Competing holes in open dynamical systems

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Abstract

We consider open dynamical systems defined on compact metric spaces with multiple shrinking holes. We study the points which are indecisive, i.e. change infinitely many times the escape hole as the holes shrink. We prove that, for transitive homeomorphisms, complete indecisiveness is generic. We provide examples of applications of the results.
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1 Introduction

In the last decades, there has been a considerable amount of attention towards dynamical systems whose points may escape outside the phase space (a seminal work is [10]). These systems are usually called *open* or *leaky* dynamical systems or even, more informally, systems with "holes". The idea has generally been to study the transient phase of a system (X, f) in which there is a hole, typically an open, connected subset $H \subset X$, such that the points arriving at H leave the space X forever. The dynamics of open systems has been investigated both from the measurable and the topological point of view. To illustrate some typical research directions, [14, 1, 5] study, for some classes of chaotic systems, the importance of the position of the hole in determining the corresponding escape rate – that is, roughly speaking, the exponential increase rate of the measure of points hitting the hole; [3] characterizes holes with maximal escape rate in the symbolic case; [12] studies a natural (albeit rather technical) notion of "critical size" for holes in the case of doubling maps of the interval; [11] develops many results for conformal maps, in particular computing the escape rates for some uniformly contracting conformal Markov systems. In some cases (see [2] for a review of applications of open dynamical systems), the holes $H^1, \dots, H^N$ can be seen as accesses to otherwise unknown, invariant regions of the phase space, and thus in fact to new dynamical systems

$$(\hat{X}^i, f^i), \quad i = 1, \dots N.$$

It becomes relevant, then, which hole the point escapes through, since this determines which new region of the phase space will be explored.
A very common class of questions concerns what happens when the size of the holes becomes vanishingly small (see for instance [4, 7, 6]). In particular, for every point, one can wonder which hole (if any) "prevails" among the competing shrinking holes, that is which hole is eventually selected as the escape hole when their size goes to zero. For some points x, we can have a well determined behavior, in the sense that we know which of the systems $(\hat{X}^i, f^i)$ we should look

at, eventually, to further track the dynamics of x. But it may well be that some points are indecisive, that is they change infinitely many times their escape hole as the radii go to zero.
Our main result (Theorem A) is that complete indecisiveness is generic for transitive homeomorphisms of a compact metric space, in the sense that only a topologically negligible set of points and of holes' choices lead to a well determined behavior as the size of the holes goes to zero. Somewhat surprisingly, this is independent of the convergence rates to zero of the holes' radii: no matter how fast a particular hole shrinks with respect to the other ones, it will still be selected infinitely many times as the escape hole by a co-meager set of points. After providing some examples, we conclude the work proving that decisive points are dense in the symbolic case.

2 Decisive and indecisive points

Throughout, $\mathcal{M}$ is a compact metric space with no isolated points and metric $d : \mathcal{M}^2 \to \mathbb{R}_{[0,+\infty)}$, while $f : \mathcal{M} \circlearrowleft$ is a homeomorphism, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1. *We denote by $\mathcal{O}(x)$ the f-orbit of $x \in \mathcal{M}$, that is*

$$\mathcal{O}(x) := \{f^k(x) : k \in \mathbb{Z}\}.$$

We denote by $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$, respectively, the forward and backward orbit of x under f, that is:

$$\mathcal{O}^+(x) := \{f^k(x) : k \in \mathbb{N}\} \qquad \mathcal{O}^-(x) := \{f^k(x) : k \in \mathbb{Z} \setminus \mathbb{N}_0\}.$$

Notice that, in Definition 1, the forward and the backward orbits do not contain the point x unless x is a periodic point.

