to X$ is a map. The space X is the base space, the space E is the total space, and the map π is called the projection of the bundle. For each $x \in X$, the space $\pi^{-1}(x)$ is called the fiber of the bundle over $x \in X$.

Intuitively, one thinks of a bundle as union of fibers $\pi^{-1}(x)$ for $x \in X$ parameterised by X and glued together by topology of space E.

Definition 16. A cross-section of a bundle (E, π, X) is a map $\kappa : X \to E$ such that $\pi \cdot \kappa = 1_X$.

It means, a cross section is a map $\kappa : X \to E$ such that $\kappa(x) \in \kappa^{-1}(x)$, the fiber over x, for each $x \in X$.

2.8 Discrete Morse Theory

DMT over a finite simplicial complex $\mathcal{K}$, with $\mathcal{L}_d$ the simplices of dimension d. Discrete Morse function f assigns a single real number to each simplex of $\mathcal{K}$. A function $\mathsf{f} : \mathcal{K} \to \mathbb{R}$ be a discrete Morse function, if and only if, for every d dimensional simplex $\sigma^d \in \mathcal{K}$, the following two conditions hold: $|\{\tau^{(d+1)} > \sigma^{(d)} \mid \mathsf{f}(\tau^{(d+1)}) \leq \mathsf{f}(\sigma^{(d)})\}| \leq 1$ and $|\{\tau^{(d-1)} < \sigma^{(d)} \mid \mathsf{f}(\tau^{(d-1)}) \geq \mathsf{f}(\sigma^{(d)})\}| \leq 1$. σ is called a critical simplex if such conditions don't hold. The function decreases from 2-simplex to one simplex and vice-versa in reverse direction. Forman's beautiful papers [25] are the standard references on this subject. There is a relation between critical simplices and collapsibility of simplicial complex. If there is no critical simplices in an interval say $[a, b]$ then it is collapsible to a vertex and the space is simply connected. Moreover, the number of critical points of some index are responsible for the topology of the underlying structure. However, one is unable to determine the efficiency of this discrete Morse function with respect to collapsibility i.e; there may be simplices that are critical that could be collapsed while still preserving the homotopy type. A simple example is that of a sphere. So, it is not only critical simplices that are significant in determining the collapsibility of topological space but also the paths between simplices. For example, if there is only one gradient path between two simplices then reversing the direction of the path cancels criticality of both the simplices. The criticality in $\mathcal{K}$ induces non-triviality in paths up to homotopy. Direction (arrows) over simplicial complex can be defined by introducing the concept of combinatorial vector field. It gives direction to each of the simplicial complex. A combinatorial vector field is a map $\mathcal{C} : \mathcal{L} \to \; \cup \{0\}$. Given such a map $\mathcal{C}$ and $\sigma \in \mathcal{L}$ with $\mathcal{C}(\sigma) \neq 0$ we draw an arrow on $\mathcal{L}$ whose tail begins at σ and which extends into $\mathcal{C}(\sigma)$. It satisfies some properties like if $\mathcal{C}$ implies that σ is always face of $\mathcal{C}(\sigma)$ then an arrow is possible. If τ is the head of an arrow then it cannot be a tail as well. Moreover every simplex is head and tail of, at most, one arrow. This field classifies each simplex in pairs and there are three disjoint possibilities viz., σ is the head of an arrow ($\sigma \in Image(\mathcal{C})$), σ is the tail of an arrow ($\mathcal{C}(\sigma) \neq 0$) and σ is neither head nor tail of any arrow ($\mathcal{C}(\sigma) \neq 0$) and $\sigma \notin Image(\mathcal{C})$ (critical simplices).The non-critical simplices occur in pairs $\sigma^{(d)} \subset \tau^{(d+1)}$ where $\mathsf{f}(\sigma^{(d)}) \geq \mathsf{f}(\tau^{(d+1)})$. It can be illustrated by drawing an *arrow* from $\sigma^{(d)}$ to $\tau^{(d+1)}$. The points that are neither heads nor tails of any arrow are exactly the critical simplices. A combinatorial vector field $\mathcal{C}$ on $\mathcal{K}$ is a collection of pairs $\sigma^{(d)} \subset \tau^{(d+1)}$ of simplices of $\mathcal{K}$ such that each simplex belongs to at most one pair of $\mathcal{C}$. If σ is not critical then there is a unique edge $e > \sigma$ with $f(e) \leq f(\sigma)$ i.e., $\mathcal{C}(\sigma) = e$ with chosen orientation. If σ is critical $\mathcal{C}(\sigma) = 0$. Given a vector field $\mathcal{C}$ on a simplicial complex $\mathcal{K}$, a $\mathcal{C}$-path is a sequence of simplices $\sigma_0^{(d)}, \tau_0^{(d+1)}, \sigma_1^{(d)}, \tau_1^{(d+1)}, \sigma_2^{(d)}, ..., \tau_r^{(d+1)}, \sigma_{r+1}^{(d)}$ such that for each $i = 0, ..., r$, $\{\sigma, \tau\} \in V$ and $\tau_i \; > \; \sigma_{i+1} \neq \sigma_i$. We say such a path is a *non-trivial closed path* if $r \geq 0$ and $\sigma_0 = \sigma_{r+1}$. If $\mathcal{C}$ is a vector field on a simplicial complex $\mathcal{K}$, $\mathcal{C}$ is the gradient vector field of some discrete function on $\mathcal{K}$ if and only if there are no non-trivial closed $\mathcal{C}$-paths i.e., Hasse diagram is acyclic and $\mathcal{K}$ is *collapsible.* If $\mathcal{C}$ has (non-trivial) closed paths, then it cannot be the gradient of a function. It corresponds to the transformation of the fundamental group of topological space.

2.9 Modelisation of Observational Data

The concept of non-locality and contextuality advocates strong correlations in the *observational data* of a *physical* experiment. A physical photon is an agency of observation in a physical experiment to collect the observational data which actively influences the system. Contextuality is property of such empirical data. Observation brings in an element of choice from outside a system. For instance, John Bell collected data from photon polarisation in a physical experiment and put forth a Bell inequality bound on the extend of correlations among data. The violation of the inequality imply strong correlations not explainable by any classical description. There is no operational meaning to observation in Turing machine [1]. Observation is reducible to a

[1]Turing's 1936 seminal paper, 'On Computable numbers, with an Application to the Entscheidungsproblem addresses one of a problem posed by Hilbert. Computation has deep roots in the foundations of mathematics. Turing's academic era was at the cross road of the foundational crises in mathematics (formalism, intuitionism and logicism) and genesis of mathematical framework of quantum physics. The discovery of paradoxes in set theory, for instance Russell's paradox or Richard's paradox, challenged the foundations of mathematics.

Cantor made a distinction between denumerable/countable infinite sets, for instance positive integers, and much larger non-denumerable/uncountably infinite sets, such as real numbers. It means most of the reals would remain inaccessible. If we throw a dart at the real number line, there is measure zero that the dart had landed exactly on the specific chosen real. Borel seeks real number is really real only if it can be expressed and uniquely defined using finite number of words. In Borel view most reals with probability one are mathematical fantasies because of no way to specify them uniquely. Most reals are inaccessible to us and would never be picked as individuals using any conceivable mathematical tools because there tools could always be explained in a language which is countably infinite set. This changes something that's true if and only if it's false, into something that's true if and only if it's unprovable! It is the intrinsic meaning of the word *computable* in Turing's seminal paper of 1936. Cantor's diagonalisation argument provoked controversies and source of paradoxes which was also used by Turing in his paper for halting problem.

The motivation to quantify such paradoxes in computability is intimately related to its connection with the foundations of mathematics and logic. The disproval of Hilbert's hypothesis to base mathematics on computational foundation gave birth to formal definition of the algorithm but with a price to avoid paradoxes. Gödel proves incompleteness of such consistent procedures through unraveling of self reference paradox. In algorithmic sense, Turing avoids paradoxes by drawing a well defined line between countable infinite sets and much larger uncountable infinite sets via Cantor's diagonalisation. The structure of the former is almost null set that lies inside the latter set which is mostly inaccessible. These results led Church to declare what it means in principle to be effectively computable. The scope of this effectiveness limits computable function to lie within computable set in order to avoid paradoxes and to harness its infinite power to practice. Turing avoided paradoxes at the cost of closing the system.

The triviality of infinite tape structure emerges from its bijective correspondence to natural numbers. Each cell of the tape can be assigned a unique name similar to address allocation.

function description in Turing-like interactive models. The data behind these experiment is not a simulated/engineered data as in strictly automated systems. The data is produced from the automated devices without any physical agency of observation, as well as, observation is passively participating in producing simulated data. The complex systems are based on inferences from empirical data which is highly complex. To state few: not ergodicity, their number of agents is ordinarily finite, non identical agents, never representable any analytic, perhaps in certain cases not even by recursive functions, no repeatable experiment gives same results. These issues are posed in the topological field theory of data programme.

Turing-like interactive models cannot explain these experimental data due to linearity of its environment, which recalls local hidden variable. The completeness imply closedness, which allows defining an effective and well-defined function accessing the unknown variables to fetch *definite* values. Topological field theory of data provides a framework to extract hidden emerging correlations among data [2] in a data space, as well as, potential semantics in field theoretic paradigm. The theory could be used to address the high correlations in the observational data and posing relevant question about the possible model of computation to express them.

The content in any cell might be unknown during computation but the known bijective pattern of addresses provides complete information to localise it. The completeness implicitly initiates an effective and well-defined function that updates or accesses the unknown variables with exactness. The process makes no distinction between algorithm and the actual real event represented by it. The result of this indistinguishability and its inherent localisation forces real event to be seen as a simulation. The disguise of triviality avoids paradoxes by expressing real events through formal languages, wiring in any unknowns in its known local structure. It restricts the significant aspect of openness and context dependency.

[2]In general relativity, the given distribution of masses provide the full geometry of spacetime using Einstein's field equations, due to inherent property of particle known as mass; which has a quantifiable value, for instance 9.1×10^{-31} kg. The dynamics of electromagnetic field (even in curved spacetime) can be known using Maxwell's equations, due to inherent property of electron known as charge; whose value is 1.6×10^{-19} coulombs. Both intrinsic properties have a quantifiable real numbers. Data is quantifiable numbers, are these numbers related to some fundamental, possibly intrinsic property? If there exists some property, then given that property one can have a general theory of complex systems which are mostly based on inferences from the empirical data.

2.10 Contextual Semantics for Computation – A Possible New Semantics

The impact of quantum physics is pervasive in cross-disciplinary sciences. For instance, biology to quantum biology, topology to quantum topology, machine learning to quantum machine learning and so on so forth. In particular, quantum physics predominates computer science in terms of speed ups and power of solving a particular set of problems faster than their classical counterpart, but do not increase expressiveness of the computation. It means, problems which are undecidable using classical computers would remain undecidable using quantum computers. In the sense of expressivity, quantum Turing machine is equivalent to the Turing machine.

Perhaps, theoretical computer science community has comparatively resisted direct *quantumisation* of the subject. By direct, we mean bit to qubit, computation to measurement and function description to matrix formulation as seen in several proposals like quantum Turing machine [21] and quantum lambda calculus [46]; but does not increase the scope of expressiveness and effective computability.[3] The generalisation[4] of quantum mechanics is a significant step to invoke mathematical structures of quantum physics in the theory of computing. One of the significant result in 2022 in quantum complexity theory was $MIP^* = RE$ which gives connection between quantum entanglement and the halting problem of Turing machine [32]; encouraging to solve open questions of cross-disciplines as well as, hints about entanglement and computability which could be accessed by foundational and mathematical understanding of entanglement or quantum contextuality.[5]There are only tools formalised in theoretical

[3]The principle of superposition might provide theoretically infinite resources but the measurement yields only one value despite of its resourcefulness. The *measurement problem* is still an important problem in the foundations and philosophy of quantum physics.

[4]The mathematical theory of operator algebra, known as von Neumann algebra plays a central role in algebraic approach to quantum theory. The generalisation of quantum postulates by Segal and Haag's algebraic approach to quantum field theory also also worth mentioning. Categorical quantum mechanics, topos quantum theory and modal interpretation of quantum mechanics are also other generalisations of quantum mechanics.

[5]The categorical quantum mechanics in particular proves to provide significant conceptual flux to computer science. The rigorous mathematical framework of quantum mechanics is insufficiently comprehensive for informatic purposes. The high level description of quantum physics opens new ways in quantum information and computation. It in turn poses new questions to the foundation of quantum mechanics. Non-locality and contextuality in particular

computer science but no semantic revolutions; the fundamental structures of computation are still missing [1], and the mathematical structures emerging from quantum physics could allow deeper investigation and novel semantic revolution in a comparatively engaging way unlike direct quantumisation of theoretical computer science.

Many prominent scientists in the theoretical computer science have been interested in physical genesis of computation in order to bridge the ever increasing dichotomy between the physical and the digital; as well as, providing a way for deeper mathematical structures in computation. The structural understanding of interaction is bedrock that connects computation to physics through an elementary concept of state and transition. The interplay between computation and physics is not new. The significance of involving physics in computation goes back to Carl Petri when in 1982 he states [41]:

> "If this approach [physical and computational ways of thinking] should turn out to be a small, but definite step towards the remote (perhaps illusory) goal of founding technology and natural sciences on a theory of information flow, the author would feel rewarded beyond merit."

Petri asserts that computation is embedded in physical processes over relativistic space-time with an elementary topology. It provides understanding of concurrent computation over a relativistic space-time. It doesn't give any account of non-locality.

During later part of 1990's, Peter Wegner was serious about the implication of quantum physics, Einstein's theory of relativity and the hidden variable theory on computability theory and the models of computation. Wegner introduced interactive realism model that was based on non-local hidden variable theory [45]. It explains the non-determinism of primary observer by the effect of secondary observations through hidden interface. It means non-locality arises due to

have been generalised to classical system using mathematical abstraction. Few significant interpretations are based on graph theory, sheaf and topos theory. The generalisations arises similar situations in apparently different settings. Abramsky's sheaf theory interpretation of local consistency and global inconsistency access contextuality in every other disciplines. The phenomenon of contextuality is pervasive which initiates new semantic of observability: the contextual semantics. A pursuit to find a common mathematical structure in all diverse manifestations to develop a widely applicable theory.

fundamental incompleteness of observer to visualise all of the parameters of the model simultaneously. The concept allows understanding computation as an open evolving system allowing observational metric to quantify the expressivity of machines. The expressivity of such interaction machine is greater than classical Turing machines. Persistent Turing machine is a minimal interactive machine but reducible to Turing machine due to inadequacy in providing the operational meaning to Wegner's observational metric.

Robin Milner pointed significance of need for new semantics to understand causality and observation in his 1991 Turing award lecture as:

> "One important topic is causal independence, which is central to Petri net theory. Now, two processes – each having concurrent components – may be indistinguishable to an external observer, and yet differ in the causal relationship among their observed actions. Observational semantics ignores causality. [...] This topic is too complex to tackle here: I mention it only to show that concurrency does indeed raise new semantic questions"

The ubiquitous computing where computing is made possible anywhere and anytime could pose new challenges because it deals with Big data. Big data is huge data with high dimensionality and complex interrelationships. Milner observes possibility of non-locality in this new paradigm of computing. He says while introducing his bigraph model:

> "It may be argued that to allow arcs to link nodes which are distant cousins, i.e. enclosed within distinct parent nodes arbitrarily far apart in the nesting structure, is contrary to reality. But we wish to model not only the reality, e.g. how communication is implemented, but also the fiction –e.g. *'action-at-a-distance'*– which the world wide web permits us to adopt. By embracing both views in the same model, one can hope to validate complex communication protocols in a mobile environment."

The interplay between physics and computation is significant for the fundamental science of information dynamics in particular quantum physics.

> "After all, in physics there are great theories which transcend mere tool-kits. We largely lack such theories, in Computer Science as a whole, and in concurrency and process calculus in particular. Is this unavoidable, as part of the nature of our subject, or will such theories emerge?[...] There is, perhaps, more prospect for guidance in finding fundamental notions of process, information flow, etc. from the rapidly developing interface between Computer Science and Physics, which has grown up around quantum informatics."

Moreover, Abramsky points out lack of non-local information flow while describing Petri's net model influenced by ideas of physics.

> "We have already discussed how Petri's development of Net theory was influenced by ideas from physics, and indeed provides some of the ingredients of a discrete physics. (One feature conspicuously lacking there is an account of the non-local information flows arising from entangled states, which play a key role in quantum informatics. Locality is so plausible to us — and yet, at a fundamental physical level, apparently so wrong!)"

Computation over a Turing machine is an abstract mathematical notion, independent of the laws of physics even though implicitly assumes the classical Newtonian physical laws. As Deutsch neatly sums it up [22]:

> "Turing hoped that his abstracted-paper-tape model was so simple, so transparent and well defined, that it would not depend on any assumptions about physics that could conceivably be falsified, and therefore that it could become the basis of an abstract theory of computation that was independent of the underlying physics. 'He thought,' as Feynman once put it, 'that he understood paper.' But he was mistaken. Real, quantum- mechanical paper is wildly different from the abstract stuff that the Turing machine uses. The Turing machine is entirely classical"

As also pointed out by Gregory Chaitin [19]:

> "Formal languages avoid the paradoxes by removing the ambiguities of natural languages. The paradoxes are eliminated, but there is a price. Paradoxical natural languages are evolving open systems. Artificial languages are static closed systems subject to limitative meta-theorems. You avoid the paradoxes, but you are left with a corpse!"

Abramsky also pointed out that the fundamental structure of computation is still missing and these generalisations and geometry could play a significant role to understand them [1].

We anticipate that the topological field theory of data could be a potential candidate and a step ahead to lead the *semantic revolution* for expressivity of deeper mathematical structures of computation, as in the case of generalisations of quantum physics, which could allow community of theoretical computer science to further access the quantum physics in a more compatible ways; as well as facilitate further influxes between theoretical computer science and quantum physics. We hope the proposed theory could provide implementation and foundational principles for *differently* [6]-than-Turing paradigm similar to invention of a physical device.

Relevant Related work

The notion of context is pervasive in both computation and quantum physics. The concept can be seen in the models of computation and quantum contextuality respectively. *Computable topology* is an exclusive broad community that studies the topological and algebraic structure of computation. The topological aspect of quantum contextuality on the other side has also been extensively developed [4, 38]. The generalisation of quantum contextuality broadens the scope to understand *structure beneath the collective context* which emerges topological models of computation [23]. The relevant applications of contextual semantics in computer science include relational database [3], natural language semantics [5], robust constraint satisfaction [7] and logic [2]. Recently, a basic

[6]Beyond would mean expressivity of novel computation/patterns using novel mathematical structure unlike strict functional description of computation. The word *differently* was proposed by Giuseppe Longo in series of my interaction with him in his house in Piombino Italy.

qualitative element of contextuality has been incorporated in the PTM [36]. The formal connection between interactive computation and quantum physics is explored on the basis of *interaction as observation* which could enable deeper look in the foundations of computability. The interested readers may also refer to the joint work of Merelli and Rasetti for background of our approach [42].

Chapter 3

Contextual Semantics

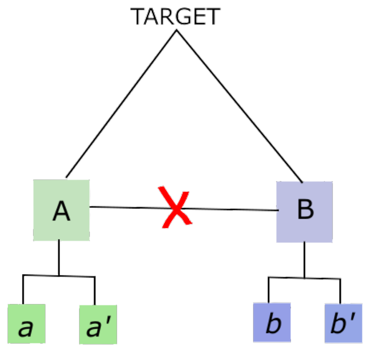
The section introduces contextual semantics and sheaf-based interpretation of the empirical model based on the observational data. The topological aspect of sheaves advocate global context dependency which is introduced as topological model of computation. We start with basic ideas to explore the roadmap for the expressiveness of contextual semantics machinery. The main question to address is, *What could be the underlying model of computation that can express the behaviour of the Bell-like models of quantum physics?* Precisely, *what could be computational generalisation of empirical models?*

3.1 The Setting

We start with basic idea of non-locality and contextuality to explore the roadmap for the expressiveness of contextual semantics machinery.

3.1.1 Contextual Semantics

Given two agents A and B, which are space-like separated over a distributed network; both agents have two local bit registers: A with a and $a^{'}$; B with b and $b^{'}$; each of them can store values either 0 or 1. When the registers contain a value (0 or 1), the agents observe and transmit it to some target. For example, A can *choose* its register $a^{'}$ and *observe* 0 and simultaneously B can *choose* b and *observe* 1, which comprises basic event of the system. The frequency of similar events represent the probability of each event as shown in the table on right side along with its basic set-up on left side of Figure 3.1. Note that each probability p_i is greater than zero.



A,B	(00)	(01)	(10)	(11)
(a,b)	p_1	p_2	p_3	p_4
(a,b')	0	p_5	p_6	p_7
(a',b)	0	p_8	p_9	p_{10}
(a',b')	p_{11}	p_{12}	p_{13}	0

Figure 3.1 On the left is the basic set-up and on the right is the probability distribution table of all events.

The entry in the table means, if A looks at a and B looks at b, then the probability that A sees 0 and B sees 1 is p_5. $\{a, a^{'}, b, b^{'}\}$ is the measurement set, $\{(a, b), (a, b^{'}), (a^{'}, b), (a^{'}, b^{'})\}$ are the measurement contexts, in short contexts, and 0 and 1 are the outputs observed.

Instead of probabilities, one can talk about possibilities of events, i.e. if the table has some probability p_i between any contexts then we assign 1 to the entry, otherwise 0, as shown in table on the left of Figure 3.2. The table can be described as a simplicial complex $\mathcal{K}$ which represent contexts, associated with fibers which represent outcomes; the bundle diagram as shown in the right side of Figure 3.2. Its corresponding planar (linearised) diagram is shown in Figure 3.3. for a better view.

The base of the fiber diagram is a rectangle which consists of the measurement set $\{a, a^{'}, b, b^{'}\}$ and the fibers consist of possible outputs i.e. 0 and 1. To each context of the measurement set, one assigns its two output values (0 and 1) as a fiber associated with it. Based on any context from the table, there is an edge between their corresponding outputs in the fibers, if and only if, it is a possibilistic event with value 1 in the table. For example, if one chooses context $(a^{'}, b^{'})$ at (11), i.e, output of $a^{'}$ is 1 and that of $b^{'}$ is 1 from table in Figure 3.2, then there is no edge between their corresponding fibers because the event is a non-possibilistic event as shown at corresponding entry of the

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0

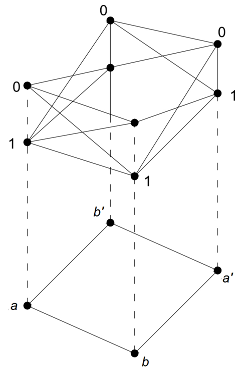


Figure 3.2 On the left is possibilistic table (here, Hardy model as example) and on the right is its corresponding fiber diagram: base consisting of contexts and fibers consists of outputs.

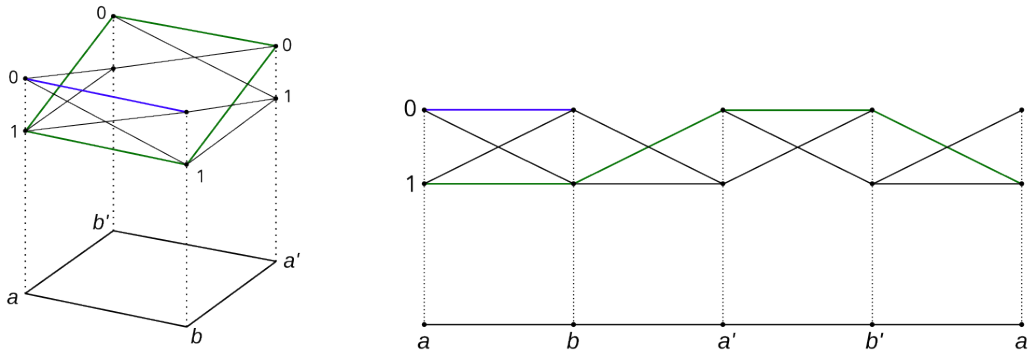


Figure 3.3 Planar diagram of Hardy Model.

table. One closes edges between values in fiber if they can appear together as possible joint outcome over $\mathcal{K}$.