In the following, $N \geq 2$ is a natural number and we denote by $[[a, b]]$ the integer interval $\{x \in \mathbb{Z} : a \leq x \leq b\}$. For every $i \in [[1, N]]$, let p_i be pairwise distinct points and let $\{\rho^i_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that

$$\lim_{n\to\infty} \rho^i_n = 0. \tag{1}$$

We call *infinitesimal* a sequence verifying (1). For every $i \in [[1, N]]$, we will denote by B^i_n the open ball centered in p_i with radius ρ^i_n. Therefore, $\{B^i_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of nested open balls. By changing finitely many values ρ^i_n if necessary, we will assume that the balls $B^1_n, \ldots, B^N_n$ are pairwise disjoint for all n.
We will often refer to these balls as the "holes" in $\mathcal{M}$. Finally, we collect the information about the holes and their shrinking behavior in the set:

$$\mathfrak{S} := \{p_i, \{\rho^i_n\}, 1 \leq i \leq N, n \in \mathbb{N}\}. \tag{2}$$

Definition 2. *Herein, by* open dynamical system*, we mean a 4-tuple* $(\mathcal{M}, d, f, \mathfrak{S})$ *under the assumptions made above.*

Let us now define some subsets of $\mathcal{M}$ based on which hole (if any) is chosen eventually when the radii go to zero. Roughly speaking, we will distinguish between points whose forward orbit:

1. eventually does not intersect a particular hole (or any hole);
2. eventually intersects one particular hole without having intersected before any other one;

3. for infinitely many values of the holes' radii, visit a particular hole without having visited any other one before.

where "eventually" is meant with respect to the index n which controls the shrinking of the balls. More precisely, we define the subsets

$$\mathfrak{T}^1(i), \mathfrak{T}^2(i), \mathfrak{T}(i) \subseteq \mathcal{M} \ (i \in \{1, \dots, N\}) \quad , \quad \mathfrak{T}^1, \mathfrak{T}^2, \mathfrak{T}^3, \mathfrak{T} \subseteq \mathcal{M}$$

as follows.
In the following, we adopt the convention $\min \emptyset = +\infty$.

Definition 3. *For $i \in \{1, \dots, N\}$, a point $x \in \mathcal{M}$ belongs respectively to $\mathfrak{T}^1(i)$ and to $\mathfrak{T}^1$ if there exists $c \in \mathbb{N}$ such that*

$$\mathcal{O}^+(x) \cap B_c^i = \emptyset \quad \text{and} \quad \mathcal{O}^+(x) \cap \bigcup_{j=1}^{N} B_c^j = \emptyset. \tag{3}$$

For $i \in \{1, \dots, N\}$, a point $x \in \mathcal{M}$ belongs to $\mathfrak{T}^2(i)$ if there is $c \in \mathbb{N}$ such that, for all $n > c$,

$$\min\{k \in \mathbb{N} : f^k(x) \in B_n^i\} < \min\{k \in \mathbb{N} : f^k(x) \in B_n^j\} \quad \textit{for all } j \neq i. \tag{4}$$

We set $\mathfrak{T}^2 := \bigcup_{j=1}^{N} \mathfrak{T}^2(j)$. We will call the points in $\mathfrak{T}^1 \cup \mathfrak{T}^2$ decisive points.
For $i \in \{1, \dots, N\}$, a point $x \in \mathcal{M}$ belongs to $\mathfrak{T}(i)$ if the following inequality holds for infinitely many integers n

$$\min\{k \in \mathbb{N} : f^k(x) \in B_n^i\} < \min\{k \in \mathbb{N} : f^k(x) \in B_n^j\} \quad \textit{for all } j \neq i.$$

A point $x \in \mathcal{M}$ belongs to $\mathfrak{T}^3$ if and only if there are $i_1, i_2 \in \{1, \dots, N\}$ $(i_1 \neq i_2)$ such that $x \in \mathfrak{T}(i_1) \cap \mathfrak{T}(i_2)$. We will call the points in $\mathfrak{T}^3$ indecisive points.
Finally, we say that x belongs to $\mathfrak{T}$ if and only if

$$x \in \bigcap_{j=1}^{N} \mathfrak{T}(j).$$

Note that, in the above, a more precise notation would be $\mathfrak{T}^j(\mathcal{M}, d, f, \mathfrak{S}, i)$, for $j = 1, 2$. However, for shortness, we will use the notation $\mathfrak{T}^j(i)$ when the open dynamical system $(\mathcal{M}, d, f, \mathfrak{S})$ is clear from the context.