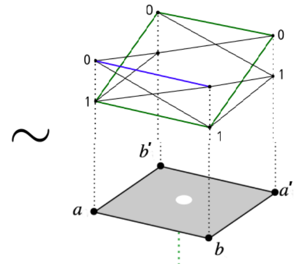
Remark 4. *It is worth noting that the table is not randomly generated. It is based on various physical and mathematical constraints. For example, the sum of probabilities for each context subject to different outputs should be equal to one;* $p_1 + p_2 + p_3 + p_4 = 1$. *So, the table cannot reproduce* any *behaviour, e.g., in which all entries are zero. Moreover, if we change one of the entries of the table it affects the table as a whole. The physical constraint in the background includes relativistic constraint and choice of agents to select the measurement setting which correspond to no-signalling and lambda independence respectively* *in sheaf-based approach. The table is the synthesis of background observational data from a physical Bell-type experiment.*

There is one context (a, b) in the table which is locally possible at (00) but globally not possible, i.e. when we extend it through different contexts over the fiber space denoted in blue in Figure 3.3, it doesn't form a closed loop because the contexts over its base space $\mathcal{K}$ are not simultaneously satisfiable which is expressed as non-collapsibility of $\mathcal{K}$. The table expresses possibilistic behaviour, non-possibilistic behaviours and the $LC - GI$ behaviour.

Local consistency means if an outcome from A point of view is possible in a given context then it should stay possible even if B were to change which measurement it makes, i.e., free choice of measurement. It means there is always a way to continue from any edge over the base of bundle diagram. For example, if A chooses a then whatever B chooses, whether b or $b^{'}$, there is an edge between both possibilities. Global consistency means one can form a complete closed path that assigns a unique value to each context. For example, the green line in Figure 3.3 forms a loop. The loop means all contexts are simultaneously satisfiable which is essence of quantum contextuality. It means the simplicial complex representing the contexts is acyclic i.e., collapsible.

The behaviour of the table cannot be explained by a classical source. Linear tape structure of the Turing machine resembles a local hidden variables in the background. It turns out that no such variables can express contextual behaviour due to the Bell theorem. It means Turing machine cannot emulate the behaviour due to linear tape structure of the environment. *What could be the underlying model of computation that can express the behaviour of the Bell-like models of quantum physics?* Precisely, *what could be computational generalisation of empirical models?* A possible explanation for $LC - GI$ is a topological representation of the linear environment of Turing machine to include global context dependency during computation. The linear base space of bundle diagram is replaced by a topological representation of environment as shown in Figure 3.4 which is formalised throughout the paper as TIM computational paradigm, isomorphic to the structure of fiber bundle.

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0



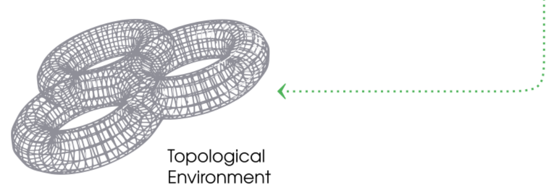


Figure 3.4 Topological explanation of the set-up for TIM computational paradigm.

3.1.2 The Empirical Model

The particular Bell-type set-up can be generalised using sheaf theory. We give a rigour free conceptual notion of this generalisation and its interplay with the empirical model.

Basic idea of sheaf theory: Sheaves comprise of two components that operate together: the horizontal topological component over which its vertical algebraic type resides. A basic example is sheaf *of* vector space *on* abstract simplicial complex. The individual sheaf is called a stalk. The idea is to move between vectors in each vectors space (stalk) via linear maps in the vertical component together moving from higher simplices to 0-simplex on its horizontal topological component.

Suppose X is a topological space with open set $\mathcal{U}$ *covering* X. Let $\mathcal{U}_i$ be open subsets of $\mathcal{U}$, where i denotes the number of open subsets of $\mathcal{U}$ in order to cover it. Sheaf theory operates on a function r defined on $\mathcal{U}$ of X and restricts to functions $\restriction r\mathcal{U}_i$ defined on $\mathcal{U}_i \subset \mathcal{U}$. $\mathcal{U}$ is recovered by gluing together the restrictions to $\mathcal{U}_i$. The *restriction-gluing* is main theme of sheaf theory. A (pre)sheaf $\mathcal{O}$ on X can be described as a rule/function which assigns to each point, a set O consisting of the elements of the functions at that point defined by its neighbourhood, i.e., $\mathcal{O}(\mathcal{U}) := O^{\mathcal{U}}$. The set O turns out to be output set in sheaf generalisation of empirical model. The sets $O^{\mathcal{U}_i}$ for all $\mathcal{U}_i$ can then be glued together by a suitable topology so as to form bundle projected onto

X. The individual *nice* function for this sheaf is then a cross-section of the projection of this bundle. The combinatorial representation of X is a finite simplicial complex $\mathcal{K}$. When X is discrete, the presheaf $\mathcal{O}$ is trivially a sheaf. In fact, every presheaf can be uniquely realised as a sheaf. A section of a sheaf is an element $\kappa \in \prod_{\sigma,\tau \in \mathcal{K}} \mathcal{O}(\sigma)$ such that $\mathcal{O}(\sigma \to \tau)\kappa(\sigma) = \kappa(\tau)\ \forall\ \sigma \subseteq \tau$, i.e., σ is the face of τ and $\kappa(\sigma)$, $\kappa(\tau)$ are elements of O. The map $\mathcal{O}(\sigma \to \tau)$ is a linear map that transforms matrix $\kappa(\sigma)$ to $\kappa(\tau)$ over vertical component of sheaf given its corresponding collapsible or acyclic $\mathcal{K}$, i.e., the cochain complex is exact.

The *cohomological obstructions* which arise in sheaf theory is the main mathematical concept to express phenomena of non-locality and contextuality. The obstructions arise due to non-simply connected regions of the topological space which has non-zero or non-vanishing cohomology groups. There could locally exist a section κ via linear map but globally non-extendible due to non-simply connected X.

Generalisation of empirical models through sheaf structure: Quantum contextuality restricts the simultaneous assignment of values to all measurements. This feature is captured by introducing the measurement cover $\mathcal{U}$ which covers the space X. The subsets of $\mathcal{U}$ denoted by $\mathcal{U}_i$ represents the set of compatible families of measurement contexts, i.e., those which can be measured simultaneously with, $\bigcup_{\mathcal{U}_i \in \mathcal{U}} \mathcal{U}_i = X$. The combinatorial description of X is represented to be a simplicial complex $\mathcal{K}$ which is homeomorphic to X. It comprises of finite measurement set representing all the available measurements in a general experiment. Each measurement $\mathcal{U} \in X$ produces an outcome in a set $O_{\mathcal{U}}$. The measurement set, the measurement contexts and the outcome set constitute the measurement scenario. X is equipped with a discrete topology and defines sheaf of events which relates open subsets $\mathcal{U}$ of X to the space of sets O.

A *section over* $\mathcal{U}$ is a function $\kappa : \mathcal{U} \to O$ that describes the event in which the measurements in $\mathcal{U}$ were performed, and the outcome $\kappa(m)$ was *observed* for each $m \in \mathcal{U}$. The function κ is generalised mathematically in a *sheaf structure* that assigns set of sections over O to each *measurement context* $\mathcal{U}$. It is represented topologically as a bundle diagram as shown in Figure 3.3. There exists a unique section for a family of subset in $\mathcal{U}$ if its corresponding family of sections is *compatible* and has simultaneous existence i.e., $\restriction \kappa_i \mathcal{U}_i \cap \mathcal{U}_j$

$= \restriction \kappa_j \mathcal{U}_i \cap \mathcal{U}_j$. For brevity, the non-collapsibility of the simplicial complex constrains the extendability of contexts to global section due to cohomological obstructions [6]

Measurement scenarios and the sheaf of events define the experiment setting and are therefore independent of any physical system. The application of this sheaf scenario on an actual physical system is captured by the notion of empirical model. The empirical model **e** in this sense is a compatible family $\{\mathrm{p}_{\mathcal{U}_i}\}_{\mathcal{U}_i \in \mathcal{U}}$, where $\mathrm{p}_{\mathcal{U}_i}$ is a probability distribution on $\mathcal{O}(\mathcal{U})$. The compatibility of the family is independent of the choice of $\mathcal{U}$. The model **e** is possibilistic expandable if and only if every section is a member of compatible family. It is possibilistically non-extendable if for some section κ (cross-section), there is no such family. The model **e** is strongly contextual if for every κ there is no such family. A simple understanding of this mathematical setting is comprehensively expressed in the tabular representation of the empirical models, for instance, the Hardy table shown in Figure 6.1. The sheaf theoretic generalises the strong forms of non-classical behaviour exhibited by quantum physics.

3.2 Topological model of computation

The computational synthesis of quantum contextuality is that computation depends on the global contexts. Topological model of computation involves global contexts as discrete topological space. We provide an idea of constructing a topological space from the overall executions of the processes. Consider a process specification ($\mathrm{P_a}$; $\mathrm{P_b}$; $\mathrm{V_b}$; $\mathrm{V_a}$) || ($\mathrm{P_b}$; $\mathrm{P_a}$; $\mathrm{V_a}$; $\mathrm{V_b}$), where $\mathrm{P_a}$, $\mathrm{P_b}$ or $\mathrm{V_a}$, $\mathrm{V_b}$ means locking or releasing a resource a and b respectively. The topological space associated with the process execution, known as *swiss flag*, is shown in Figure 3.5.

The diagram in Figure 3.5 is the discrete space of all possible executions of the given processes. It consists of the computation $\mathcal{V}_1 \cdot \mathrm{P_a} \rightarrow \mathcal{V}_2$, the commuting contexts executing concurrently like $\{\mathcal{V}_1\mathcal{V}_2\mathcal{V}_6\mathcal{V}_7\}$ and the global contexts that forms the overall structure *woven* through all contexts. All executions are infeasible due to unavailability of resources except $\{\mathcal{V}_1\mathcal{V}_2\mathcal{V}_6\mathcal{V}_7\}$, $\{\mathcal{V}_4\mathcal{V}_5\mathcal{V}_{10}\mathcal{V}_9\}$, $\{\mathcal{V}_{21}\mathcal{V}_{22}\mathcal{V}_{17}\mathcal{V}_{16}\}$ and $\{\mathcal{V}_{19}\mathcal{V}_{20}\mathcal{V}_{25}\mathcal{V}_{24}\}$. The infeasible and feasible computation correspond to possibilistic and non-possibilistic behaviour of

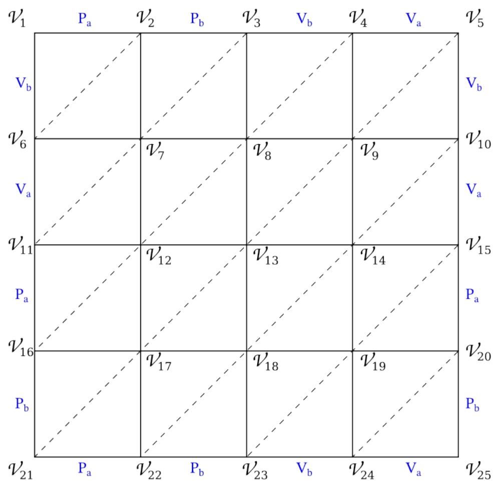


Figure 3.5 Topological space associated with the process execution.

empirical models. The relation between feasibility of the speciifed simplices and the process execution can be understood by defining a resource potential r to have maximum capacity of say 1; so, $r(V_1)(a) = r(V_1)b = 1$ in the beginning. Since, the transition $V_1 \to V_2$ is labelled P_a, and the action of P_a is to decrement the number of resources of a, so $r(V_2)a = 0$ and $r(V_2)b = 1$. Following the parallel execution in this way allows these four squares to have r=1 given the order of executions. Further, say at V_4, the resource b is released and hence r=1, i.e., $r(V_4)b = 1$, so makes a transition to V_9. [1] The further details of this example could be seen in [23].

The local partial orders among processes *always* admits a consistent global order; it could be further access [24] to understand the always global consistency of the model and hence unable to express local consistency-global inconsistency $LC - GI$. The model cannot explain LC-GI behaviour of the empirical model because of its causality condition. The causality in the model is expressed as a loop free (acyclic) discrete space (locally partial ordered space) in which every feasible local process *always* admits to a global structure.

The two key observations about the above discrete topological space are worth nothing here:

[1] For further details, chapter 3 of the book [23] is recommended. For a brief understanding, reader can access page 1-5 of this book.

1. Firstly, the discrete space is not constructed from observational data output from a physical experiment. In the background of the empirical model is data of observations which is output from Bell-type experiments using photons as an operational agency of observation. The space is based on a priori defined program not data unlike empirical model of quantum physics — *non-empirical*.

2. Secondly, the discrete space associated with the program expresses all of its possible executions. The possibility to know well defined a priori executions of a process is because the structure of space doesn't change — *static*.

We address the first point by constructing a discrete topological space from PTM computation using topological data analysis. It is not based on a priori execution of program but data. The observational data output from the Bell-type experiments is generalised and condensed in tables of empirical models. The construction of discrete space in our work is based on the data from the tables which is implicitly empirical unlike simulated/engineered data.

The second point is addressed by allowing computation to change the topology of the constructed discrete space which in turn constrains the computation in a feedback loop. It allows non-trivial loops to emerge in a causal way and facilitates the definition of an open environment responsible for $LC - GI$.

3.3 The Roadmap

The first step is to formalise an interactive machine that can express contextual behaviour in Section 4.1. The second step is to show that this machine can emulate empirical model in Section 5.1. The third step is the expressiveness of contextual semantics machinery in Section 5.1. The fourth step is to emulate examples of empirical models of quantum physics over the machine in Section 6.1. Models including Hardy model in Subsection 6.2.2, Kochen Specker model in Subsection 6.2.3, Mermin-Peres Magic Square in Subsection 6.2.6, Popescu-Rohrlich Boxes in Subsection 6.2.5 and Greenberger-Horne-Zeilinger model in Subsection 6.2.4.

Chapter 4

Topological Interactive Machine

The chapter introduces the topological interactive machine and its structural isomorphism to the fiber bundle.

4.1 Topological Interactive Machine

Persistent Turing machine is a minimal extension of Turing machine which captures the idea of interactive computation. It is a multi-tape Turing machine with a persistent work tape whose content is preserved between successive Turing machine computations. The *work tape* is non-observable but affects the PTM sequential computation. PTM's environment is a mapping from PTM computation to its corresponding feasible set of equivalence classes which is always feasible due to linearity. It doesn't take into account the observational postulate/metric of global contexts during its computation. As a result, PTM cannot express contextual behaviour even though conceptual non-observability of environment provides a qualitative description of contextuality.

The work-tape of PTM should be given a topological meaning in order to quantify the classical postulate of quantum contextuality. The discrete space results from empirical data unlike program. It means the space cannot be constructed a priori due to its non-static structure.

Empirical data: For brevity, consider each function q has a probabilistic measure to transform input i to output o based on the context. The total PTM computation (behaviour) $q : i \to o$ produces empirical data in a given time t interval. Each $q_t : i \to o$ outputs *data* that is represented as an element in an arbitrary set Z. This *set* Z can be analysed by persistent homology, a procedure used in topological data analysis, to construct a topological space (environment) from PTM computations [17]. The *data* in our paper is observational data of a physical experiment which is condensed in a tabular form of empirical models of quantum physics.

Dynamic (non-static): In order to allow computation to change the topology of contexts and vice-versa, we need to explicitly *extract out* the computation from the structure. The abstraction is carried out by associating a *state space* (consisting of states) to each node of the discrete topological space, here simplicial complex. It gives an element of (free) choice to each node to make any possible transition between state spaces irrespective of $\mathcal{K}$. A transition between states in the state spaces is only permissible when their underlying simplices allow it. The computable function is generalised to partial function which *discovers* the transition in the presence of contexts. The computation is further allowed to affect the simplices of the simplicial complex which in turn changes its topology. The partial function f that transforms input i to output o at given time t, i.e., $f_t : i \to o$ is characterised by its corresponding set of path over the topological environment.

The PTM is isomorphic to a general class of effective transition systems called interactive transition systems. It means, PTM is abstraction of a process associated with a topological environment unlike work-tape. The input-output tape with states is a process (state space) entangled to each node of the discrete topological environment.

The topological environment realised as simplicial complex encodes global contexts that represent structure constructed from PTM computation which represent function. Both function and structure are described as a unique algebraic object which has a mathematical description of a fiber bundle.

Definition 17. (**Topological Environment** [36]) The topological environment $\mathcal{E}$ is the simplicial complex $\mathcal{K}$ which is constructed from the set Z consisting of the PTM computations between states in the state space $\mathbf{S}_t$ available at a given time t using topological data analysis.

Let $\mathcal{E} \sim X$ be a topological environment with $\mathcal{U}_i$ and $\mathcal{U}_j$ as its open sets which consist of all the global contexts. $\mathcal{E}$ can be combinatorially described as a simplicial complex $\mathcal{K}$. When $\mathcal{U}_i \cap \mathcal{U}_j$ agrees at the overlap there is an edge between vertices of $\mathcal{K}$. $\mathcal{K}$ contains a subset of simplices known as critical simplices CS where $\mathcal{U}_i \cap \mathcal{U}_j$ of $\mathcal{E}$ doesn't agree. Let $\mathcal{V}$ be a flexible state variable which has a dual characterisation. First, it possess a set of values (of different state of behaviour) represented as a state space $\mathbf{S}$ comprising of set of states. Second, it is an embedding in $\mathcal{E} \sim \mathcal{K}$ such that each vertex of $\mathcal{K}$ is an element of $\mathcal{V}$. The dual use of $\mathcal{V}$ associates local computation with the global topological environment. The state space $\mathbf{S}$ possess a set of states s associated to each $\mathcal{V}$ of $\mathcal{K}$. The map π assigns $\mathbf{S}$ to each $\mathcal{V}$, like, each $\mathbf{S}_i$, where i is any state space in $\mathbf{S}$, to each $\mathcal{V}_i$, where i is any vertex of $\mathcal{K}$. $\mathbf{S}$ can have different number of states at each $\mathcal{V}$. The inverse of π map of any state $s_i \in s$ contained in $\mathbf{S}_i$ is a particular vertex of $\mathcal{K}$, i.e., $\pi^{-}(s_i) = \mathcal{V}_i$. The embedding Emb of π^{-1} of any state $s \in \mathbf{S}$ i.e., $Emb\ \{\pi^{-}(s_i)\}$ is the set of all simplices of $\mathcal{K}$ having projection of $\pi^{-}(s)$ (which is a vertex) as its element. An action $a_i \in A$ is admissible between any permutation of state s_i and s_j of $\mathbf{S}_i$ and $\mathbf{S}_j$ respectively if the condition $Emb\ \{\pi^{-}(s_i)\} \cap CS = \emptyset$ is satisfied. Action a_i happens in a given subset of simplices of $\mathcal{K}$ which represents the state of the environment. For simplicity, set a bound on the cardinality of state space to some natural number $\mathbb{N}$. A computational step $s_i \xrightarrow{a_i} s_j$ would reduce cardinality of state space of $\mathbf{S}_i$ containing s_i to one and $\mathbf{S}_j$ containing s_j gains one, similar to bio-inspired models of computation. If the cardinality $|\mathbf{S}_i|$ is less than a chosen bound then it turns the generic simplices to CS. The other map δ observes the effect of A on $\mathcal{K}$ and iterates the environment based on overall computations during that computational step. The iterated $\mathcal{K}$ is the new environment for the next computation.

Concisely, the computational process goes as: Suppose t is time and $\mathbb{N}$ is a natural number. At $t = start$, $|\mathbf{S}| > \mathbb{N}$ and $|CS| = \emptyset$. At $t = t+1$, $s_i \xrightarrow{a_i} s_j$ for action $a_i \in A$. Check, if $Emb\ \{\pi^{-}(s_i)\} \cap CS = \emptyset$ then A is allowed, else not allowed. For a transition $s_i \rightarrow s_j$; delete s_i from $\mathbf{S_i}$ and add s_i to $\mathbf{S_j}$. At $t = t+1$, if $|\mathbf{S}| < \mathbb{N}$ in $\mathbf{S_i}$ then turn simplex to CS. Iterate the simplicial complex using (generalised) DMT.

First, we would like to give an example of fiber bundle to understand its structure; and give an example based on the machine to clarify its working and definition; as well as to see its isomorphism with fiber bundle structure.

4.2 Fiber Bundle Example

Let's take a simple example of a cylinder in order to understand fiber bundles. The definition has been given in Chapter 2. For the examples we use simply written alphabets to denote any mathematical structure, like V denotes vertices in examples and S denotes state space instead of $\mathcal{V}$ and $\mathbf{S}$ to express that we are dealing with an example not generalisation.

We choose one particular circle S_1 denoted in red in the middle of this hollow cylinder as in Figure 4.1. With S_1 in place, every point on the cylinder, like point p, can be given a coordinate system. One can imagine a coordinate system by associating point p with point on the S_1 and height above it. If we take height at S_1 as zero, the height above S_1 as positive and below as negative. So, p has two coordinates (a, b), $a \in S_1$ and $b \in \mathbb{R}$. The ordered pair on the surface of the cylinder identifies every point on it. The cylinder can be broken down into three separate spaces ; the whole cylinder as the total space E, S_1 as the base space and $\mathbb{R}$ as fibers. The idea is that the total space can be constructed from two different spaces; base space and fiber space; i,e., $E = S_1 \times \mathbb{R}$. The ordered pair clearly places every point in the total space of the cylinder as a point that has a specific location on the base space S_1 and specific location in the fiber.

Notice that every point on the cylinder is an ordered pair of S_1 and $\mathbb{R}$. A particular fiber $\mathbb{R}_a$ associated with a is really not the same as other fiber $\mathbb{R}_b$ associated with other arbitrary point, say q; but both are isomorphic to generic $\mathbb{R}$. [1] Each fiber is associated with each point in the total space via projection map π, i.e., $\pi : E \to B$ with $\pi(a, b) = a$ and $\pi^{-1}(a) = \mathbb{R}_a$. Take another point q, with coordinates (c, d), so, $\pi(c, d) = c$ and $\pi^{-1}(c) = \mathbb{R}_c$. Both $\mathbb{R}_a$ and $\mathbb{R}_c$ are different sets, at the same time diffeomorphic to the generic fiber $\mathbb{R}$, the local

[1] There could be a kind of confusion when we talk about fiber; we can talk about a fiber at a point or standard fiber for which all the fibers are identical copies, i.e., diffeomorphic.

trivialisation condition; there should always be a way to be able to divide the total space locally, i.e., the image of E has to be diffeomorphic to $S_i \times \mathbb{R}$.

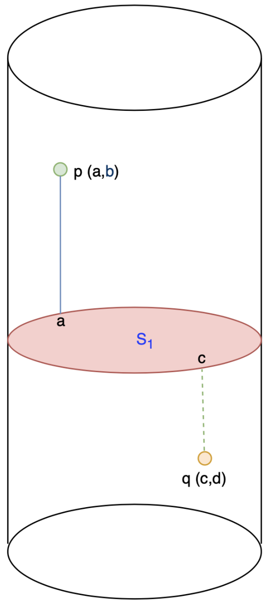


Figure 4.1 An Example

The construction of the total space of a cylinder is the cartesian product of the base space and the fibers, i.e., it is has a product topology because is a trivial space; It is locally a product space and globally a product space; but in general situations the total space could be complex. Imagine the imaginary lines perpendicular to S_1 has a infinitesimal thickness, with infinitesimal thickness of $a \in S_1$ and length of $\mathbb{R}_a$, i.e., $S_1 \times \mathbb{R}_a$ locally and globally gluing together these seemingly infinite strips, unions and intersections yield the global topology simply as global cartesian product. This is *local consistency and global consistency* because of linear global structure.

For a general situation, any given subset U_i of B that is part of its cover, there is a pre-image in E, as shown in the Figure 4.2. The image of E is diffeomorphic to $U_i \times F$ due to local trivialisation condition. We skip the technical details here for the sake of brevity. Any local (open) region of E is diffeomorphic to product of open sets in B and fiber F via π and ψ respectively. There will be other open set, say U_j in B with its pre-image in E and corresponding diffeormorphism but under different mapping, here σ. The two different diffeomorphisms, i.e., $U_i \times F$ and $U_j \times F$ don't have to necessarily agree over the overlap at $U_i \cap U_j$, as shown in Figure 4.2, as a dotted red line. A point in the intersection under one map could land in a completely different place in other map. We define a map from $F \to F$ that converts the two different local trivialisations into one another; resolving intersections; a map from F to itself, converting F that is generated via ψ mapping to be able to translate to σ mapping, i.e., $\psi \cdot \sigma^{-1}$. Formally, it is known as structure group of the fiber bundle, for cylinder its identity. This is local consistency and global inconsistency; there is a local trivialisation but globally no action between two fibers because of non-linearity in B. It is always possible to describe any region of the E as the cartesian product of B and the standard fiber F region; but the mapping between the different cartesian products with mutual non-empty intersection is not always possible.

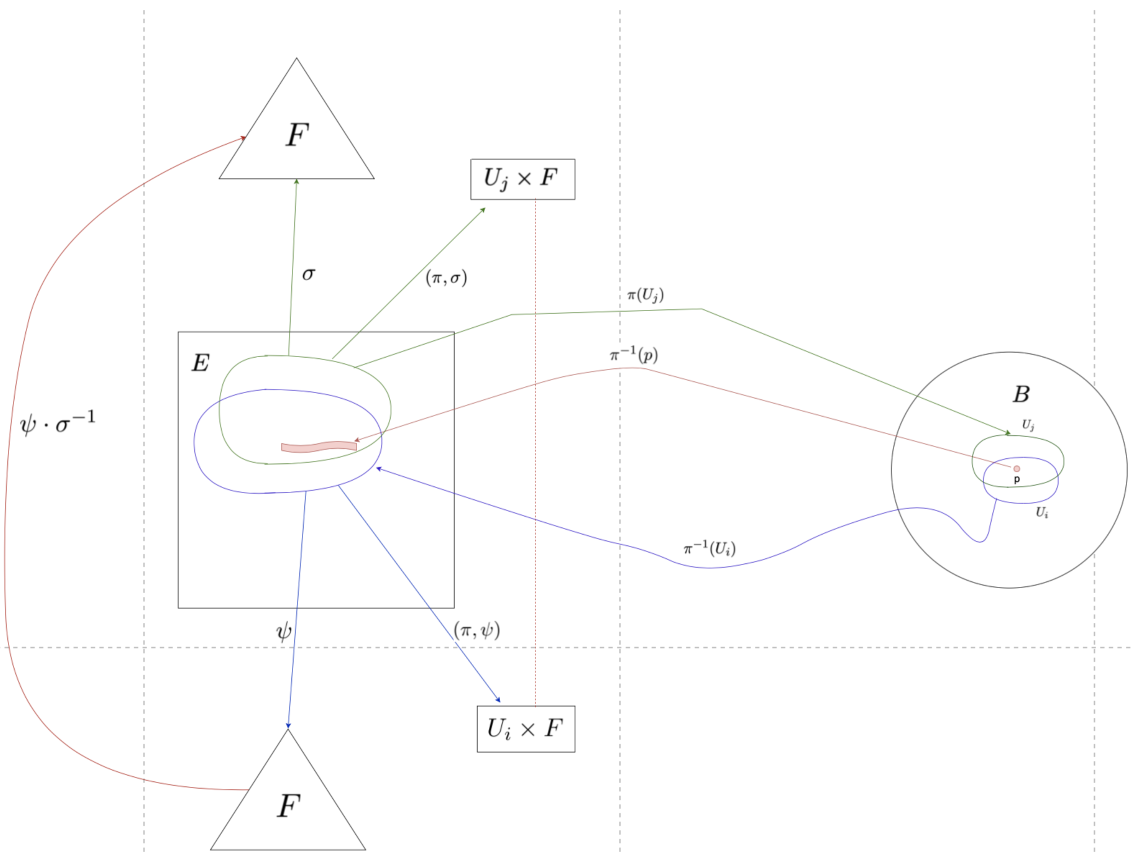


Figure 4.2 Fiber Bundle

4.3 Illustrative Working of Topological Interactive Machine

Now, Let's take a simple example to understand the working of the machine.

Given a simplicial complex K with four 0-simplices $\{V_1, V_2, V_3, V_4\}$, five 1-simplices $\{V_{12}, V_{14}, V_{23}, V_{13}, V_{34}\}$ and two 2-simplices $\{V_{123}, V_{134}\}$, as shown in the Figure 4.3. Here, V_{12} means an edge between V_1 and V_2 and similarly for other notations. Notice, there is a colour coding [2] of each simplices $K_i \in K$: green and red, which is based on discrete Morse theory; but in order to abstract the complexity, lets assign weight to each simplices K_i.

$$w(K_i) = \begin{cases} 1, & \text{if } K_i \text{ is generic simplices: Green} \\ 0, & \text{if } K_i \text{ is critical: Red} \end{cases}$$

Each vertex of the K is a hyper-space, i.e., it has some additional information (as in bigraph in computation or fat-graphs in gauge field theory). The additional information [3] is represented as a fiber S_i attached to each vertex V_i via projection map π_i, i.e., $\pi_i : S_i \to V_i$. In the given example 4.3, we have four fibers (state spaces) $\{S_1, S_2, S_3, S_4\}$, with $|S_1| = 2$, i.e., S_1 consists of two states s_1 and s_2. Similarly, $|S_2| = 3$,$|S_3| = 1$ and $|S_4| = 1$. We also set minimum bound to each S to be able to compute, here $|S| > 1$, i.e., every S_i should contain at least two states to take part in computation. For example, $|S_3| = |S_4| = 1$ means it cannot take part in computation by setting its corresponding simplices V_3, V_4 and V_{34} red colour. In our example, we have two checks; first is to observe $w(K_i)$ for action A between two state spaces and second is $|S| > \mathbb{N}$

[2] K is iteratively constructed from the given information of the empirical model; expressing the class of behaviours, i.e., possibilistic, non-possibilistic and locally consistent-globally inconsistent events which corresponds to feasible, infeasible and contextual computation/behaviour respectively. The non-possibilistic event is represented by the red simplex, the possibilistic event via black and the contextual event by blue using discrete Morse theory. The point is that the weight 0 or 1 (red or green) is based on the model and in the example we don't bother ourselves about how to colour code them.

[3] For example, the additional information could be output space (set) of each agent or intentionality of each agent in a global environment. It gives element of choice to the agents to select any element from its set, otherwise if for example the global environment actually determines the choices, then it becomes a self-fulfilling prophecy; and any behaviour can be trivially reproduced.

changing $w(K_i)$. It means, computation between two state spaces is constrained by K and vice-versa.

We can write π instead of π_i and distinguish it with the associated set of (V_i, S_i), i.e., one can say the map π at the particular V_i and S_i; but here we prefer subscript with π to make it more clear.

- V is a set of vertices of a simplicial complex K, and each S_i is a finite set which is a fiber associated with each $V_i \in V$ via projection map π; but π is not a bundle map from union of all S_i to the set of vertices V. π further identifies the set of simplices (open neighbourhood) associated with its particular fiber/vertex via embedding function (mentioned below) to decide whether the set contains any red simplex; and prepare for next step which is action between fibers, i.e., if there exists a red simplices in embedding of π, then the corresponding action would be infeasible, otherwise feasible. It could be equivalent to a bundle map in trivial fiber bundles; where the space is simply connected, i.e., its associated simplicial complex is collapsible. The bundle map in fiber bundle is of the type (π, ψ) or (π, σ), as shown in the Figure 4.2; with π and ψ having similar meaning as π and A.

Now, let us further explain the example. The inverse map of π of any state $s_i \in S_i$ is a particular vertex $V_i \in V$, i,e., $\pi^{-1}(s_i) = V_i$. The embedding Emb of any inverse projection of π corresponding to a particular $s \in S$, i.e. $Emb\{\pi^{-1}(s_i)\}$ is the set of all K_i of K having projection of $\pi^{-1}(s)$ as its element.

Here, lets choose S_1, first we would check $|S_1| = 2$, which satisfies our given bound on S ($|S| > 1$). Moreover, $\pi^{-1}(s_1) = V_1$, which means $Emb(V_1) = \{V_1, V_{12}, V_{13}, V_{14}\}$. Now, check the $w(V_i)$, here, weight of all simplices is 1, i.e., $Emb(V_i) \cap Red = \emptyset$. It means, state s_1 can make transition to any of the states of S_2, i.e., there is an admissible action, say $a_{15} \in A$ between $s_1 \xrightarrow{a_{15}} s_5$.

- The action A can be seen in both equivalent ways; as a set of relations or set of pair of states. Since A depends on $w(K_i)$ and $|S|$, so we prefer to focus on the pair of states.

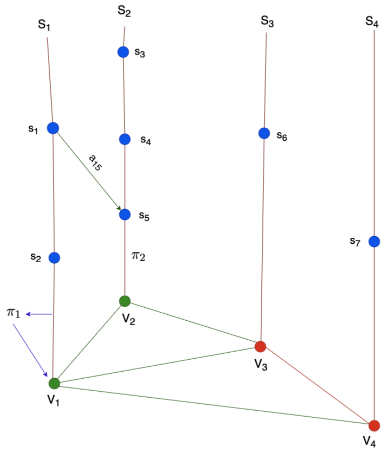


Figure 4.3 A Simple Example.

After action A, here a_{15} between s_1 and s_5, both conditions are re-checked; i.e., $|S|$ and $w(K_i)$. Here, $|S_1| = 1$, $|S_2| = 2$ and hence, $w(V_1)$ turns red which means in next iteration it cannot take part in the computation. This is the role of δ map.

- The map δ is a map from $V \times S \to V \times S$; which takes vertex in K and its respective fiber S to another vertex and fiber as shown in the Figure 4.5; $\delta_1 : V_1 \times S_1 \to V_1^{'} \times S_1^{'}$ and $\delta_2 : V_2 \times S_2 \to V_2^{'} \times S_2^{'}$. δ map updates each fiber and its corresponding simplex based on global consistency condition. [4] The updating process of δ and its comparison to fiber bundle can further be referred to figure 4.6, such that the squares in figure 4.7 commutes. It is like a simplicial map preserving fibers, taking a region in the total space and glues it together globally. [5]

The square in left hand side of the figure 4.7 is the pre-conditions (a local consistency) of our example 4.3; which is to use projection maps and observing action. Fix a point in the square, say V_j, one can project the image of S_i via π_i reaching V_i and then reaching V_j via V_{ij} (edge is green); or S_i makes an

[4]In fiber bundle, it is always possible that the total space is locally homeomorphic to a cartesian product but globally not always possible equipping it with non-trivial topology.

[5]The point in the fiber space S has dual characterisation; as a fiber as well as a point in the total space. we masquerade total space here in its explicit manifestation because it is base space modulo fiber space and constructed likewise.

action A_{ij} which outputs S_j, which can further project its image via π_j finally reaching V_j, i.e., $V_{ij} \cdot \pi_i = \pi_j \cdot A_{ij}$, once conditions are satisfied.

The square on right commutes for updating step, checking the post-condition. With slight abuse in notation, (π_i, A_i) sends $V_i \times S_i$ to S_i via A_i, as well as projects the image V_i via π_i. Similarly, $V_j \times S_j$ to S_j via A_j, as well as projects the image V_j via π_j; via (π_j, A_j). Notice that there are two ways to visualise a point in the state space (just as in fiber bundles); first as a set whose elements consist of states which are projected to each vertex using π (state/fiber space); second the same point has a dual characterisation as being part of the total space (which we intend to construct from the base space modulo the fiber space) as discussed in the fiber bundle example 4.6. So, the cross product between S_i's and V_i's is in the total space. $V_i \times S_i$ and $V_j \times S_j$ are formed from different local trivialisation maps and is locally always possible to slice the total space in these thin lines (like imagine as thin perpendicular lines of specific thickness in case of cylinder); but these different slices can be merged in the total space if there is a map δ between them. It means, the total space can be glued together as the base space agrees on the intersection.

There is one more thing to account here; in our case the model is a priori given (even though it is not mandatory for our machine in principle; but would require further extensions to include situations where phase space is not given; for such case, like in natural computing and biological modelling, the machine would be semi-dynamical.), so we know locally which computation is feasible, observed as an action A_{ij}; but there is an element of choice to all agents (as hyper-vertices) to take part in any computation; as well as any change in the environment at arbitrary far apart contexts,(say, any simplex turns red), then it could change the homotopy type of the space (We do not formalise the openness in the thesis but introduce the concept which allows non-trivial loops in our case to provide a causal explaination for $LC - GI$) Intuitively, let's say S_i observes its neighbourhood as embedding and decides to make an action after satisfying the conditions; but at the same time S_j (where S_i chose to make an action), in turn decided to make transition to other state space and eventually turned $|S_j|$ less than the required number $\mathbb{N}$. So, locally for S_i satisfied the conditions but globally it is not possible due to concurrent choice of S_j. Since

$V_i \times S_i$ is an ordered pair in the total space; as well as implicitly if you consider a standard fiber S (as in fiber bundles), the action A_{ij} is $A_i : S_i \to S$ and similarly from other side $A_j : S_j \to S$; which means $\delta = A_j \cdot A_i^{-1}$ converting two different local trivialisation into one another, and further validating the action and the intersection globally. After, the conditions are checked, i.e., $|S|$ and $w(K)$ which updates the vertices and their corresponding fibers for the next iteration. Such iterations form a 3-dimensional cubical structure representing the ongoing computation as iteration of these cubes as in Figure 4.5. Imagine a dotted line between $V_i \times S_i$ and S_i, $V_j \times S_j$ and S_j, V_i and V_i and V_j and V_j. The left cube would check conditions as well as iterate all the fiber bundle mappings and its commutation as in the Diagram 4.6, In this way, updating each simplex and fiber as shown in a different way in Figure 4.5.

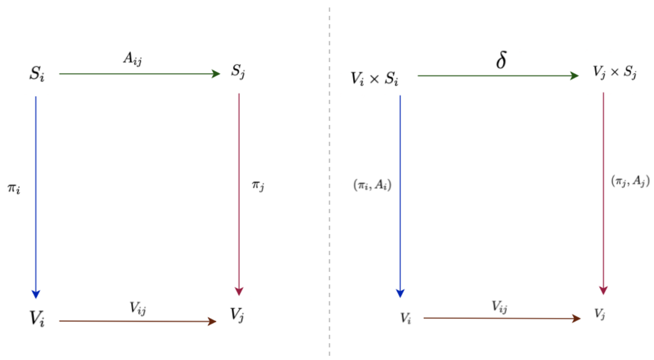


Figure 4.4 Pre-condition on left and Post-condition on right

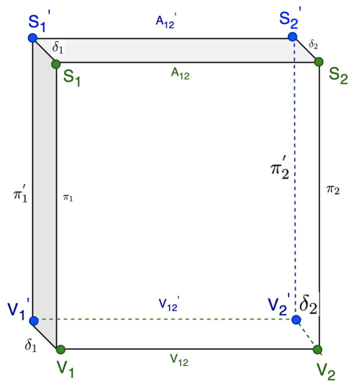


Figure 4.5 Computational Cube

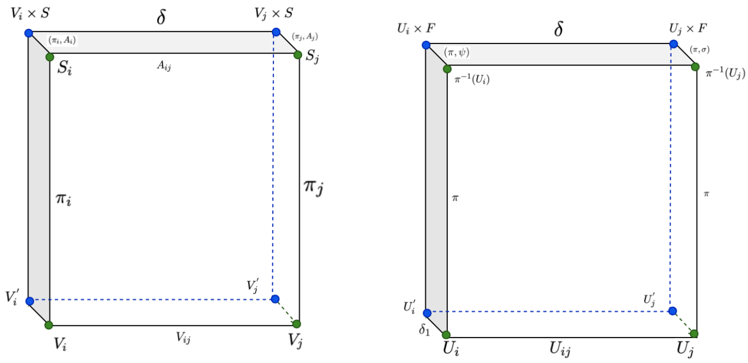


Figure 4.6 Topological Interactive machine isomorphic to fiber bundle structure.

4.4 Isomorphism between TIM and Fiber Bundle

The right cube represent a complete iteration of a fiber bundle via different maps with slight abuse of notation 4.6 (repetitive maps are omitted and those which are not important for now). We have two open sets U_i and U_j whose pre-image is in the total space E as $\pi^{-1}U_i$ and $\pi^{-1}U_j$ respectively. It has two values; one is its projection (via π) onto U_i and other is the height which should be equal to a standard fiber F via map ψ. Informally, the strip should be isomorphic to the cartesian product of $U_i \times F$ via (π, ψ); which is local trivialisation condition. These stripes glue together to form whole space. The process is similar for other open neighbourhood U_j, which maps to $U_j \times F$ via (π, σ). Now, the intersection between U_i and U_j denoted as U_{ij} is compatible if there is a feasible map between $U_i \times F \rightarrow U_j \times F$ via δ map. The δ map is compatible with the intersection then the $U_i \times F$ and $U_j \times F$ will be projected back to the same U_i and U_j respectively, else different open set, say U_i' and U_i'. For further understanding, you may refer to figure 4.1. The left cube represent the computation and update process in the topological interactive machine isomorphic to the fiber bundles. Here, V_{ij} has given weight w determining whether there is an edge between two vertices V_i and V_j; represented as red or green using discrete Morse theory. One can follow same path as fiber bundle and δ determines whether the vertices V_i and V_j are projected to itself with same weight; or other vertices, say V_i' and V_j'. The updating process via δ can be seen in the Figure 4.5. It should be noted that any π^{-1} map is an ordered pair; with one element from the base space and other element from fiber F. It should have been say $U_i \times \pi^{-1}(U_i)$ in case of right cube and $V_i \times S_i$ in left cube; but $\pi^{-1}(U_i)$ or S_i should be isomorphic copies of standard fiber F or S respectively.

Definition 18. (**TIM**) A Topological interactive machine (see Figure 4.8) is a sextuple $(\mathcal{V}, \mathcal{K}, \mathbf{S}, \pi, A, \delta)$ where:

- $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathbb{N}}$ is a finite (possibly unbounded) set of flexible state variables embedded in the topological environment $\mathcal{E}$ such that each vertex of its combinatorial space $\mathcal{K}$ is an element of $\mathcal{V}$.
- State space $\mathbf{S} = \{\mathbf{S}_i\}_{i \in \mathbb{N}}$ consisting of a finite set of processes each containing a finite set of states s.
- $\pi : \mathbf{S} \to \mathcal{V}$ assigns to each $\mathcal{V}_i$ a state space $\mathbf{S}_i$.[6] Each element $s_i \in \mathbf{S}_i$ observes its corresponding simplex of $\mathcal{K}$; as well as its embedding *Emb* [7]. Any action A between states of $\mathbf{S}_i$ and $\mathbf{S}_j$ is allowed only if *Emb* $\{\pi^{-}(s_i)\} \cap CS = \emptyset$ and $|\mathbf{S} > \mathbb{N}|$ such that the left square in Figure 4.7 commutes. It is the pre-condition expressing local consistency.
- A is the set of actions $\mathtt{A}$ defined as is a mapping function $\mathtt{A} : \mathbf{S} \to 2^{\mathbf{S}}$ that maps each state $s \in \mathbf{S}$ to an element of the power set of $\mathbf{S}$. .
- $\delta : \mathcal{V} \times \mathbf{S} \to \mathcal{V} \times \mathbf{S}$ takes vertices in $\mathcal{K}$ and states in $\mathbf{S}$ [8] and maps to another $\mathcal{K}$ and $\mathbf{S}$ such that the square on right side of the Figure 4.7 commutes. It is the post-condition expressing global consistency.

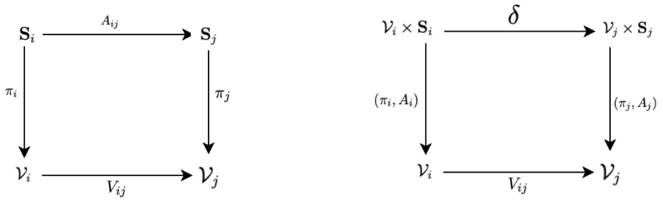


Figure 4.7 Local consistency on the left and global consistency on the right

The commutation of squares are already discussed before.

[6] π map projects the state space onto each vertex, we skip using the subscript, such as π_i meaning a map from S_i to V_i, because the subscript is implicit as one could talks about a particular fiber at a specific vertex.

[7] neighbourhood $\mathcal{U}_i \cap \mathcal{U}_j$ of $\mathcal{E}$.

[8] takes a region in the total space and glues it together. Imagine the cartesian product of the vertices and the fiber as a thin line of infinitesimal thickness which are glued together to make the total space. The point in $\mathbf{S}$ has dual characterisation; as a fiber as well as a point in the total space. we masquerade total space here in its explicit manifestation because it is base space modulo fiber space and constructed likewise.

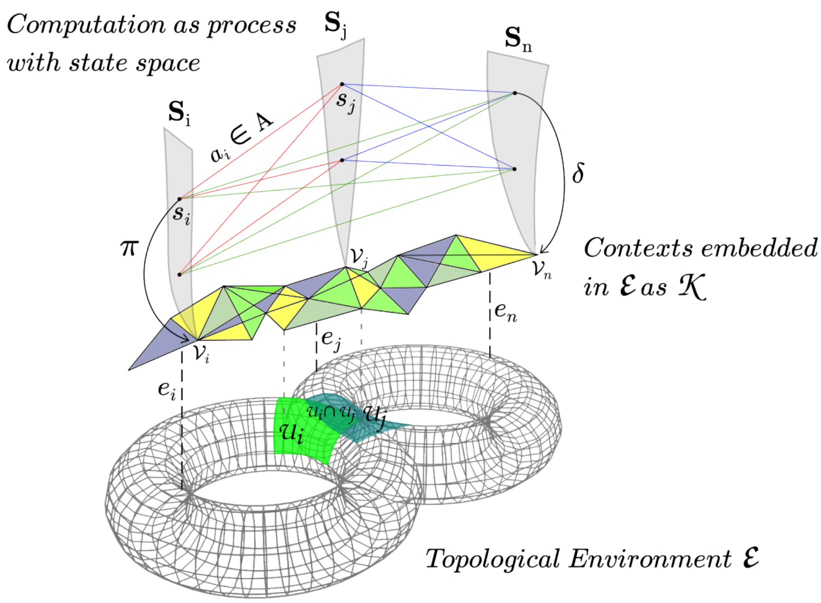


Figure 4.8 Topological Interactive Machine.

TIM computational paradigm facilitates the definition of the open environment in which the global environment constrains the computation which in turn changes its topology. TIM is isomorphic to the structure of fiber bundle. We would provide an equivalence between the definition of TIM and the fiber bundle. First, we define a general concise informal definition of a fiber bundle and then compare it with TIM.