Remark 4. *We remark that Eq.* (1) *and* (3) *imply*

$$\mathcal{O}^+(x) \cap B_n^i = \emptyset, \quad \textit{for all } n \geq c$$

since every sequence $\{B_n^i\}_{n \in \mathbb{N}}$ is decreasing with respect to set inclusion. Therefore, $\mathfrak{T}^1(i)$ is precisely the set of points x such that $p_i \notin \overline{\mathcal{O}^+(x)}$ and thus $\mathfrak{T}^1$ is the set of points x such that $\{p_1, \dots, p_N\} \cap \overline{\mathcal{O}^+(x)} = \emptyset$. Finally, $\mathfrak{T}^1 = \cap_{i=1}^{N} \mathfrak{T}^1(i)$.

Clearly, $\mathfrak{T} \subseteq \mathfrak{T}^3$.

Definition 5. *Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of sets. We define*

$$\liminf_{n \to \infty} S_n := \bigcup_{n \geq 1} \bigcap_{j \geq n} S_j \quad , \quad \limsup_{n \to \infty} S_n := \bigcap_{n \geq 1} \bigcup_{j \geq n} S_j.$$

If for some set S we have $\liminf_{n \to \infty} S_n = \limsup_{n \to \infty} S_n = S$, we say that $\lim_{n \to \infty} S_n = S$.

Remark 6. *For $i \in \{1,\dots,N\}$, the set $\mathfrak{T}(i)$ can be written as*

$$\mathfrak{T}(i) = \limsup_{n\to\infty} \left(\bigcup_{m\geq 1} \left(f^{-m}(B_n^i) \setminus \left(\cup_{k=1}^m \cup_{j\neq i} f^{-k}(B_n^j) \right) \right) \right). \tag{5}$$

Definition 7. *We say that a subset S of a topological space X is* meager *if there exist a countable family $\{S_i\}_{i\in\mathbb{N}}$ of subsets of X such that $\left(\overline{S_i}\right)^\circ = \emptyset$ for every $i \in \mathbb{N}$ and $S = \cup_{i\in\mathbb{N}} S_i$. We say that a set $S \subseteq X$ is* co-meager *if $X \setminus S$ is meager.*

Definition 8. *We say that $(\mathcal{M}, d, f, \mathfrak{S})$ is* completely indecisive *if, for every set of positive, decreasing, infinitesimal sequences $\{\rho_n^i\}_{n\in\mathbb{N}}$ $(i = 1,\dots,N)$, the set $\mathfrak{T}$* is co-meager*.*

Notice that, for every $i \in [[1, N]]$, the set $\mathfrak{T}$ cannot contain $\mathcal{O}^-(p_i)$, so it cannot be $\mathfrak{T} = \mathcal{M}$.
The definitions of the sets $\mathfrak{T}^i$ can be naturally expressed in terms of certain sequences which we will associate to each point of $\mathcal{M}$.

Definition 9. *For $x \in \mathcal{M}$, we define the sequence $\{s_x(n)\}_n$ of numbers between 0 and N as follows. We let $s_x(n)$ be 0 when $\mathcal{O}^+(x)$ does not intersect B_n^i for all $i = 1,\dots,N$, otherwise let $s_x(n)$ be the unique index $i \in \{1,\dots,N\}$ such that Eq.(4) holds.*

Note that for every n the index i of Definition 9 is well defined since the balls $B_n^1,\dots,B_n^N$ are disjoint.

Lemma 10. *The following equivalences hold:*

1. $x \in \mathfrak{T}^1 \iff s_x(c) = 0$ *for some $c \in \mathbb{N}$ (thus $s_x(n) \to 0$ as $n \to \infty$).*
2. $x \in \mathfrak{T}^2(i) \iff s_x(n) \to i$ *as $n \to \infty$.*
3. $x \in \mathfrak{T}^2 \iff s_x(n)$ *converges to some $i \in \{1,\dots,N\}$ when $n \to \infty$.*
4. $x \in \mathfrak{T}^3 \iff s_x(n)$ *is not convergent.*
5. $x \in \mathfrak{T}(i) \iff$ *a subsequence of $\{s_x(n)\}_n$ converges to i when $n \to \infty$.*
6. $x \in \mathfrak{T} \iff \{s_x(n)\}_n$ *has subsequences convergent to i for all $i = 1,\dots,N$.*