Fiber bundle: Suppose X is a topological space. F-bundle with fiber(s) F over X is a total space E and a surjective map $\pi : E \to X$ with the property that X admits an open cover $\mathcal{U}_i$ such that for each i there is a homeomorphism $\psi : \pi^{-1}(\mathcal{U}_i) \to F \times \mathcal{U}_i$ with projection $\theta : F \times \mathcal{U}_i \to \mathcal{U}_i$ back to X such that all maps commute, i.e., $\theta \cdot \psi = \pi$. Moreover, for $\mathcal{U}_i, \mathcal{U}_j$ the transition functions belong to the structure group $\mathcal{G}$ of the fiber bundle, i.e., $\psi_j^{-1} \cdot \psi_i : F \times (\mathcal{U}_i \cap \mathcal{U}_j) \to F \times (\mathcal{U}_i \cap \mathcal{U}_j)$.

Transition from local to global section: The fiber bundle is locally a product space but globally can have a different topology. It gives a passage from local to global in the same sense as in sheaf theory. A section κ of a fiber bundle gives an element of the fiber over every point of X. Its a map $\kappa : X \to E$ such that $\pi \cdot \kappa$ is identity in X, i.e., for every $x \in X$, $\kappa(x)$ is an element of the fiber over x. For example, in a special case of a vector bundle, the zero section is defined when $\kappa(x) = 0$. The value of κ is sought in each fiber of the vector bundle which looks like a horizontal line – cross-section of a fiber bundle. A section of the fiber bundle means to specify the exact fiber state in each fibers of the bundle through local gauge transformation. These local sections can be extended to global sections only if the bundle is trivial. The local sections in non-trivial spaces cannot be globally extended: $LC - GI$.

The fiber bundle structure has also an equivalent description in terms of covering spaces. The covering spaces of X with fiber F are classified by the *action* of fundamental group $\pi(X)$ on F, i.e., $\pi(X) \to Aut(F)$. F consists of a set and $Aut(F)$ is the group of permutations of this set which constitutes E. The fiber bundle has two components: the group of permutations of states of F and its interplay with X. The interplay is the essence of Grothendieck's Galois theory. It gives a way to reconstruct the fundamental group of X as a group of transformations of the universal covering spaces which is same as the automorphisms of the fiber over X.

Comparison: In TIM, $\mathcal{E} \sim \mathcal{K}$ replaces X. $\mathbf{S}$ replaces F with same projection map π. Map A replaces ψ which is between two fibers $\pi^{-1}(\mathcal{U}_i)$ and $\mathcal{U}_i \times F$ replaced with $\mathbf{S}_i$ and $\mathbf{S}_j$ respectively. The map θ quantifies the overlap $\mathcal{U}_i \cap \mathcal{U}_j$ through π and ψ. In a discrete set-up, when $\mathcal{U}_i \cap \mathcal{U}_j$ agrees at overlap there is an edge between corresponding vertices of $\mathcal{K}$. $\mathcal{K}$ contains CS where the intersection doesn't agree. The simplices of $\mathcal{K}$ are classified using DMT into generic simplices and CS. The condition of overlap is a prior condition for an admissible action A. The map δ replaces θ which then quantifies the overlap in a computational way. Based on cardinality bound on states in $\mathbf{S}$ due to A, δ recalibrates its corresponding edge to generic or CS. It allows iteration of $\mathcal{K}$ and generate new environment $\mathcal{K}'$ for next computational step such that $\delta \cdot A = \pi$. The structure group G acting on bundle is the gauge group $\mathcal{G}$. TIM has a mathematical structure of the fiber bundle.

Local consistency-global inconsistency: The local gauge transformations are characterised by the action A between states of any state spaces which is the cross-section of TIM. The global extension of local section depends on the *type* of underlying simplices of $\mathcal{K}$. Simplices of $\mathcal{K}$ are classified into CS, generic and non-trivial simplices (allowing loops) using DMT which corresponds to $LC - GC$, $LI - GI$ and $LC - GI$ respectively.

4.5 Bell-type Scenario and Fiber bundle

In the background of empirical model is the data of observation from a physical Bell-type experiment. Bell scenario consists of system/agents N. At each site, agent makes an observation from the measurement set M to observe the output O with a probability. It is characterised by a triplet (N, M, O). Statistics is calculated from the data obtained from any permutation between the triplet. The total probability space is the behaviour space which has a topological and geometric representation [16]. Each permutation corresponds to specific region in the space either local, quantum or no-signalling region. Whatever happens in the triplet equivalently affects its topology.

Both triplet and topology can be associated in the following way: to each input set of each agent one can assign all values of output set. Each element of the input set can be represented as a vertex of a simplicial complex embedded in a topological space. The number of possible outcomes per input/vertex is represented as a fiber space. The structure is isomorphic to the fiber bundle.

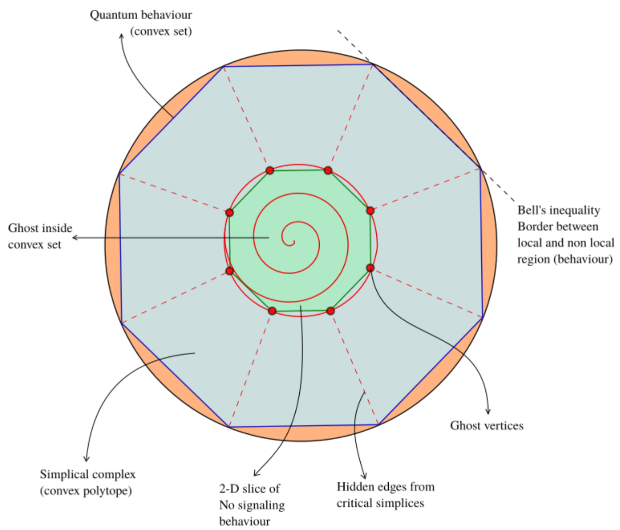


Figure 4.9 Topological realisation of Bell-type correlations categorised as the classical region, the quantum region and the no-signalling region.

Chapter 5

Empirical Model $\equiv$ Topological Interactive Machine

The chapter is based on main results of the thesis. We prove how empirical model of quantum physics can be emulated over topological interactive machine which has structure of fiber bundle; as well as, the relation between strong collapsibility and contextuality using discrete Morse theory. The group representing the machine has a peculiar structure which encodes contextual semantics.

5.1 Empirical Model $\equiv$ TIM

The computation part of TIM represented as process is formalised as symmetry group and the topological environment is formalised as fundamental group. The fundamental group is expressed as paths on a discrete topological space, here simplicial complex, using DMT. The symmetry group considers all permutations between states of any state spaces. Both groups collectively act on fiber bundle. The structure group acting on a principle fiber bundle is called a gauge group. It is a semi-direct product of symmetry group and fundamental group. Gauge group is the measure of contextual semantics in TIM.

Proposition 2. *A Topological interactive machine emulates behaviour bisimilarity of the empirical model* $\mathbf{e}$ *isomorphic to the structure of fiber bundle.*

Proof. Empirical model is based on the sheaf theoretic structure [4, 6]. There exists an adjoint functor between the category of sheaves and the category of fiber bundles on a given topological space [35] [In particular, Theorem 2 of Section 5 of Chapter II]. TIM is isomorphic to the structure of fiber bundle [1] by Definition 18. It implies that TIM emulates behavioural bisimulation of the empirical model. The contextual behaviour of the empirical model is quantified as a gauge group acting on the TIM. The computational machine-theoretic generalisation of empirical models in TIM paradigm is given in Section 6.1.

□

Explanation: The sheaf of sets $\mathcal{O}$ over a topological space X via σ is locally homeomorphic with X unlike in fiber bundle the space is not locally homeomorphic with base space B, but locally homeomorphic to $U \times F$, for suitable open subset $U \subseteq B$. So, naively one needs to weaken the condition in the fiber bundles and restricting it to local homeomorphism to U and in the other case adding to sheaf of sections and defining a topology on it would extend its local homeomorphism as in fiber bundles. Infact, it is one of the equivalence between a particular bundle and sheaves. It has been discussed in https://johncarlosbaez.wordpress.com/2020/01/07/topos-theory-part-2/ and Chapter 2 of ref [35]. Every sheaf comes from bundles and conversely every sheaf gives rise to a bundle in an informal sense. Infact, the set of local sections of fiber bundle form sheaf over the topological space. Every sheaf is a sheaf of cross-sections. Few celebrated results based on correspondence between sheaves and bundle include Serre-Swam theorem, a correspondence between vector bundles and particular sheaves; also the correspondence between étale bundles and sheaves. The structural equivalence (i.e., a function defined over topological space) between bundles and sheaves; as well as the their equivalent way to measure contextuality, i.e., both structures quantify contextuality as obstructions to the existence of global sections; is more significant in our context.

A key tool in computing homology is long exact sequences. Every homology theory that satisfy the homology axioms must give a long exact sequence for every pair of space, say (X, A) which involves relative homology groups, i.e.,

[1]

$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A)\cdots$. We can however turn it into sequence of absolute homology groups, i.e, $\cdots \to H_n(A) \to H_n(X) \to H_n(X/A)\cdots$ using excision theorem to relate the relative homology group with the reduced homology group of the quotient, $H_n(X,A) \sim H_n(X/A)$. So, the short sequence of spaces $0 \to A \xrightarrow{f} X \xrightarrow{g} X/A \to 0$ leads to long exact spaces in homology. (The short exact sequence can split if there exists a homomorphism $h : C \to B$ such that the composition $g \cdot h$ is the identity map on C which gives it the structure of direct sum, i.e., $B \sim A \times C$.) The concept has been used in contextuality in the sense that if the connecting homomorphism (denoted as γ) in long exact sequence (by applying snake lemma) at a specific context C and corresponding section say r is equal to zero, i.e., $\gamma_C(r) = 0$; there exist a compatible family such that the local sections can be globally extended.

For a pair (E,F) we also have long exact sequence of homotopy group, the relative homotopy group, i.e. $\cdots \to \pi_n(F,e) \to \pi_n(E,e) \to \pi_n(E,F,e)$. The question is when we have short exact sequence of spaces, can we use excision to get along long exact sequence of absolute homotopy group? It is not possible because we no longer have excision in case of homotopy groups but we can instead of looking at quotient maps we look at fiber bundles, generally maps which satisfy homotopy lifting property and can give long exact sequence of a specific context. The dependence on class of path of the space makes it difficult to use excision directly, in particular in discrete setting there can be multiple paths for a function to surf through; as well a critical simplices give many possibilities for a gradient flow, making it difficult to cut out a segment trivially, instead using fiber bundle, in which segments are glued based on path lifting property. In this sense, existence of the universal covering space (which is an example of the fiber bundle) means vanishing γ and hence allowing local relations to be globally extended

Moreover, Bell scenario has a computing operational sites with associated topology which could be seen as a fiber bundle to express behavioural bisimulation of empirical models. The community of theoretical computer science is more interested in the expressiveness of the class of behaviour of the empirical models, in particular $LC - GI$. Bell scenario consisting of systems/agents $\mathscr{N}$. At each site, agent makes a measurement from a measurement set $\mathscr{M}$ to observe output from output set $\mathscr{O}$ with a probability. Bell scenario is characterised by triplet $(\mathscr{N}, \mathscr{M}, \mathscr{O})$. Statistics is calculated from the data obtained from any permutations between the triplet. The total probability space can be represented as a behaviour space with topological realisation. Each permutation corresponds to specific region in the space either classical, quantum or no-signalling region. Whatever happens in the triplet equivalently affects it's topology. Both triplet and topology can be entangled in the following way: to each input set of each agent, assign all values of the output set. Each element of input set can be represented as a vertices of simplicial complex embedded in topological space. The number of possible outcomes per input/vertex is represented as fiber space. The structure is isomorphic to fiber bundle. The reader may access Appendix A.3 for further intuition.

Symmetry group encode PTM computation

A representation of a finite group $\mathcal{F}$ over field $\mathbb{F}$ is a homomorphism $\rho : \mathcal{F} \to GL(V)(= Aut(V))$ where V is a vector space of finite dimension over $\mathbb{F}$. $GL(V)$ is general linear group and $Aut(V)$ is the automorphism group of V, i.e., group of linear invertible transformations $T : V \to V = GL(V)$. The vector space V is called the representation space of $\mathcal{F}$. The representation theory gives a way to associate every element of group to matrices and group operation to matrix operation. The vector space is an abelian group under addition as is the state space under transformation/function as permutation group. The group-theoretic representation of regular languages is expressed as automaton which is central to formal language theory and a way to see vector space as state space.

Remark 5. *There exists no invertible transformation T between irreducible representation of different dimensions.*

Theorem 4. *Let $\mathcal{F}$ be a finite group, $\mathbf{D}_i$ and $\mathbf{D}_j$ be irreducible representations of state space $\mathbf{S}_i$ and $\mathbf{S}_j$ over state set s respectively. There exists an admissible linear transformation (computation) $\mathbf{T}$ from $\mathbf{S}_i$ to $\mathbf{S}_j$ i.e., transition $\mathbf{T} : \mathbf{S}_i \to \mathbf{S}_j$ if and only if $\mathbf{T} \cdot \mathbf{D}_i(g) = \mathbf{D}_j(g) \cdot \mathbf{T}$ for all $g \in \mathcal{F}$, else it is a forbidden transition or contextual computation.*

Proof. Schur lemma deals with the equivalence of irreducible representations. For a finite group $\mathcal{F}$; and $\mathbf{D}_i$ and $\mathbf{D}_j$ be irreducible representations of state space $\mathbf{S}_i$ and $\mathbf{S}_j$ over s. Suppose we are able to construct a linear transformation $\mathbf{T} : \mathbf{S}_i \to \mathbf{S}_j$ such that $\mathbf{T} \cdot \mathbf{D}_i(g) = \mathbf{D}_j(g) \cdot \mathbf{T}$ for all $g \in \mathcal{F}$ condition holds i.e., $\mathbf{T}$ intertwines the two irreducible representations then there are three possibilities as per Schur criteria.

1. $\mathbf{T} = 0$ then $\mathbf{D}_i$ and $\mathbf{D}_j$ are not equivalent and transitions are forbidden like deadlocks.
2. $\mathbf{T} \neq 0$ and $\mathbf{T}$ is a singular transformation then $\mathbf{D}_i$ and $\mathbf{D}_j$ are not equivalent and transition responsible for locally consistent and globally consistent computation.
3. $\mathbf{T} \neq 0$ and $\mathbf{T}$ is a non-singular (invertible) transformation, $\mathbf{D}_i$ and $\mathbf{D}_j$ are equivalent and transition is admissible.

□

The local transformations $\mathbf{T}$ between $\mathbf{S}$ are carried out in the presence of underlying simplices of $\mathcal{K}$. The singular transformation over $\mathbf{S}$ is locally feasible but globally infeasible due to the presence of non-trivial loops in $\mathcal{K}$.

Fundamental group encode topological environment

Theorem 5. *The non-trivial fundamental group of the topological environment $\mathcal{E}$ implies contextual semantics in TIM computational paradigm.*

Proof. $\pi(\mathcal{E}) = 0$ corresponds to simply-connected space. The simply connected spaces are product space in which every local section can be extended to global section by the definition of TIM. The local sections in non-trivial spaces cannot be globally extended: $LC - GI$, which means the fundamental group of $\mathcal{E}$ is non-trivial. It infers non-vanishing homotopy groups. $\mathcal{K}$ quantifies $\mathcal{E}$ as a set of paths in discrete setting using DMT as discussed below.

□

5.2 Strong Collapsibility and Contextual Semantics

The Theorem 5 gives a general description about contextual semantics of TIM. The machine is not based on $\mathcal{E}$ but its equivalent discrete counterpart $\mathcal{K}$. The combinatorial structure of $\mathcal{E} \sim \mathcal{K}$ representing $\mathbf{e}$ encodes information about its contextual semantics as homotopy classes. The homotopy information of $\mathcal{K}$ is quantified using DMT.

The equivalent expressiveness of explicit topological environment encoding contextual semantics is based on strong collapse of its directed simplicial complex $\vec{\mathcal{K}}$, constraining the feasibility of computations over $\mathbf{S}$ unlike PTM. There is a relation between collapsibility of a complex and contextuality. The fundamental group is expressed in discrete setting as set of paths over simplicial complex using DMT which constrain transformation/computation of Theorem 5.1. It provides homotopy information about the space which is a measure of contextuality. We introduce concept of strong collapses and their relation with contextual semantics.

Strong collapsibility $\implies$ Locality / Non-contextuality The topological environment is combinatorially represented as simplicial complex and its collapsibility is characterised with contextual semantics of $\mathbf{e}$. $\mathcal{K}$ is immersed in $\mathcal{E}$ and collapsibility of $\mathcal{K}$ doesn't infer contractibility of $\mathcal{E}$. There exist finite spaces that are not homotopy equivalent to their associated complexes. We introduce *strong collapses* such that the contractibility of topological spaces is equivalent to strong collapsibility of ordered simplicial complex $\vec{\mathcal{K}}$. The direction to $\mathcal{K}$ is given via discrete Morse function $\mathfrak{f}$ of DMT and classifies its vertices and edges as critical and generic simplices. Forman's beautiful paper is the standard reference on this subject [25]. A very short account on DMT is also provided in Chapter 2. Moreover, the paths are classified as non-closed paths (gradient) and non-trivial closed paths. Strong collapses infer no critical simplices and non-trivial closed paths in $\vec{\mathcal{K}}$ which imply the existence of global section of $\mathbf{e}$ and factorability of topological environment.

Definition 19. Let $\mathbf{e}$ be an empirical model and $\vec{\mathcal{K}}$ its associated directed simplicial complex over discrete Morse function $\mathfrak{f}$. The $\mathtt{n}$ non-possibilistic events of **e** are represented by `m` critical simplices $\mathcal{CS}$ where $\mathtt{m} \geq \mathtt{n}$ and incompatible family of sections by class of non-trivial closed paths $\mathcal{CP}$ (see Figure 5.1).

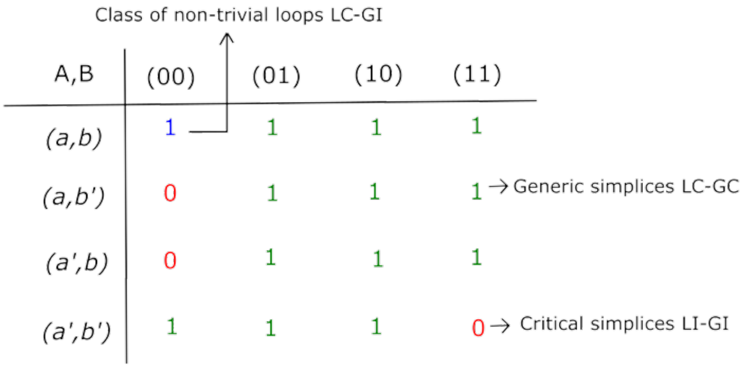


Figure 5.1 Topological characterisation of Hardy Table.

Definition 20. Let $\vec{\mathcal{K}}$ and $\vec{\mathcal{L}}$ be directed simplicial complexes. We say that there is a **strong collapse** from $\vec{\mathcal{K}}$ to $\vec{\mathcal{L}}$ (indicated with $\vec{\mathcal{K}} \searrow \vec{\mathcal{L}}$) if there are no critical simplices $\mathcal{CS}$ and non-trivial closed paths $\mathcal{CP}$ from $\vec{\mathcal{K}}$ to $\vec{\mathcal{L}}$.

If there exist no critical simplices in an interval say $[a, b]$ then it is collapsible to a null vertex $\{\emptyset\}$. But it is not the sufficient condition for contractibility of its associated $\mathcal{E}$. The obstruction to the existence of global section are holes in topological space that correspond to $\mathcal{CS}$. These induce cyclicity in simplicial complex and are responsible for its non-collapsibility up to homotopy. The non-existence of $(\mathcal{CS}, \mathcal{CS})$ signifies collapsibility of $\mathcal{K}$. It infers that corresponding empirical model has no non-possibilistic events and incompatible families of paths. These are *strong collapses* of $\vec{\mathcal{K}}$. The cyclicity in $\mathcal{K}$ is allowed in open environment of TIM computational paradigm.

Lemma 1. *If a directed simplicial complex $\vec{\mathcal{K}}$ is strong collapsible then its associated finite topological space $X \sim \mathcal{E}$ is contractible.*

Proof. Strong collapses imply no $\mathcal{CS}$ and $\mathcal{CP}$ in $\vec{\mathcal{K}}$. The non-existence of $\mathcal{CS}$ signifies collapsibility [25]. So we need to only prove relation of $\mathcal{CP}$ and collapsibility. Suppose we have discrete Morse function $\mathfrak{f}$ and a gradient path

$$\sigma_0^{(d)}, \tau_0^{(d+1)}, \sigma_1^{(d)}, \tau_1^{(d+1)}, \sigma_2^{(d)}, \ldots\ldots, \tau_r^{(d+1)}, \sigma_{r+1}^{(d)}$$

such that for each $i = 0, ..., r$, $\{\sigma, \tau\} \in V$ and $\tau_i > \sigma_{i+1} \neq \sigma_i$. Let this sequence be a *non-trivial closed path*, meaning that $r \geq 0$ and $\sigma_0 = \sigma_{r+1}$. Then, by definition of function of DMT we have $\mathsf{f}(\sigma_0^{(d)}) \geq (\tau_0^{(d+1)}) > (\sigma_1^{(d)}) \geq (\tau_1^{(d+1)}) > ....(\sigma_r^{(d)}) \geq (\tau_r^{(d+1)}) > (\sigma_{r+1}^{(d)}) = \sigma_0^{(d)})$ which is a contradiction. It could be only possible when we allow a transformation of fundamental group of X changing $\mathcal{E}$ to new topological environment $\mathcal{E}'$ which facilitates openness in TIM. □

Theorem 6. *If a directed simplicial complex $\vec{\mathcal{K}}$ is not strong collapsible then $\vec{\mathcal{K}}$ is contextual in nature.*

Proof. From Proposition 2, the empirical model is represented as $\mathcal{E}$ which is part of a fiber bundle description. Non-contractibility of $\mathcal{E}$ infers non-locality and contextuality. Under *strong* collapses, contractibility of $\mathcal{E}$ is equivalent to collapsibility of $\vec{\mathcal{K}}$. The acyclicity of $\mathcal{K}$ is the extendibility of every compatible family to a global section. Lemma 1 defines strong condition for $\vec{\mathcal{K}}$ to be strong collapsible. The existence of $(\mathcal{CS}, \mathcal{CP})$ in $\vec{K}$ is non-collapsibility to $\{\emptyset\}$. It infers existence of obstructions to global section as $\mathcal{CS}$ and induces incompatible family of paths as $\mathcal{CP}$. The corresponding cyclic Hasse diagram of $\vec{\mathcal{K}}$ infers contextuality.

□

Lemma 2. *Let* $\mathbf{S}_i \overset{\phi}{\rightarrowtail\twoheadrightarrow} \mathbf{S}_j$ *be permutations between state spaces* $\mathbf{S}_i$ *and* $\mathbf{S}_j$*, and* $\mathcal{V}_i$ *and* $\mathcal{V}_j$ *be two vertices of a simplicial complex* $\mathcal{K}$*. If the square*

$$\begin{array}{ccc} \mathbf{S}_i & \overset{\phi}{\rightarrowtail\twoheadrightarrow} & \mathbf{S}_j \\ \eta\uparrow & & \mu\uparrow \\ \mathcal{V}_i & \xrightarrow{\psi} & \mathcal{V}_j \end{array}$$

commutes i.e. $\eta \cdot \phi = \psi \cdot \mu = e$ *then the associated* $\mathcal{K}$ *of the model is collapsible and the global section exists.*

Gauge group encodes contextual semantics

Lemma 3. *The gauge group* $\mathcal{G}$ *of fiber bundle encodes contextual semantics as semi-direct product of symmetry group* S_n *of state space and fundamental group of discrete topological space* $\pi(\mathcal{K})$ *of TIM, i.e.,* $\mathcal{G} = \mathrm{S}_n \rtimes \pi(\mathcal{K})$*.*

Proof. By definition, the gauge group $\mathcal{G}$ is the structure group of the TIM. It has two components: symmetry group acting on PTM computation and fundamental group of $\mathcal{E}$ quantified in terms of paths over $\mathcal{K}$ using DMT. The interplay between both groups is described extensively in Section 5.1 and Section 6.1 respectively. So, $\mathcal{G} = \mathrm{S}_n \rtimes \pi(\mathcal{K})$.

□

Remark 6. *The fiber bundle doesn't always admit a product topology, i.e., it cannot be always expressed as direct product of* $\mathcal{U} \times F$. *The direct product* $\times$ *generalises to the semi-direct product* $\rtimes$. *A computation* $s_i \xrightarrow{a_i} s_j$ *over state space is subjected to* $\mathcal{V}_i \to \mathcal{V}_j$ *over* $\mathcal{K}$ *in TIM paradigm. The effective computation cannot always be associated with linear environment unlike Turing computation. It is the computational meaning of the semi-direct product in mathematical sense.*

Remark 7. *The structure of fiber bundle is pervasive in physics. The structure is realised broadly in topological field theories like Yang Mill theory, quantum theory and standard model of the known universe. In particular, the topology of fiber bundle provides inseparable holistic paradigm which is known to reproduce non-locality [37].*

Remark 8. *The categories of paths based on* DMT *is same as flow categories defined over a topological space based on de Rham cohomology via Hodge decomposition. The gradient flow,curl flow and harmonic flow corresponding to* $LC - GC$, $LI - GI$ *and* $LC - GI$.

Logical equality condition for contextual semantics in TIM: Hidden variables are realised as finite topological environment $\mathcal{E}$ that gives to each vertex $\mathcal{V}_i$ a random environmental variable e_i. $\mathcal{V}_i$ is represented as a flexible state variable in TIM that is immersed in $\mathcal{E}$ realised as a combinatorial structure of a simplicial complex $\mathcal{K}$. The state space $\mathbf{S}$ consists of a possible set of states s_i that flexible variables hold. The composition of environmental variables and flexible variables form*logical equality* that is satisfied by the $s_i \in \mathbf{S}$ over some field $\mathbf{F}$. The logical equality is of the form, $\mathbf{p} = e_1\mathcal{V}_1 + e_2\mathcal{V}_2 + e_3\mathcal{V}_3 + ...+ e_n\mathcal{V}_n = \mathcal{I}$ where set $(e_1, e_2, ...e_n) \in \mathcal{E}$ are hidden environmental variables and $(\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n)$ as flexible variables that correspond to vertices of $\mathcal{K}$ and $\mathcal{I}$ is the model-specific topological invariant. The solution or possible states s of each $\mathcal{V}$ in the equality are contained in **S**, possibly a fiber over each $\mathcal{V}$ through projection map π of TIM. The automorphism between states in **S** affect the ambient environment of TIM that can be represented by symmetry group S_n. It represents invariance of the topological environment by the automorphism over states pace. Whatever happens in **S** equivalently affects topological environment $\mathcal{E}$.

Proposition 3. *Violation of logical equality implies contextual semantics in TIM.*

The non-solvability of logical equality by different permutations between states in **S** implies non-locality and contextuality in TIM. The logical equality can be represented by Symmetry group S_n. Abramsky has already provided a relation between the existence of global sections and solutions of linear systems along with examples in his sheaf theoretic interpretation [4].

Remark 9. *The given set of states are separable if the logical equality condition is satisfied, otherwise entangled. The geometric and topological interpretation of entangled statesin the fiber bundle approach corresponds to homotopy class [37]. During compositionality of processes and their interaction, the strange loop mitigate through the structure of the environment which adds up at least one extra edge to the model non-trivially, quantifying entangled correlations in classical sheaf theoretic sense.*

Remark 10. *It may seem unclear that the classical Galois group applies to solutions of polynomials while the non-local inequalities are linear. There are different descriptions of Galois categories, in particular, its interpretation in terms of coverings and fundamental group, topologically relates solvability. It is expressed by logical equality which is an interplay between vector spaces and topological spaces. The* polynomiality *is* absorbed *in combinatorics of space which is characterised as simplices $\mathcal{V}$ immersed in $\mathcal{E}$ whose structure* breaks *its solution.*

Remark 11. *The polynomial provides a combinatorial description of the topological space. The variables represent vertices and the exponent represents in* how many ways *it is linked. It constructs the simplicial complex in an ambient topological environment represented as coefficients. It is related to the polynomial functor construction of a dynamical system.*

Remark 12. *The polynomiality can be understood in the context of imbeddability of partial algebra into commutative algebra which quantifies contextuality in Kochen-Specker sense.*

Remark 13. *The subtle reason behind the possibility to map sheaf theoretic interpretation of quantum contextuality in TIM is due to gauge theoretic structure of the machine.*

The fundamental idea to classify a topological space is based on the concept of symmetry. A connected topological space invariant under symmetry group can't change under any permutation unless going through non-trivial region. In this sense, contractible spaces retain symmetry while the existence of the non-trivial region in topological space is a *symmetry breaking process* in gauge theoretic perspective.

Remark 14. *The computation part of TIM, represented as process endowed with state space is formalised as Galois group entangled to topological environment formalised as fundamental group. Equivalently,* $e_1\mathcal{V}_1 + e_2\mathcal{V}_2 + e_3\mathcal{V}_3 + ... + e_n\mathcal{V}_n \rightarrow$ *Symmetry group* S_n *and* $\mathcal{I} \rightarrow$ *fundamental group of topological environment* $\pi(\mathcal{E})$ *of logical equality.*

5.3 Grothendieck *Gesture*

The formalisation could also be seen in terms of Grothendieck Galois theory. We do not intend to formalise the concept but just to provide a slight gesture. The fundamental theorem of covering spaces states that for any *nice* topological space, the fundamental group at a point can be reconstructed as a group of deck transformations of the universal covering space; which is same as automorphism of the fiber over that point of the projection map. One could extend the scope of fundamental group to fundamental groupoid. Grothendieck Galois theory gives such correspondence between covering space of X. The fibers is a set of some field, say, field of natural numbers, which is state space in our case; associated with field extension in Galois sense which is the action between states, given the fundamental group of X which is the environment. For interested readers, following is a further explaination.

Definition 21. (Action) A simple extension is a field extension $[\mathbf{F} : \mathbf{E}]$ such that $\mathbf{F} = \mathbf{E}(a_1)$ for some $a_1 \in \mathtt{A}$, $\mathbf{E} \in \mathbf{S}_i$ and $\mathbf{F} \in \mathbf{S}_j$

The solvability of logical equality over $\mathcal{K}$ is to extend field $\mathbf{E}$ to $\mathbf{E}(a_1) = \mathbf{F}$ by *adjoining* element a_1 to subfield $\mathbf{E}$. All fields contain admissible states such that $\mathbf{E} \overset{a_1}{\rightarrow} \mathbf{F}$ corresponding with the casual structure of ambient environment $\mathcal{E}$.

Definition 22. (Admissibility) A simple field extension $[\mathbf{F} : \mathbf{E}]$ is an admissible transition from state-spaces over field $\mathbf{E}$ to $\mathbf{F}$ such that $\mathbf{F} = \mathbf{E}(a_1)$ for some $a_1 \in \mathtt{A}$ i.e., $\mathbf{E} \overset{a_1}{\rightarrow} \mathbf{F}$.

Let $\mathbf{S}_i$ and $\mathbf{S}_j$ be state-space over field $\mathbf{E}$ and $\mathbf{F}$ respectively. The field extension$[\mathbf{F} : \mathbf{E}]$ is an admissible transition from $\mathbf{S}_i$ and $\mathbf{S}_j$ such that a_1 is an element in $\mathbf{F}$ that adjoins with $\mathbf{E}$ for transition. Let m and n be cardinalities of the set of states of $\mathbf{S}_i$ and $\mathbf{S}_j$ respectively. We are interested in the permutation σ between set elements over some field and its effect on topology of the base space. *Field extensions imply admissible transitions between states over* $\mathbf{F}$.

Definition 23. An extension $\mathbf{F}$ is finite, normal and separable over $\mathbf{E}$ if $\mathbf{E}$ completely splits in distinct linear factors in $\mathbf{F}$.

Definition 24. A finite, normal and separable extension $[\mathbf{F} : \mathbf{E}]$ is called Galois extension denoted as $\text{Gal}[\mathbf{F} : \mathbf{E}] = Aut[\mathbf{F} : \mathbf{E}]$.

Definition 25. (Permutation between states in state space of processes) Any permutation ω of some field $\mathbf{F}$ over a set of states $\omega = (s_1, , ..., s_n)$ over $\mathbf{S}$ defined by $\omega f((s_1, ..., s_n) = (\omega s_1, ..., \omega s_n)$ for each function f of $(s_1, , ..., s_n)$ extends to bijection ω of $\mathbf{F}((s_1, ..., s_n)$ this bijection is an automorphism of $\mathbf{F}((s_1, ..., s_n)$.

The most important property of $\mathbf{F}(s_i)$ is that it is symmetric with respect to s_i, in the sense that any permutation ω of set a_i extends to a bijection ω of a_i and is an automorphism of s_i. It is just a permutation of states in $\mathbf{S}$ and its effects on logical equality. Let $\mathbf{S}_i$ and $\mathbf{S}_j$ be state spaces over fields $\mathbf{E}$ and $\mathbf{F}$ and n and m be the cardinality of state set $s \in \mathbf{S}$ then total permutation between $\mathbf{S}_i$ and $\mathbf{S}_j$ is $(n + m)!$. It represents all possible permutations of two sets. In this sense, the directed edge satisfying conditions at $\mathcal{U}_i \cap \mathcal{U}_j$ between two vertices $\mathcal{V}_i$ and $\mathcal{V}_j$ in $\mathcal{K}$ lifts to many possible permutations of states $s \in \mathbf{S}$. If the permutations retain symmetry then there exists a transition function, an action A between state spaces $\mathbf{S}$.

Definition 26. Let $[\mathbf{F} : \mathbf{E}]$ be any field, the automorphisms ω of $\mathbf{F}$ fixing all elements of $\mathbf{E}$ forms the Galois group of $\mathbf{F}$ over $\mathbf{E}$ denoted as $\text{Gal}(\mathbf{E}/\mathbf{F})$.

Lemma 4. *Let $\mathbf{S}_i$ and $\mathbf{S}_j$ be state spaces over fields $\mathbf{E}$ and $\mathbf{F}$ respectively. If the permutation σ of their field extension $[\mathbf{F} : \mathbf{E}]$ satisfies the logical equality $\mathbf{p}$ over simplicial complex $\mathcal{K}$ then there is an admissible transition from $\mathbf{S}_i$ and $\mathbf{S}_j$.*

Definition 27. A Galois group $\text{Gal}(\mathbf{p}/\mathbf{F})$ is solvable over logical equality $\mathbf{p}$ if there exists a sequence of subgroups

$$\text{Gal}(\mathbf{p}/\mathbf{F}) = \text{G}_0 \supset \text{G}_1 \supset \text{G}_2 \supset, ..., \supset \text{G}_t = \mathfrak{e}$$

such that G_n is normal inside G_{n-1} and $\text{G}_i/\text{G}_{i-1}$ is abelian.

Theorem 7. *There exists global section for an equation $\mathbf{p} = (e_1\mathcal{V}_1 + e_2\mathcal{V}_2 + e_3\mathcal{V}_3 + ... + e_n\mathcal{V}_n = \mathcal{I}) = \mathfrak{e}$ over field $\mathbf{F}$ if and only if its Galois group $\text{Gal}(\mathbf{p}/\mathbf{F})$ is solvable.*

Proof. Let $\mathbf{F}$ denote set of intermediate Galois extensions and G_i set of subgroups of Galois group then we define two maps $\dagger : \mathbf{F}_i \to \text{G}_i$ and $\Delta : \text{G}_i \to \mathbf{F}_i$ which reverses inclusion. These two maps constitute the Galois correspondence between $\mathbf{F}_i$ and G_i. The solution field corresponds to identity containing the solution of the logical equation.

□

Corollary 1. *If the Galois group $\text{Gal}(\mathbf{p}/\mathbf{F})$ is non-solvable then TIM is contextual in nature.*

Corollary 2. *If the subgroups G_i of the Galois group are abelian then the corresponding simplicial complex $\mathcal{K}$ is collapsible and the global section exists.*

Fundamental group encode topological environment

Theorem 8. *If the fundamental group of a topological environment $\mathcal{E}$ is non-trivial then TIM is contextual in nature.*

Proof. $\pi(\mathcal{E}) = 0$ corresponds to simply-connected space. There is a bijection between the collection of all isomorphism classes of connected covering space of $\mathcal{E}$ and the conjugacy classes of subgroups of $\pi(\mathcal{E})$: the Galois correspondence. $\mathcal{E}$ with fiber $\mathbf{S}$ are classified by transitive action of $\pi(\mathcal{E})$ on $\mathbf{S}$ that is $\pi(\mathcal{E}) \to Aut(\mathbf{S})$. It assigns to each Galois cover the subgroup of $\pi(\mathcal{E})$. The normal and abelian subgroups of $\pi(\mathcal{E})$ infer connected covering space. For a simply connected space, the $\mathrm{Gal}(\mathbf{p}/\mathbf{F}) \simeq \pi(\mathcal{E})$ that is is identity e and trivial space.

□

Lemma 5. *Let* $\mathbf{S}_i \overset{\phi}{\rightarrowtail\!\!\!\twoheadrightarrow} \mathbf{S}_j$ *be permutations between state spaces* $\mathbf{S}_i$ *and* $\mathbf{S}_j$*; and* $\mathcal{V}_i$ *and* $\mathcal{V}_j$ *be two vertices of a simplicial complex* $\mathcal{K}$*. If the square*

$$\begin{array}{ccc} \mathbf{S}_i & \overset{\phi}{\dashrightarrow} & \mathbf{S}_j \\ {\scriptstyle\eta}\uparrow & & \uparrow{\scriptstyle\mu} \\ \mathcal{V}_i & \overset{\psi}{\longrightarrow} & \mathcal{V}_j \end{array}$$

commutes i.e. $\eta \cdot \phi = \psi \cdot \mu = e$ *then the associated* $\mathcal{K}$ *of the model is collapsible and the global section exists.*

Remark 15. *If a cycle is formed over state space then it has a non-solvable Galois group. The adjoin element to each field extension will lead to a contradiction. It is equivalent to the question of strong collapsibility of topological environment* $\mathcal{E}$ *and further explained in Section 6.1.*

Chapter 6

Contextual Semantics Machinery: Examples

The chapter provides various examples for the proposed general computational framework based on main results of the proceeding chapter. It provides computational understanding of empirical models in the foundations of quantum physics. The models include: Hardy Model,KochenSpeckerModel, Mermin-Peres Magic Square, Popescu-Rohrlich Boxes and Greenberger-Horne-Zeilinger Model.

6.1 Contextual Semantics Machinery: Examples

The section generalises empirical models of quantum physics in a computational way through TIM paradigm. The only information given is the possibilistic table of models. We provide examples of Hardy model, Kochen Specker model, Mermin-Peres Magic Square, Popescu-Rohrlich Boxes and Greenberger-Horne-Zeilinger Model.

TIM has computational space and topological environment which are mathematically represented as fiber (state) space and discrete topological space respectively in a fiber bundle. The gauge group measures contextuality which is a semi direct product of symmetry group and fundamental group. DMT

quantifies the fundamental group in a discrete setting. The idea is to construct the total space associated with empirical models in terms of base space modulo fiber space; but we prefer to keep it in terms of semi-direct product for computational purpose so that it could reflect the TIM computation.

We would like to provide you with few insights to facilitate understanding about the results of the Hardy model. We are given the table of the Hardy model and what interests us is the different class of behaviour that the model is able to express; feasible, infeasible and contextual ($LC - GI$) corresponding to possibilistic, non-possibilistic and $LC-GI$ events; the behavioural bisimulation. The model has a sheaf structure with essentially two components; the output space and the measurement contexts; and based on whether an event is possible or non-possible the function (section) moves around. From a theoretical computer science perspective, we are not only interested to express this class of behaviour; but also the novelty it can bring in to the model of computation expressing it.

At first, we accessed Turing-like interactive models to express this class of behaviour; but due to Bell-Kochen-Specker theorem, the tape structure recalls local hidden variables. After, as we known that contextuality is a topological property, we accessed topological and geometric models of computation but the model expresses feasible and infeasible behaviour (as shown in the example of process specification above) but not $LC - GI$ behaviour. Any local partial orders among processes *always* admits a consistent global order. The consistency comes in by restricting processes to local partially ordered spaces which avoid non-trivial loops: a causality condition expressed as a loop free and fixed simplicial complex. In order to express $LC - GI$ behaviour, there should be a possibility to allow non-trivial loops in a casual way. One way is to allow topological structure constrain the computation (as in case of topological models); as well as vice-versa, i.e., allow computation to change the topology of the environment; which leads us to a novel concept of openness unlike Turing-like interactive models. It means expressiveness of LC-GI in a model of computation requires openness, otherwise the non-trivial loops would fall into a paradox.

There are few points to say here.

1. First, non-locality and contextuality as obstructions to the existence of global section is very well understood in continuous spaces, as in the example of fiber bundle, there is always a way to locally go from $\pi^{-1}(U_i) \to U_i \times F$ or $\pi^{-1}(U_j) \to U_j \times F$ which is local trivialisation condition; but globally it is not always possible, here, $U_i \times F \to U_j \times F$. So, in continuous spaces, a hole as an obstruction constrain its global section, in particular at intersections. On the other hand, we have a priori given discrete space in computation, (considering executions of processes as paths over space and equivalence between them in terms of homotopy, not important to discuss here.); as in the process specification example, we already know all of the possible execution paths of the program; as well as feasible and infeasible regions. The red simplices means infeasible and the green means feasible; but openness facilitates new patterns to emerge like $LC - GI$. [1]

2. Secondly, as in case of Bell scenario there is an element of choice for each agent to choose from the register and send it to the target (which is essentially a computation) independent of the hidden variables, otherwise, every behaviour can be trivially reproduced if hidden variables actually would determine the measurement setting. The topological environment constrain the computation but does not influence the choice of each fiber so as to choose from its states and make any possible transition. So there is an element of choice involved; as well as both structures; the fibers and the topological space work in synergy.

[1] In topological interactive machine, the topological structure could be a priori given, but we essentially could allow the space to change during computation – openness; as we saw in the example of TIM working; that the weight of the edge depends on the cardinality of state space. After computation, $|S_1| = 1$ which is less than the minimum bound on the cardinality of S, hence turns its vertex red for next iteration; as well as changing one of the simplices in the topological space would re-iterates globally (due to constrains from discrete Morse theory), which could in turn change the homotopy type of the space. The paradigm allows non-trivial loops which corresponds to $LC - GI$ behaviour. It is worth mentioning here that a general formalisation of the concept of openness is not in the scope of my current work. The concept finds its natural meaning in the biological context; whose complexity could be modelled using extended Grothendieck structures. The fiber bundle and topological interactive machine are semi-dynamical systems. There are general mathematical theories like generalisations of discrete Morse theory to address the openness which would be one of the future goal of this work. Here, openness suffice as a causal explaination of non-trivial loops; without pondering on its formalisation.

6.2 The Overview

We are given Hardy table; First, from the context set $\{a, b, a^{'}, b^{'}\}$, we construct a simplicial complex, say K_1 by considering its power set which has a discrete topology as in Figure 6.2. Notice every simplex of K_1 is feasible; one could randomly pick any non-possibilistic context say $ab^{'}$ at (00) (between their fibers which we didn't consider yet) and mark it red. But it would be obsolete because there is no condition that can make any simplex red or green; as well as there is no element of choice given to the fibers (output space); because if one makes an edge as red between $ab^{'}$, it would mean the each fiber would consist of only 0 (i.e., at (00)), because at (11) the given context is possible. The idea is to construct the topological space iteratively so that it can express all class of behaviour of the table; as well as at every iteration, there is should be a rule to change the simplices as red not randomly. In order to apply the rule, we use discrete Morse theory (which gives a function that assigns to each simplices a real number under the condition that as we move from lower to higher simplices, the number should increase; and where the number decrease the simplex is turned to critical, the red one) by identifying the boundary of the simplicial complex K_1 of Figure 6.2 and it changes to other simplicial complex K_2 as in Figure 6.3. There are other reasons for this identification but we skip for now for the sake of brevity. Now, given K_2, one applies the rule using discrete Morse theory and it turns that four 0-simplices, two 1-simplices and one 2-simplex turned red (based on discrete Morse function) as in Figure 6.4. It is quite evident because if one evaluates the Euler characteristic of the space it turns out to be zero, which is same as 1-torus; as well as there is an CW attachment map between red simplices which are homeomorphic to 1-torus; Each simplex is assigned a weight. When we compare the red simplices to the table, we find that only two non-possibilistic contexts $ab^{'}$ and $a^{'}b^{'}$ are turned red. (In the machine, we have synergy between the fiber space and the discrete topological space; so any simplex turned red could be skipped because in irreducible fiber space (attached in the meanwhile) those simplices might not be considered; as well as those which are not in the table. So the total space associated with the Hardy model will be the quotient space of given topological space modulo the fiber space; but we kept it in terms of semi-direct product to reflect the computational paradigm of TIM.) We start another iteration and which takes other non-possibilistic event $a^{'}b$ as shown in Figure 6.5. Similarly, as a check its Euler characteristic is -2, the 2-torus; which can be independently also constructed from the CW mapping between given n-simplices. All the non-possibilistic events are expressed on the discrete space which is homeomorphic to the 2-torus except the strange context (ab) which is $LC - GI$. The weaving of contexts is very peculiar, i.e.,the construction of surface invokes similar iterations even after trying different permutations of the contexts (with an intention to see if we can get minimum surface to express all classes of behaviour; moreover, using discrete Morse theory we can get the maximum number of holed surface due to the fact that critical simplices are greater than the Betti number of the surface.). The context (a, b) can't be expressed as a critical simplex because it should be possible when it observes its corresponding simplices via π map. So, we further iterate and we get another simplicial complex as shown in Figure 6.6. As shown, there is a possibility of non-trivial loops due to equidistant critical points. The edge joining vertices $a^{'}$ and $a^{'}$ (marked with a red arrow over their edge) has a choice to go in either directions; which facilitates non-trivial loops (which finds its explanation due to openness) marked in blue. LC-GI would mean an edge between (ab) at (00) is possible because there is no critical simplices but due to non-trivial loop it is not possible. As a result, the final space is homeomorphic to a 3-torus, which will be our base space of the Hardy model. It is measured by its fundamental group quantified via discrete Morse theory in discrete setting. This explains theorem 11.

The fiber space consists of the total possible outcomes. From our first example 4.3, the state space S_1 has two states s_1 and s_2, and if the conditions are satisfied then any state $s_i \in S_1$ can make action to any state in S_2. The fiber space can have permutations between different states in each fiber S_i as shown in Figure 4.8. In the particular example, there are four state spaces with permutations between their respective state; which would also be equal to the number of vertices (because π assigns to each vertex a fiber space). So, the measurement set (or respective fibers) in Hardy model are four, $\{a, b, a^{'}, b^{'}\}$ represented as a regular tetrahedron. Each symmetry of the tetrahedron corresponds to a permutation of its states in the state space (0 and 1 in each fiber). It accounts all the 24 symmetries of the tetrahedron along different planes. The group of all self-isometries that sends a regular tetrahedron to itself is isomorphic to S_4 symmetry group. One can imagine fiber associated to each vertex of its base space homeomorphic to 3-torus; but we are interested in its irreducible class of behaviour, because other behaviour would categorise into the irreducible ones. After this, we follow the standard mathematics for irreducible representation of S_4. This explain theorem 10.