Proof. To prove item 1, note that $x \in \mathfrak{T}^1$ means that there exists c such that $\mathcal{O}^+(x)$ does not intersect B_c^j for any j, which is the definition of $s_x(c)$ being 0. If this is satisfied for some c, then it is also satisfied for all $n \geq c$, because $B_n^j \subseteq B_c^j$.
The equivalences in items 2 and 5 are restatements of Definition 3. To prove item 3, note that any $x \in X$ satisfies $x \in \mathfrak{T}^2$ if and only if $x \in \mathfrak{T}^2(i)$ for some $i \in \{1,\dots,N\}$, which is equivalent by item 2 to $s_x(n)$ converging to some value $i \in \{1,\dots,N\}$ as $n \to \infty$. Similarly, to prove item 6 note that any $x \in X$ satisfies $x \in \mathfrak{T}$ if and only if $x \in \mathfrak{T}(i)$ for all $i = 1,\dots,N$, which is equivalent to $\{s_x(n)\}_n$ taking all the values $i = 1,\dots,N$ infinitely many times for all $i = 1,\dots,N$.
Finally, to prove item 4, note that for any $x \in X$ the sequence $s_x : \mathbb{N} \to \{0, 1,\dots,N\}$ does not converge if and only if it takes at least two different values $i_1, i_2 \in \{0,\dots,N\}$ infinitely many times. By item 5, we have $x \in \mathfrak{T}(i)$ if and only if $s_x(n)$ takes at least two different values $i_1, i_2 \in \{1,\dots,N\}$ for infinitely many values of n. These two conditions are equivalent due to the fact that, as we proved above, if $s_x(c) = 0$ for some c, then $s_x(n) = 0$ for all $n > c$. □

3 Results

We start by proving a very simple structure lemma.

Lemma 11. *The space $\mathcal{M}$ splits as $\mathcal{M} = \mathfrak{T}^1 \sqcup \mathfrak{T}^2 \sqcup \mathfrak{T}^3$.*

Proof. By Lemma 10, it is enough to prove that for every $x \in \mathcal{M}$, either $s_x(n) \to 0$, $s_x(n)$ converges to some value $i \in \{1, \dots, N\}$, or $s_x(n)$ is not convergent. This is clear, because the sequence $\{s_x(n)\}_n$ takes values in the set $\{0, 1, \dots, N\}$. $\square$

The sets defined in the previous section are f-invariant, if we neglect the countably many points in the orbits $\mathcal{O}(p_i)$. To prove this f-invariance, we will make use of the following lemma:

Lemma 12. *If $f(x) \notin \{p_1, \dots, p_N\}$, then $s_x(n) = s_{f(x)}(n)$ for all values of n except finitely many.*

Proof. Let $m \in \mathbb{N}$ be the smallest positive integer such that $f(x) \notin B^i_m$ for all $i \in \{1, \dots, N\}$.

Case 1: $x \in \mathfrak{T}^1$. Then, there exists a positive integer $c \geq m$ such that $s_x(n) = 0$ for all $n \geq c$. Hence $p_i \notin \overline{\mathcal{O}^+(x)}$ for all i. As $\overline{\mathcal{O}^+(x)} = \overline{\mathcal{O}^+(f(x))} \cup \{f(x)\}$ and $f(x) \notin \{p_1, \dots, p_N\}$, this is equivalent to $p_i \notin \overline{\mathcal{O}^+(f(x))}$ for all i, that is, $s_{f(x)}(n) = 0$ for all $n \geq c$.

Case 2: $x \notin \mathfrak{T}^1$. Then, for any fixed $n \geq m$, $s_x(n) = i$ for some $i \in \{1, \dots, N\}$. This means that i is the unique index such that the inequality (4) holds for x. Since $f(x) \notin B^j_n$ for all j, it follows that i is the unique index such that the inequality (4) also holds for the point $f(x)$, that is, $s_{f(x)}(n) = i$.