The fiber space in Figure 6.7 is embedded/associated with the discrete topological space in Figure 6.6 and act as a single algebraic object isomorphic to the structure of fiber bundle; as well as computationally expressed by topological interactive machine. This explains theorem 12. The Figure 6.8 further puts everything together in a broader perspective. The structure expresses all the class of behaviour of the Hardy model; possibilistic, non-possibilistic and $LC - GI$ which corresponds to feasible (black), infeasible (red) and contextual (blue) respectively. Since, Hardy Model is logically contextual so there are some path that are possible in the fiber space. The fiber bundle provides behavioural bisimulation of empirical models. One can think of TIM as a labelled transition system in a discrete topological space; in which every action depends on the feasibility of its corresponding simplices of the structure as well as could change its global configuration of the contexts from local transition. The generalisation expresses the behaviour bisimilarity of empirical models as well as the degree of contextuality; for instance in Hardy model there are set of global section but in Kochen specker there is no global section possible.

The models works as defined by TIM with two components the discrete base space and the fiber space whose quotient forms the total space. In our case, we are already given the model; which gives us an edge to know which simplices to consider and not to be accounted, for a real time model the model would require extensions. This is reason why we are comfortable with discrete Morse theory to condition the simplices because we already know different colour coding of simplices based on nature of events. It is sufficient for our case.

Remark 16. *The idea is to construct the total space associated with empirical models in terms of base space modulo fiber space; but we prefer to keep it in terms of semi-direct product for computational purpose so as to reflect the TIM computation.*

Remark 17. *TIM is a labelled transition system in a discrete topological space; in which every action depends on the feasibility of its corresponding simplices of the structure as well as could change its global configuration of the contexts from local transition. The generalisation expresses the behaviour bisimilarity of empirical models as well as the degree of contextuality; for instance in Hardy model there are set of global section but in Kochen specker there is no global section possible.*

Remark 18. *The different colour coding of simplices of the discrete topological space (based on discrete Morse theory) realised as simplicial complex are further validated or disregarded by the fiber space computations, because the total space is the discrete space modulo fiber space. Comparing it with the working of TIM (as in the example), at each iteration each state in the fiber space observes its neighbourhood via π and embedding function and decides to make the action based on the weight of its corresponding simplices (red or green); afterwards checking post-conditions at intersection and updating the weights of simplices. In this process some weights change from red to green when whole space discrete manifold modulo fiber is considered.*

The other point is about remaining models. After providing an overview of Hardy Model, I think it would be easy to understand other models. I can further facilitate; observe let's say in Kochen-Specker model there are blue dotted lines, which gives one iteration as in the Hardy case. So, the construction proofs given as concrete diagrams essentially express every iterative step, where each iteration is differentiated by vertical dotted lines in blue. The whole discrete manifold of genus= 6 represents the Kochen Specker. The model is strongly contextual with no possible path in state space as shown in figure 6.13 of thesis. Notice the horizontal dotted lines in blue and orange represented that it is a whole surface. The model expresses all class of behaviour represented in blue, black and red; as well as its irreducible fiber space invoke no section; hence strongly contextual. It is similarly proceeded in other models.

6.2.1 Computational Generalisation of Empirical Models

The general procedure has two broad steps based on results of Section 5.1: The procedure constructs a discrete topological space associated with state space based on the given information of possibilistic table of empirical models of quantum physics. Both represent TIM computational paradigm.

Topological environment The discrete topological space is constructed in the following steps:

1. A simplicial complex $\mathcal{K}$ is constructed from the measurement set of the possibilistic table by considering its power set which is equipped with a discrete topology. $\mathcal{K}$ provides all possible ways in which the basic set-up of the table can be performed.

2. $\mathcal{K}$ constructed from the first step is unable to distinguish between the same contexts because there is a set of same contexts in its combinatorial description. It would be difficult to recognise a specific context among this set which could be turned critical in the next step. To distinguish the contexts we enumerate $\mathcal{K}$ in a way so as to identify its boundary in topological sense.

3. Apply DMT on $\mathcal{K}$. It categorises $\mathcal{K}$ as non-possibilistic, possibilistic and LC-GI corresponding to critical simplices, generic simplices and non-trivial loops respectively. Loops are avoided in DMT but the possibility arises due to equidistant critical simplices.

4. Not all non-possibilistic events can be expressed in the third step because DMT constrains the way different contexts $\mathcal{K}$ are connected. So, iterate the space by constructing simplices until all of the non-possibilistic and LC-GI events are expressed. It is a brute force exhaustive approach.

PTM Computation The TIM computational paradigm addresses the permutation of the state (fiber) space and consists of the following steps:

1. A regular polyhedra is constructed based on measurement set which represents its different permutations in the state space. Each symmetry of the polyhedra corresponds to a permutation of its measurement set which is represented as its associated polyhedra group.

2. The next step is to find an irreducible representation of polyhedra group which represents its symmetries as invertible matrices forming a group.

3. The valid contexts considered are based on the contexts in the irreducible representation of the group. Any context which belongs to CS of the topological environment is not taken into account because it may not be in the set of irreducible contexts of the fiber space. Both computation and structure constrain each other.

4. Since the fiber space is attached with the base space, the computation (transformation) over the fiber space depends on the type of simplices of its underlying space. Schur lemma for irreducible representation of groups express feasible, infeasible and contextual computation based on $\mathcal{K}$.

6.2.2 The Hardy Model

Our goal is to first construct a simplicial complex $\mathcal{K}$ from the given Hardy table in Figure 6.1 and then address the permutation in state space subject to $\mathcal{K}$ following the general procedure based on main results of Section 5.1. We provide an extensive explanation of Hardy model and constructive proofs of other models are included below 6.2.7. Correspondingly, see construction of other models including Kochen Specker structure in subsection 6.14 and computation in subsection 6.13, Greenberger-Horne-Zeilinger structure in subsection 6.11 and computation in subsection 6.12 and Popescu-Rohrlich structure in subsection 6.9 and computation in subsection 6.10.

The first phase of the general procedure to construct $\mathcal{K}$ is the following:

Combinatorial description of the Hardy table A simplicial complex $\mathcal{K}$ is constructed from the measurement set $\{a, a^{'}, b, b^{'}\}$ by considering its power set which is equipped with a discrete topology as shown in Figure 6.2. $\mathcal{K}$ provides

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0

Figure 6.1 The Hardy table

all possible ways in which the basic set-up of the table can be performed. For example, one of the possible set-up would be when A selects $a^{'}$ and B selects b at time t. Afterwards at time $t^{'}$, A then selects a and B selects $b^{'}$. $\mathcal{K}$ gives more information about different possible contexts when simultaneously measurement which is not determined directly in the table.

For example, $\mathcal{K}$ takes into account three or four contexts, e.g., $ab^{'}a^{'}$. It say, A chooses a and B chooses $b^{'}$. While B chooses $b^{'}$, A in the different possible set-up chooses $a^{'}$. It can also take into account when one of the agents doesn't measure in a given round. For example, consider $\Delta a^{'}bb$ where A chooses $a^{'}$ and B chooses b, afterwards A doesn't choose anything while B chooses b. Pertinently, K takes into account all the possible ways (paths) in which the set-up of the table can be performed.

The contexts bb, $b^{'}b^{'}$, aa and $a^{'}a^{'}$ are not considered in the table therefore don't change the topological space associated with the table. We could use cell complexes (CW-complexes) that can avoid these contexts instead of $\mathcal{K}$ but both give same topological information. For example, the torus triangulated using $\mathcal{K}$ yields same number of critical simplices in each dimension using DMT as it would using cell complexes. Both have same topological realisation. The *italic* a, b representing measurement set/contexts in contextual semantics is represented in normal font a, b as its combinatorial counterpart in the diagrams which are *woven* together from set $\{a, a^{'}, b, b^{'}\}$. The original naming of contexts in different empirical models kept same in their combinatorial counterparts realised as $\mathcal{K}$.

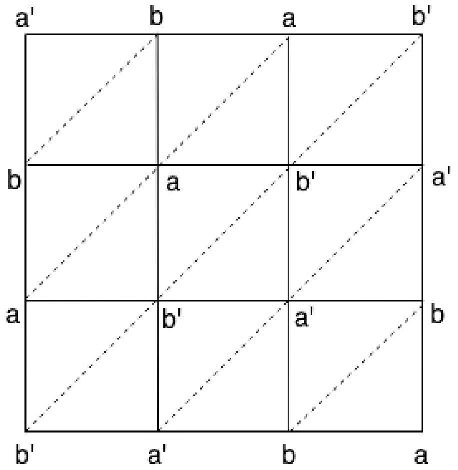


Figure 6.2 Combinatorial description of Hardy table

Element of choice We cannot associate a specific output from different possible outputs, i.e., (00), (01), (10) and (11), directly to each of the contexts, say $a^{'}b$, *a priori* from the table. The reason is that it gives no choice to the agents for *choosing* the measurement setting and in turn makes hidden variables a self-fulfilling prophecy which can reproduce any behaviour. It brings in a problem to distinguish between same contexts if one gives choice to each measurement set, i.e., each vertex is associated with all possibilities of the outputs to allow choices for agents to choose any context.

Problem of indistinguishability of same contexts: A problem arises when an element of free choice is given to agents. Let's say, $(a^{'}b)$ is non-possibilistic event then there are almost six $(a^{'}b)$ corresponding to different ways the set-up was run as shown in Figure 6.2. It is not possible to distinguish between same contexts. In TIM computational paradigm, a function $\pi : \mathbf{S} \rightarrow \mathcal{V}$, associates each state/output s in fiber to its corresponding vertex $\mathcal{V}$ of $\mathcal{K}$ known as embedding of a state in state space, i.e., $Emb(\pi^{-1}(S))$, or simply $Emb(\mathcal{V})$ as a set of all simplices containing $\mathcal{V}$ as its element. There is an edge in the fiber space, if and only if, $CS \cap Emb(\mathcal{V}) = \emptyset$. Following the definition of TIM, with slight abuse of notation, $\pi^{-1}(00) = a^{'}b$ and $Emb(a^{'}b) = \{a^{'}, b, a^{'}b, bb\}$. $CS \cap Emb(\mathcal{V}) = \{a^{'}b\} \cap \{a^{'}, b, a^{'}b, bb\} = \{a^{'}, b, a^{'}b\} \neq \emptyset$, therefore, the edge in the fiber space cannot be connected. But it would be true for all $a^{'}b$ irrespective of being possibilistic or non-possibilistic event at the given output. Say, if we take another $a^{'}b$ anywhere on K, it will not allow an edge over fibers even if they are possibilistic due to lack of indistinguishability between similar contexts. The context $a^{'}b$ at (00) is non-possibilistic event represented as critical simplices CS of $\mathcal{K}$ as shown in the table 6.1, i.e., $CS = \{a^{'}b\}$. Since each fiber has all possible values as output, i.e., 0 and 1, so it is not possible to distinguish between different $a^{'}b$ in its combinatorial description. There are many $\{a^{'}b\}$ in its corresponding $\mathcal{K}$ as can be seen in Figure 6.6.

Solution: The issue is solved by enumerating $\mathcal{K}$ in such a way that its boundaries are identified shown in dotted lines in Figure 6.3.

Explanation: The contexts are arranged in such a way that no combination of contexts along horizontal and vertical lines are repeated. Each line of the Figure 6.3 has unique context combination. We choose say context $a^{'}b$ at (00) which is non possibilistic event from the Hardy table. We set the unique path with same initial and final point $a^{'}$ due to identification carried out by enumerating $\mathcal{K}$ as in Figure 6.3. A function *surfing* over it will be possible only if the path doesn't contain CS in the sense of TIM. If there is no path possible from $a^{'}$ to itself then there is no edge over its fiber unlike other same context $a^{'}b$ at possibilistic event. In this way, same contexts are distinguished which turn $\mathcal{K}$ to be homeomorphic to a torus. It is more evident from Figure 6.4.

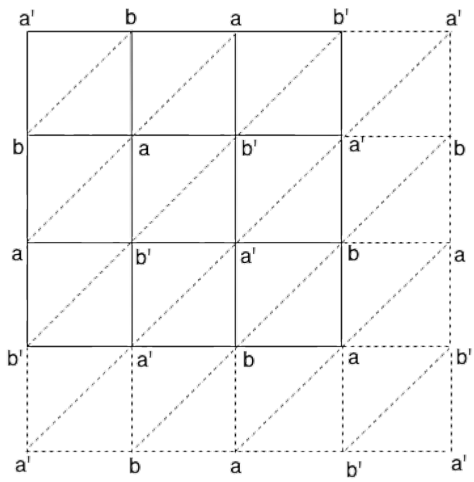


Figure 6.3 First enumeration for distinguishability of same contexts.

Enumeration based on DMT If one calculates its Euler characteristics χ, the number of vertices v are 25, edges e are 89 and faces f are 64, $\chi(\mathcal{K}) = v - e + f = 25 - 89 + 64 = 0$, which is Euler characteristic of torus. The boundary of the simplicial complex is identified. It is homeomorphic to a torus. Based on the construction of DMT, the contexts ab' and $a'b'$ are non-possibilistic shown in red as critical simplices. There is also one 2-simplex shown as critical based on three contexts at a time which is not expressed in the table as shown in Figure 6.4. Any of the critical simplices will only be considered effective based on the allowed permutation over fiber space discussed in second phase of the generalisation.

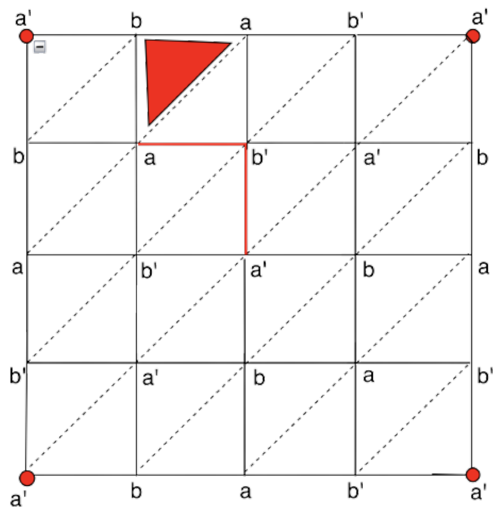


Figure 6.4 DMT on first enumeration.

The space constrains the way contexts are connected so there remains another non-possibilistic event $a'b$ which cannot be expressed in 1-torus due to constraints put by $\mathcal{K}$. As a result, we enumerate $\mathcal{K}$ to consider context $a'b$ which is a 2-torus as shown in Figure 6.5. It has $v = 32$, $e = 106$ and $f = 72$ with $\chi(\mathcal{K}) = -2$.

The 2-torus doesn't take into account the context (a, b) which is LC-GI in the table. It is the non-trivial loop over $\mathcal{K}$ which is not possible over 2-torus at context (a, b). LC-GI would mean an edge between (ab) at (00) is possible because there is no critical simplices but due to non-trivial loop it is not possible.

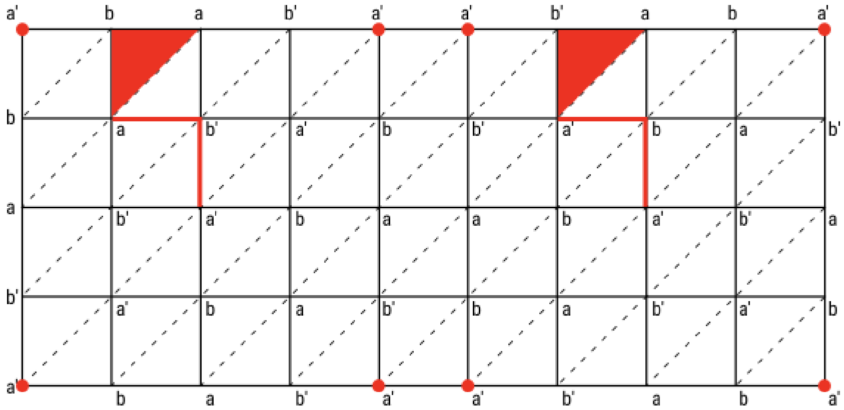


Figure 6.5 Second enumeration for containing other non-possibilistic contexts of Hardy table.

There is a possibility of non-trivial loops due to equidistant critical points. The edge joining points $a^{'}$ and $a^{'}$ marked as blue has a choice to go in either directions. Let's say, at some time it chooses different direction (marked by red arrow). It turns out that there is a possibility of a non-trivial loop for example triangle bba taking into account the context (a, b)

Following the construction, K is enumerated to be a 3-torus. $v = 48$, $e = 164$ and $f = 112$ with $\chi(\mathcal{K}) = -4$. The resulting simplicial complex $\mathcal{K}$ based on contexts of the table effectively triangulates a topological space X homeomorphic to a 3-torus as in Figure 6.6.

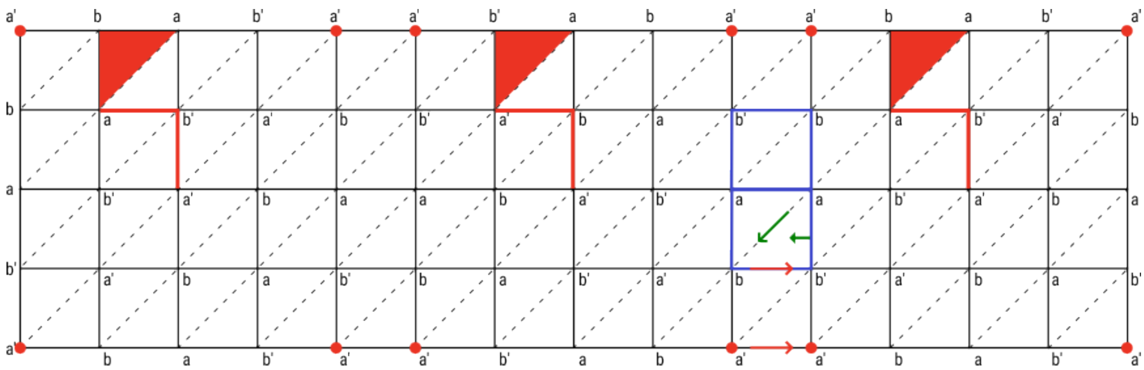


Figure 6.6 Final enumeration: Topological environment of Hardy Model. The red simplices represent infeasible computation, blue region represent possible non-trivial loops for contextual behaviour and black simplices show feasible computation.

Each vertex of the $\mathcal{K}$ realised as 3-torus has a fiber attached to it. It consists of all possible outputs providing element of choice for the agents. Broadly, the space consists of three classes of behaviour, one class is the subset of generic simplices which represents possibilistic/feasible behaviour, the second class is subset of critical simplices which represents the non-possibilistic/infeasible behaviour and third class is set of non-trivial paths which represents LC-GI behaviour. Instead of accounting all the fibers over each vertex, we are interested to extract an irreducible behaviour that can be expressed over the space.

The second phase addresses the permutation of the state space and consists of the following steps:

Fiber space for computation The measurement set $\{a, a^{'}, b, b^{'}\}$ consists of four elements which is represented as a regular tetrahedron. Each symmetry of the tetrahedron corresponds to a permutation of its four vertices which is represented as tetrahedral group. Every element of the tetrahedral group permutes the vertices of the regular tetrahedron among themselves. It accounts all the 24 symmetries of the tetrahedron along different planes. The group of all self-isometries that sends a regular tetrahedron to itself is isomorphic to S_4 symmetry group. So the group of all these symmetries corresponds to subgroups of S_4. The measurement set represented as a regular tetrahedron is shown in Figure 6.7. On the left of the figure, the tetrahedron is linearised. Let arbitrary finite groups F_i act on each edge. For example, F_1 acting on $a^{'}b$ edge would correspond to one of the symmetry operation on the regular tetrahedron. There are 5 conjugacy classes of S_4 and $|S_4| = 24$ which represent the 24 symmetries of tetrahedron. The elements of F_i are partitioned in 5 classes of S_4. Among them, 12 are forbidden symmetries.

Irreducible Representation The next step is to find an irreducible representation of S_4 which represents its 24 symmetries as invertible matrices forming a group. Each element of the group is given its corresponding matrix which is represented over the state space (fiber) as a representation of group F. The arbitrary groups F_i where $i = 6$ are subgroups of S_4. The 24 elements which are presented as matrices via homomorphism are categorised into conjugacy classes of S_4 based on the type of transformation based on Theorem 5.1. The Theorem 19 is synthesis of this step in which transformation **T** is allowed based on the Theorem 5.1 expressing computation subject to the Theorem 5 expressing global constrain.

Since the fiber space is attached with the base space as shown in 6.7 is embedded in a discrete space of Figure 6.6. So, the transformation **T** over the fiber space depends on the type of simplices of its underlying space.

TIM paradigm The transformation in fiber space is based on its corresponding path over the discrete manifold. Following the above three cases, we have, two edges $ab^{'}$, two edges $a^{'}b^{'}$, two edges $a^{'}b$ and one 2-simplex $aa^{'}b^{'}$ as critical on the discrete space which correspond to 8 infeasible transformations in fiber space shown in red. The two 2-simplex aab are not considered because they are not a valid permutation in the fiber space. There is a possibility of a non-trivial path ab which is based on one cell, formed of 4 ab edges corresponding to LC-GI shown in blue in Figure 6.8. Finally, transformations based on generic simplices correspond to feasible transformations which are in 12 over fiber space shown in green. From the subgroup perspective of S_4; the 8 elements form a dihedral group D_4, 4 elements form $\mathbb{Z}_2$ group and 12 elements form A_4 group.

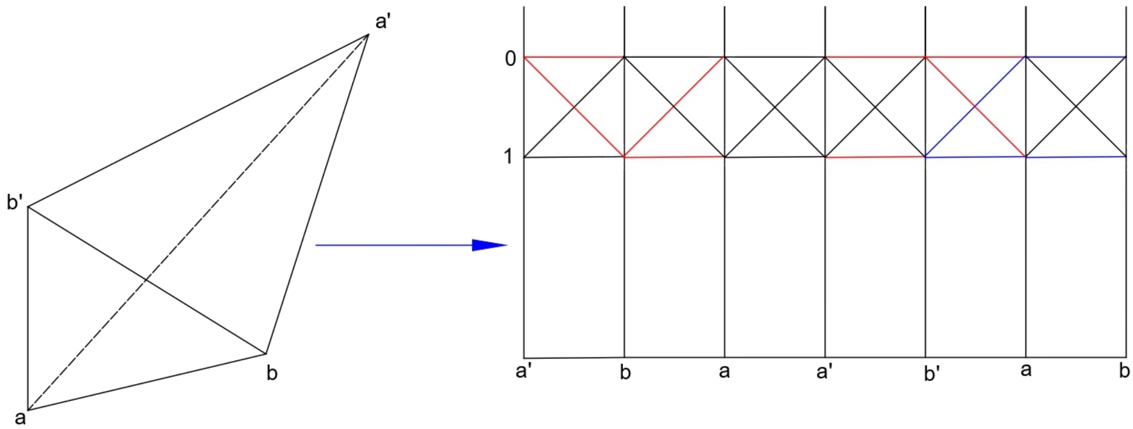


Figure 6.7 Hardy Model s logically contextual due to possible computations in state space.

Theorem 9. *Symmetry group* S_4 *represents Hardy model computation.*

Proof. Hardy table is isomorphic to character table of S_4 symmetry group. It is represented as tetrahedron over measurement set of size 4. Its permutations $4! = 24$, represent the order of S_4 that are linearised over state space. The character table of S_4 is locally consistent (LC) satisfying Schur orthogonality relations but there are inequivalent irreducible representations responsible for global inconsistency (GI). It consists of 5 conjugacy classes where 12 elements are trivial permutations that correspond to alternating group A_4 group. The tetrahedron forms three cycles, two of which are LI-GI and one cycle is non-trivial path $\mathcal{P}$ which corresponds to LC-GI. So the remaining 8 non-trivial elements form dihedral group D_4 (non-abelian) and other 4 elements form Klein group K_2 (acyclic). The S_4 explores complete representation of Hardy model with $S_4 \supset A_4 \supset D_4 \supset K_2 \supset \mathfrak{e}$ subgroup inclusion. The non-abelian structure of S_4 implies non-locality of the Hardy model. The explicit topological structure of Hardy model is described in Section 6.1 shown as Figure 6.6 isomorphic to 3-torus that is constructed from 12 non-trivial elements of S_4. □

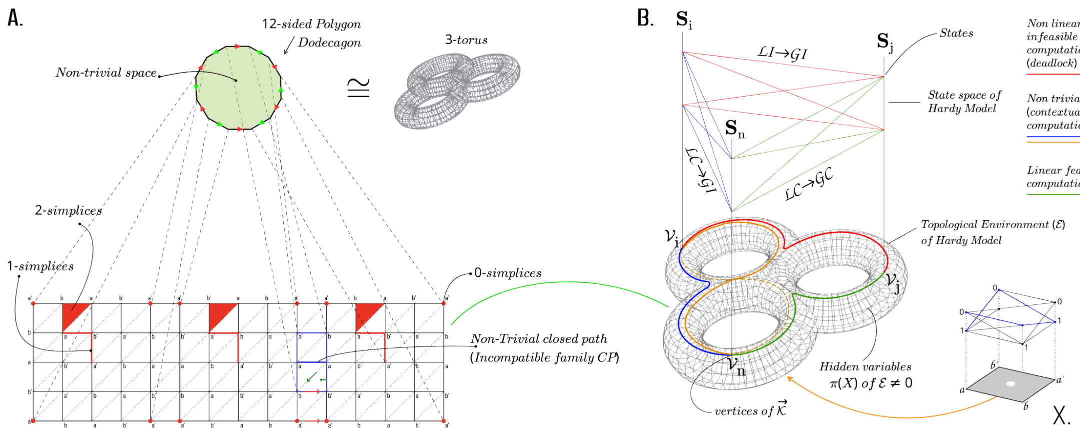


Figure 6.8 Topological realisation of the Hardy model HM as 3-torus. The right side represents the computation over hardy model with $g = 3$. The blue line $\mathcal{V}_1 \to \mathcal{V}_2$ has obstructions (two holes) that correspond to infeasible computation like a deadlock ($LI \to GI$), the green line $\mathcal{V}_2 \to \mathcal{V}_3$ has no non-linearity in between that corresponds to feasible computation ($LC \to GC$) and the red line $\mathcal{V}_3 \to \mathcal{V}_1$ goes around obstruction (holes) forming a non-trivial closed path as brown line which corresponds to locally consistent and globally inconsistent computation ($LC \to GI$). The brown arrow from X to B infers its base space as 3-torus constrains its computation.

Theorem 10. *Symmetry group* S_4 *represents Hardy model*

Proof. Proof by construction. □

Theorem 11. *Any empirical model representable by Hardy model type scenario has a topological representation of a surface of genus 3 and is logically contextual in nature.*

Theorem 12. *For the Hardy model* $\mathcal{G} = \mathrm{S}_4 \rtimes \pi(T^3)$

6.2.3 Kochen Specker Model

Theorem 13. *Dihedral group* D_5 *represents the Kochen-Specker model which represents the symmetry of the pentagonal bipyramid.*

Theorem 14. *Any empirical model representable by the Kochen-Specker type scenario has a topological representation of a surface of genus 6 and is strongly contextual in nature.*

Theorem 15. *For the Kochen-Specker model* $\mathcal{G} = \mathrm{D}_5 \rtimes \pi(T^6)$

6.2.4 Greenberger-Horne-Zeilinger Model

Theorem 16. *Octahedral group* O_h *represents the Greenberger-Horne-Zeilinger model which represents symmetry of the octahedron.*

Theorem 17. *Any empirical model representable by the Greenberger-Horne-Zeilinger type scenario has a topological representation of a surface of genus 8 and is strongly contextual in nature.*

Theorem 18. *For the Greenberger-Horne-Zeilinger model* $\mathcal{G} = \mathrm{O}_h \rtimes \pi(T^8)$

6.2.5 Popescu-Rohrlich Boxes

Theorem 19. *Symmetry group* S_4 *represents the Popescu-Rohrlich boxes which represents the symmetry of tetrahedron.*

Theorem 20. *Any empirical model representable by the Popescu-Rohrlich boxes type scenario has a topological representation of a surface of genus* 4 *and is strongly contextual in nature.*

Theorem 21. *For Popescu-Rohrlich boxes model* $\mathcal{G} = \mathrm{S}_4 \rtimes \pi(T^4)$

6.2.6 Peres-Mermin Magic Square

Theorem 22. *Octegedral group* O_h *represents the Peres-Mermin Magic Square.*

Theorem 23. *Any empirical model representable by the Peres-Mermin magic square has a topological realisation of a 1-torus* T^1*, and is contextual in nature [38].*

Theorem 24. *For the Mermin square* $\mathcal{G} = \mathrm{O}_h \rtimes \pi(T^1)$

6.2.7 Proofs by Construction

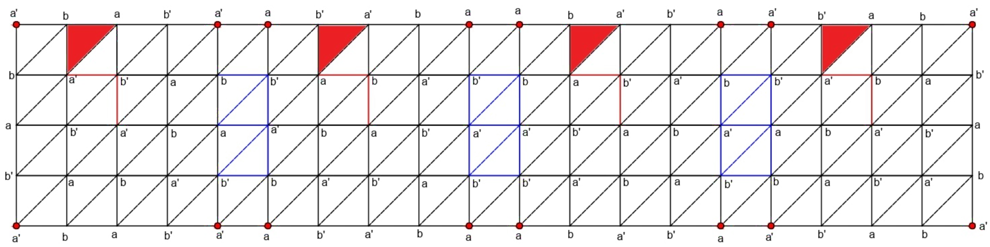


Figure 6.9 Popescu-Rohrlich box is represented by a surface with genus 4. The red simplices represent infeasible computation, blue region represent possible non-trivial loops for contextual behaviour and black simplices show feasible computation.

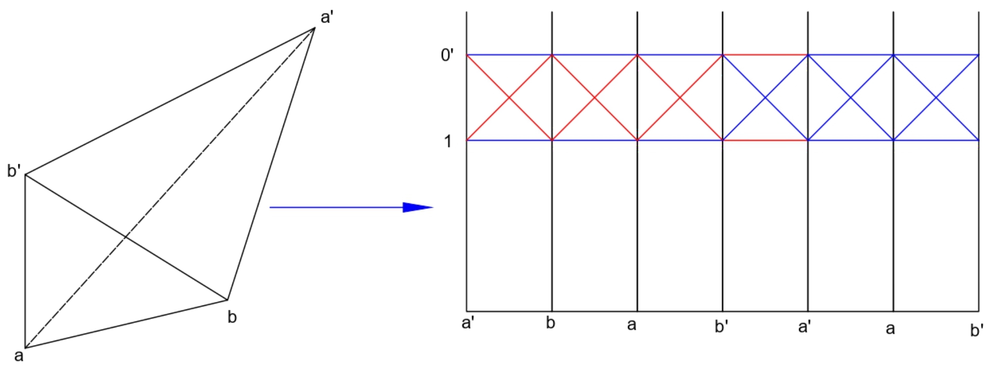


Figure 6.10 Popescu-Rohrlich computation. It is strongly contextual due to no possible paths/computation over the state space. Red paths and blue paths represent infeasible and contextual computation respectively which corresponds to the red and blue simplices of its underlying discrete surface as in Figure 6.9. The state space of hardy model is also represented by a tetrahedron but unlike Popescu-Rohrlich box it allows computation in its state space due to discrete surface of genus 3 6.7.

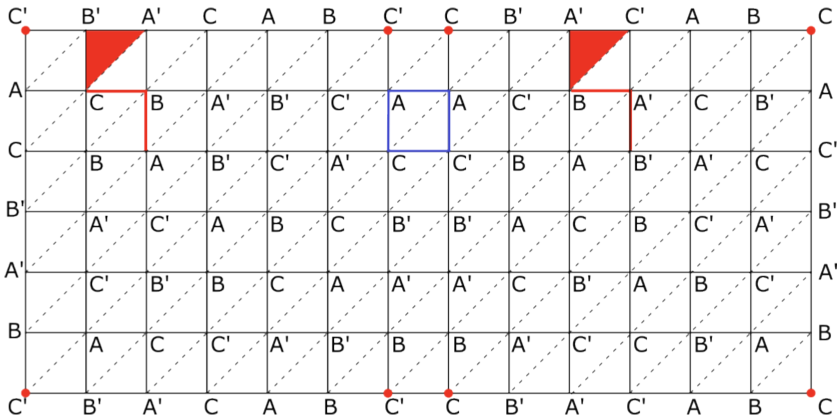


Figure 6.11 Greenberger-Horne-Zeilinger Model is represented by the surface of genus 8. The discrete surface is 4 times of the above figure but for simplicity, it is shown in a reduced way.

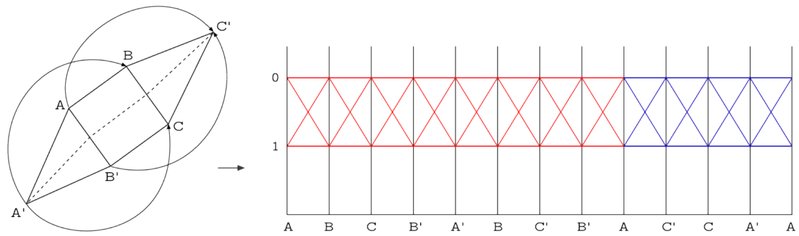


Figure 6.12 Greenberger-Horne-Zeilinger is strongly contextual with no possible computation in the state space.

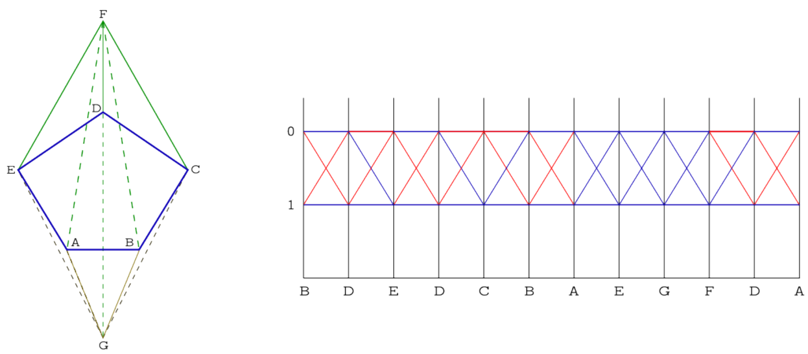


Figure 6.13 Kochen Specker model is strongly contextual with no possible computation in the state space.

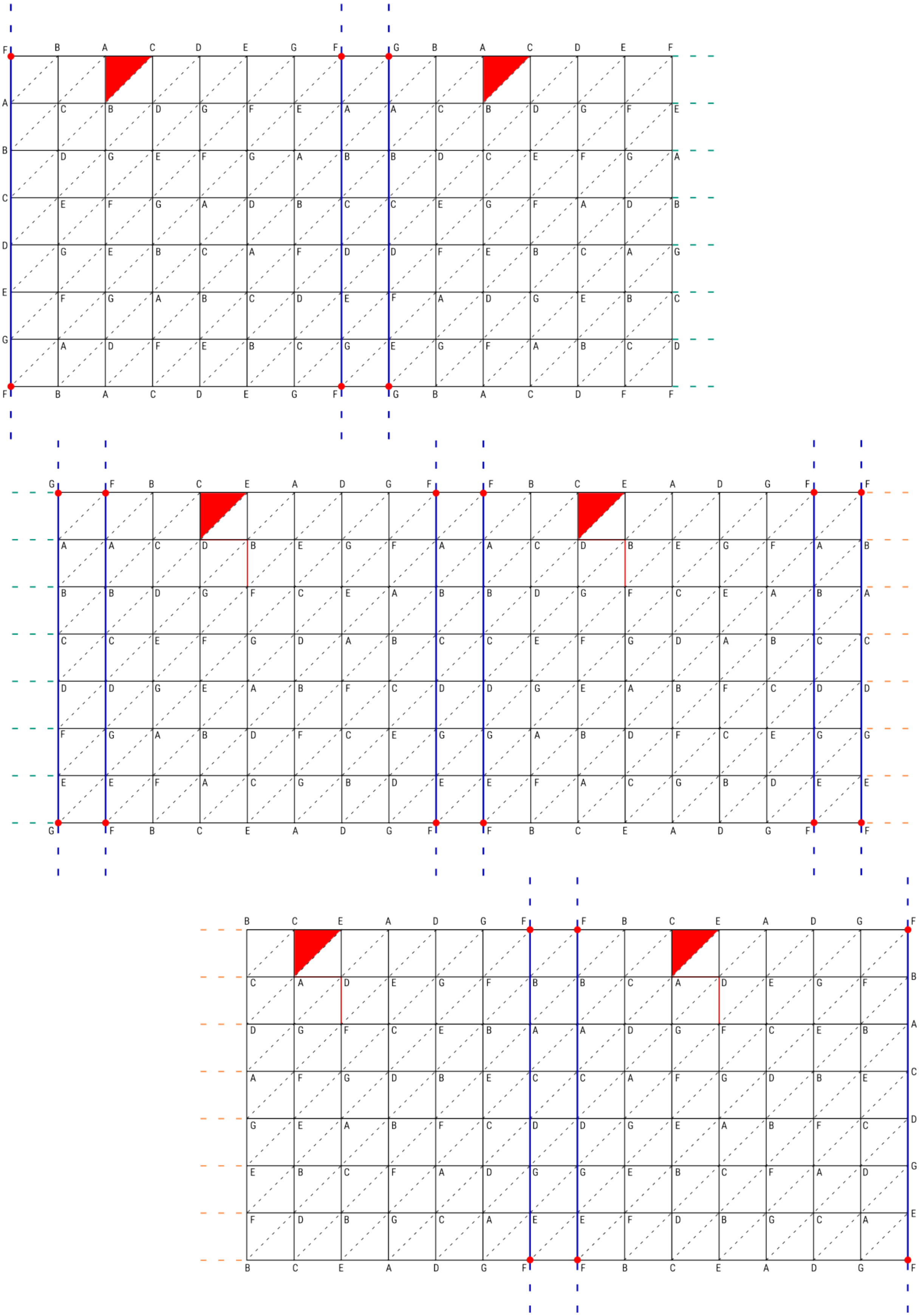


Figure 6.14 The whole discrete manifold represents the Kochen Specker model having genus= 6. The model is strongly contextual with no possible path in state space as shown in diagram 6.13.

Chapter 7

Discussion

The chapter discusses possible direction of our work. We present few possible directions of our work:

1. The preliminary results of the study can provide a way to explore the foundations of quantum advantage. The non-trivial loops in the discrete manifold are responsible for $LC - GI$ which are sufficient to construct the ambient topological space associated with computation. It means information about the class of loops could exempt most of the generic information leading to quantum advantage in the computationally hard tasks. It would be interesting to measure the computational complexity of the TIM in order to quantify quantum advantage. In particular, we would like to compute the Jones polynomial in the context of the TIM. The background mathematical structure of the TIM is operated by a topological field theoretic paradigm which makes the machine a natural candidate to compute it. The machine can use loops as a resource for quantum advantage to give a polynomial time algorithm to compute the Jones polynomial. The TIM computational paradigm based on *topological field theory of data* gives access to a lot of beautiful mathematics, for instance quantum group and Hopf algebra, and a cross-disciplinary platform [42].

2. The connection can be explored in the light of topological circuits and non-commutative linear logic which could provide implementation and programming principles for hypercomputing. The *topological insulators*

based on interaction between quasiparticles emerges from complex interaction between large numbers of other particles for ultrafast computation. It is similar to the TIM computational model in which an action emerges given the global contexts realised as the topological environment. In theory, a highly correlated system gives rise to *exotic states* in condensed matter physics. Exotic states follow topological field description which forms the backbone of the TIM. The circuit machinery based on exotic state, for instance superconducting and topological circuits, could be seen in the light of the TIM.

3. It would be interesting to explore questions like, *can we interpret the significant aspects of quantum mechanics over TIM?* The study expects to initiate discussions on *paradoxical* reasoning in non-quantum computation.

4. The connection could provide a novel approach to compositionality. It can also address novel concepts like intentionality and perception in the nature-inspired model of computation and artificial intelligence with applications to emulate the *biological cell.* and neuroscience.

5. Reaction systems have been studied as a formal framework for modelling system-environment interaction in biological systems as well as nature-inspired models of interactive computations. A formal reaction system is a triplet that represents the reactants R, inhibitors I and the product P. Whenever all reactants and none of the inhibitors are present, the reaction yields the product. Moreover, there is a universal set of elements that can enter the system from the outside environment and interact with the reaction process at any given time. In most methods [27, 26, 20], the evolution of a reaction system through inclusion of environmental variables have not been considered. Ehrenfeucht and Rozenberg introduced a formal description of biochemical interactions within a bounded porous membrane that can interact with the environment [9, 15]. The initial formal model for reaction system was set-theoretic [9] and was extended to graph-based reaction systems based on *surfing* within *universe* graph through its subgraphs, creating trajectories that provide information about behaviour of the reaction systems [34]. In particular, Jonoska et al. represents the reaction system as a transition (directed) graph with vertices as states and directed edge as transitions that takes into account

the environmental input. The representation of a reaction system as a directed graph includes environmental variables that influences behaviour of the reaction system [28] and in turn influences the environment during its evolution [13]. The dynamics of a reaction system is defined as interactive process. In one step of the process, all reactions enabled by the current state are applied simultaneously and the union of their product graphs forms the successor state. In this line of research, an important theme concerning the *structure* of the system vs its behaviour was left open, which in case of graph-based reaction systems is translated as the possible influence of the structure of reactions on interactive processes. Moreover, the *context* is assumed to contribute arbitrarily in a reaction system, but in practice it correlates to the reaction system in a specific way due to its structure. The structure of context has been studied in network-based reaction systems where the context has a topological structure which originates from a network of reactions. Such a network is formalised as a graph with reaction systems residing at its nodes, where each reaction system contributes to define the context of all its neighbours. In this sense, the functioning of a reaction system is influenced by the *trivial* structure of its underlying graph [13, 14] and the interactive network process is given by a vector of individual interactive processes of reaction systems residing at the network nodes.

We extend the scope of this open question by topology-based interpretation of environment of network based reaction system in terms of $S[B]$ paradigm, introduced by of one of author EM. B is the behaviour of reaction system that represent local (internal) computation embedded in an ambient structure S that represents topological environment $\mathcal{X}$. Its homotopy information provide equivalence between reaction systems as well as biological interpretation of the topological reaction systems.

6. We expect the results provide a way to answer open question of Terry Tao: can a 'group' be a universal Turing machine?

7. The immediate study will allow us to obtain stronger results in terms of applications of Rice's theorem that relates decidability to contextual semantics, leading to conjecture about an automaton accepting language

of Hardy model is undecidable. It may provide relation between contextual semantics and undecidability.

8. The connection could provide a deeper structure of compositionality.

9. The results could correspond perfect zero knowledge and probabilistically checkable proofs with strong contextuality and non-trivial element of machine respectively in the complexity theory.

10. Given an empirical model, we were able to extract a topological space which is part of the computational model; what about reverse, i..e., given non-trivial topological space of more non-trivial loops, can we construct the table of correlations? It could allow predicting new possible class of empirical models that could be in future discovered by some physical experiment.

Bibliography

[1] S. Abramksy. What are the fundamental structures of concurrency? we still don't know! *Electron. Notes Theor. Comput. Sci.*, 162:37–41, 2006.

[2] S. Abramksy and R.S. Barbosa. The logic of contextuality. In *Proceedings of 29th EACSL Annual Conference on Computer Science Logic (CSL 2021) LIPIcs*, volume 183, 2021.

[3] S. Abramsky. *Relational Databases and Bell's Theorem, Lecture Notes in Computer Science*, volume 8000. Springer, Berlin, Heidelberg, 2011. ISBN 978-3-642-41660-6.

[4] S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(113036), 2011.

[5] S. Abramsky and M. Sadrzadeh. Semantic unification a sheaf theoretic approach to natural language. In *Categories and Types in Logic, Language, and Physics:Essays Dedicated to Jim Lambekon the occasion of his 90th birthday, LNCS*, volume 8222, pages 1–13, 2014.

[6] S. Abramsky, S. Mansfield, and R. S. Barbosa. The cohomology of non-locality and contextuality. In *Proc. 8th Int. Workshop on Quantum Physics and Logic (Electronic proc. In Theoretical Computer Science), (Nijmegen, Netherlands, 27–29 October)*, volume 95, pages 1–14, 2011.

[7] S. Abramsky, G. Gottlob, and P. G. Kolaitis. Robust constraint satisfaction and local hidden variables in quantum mechanics. In *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence*, pages 440–446, 2013.

[8] Samson Abramsky, Esfandiar Haghverdi, and Philip J. Scott. Geometry of interaction and linear combinatory algebras. *Mathematical Structures in Computer Science*, 12:625 – 665, 2002. URL https://api.semanticscholar.org/CorpusID:16756583.

[9] G. Rozenberg A.Ehrenfeucht. Reaction systems. *Fundamenta Informaticae*, pages 263–280, 2007.

[10] J. Baez and J. Master. Open Petri nets. *Mathematical Structures in Computer Science*, 30(3):314–341, 2020.

[11] R. B. Barbosa. *Contextuality in quantum mechanics and beyond.* PhD thesis, University of Oxford, 2015.

[12] J.S Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1:195–200, 1964.

[13] P. Bottoni, A. Labella, and G. Rozenberg. Reaction systems with influence on environment. *Journal of Membrane Computing*, 1:3–19, 2019.

[14] P. Bottoni, A. Labella, and G. Rozenberg. Networks of reaction systems. *Int. J. Found. Comput. Sci.*, 31:53–71, 2020.

[15] R. Brijder, A.Ehrenfeucht, M. Main, and G. Rozenberg. A tour of reaction systems. *Int.J.Found.Comput.Sci.*, 22(1499-1517), 2011.

[16] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner. Bell nonlocality. *Rev. Mod. Phys*, 86(419), 2014.

[17] G. Carlsson. Topology and data. *Bulletin of the American Mathematical Society*, 46(2), 2009.

[18] G. Carù. On the cohomology of contextuality. In *QPL*, 2016.

[19] Gregory J. Chaitin. Meta math!: The quest for omega. 2004. URL https://api.semanticscholar.org/CorpusID:119646414.

[20] A. Dennunzio, E. Formenti, L. Manzoni, and A. Porreca. Reachability in resource-bounded reaction systems. In *LATA*, 2016.

[21] David Deutsch. Quantum theory, the church–turing principle and the universal quantum computer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 400:117 – 97, 1985. URL https://api.semanticscholar.org/CorpusID:1438116.

[22] David Deutsch. The fabric of reality : the science of parallel universes– and its implications. 1997. URL https://api.semanticscholar.org/CorpusID:60748989.

[23] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram, and M. Raussen. *Directed Algebraic Topology and Concurrency.* Springer International Publishing, 2016. ISBN 978-3-319-15398-8.