$\square$

Corollary 13. *Let $\mathcal{O} := \bigcup_{i=1}^N \mathcal{O}(p_i)$, and fix $i \in \{1, \dots, N\}$. Then $f\left(\mathfrak{T}^1(i) \setminus \mathcal{O}\right) = \mathfrak{T}^1(i) \setminus \mathcal{O}$. The same holds if we substitute $\mathfrak{T}^1(i)$ by $\mathfrak{T}^1, \mathfrak{T}^2(i), \mathfrak{T}^2, \mathfrak{T}^3, \mathfrak{T}(i)$ or $\mathfrak{T}$.*

Proof. Remember that $x \in \mathfrak{T}^1(i)$ if and only if $p_i \notin \overline{\mathcal{O}^+(x)}$. So, as $\overline{\mathcal{O}^+(x)} = \overline{\mathcal{O}^+(f(x))} \cup \{f(x)\}$, any $x \notin \mathcal{O}$ satisfies that $x \in \mathfrak{T}^1(i) \iff f(x) \in \mathfrak{T}^1(i)$.
For any other set $X \in \{\mathfrak{T}^1, \mathfrak{T}^2(i), \mathfrak{T}^2, \mathfrak{T}^3, \mathfrak{T}(i), \mathfrak{T}\}$, note that if $x \notin \mathcal{O}$ then Lemma 12 implies that $s_x(n) = s_{f(x)}(n)$ for big enough n, so we conclude by Lemma 10 that $x \in X$ if and only if $f(x) \in X$. $\square$

The following Lemma shows that the points which, for infinitely many positive integers n, visit the ball B^i_n without having visited before any other ball B^j_n $(j \neq i)$ form a co-meager set under some additional assumptions.

Lemma 14. *Suppose that the points $p_1, \dots, p_N \in \mathcal{M}$ have pairwise disjoint orbits, that is*

$$\mathcal{O}(p_i) \cap \mathcal{O}(p_j) = \emptyset \quad \text{for } i \neq j. \tag{6}$$

Suppose that $\overline{\mathcal{O}^-(p_i)} = \mathcal{M}$ for some $i \in \{1, \dots, N\}$. Then, for every set of positive, decreasing, infinitesimal sequences $\{\rho^j_n\}_{n \in \mathbb{N}}$ $(j = 1, \dots, N)$, the set $\mathfrak{T}(i)$ is co-meager.

Proof. Recalling Remark 6, the set $\mathfrak{T}(i)$ can be written as

$$\limsup_{n \to \infty}(B_n) = \cap_{K \geq 1} \cup_{n \geq K} B_n$$

where $B_n := \bigcup_{m=1}^{\infty} \left(f^{-m}(B_n^i) \setminus (\cup_{k=1}^{m} \cup_{j\neq i} f^{-k}(B_n^j))\right)$. Set

$$C_n := \bigcup_{m=1}^{\infty} \left(f^{-m}(B_n^i) \setminus (\cup_{k=1}^{m} \cup_{j\neq i} f^{-k}\left(\overline{B_n^j}\right)\right).$$

Clearly C_n is open and $C_n \subseteq B_n$. Therefore,

$$\mathfrak{T}(i) \supseteq \bigcap_{K\geq 1} \bigcup_{n\geq K} C_n.$$

As $\mathcal{M}$ is a Baire space, it will be enough, by Baire's theorem, to prove that for each K, the set $\cup_{n\geq K} C_n$ is open and dense. It is open since C_n is open for all n. It is dense because, as we will prove next, it contains $\mathcal{O}^-(p_i)$.
For any fixed $m \in \mathbb{N}$, the point $f^{-m}(p_i)$ is contained in $f^{-m}(B_n^i)$ for all n. However, as the orbits of the p_i are disjoint, for big enough n the point $f^{-m}(p_i)$ is not contained in $\cup_{k=1}^{m} \cup_{j\neq i} f^{-k}\left(\overline{B_n^j}\right)$. So for big enough n, $f^{-m}(p_i)$ belongs to C_n, which proves that $\mathcal{O}^-(p_i) \subseteq \cup_{n\geq K} C_n$, as we wanted. □