[24] Lisbeth Fajstrup, Martin Raussen, and Éric Goubault. Algebraic topology and concurrency. *Theor. Comput. Sci.*, 357:241–278, 2006.

[25] R. Forman. Morse theory for cell complexes. *Advances in mathematics*, 134:90–145, 1998.

[26] E. Formenti, L. Manzoni, and A. Porreca. On the complexity of occurrence and convergence problems in reaction systems. *Natural Computing*, 14: 185–191, 2014.

[27] E. Formenti, L. Manzoni, and A. Porreca. Cycles and global attractors of reaction systems. In *DCFS*, 2014.

[28] Daniela Genova, H. J. Hoogeboom, and N. Jonoska. A graph isomorphism condition and equivalence of reaction systems. *Theoretical computer science*, 701:109–119, 2017.

[29] Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987. URL https://api.semanticscholar.org/CorpusID:40737441.

[30] Jean-Yves Girard. Geometry of interaction 1: Interpretation of system f. *Studies in logic and the foundations of mathematics*, 127:221–260, 1989. URL https://api.semanticscholar.org/CorpusID:117895764.

[31] D. Goldin, S. A. Smolka, P. C. Attie, and E. L. Sonderegger. Turing machines, transition systems, and interaction. *Information and Computation*, 194(2):101–128, 2004.

[32] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry S. Yuen. Mip* = re. *Communications of the ACM*, 64:131 – 138, 2020. URL https://api.semanticscholar.org/CorpusID:210165045.

[33] S. Kochen and E.P. Specker. The problem of hidden variables in quantum mechanics. *Indiana University Mathematics Journal*, 17:59–87, 1967.

[34] H. Kreowski and G. Rozenberg. Graph transformation through graph surfing in reaction systems. *J. Log. Algebraic Methods Program.*, 109, 2019.

[35] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: a first introduction to topos theory.* Springer New York, NY, 1992.

[36] E. Merelli and A. Wasilewska. Topological interpretation of interactive computation. In *Lecture Notes in Computer Science*, volume 11500, pages 205–224, 2019.

[37] Y. Ne'eman. Classical geometric resolution of the Einestein-Poldosky-Rosen paradox. *Proc. Natl. Acad. Sci. USA*, 80:7051–7053, 1983.

[38] C. Okay and R. Raussendorf. Homotopical approach to quantum contextuality. *Quantum*, 4:217, 2020.

[39] Judea Pearl. *Causality Models, Reasoning, and Inference.* Cambridge University Press, 2009.

[40] Roger Penrose. On the cohomology of impossible figures. *Leonardo*, 25: 245–247, 1993.

[41] C.A. Petri. State-transition structures in physics and in computation. *International Journal of Theoretical Physics*, 21(12), 1982.

[42] M. Rasetti and E. Merelli. *Topological Field Theory of Data: Mining data beyond complex networks, In Advances in Disordered Systems, Random Processes and Some Applications*. Cambridge University Press, 2016. ISBN 9781316403877.

[43] R.Milner. In *The Space and Motion of Communicating Agents*, 2009.

[44] P. Wegner. Why interaction is more powerful than algorithms. *Commun. ACM*, 40(5):80–91, 1997.

[45] P. Wegner. Towards empirical computer science. *The Monist*, 82:58–108, 1999.

[46] Margherita Zorzi. On quantum lambda calculi: a foundational perspective. *Mathematical Structures in Computer Science*, 26:1107 – 1195, 2014. URL https://api.semanticscholar.org/CorpusID:30158098.

Appendix A

A.1 Genesis of Interaction as Observation

The interaction of light and matter started a fundamental shift in the classical understanding of the nature of reality. The data of observation from spectroscopy confirms that the light and the matter couples together in an irreducibly holistic way. It started with an error, known as the ultraviolet catastrophe, reflected from the data of the black body radiation. A black body is an idealised physical body that could absorbs/emit all the spectrum of the electromagnetic radiation. Rayleigh-Jean law approximates the spectra of electromagnetic radiation (light) as a function of wavelength from a black body (matter) at a given temperature. It means an ideal black body at equilibrium would emit unbounded energy at shortest wavelengths; the ultraviolet region of the spectrum. The explanation agrees with the experimental data at large wavelengths but strongly disagrees at short wavelengths. The interaction of light and matter in the ultraviolet region remained open question till the end of the 19th century.

In 1901, at the beginning of the 20th century, the German physicist Max Plank derives a new equation to describe the overall behaviour of the black body. Plank discovered that action is quantised in microcosm. Action quantifies change/evolution of a physical system over time. The energy E of a photon is proportional to the frequency ν, i.e., $E = h\nu$ where h is the Plank's constant. The ratio E/ν is a constant and independent of the substance absorbing/emitting the photon. It means E and ν cannot take any value but admissible values that satisfy their ratio to h. The quantisation of action implies that in nature, change occurs in small steps. Plank was awarded Nobel Prize in 1918 for the establishment and development of the theory of elementary quantum of action h. It opened a new way of thinking about nature of reality unlike Newtonian and Leibniz continuum. Plank himself pointed towards this fundamental shift in his Nobel Prize lecture:

> "The quantum of action must play a fundamental role in physics, and here was something entirely new, never before heard of, which seemed called upon to basically revise all our physical thinking, built as this was, since the establishment of the infinitesimal calculus by Leibniz and Newton, upon the acceptance of the continuity of all causative connections."

In the spring of 1913, Neil Bohr in his famous paper, 'On the Constitution of Atoms and Molecules', explains Rutherford's atomic instability[1] using quantum of action which won him Nobel prize in 1922. Bohr postulates atom to have discrete levels in which electrons move. The transitions between these level are admissible in accordance to h and ν. It led to a question: *Why do elementary particles in atom occupy only certain admissible levels?* Bohr gave proper meaning to h which afterwards opened a new door for a radically new understanding of nature – the quantum leap. Quantum physics is fundamentally different from Newtonian philosophy of physics and Einstein's theory of relativity.

The discovery of h initiated debates about fundamental concept of causality in physics. Bohr's interpretation of h abandons the idea of the independent causal behaviour of quantum system. The whole dynamics of a classical system can be known with definiteness given the initial conditions and the function that relates different variables. We can evaluate values of any physical quantities at any time using calculus machinery. In particular any effect can be traced back to its cause due to inherent continuity of space over which the events take place unlike h. The quantum of action h restricts definition of a well-defined function that could provide overall evolution of a quantum system. Bohr's interpretation of h shows ineffectiveness of functional description to symbolise quantum system. The function description is replaced by *act of observing*.

[1]The problem in Rutherford's model of atom was that the electron in the model would loose energy while revolving around the nucleus due to emission of electromagnetic radiations. It would rapidly spiral inwards and collapse in no time.

Observation in turn restricts assignment of simultaneous values to all variables of a system. It offers an *element of choice* to the observer in order to choose a variable from the set of variable to yield a value. The irreducible indivisibility of h implies renunciation of causal ordering. Bohr elaborates relation of h and causal mode of description in his 1929 lecture on, 'The Quantum of Action and the Description of Nature' on the 50th doctoral anniversary of Max Planck.

> "[T]he application of [classical mechanics and electromagnetic theory] to atomic problems was destined to reveal a hitherto unnoticed limitation that found its expression in Planck's discovery of the so-called quantum of action, which imposes upon individual atomic processes an element of discontinuity quite foreign to the fundamental principles of classical physics, according to which all action may vary in a continuous manner. The quantum of action has become increasingly indispensable in the ordering of our experimental knowledge of the properties of atoms. This goal has not been attained, still, without a renunciation of the causal spacetime mode of description that characterises the classical physical theories which have experienced such a profound clarification through the theory of relativity."

The conceptual understanding of quantum physics is centred on *interaction as observation*. The significance of observation can be seen in Bohr's concept of complementarity and Heisenberg's principle of uncertainty, who were among the founders of the well known Copenhagen interpretation of quantum physics. On 16th September 1927, Bohr presented his idea about concept of complementarity in a lecture, 'The Quantum Postulate and the Recent Development of Atomic Theory', at the conference held in Como Italy in the memory of the 100th anniversary of death of Alessandro Volta. Complementarity states that observation interferes the system in an uncontrollable way which was a consequence of indivisibility of h. The system cum observation forms an inseparable unity. The act of observing restricts the simultaneous values for orthogonal conjugate quantities of the system. For example, in order to measure/observe certain value of position of an electron, a short light wave is hit on it. In such a collision, the electron suffers a recoil which uncertain simultaneous value of its momentum –the Heisenberg's 1927 uncertainty principle.

The uncertainty means that value to a quantity cannot be given *a priori* rather its discovered by observation. The matrix transformation is essentially a function but the concept of observation is prominent in the form of external choice which depend on internal states of the observer and the cumulative contexts of the experiment. The uncontrolled act of observations allows values to be probabilistic in nature which is the source of statistical character of the quantum theory. Accordingly, an independent reality in the ordinary physical sense can neither be ascribed to the phenomenon nor to the agencies of observation. This new way to approach reality was not comfortable to the scientific community. In 1935, Albert **E**instein along with his postdoctoral researchers Boris **P**odolsky and Nathan **R**osen published a paper popularly known for EPR paradox to expose the incompleteness of quantum physics to describe physical reality. Einstein felt that the new approach has renounced the historic task of natural science to give explanation of different aspects of nature independent of observation.

Einstein conducted a thought experiment in which measurement/observation on one system provides definite values of any variables of other system which had interacted in past unlike Heisenberg's uncertainty and Bohr's complementarity. It allows paradoxical *instantaneous action* between space-like separated systems, violating principle of causality, or more generally local realism. Einstein states his criteria for reality in EPR paper as:

> "If, without in any way disturbing a system, we can predict with certainty (re., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding of this physical quantity. [..] Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality."

Einstein's logical disjunction in the paper states that if quantum physics is complete then quantities can have simultaneous value which turn to be a paradox in classical sense. The systems are entangled together in a non-local way which is responsible for the high correlation. Einstein called this non-triviality as *spooky action at a distance* which laid foundation of quantum information theory. Einstein believed that quantum physics was true but incomplete description of reality. He hypothesised existence of *hidden variables* that could explain the non-local effect. Bohr responded to Einstein and it went on as a series of debates popularly known as Bohr-Einstein debates.

Bohr main argument was interaction as observation and significance of experimental data to yield explanation unlike a priori philosophical conception. Bohr observes:

> "The extend to which an unambiguous meaning can be attributed to such an expression as 'physical reality' cannot of course be deduced from a priori physical conceptions but must be founded on a direct appeal to experiments and measurements. [...] – the description of the phenomenon to be studied by necessity of a final renunciation of the classical the experimental arrangement concerned, of ideal of causality and a radical revision of our which the passing of the particle through the attitude towards the problem of physical reality."

In 1964, John Stewart Bell introduced a condition known as Bell inequality. The condition would satisfy if local hidden variables exists else violates Bell inequality. Bell proved that correlations between quantum systems cannot be reproduced by any theory of local hidden variables. A variety of theorems exist which impose restrictions on existence of hidden variable[2], for instance Gleason's theorem, Von Neumann Proof and Kochen Specker theorem. Several experiments were carried out to test quantum mechanics in connection to local realism theory. Stuart J. Freedman and John Clauser carried out first Bell test followed by Alain aspect and his team until 2015 closure of loopholes in Bell test by Anton Zeilinger.

Last year in 2022, Nobel prize was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger for experiments with entangled photons which establishes the violation of Bell inequalities and pioneering quantum information science.

[2]From the interactive dialogue between leading figures of the time various interpretations of quantum physics emerged, for instance Bohmian quantum mechanics. It provides causal explanation of quantum correlations based on non-local hidden variables. Even though not widely accepted in community provides novel approach to the foundations of quantum physics.

A.2 Condition for Locality

In Bell scenario the outcomes are not statistically independent from each other ($p(m_1m_2\ |s_1s_2) \neq p(m_1|s_1)\ p(m_2|s_2)$). Even though the two systems may be spacelike separated, the existence of such correlations is not mysterious. In particular, it does not necessarily imply some kind of direct influence of one system on the other. These correlations may simply reveal some interdependence between the two systems which was established when they interacted in the past. These correspond to correlations of local realism. The assumption of locality implies that we should be able to identify a set of past factors, described by some variable λ, having a joint causal influence on both outcomes, and which fully account for the dependence between m_1 and m_2. Once all such factors have been taken into account, the *residual indeterminacies* about the outcomes must now be decoupled; that is, the probabilities for m_1 and m_2 should factorise $p(m_1m_2\ |s_1s_2,\ \lambda) = p(m_1|s_1,\ \lambda)\ p(m_2|s_2,\ \lambda)$. This *factorisability* condition simply expresses the fact that we have found an explanation according to which the probability for m_1 depends only on the past variable λ and on the local measurement s_1, but not on the distant measurement and outcome, and similarly for the probability to obtain m_2. The condition for locality for all distribution of λ can be written as $\int_\Lambda d\lambda\ q(\lambda)\ p(m_1|s_1,\ \lambda)\ p(m_2|s_2\ \lambda)$ where it is implicitly assumed that the measurements s_1 and s_2 can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda|s_1, s_2 = q(\lambda))$. This decomposition represents a precise condition for locality in the context of Bell experiments. It is worth noting that no assumptions of determinism or of a "classical behavior" is being involved: it is assumed that m_1 (and similarly m_2) is only probabilistically determined by the measurement s_1 and the variable λ, with no restrictions on the physical laws governing this causal relation. Locality is the crucial assumption behind above decomposition. In relativistic terms, it is the requirement that events in one region of space-time should not influence events in spacelike separated regions. But it is now a straightforward mathematical theorem that the predictions of quantum theory for certain experiments involving entangled particles do not admit a decomposition of the form above[2]. In discrete form locality condition in terms of the single site probabilities can be written as a convex combination over all deterministic maps, i.e. $p(m|s,\lambda) = \sum_{g_j} p_{g_j} \delta^{m_j}_{g_j(s_j)}$ where

[2]From CHSH inequality if probability satisfy locality decomposition we have $S \leq 2$. But for entangled pair of photons there is a contradiction $S = 2\sqrt{2} > 2$ Hence contradiction with the locality constraint. It establishes nonlocal character of quantum theory

g_j are the single site maps. The decomposition equation can be rewritten as

$$p(m|s) = \sum_{g_1,g_2,\cdots g_n} p_{g_1,g_2,\ldots.g_n} \prod_{j}^{\mathscr{N}} \delta^{m_j}_{g_j(s_j)}$$

taking convex combination of all combinations of single site map g_j so that $p_{g_1,g_2,\ldots.g_n} \geq 0$ and $\sum_{g_1,g_2,\cdots g_n} = 1$ is satisfied. The decomposition is not unique; uniqueness is only guaranteed when $\sum_{g_1,g_2,\cdots g_n} = 1$ for a particular choice of single site maps. Bell's theorem violates this condition depicting nonlocal behaviour.

A.3 Bell Scenario and Fiber bundle

Consider $\mathscr{N}$ correlated space-like separated system. Each j^{th} site for $j \in \{1, 2, ..., \mathscr{N}\}$ makes a measurement $\mathscr{M}_{s_j}$ from a choice of c_j measurements, where $s_j \in \{1, 2, ..., c_j\}$ labels the choice of measurement and can be expressed as cyclic group c_j denoted as $\mathbb{Z}_{c_j}$. Each measurement $\mathscr{M}_{s_j}$ has $d(s_j)$ possible outcomes $\mathscr{O}_{m(s_j)}$, where $m(s_j) \in \{1, 2, ..., d(s_j)\}$ can be expressed as an element of cyclic group of $d(s_j)$, denoted as $\mathbb{Z}_{d(s_j)}$. In operational terms, each j^{th} box takes an input s_j and returns an output $m(s_j)$. Assume that $d(s_j) = d_j$ for simplicity. The number of possible measurements $\mathscr{M}_{s_j}$ may be different for each measurement choice s_j at each site j, and that number of possible outcomes $\mathscr{O}_{m(s_j)}$ may be different for each measurement $\mathscr{M}_{s_j}$ on system j. Such a Bell scenario is characterised by the triple $(\mathscr{N}, \mathscr{M}, \mathscr{O})$, where $\mathscr{M} = \{\mathscr{M}_{s_j}\}$; $j \in \mathscr{N}$; $s_j \in c_{j_{\{j \in \mathbb{N}\}}}$ and $\mathscr{O} = \{\mathscr{O}_m(s_j)\}$, where $m_j \in d_j$ and $\{j \in \mathbb{N}\}$. So the number of possible outcomes per measurement can be represented as table $\mathscr{O} = (\mathscr{O}_1^{\mathscr{M}_1}, ..., \mathscr{O}_1^{\mathscr{M}_{c_1}}); ...; (\mathscr{O}_{\mathscr{N}}^{\mathscr{M}_1}, ..., \mathscr{O}_{\mathscr{N}}^{\mathscr{M}_{c_j}})$. The choice of measurement setting is independent i.e. the choice of measurement is completely random. So there can be any possible permutations between $(\mathscr{N}, \mathscr{M}, \mathscr{O})$. In practice, it is interesting to work on subsets of different scenarios. Different permutations of subsets produce different behavior that belong to any region of the affine space. Given non-empty subset $\mathscr{J} \subseteq \{1, 2, ..., \mathscr{N}\}$ of all $\mathscr{N}$ with $|\mathscr{J}|$ being the number of parties in the subset, the output of subset is written as $m^{(j|j \in \mathscr{J})}$ with input subset as $s^{(j|j \in \mathscr{J})}$. Statistics are calculated from the data obtained from the parties denoted as conditional probabilities $p(m^{j|j \in \mathscr{N}}|\mathbf{s})$, the probability of obtaining output $m^{j|j \in \mathscr{N}}$ given input(s) $\mathbf{s}$. The conditional probabilities $p(m|s)$ are stochastic maps producing the map, or function $g : \bigotimes_1^{\mathscr{N}} \mathbb{Z}_{c_j} \to \bigotimes_1^{\mathscr{N}} \mathbb{Z}_{d_j}$ with some probability.These probabilities are transition functions. The joint probability of obtaining the outcomes $(m_1, ...m_n)$ given the measurement settings $(s_1, ..., s_n)$ will be denoted by $p_{m_1 \ldots m_n}|_{s_1 \ldots . s_n}$. These

$t = \prod_{i=1}^{\mathscr{N}}(\sum_{j=1}^{\mathscr{M}_i} \mathscr{O}_i^j)$ probabilities form the components of a vector $\vec{p}$ in affine space $\mathbb{R}^t$. The different permutations of $(\mathscr{N}, \mathscr{M}, \mathscr{O})$ form a space of correlations or behaviour that can be interpreted in topological perspective. The permutations in group-theoretic sense can be represented by symmetry group $\mathscr{G}_s$. It classifies states within the space of states. The probability of each run forms a space of correlations among events, we call as *behavior space* as used first by Tsirelson. The full behaviour space is an affine space, classified as local $\mathscr{L}$, quantum $\mathscr{Q}$ and no-signalling $\mathscr{N}\mathscr{S}$ region. Affine space $\mathbb{R}^{\mathscr{M}^2\mathscr{O}^2}$ provides a framework to analyse behaviour space in topological perspective. It is formed from all joint probabilities of $(\mathscr{N}, \mathscr{M}, \mathscr{O})$, denoting total behaviour $\mathscr{X}_B$ and refers the set $p = \{p(m|s)\}$ of all these probabilities as a *behaviour*. This space can be given a discrete topological realisation. It has been extensively studied in the prospectus of the topological methods for contextuality, for instance, sheaf cohomology and homotopy. Whatever happens in the triplet $(\mathscr{N}, \mathscr{M}, \mathscr{O})$ equivalently affects the behaviour space equipped with geometry/topology. Both triplet and topology can be entangled in the following way: to each input set of each agent, assign all values of the output set. Each element of input set can be represented as a vertices of simplicial complex embedded in topological space. The number of possible outcomes per input/vertex is represented as fiber space. The structure is isomorphic to fiber bundle.

A.4 A Dream!

The mathematical model of computation, to be known as topological interactive machine, has been defined in a very specific context of the problem addressed in the thesis. The results are preliminary and I can see a long road to put the machine in a broader context. We preferred *machinery* instead of *model* for implementation reasons. The machine could prove subtle relations with topological states of matter; in future might provide exotic substances to realise the machine as our computing devices today. The figure A.1 is my imaginative realisation of the working of machine, in case it could be realised! A dream!

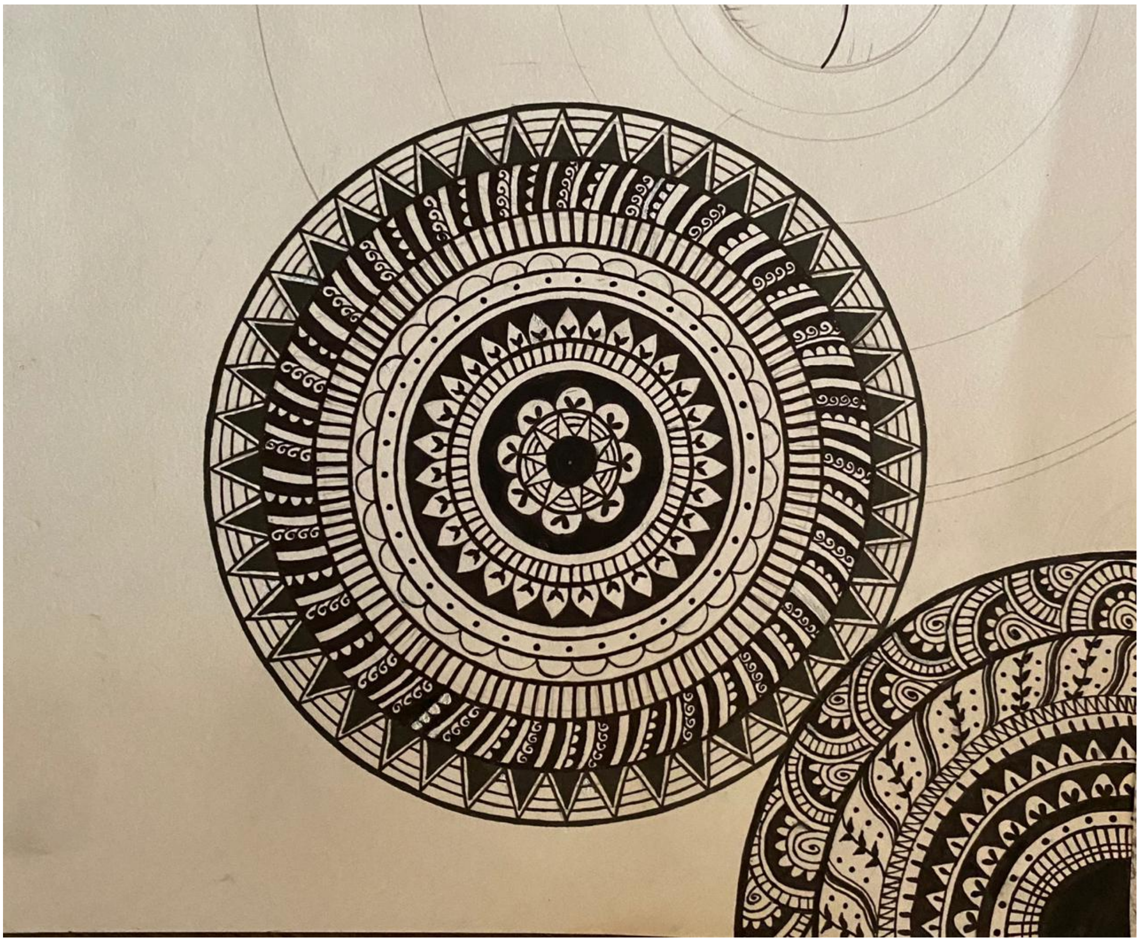


Figure A.1 Courtesy: Farisa Masood Purra.