Lemma 15. *If $p_1, \ldots, p_N \in \mathcal{M}$ have pairwise disjoint orbits and, for every $i \in [[1, N]]$, we have $\overline{\mathcal{O}^-(p_i)} = \mathcal{M}$, then $(\mathcal{M}, d, f, \mathfrak{S})$ is completely indecisive.*

Proof. By Lemma 14, $\mathfrak{T}(i)$ is co-meager for every $i \in [[1, N]]$ and every set of positive, decreasing, infinitesimal sequences $\{\rho_n^i\}_{n\in\mathbb{N}}$ $(i = 1, \ldots, N)$. Since finite intersections of co-meager sets are co-meager, so is $\mathfrak{T} = \cap_{i=1,\ldots,N} \mathfrak{T}(i)$. □

We recall that a homeomorphism $f : \mathcal{M} \circlearrowleft$ is called *transitive* if $\overline{\mathcal{O}^+(x)} = \mathcal{M}$ for some $x \in \mathcal{M}$. This definition of transitivity is equivalent, under the hypotheses that $\mathcal{M}$ is compact metric with no isolated points, to the topological definition of transitivity, that is:

$$\text{For every nonempty and open } A, B \subset \mathcal{M},\ \exists k > 0 : f^k(A) \cap B \neq \emptyset$$

(see e.g. [13]), and it also implies that there is backward transitivity, i.e. the existence of a point y such that $\overline{\mathcal{O}^-(y)} = \mathcal{M}$ (see e.g. [9], p. 70). We are now ready to give our main result.

Theorem A. *Let $\mathcal{M}$ be a compact metric space with no isolated points and $f : \mathcal{M} \circlearrowleft$ a transitive homeomorphism. For every $N \geq 2$, there exists a co-meager set $X \subseteq \mathcal{M}^N$ such that, for every $(p_1, \ldots, p_N) \in X$, the open dynamical system $(\mathcal{M}, d, f, \mathfrak{S})$ is completely indecisive.*

Proof. For every $k \in \mathbb{Z}$, $h, j \in [[1, N]]$, $j \neq h$, let us define the set

$$A_k^{h,j} := \{(x_1, \ldots, x_N) \in \mathcal{M}^N : x_h = f^k(x_j)\},$$

that is, the set of all N-tuples such that the h-th component is the f^k-image of the j-th component. Since $\mathcal{M}$ has no isolated points, the set $A_k^{h,j}$ is nowhere dense in $\mathcal{M}^N$ if $f^k \equiv \mathrm{Id}_{\mathcal{M}}$, as in this case $A_k^{h,j}$ is simply the set $E^{h,j} := \{(x_1, \ldots, x_N) \in \mathcal{M}^N : x_h = x_j\}$. Defining the homeomorphism $\iota_j : \mathcal{M}^N \circlearrowleft$ as:

$$\iota_j = \underbrace{\mathrm{Id}_{\mathcal{M}} \times \ldots \times \mathrm{Id}_{\mathcal{M}}}_{j-1 \text{ times}} \times f^k \times \underbrace{\mathrm{Id}_{\mathcal{M}} \times \ldots \times \mathrm{Id}_{\mathcal{M}}}_{N-j \text{ times}},$$

we have clearly $\iota_j(E^{h,j}) = A_k^{h,j}$, and since being nowhere dense is preserved by homeomorphisms, $A_k^{h,j}$ is always such in our assumptions.
Therefore,

$$A := \bigcup_{\substack{k\in\mathbb{Z} \\ h\in[[1,N]] \\ j\in[[1,N]],\ j\neq h}} A_k^{h,j}$$

is a meager set. By construction, every N-tuple in the co-meager set $D := \mathcal{M}^N \setminus A$ is such that its components have pairwise disjoint f-orbits.
Since the system $(\mathcal{M}, f)$ is transitive, the set $Y \subset \mathcal{M}$ of points x such that $\overline{\mathcal{O}^-(x)} = \mathcal{M}$ is co-meager (see for instance [8], Theorem 2.9). Therefore, $X := Y^N \cap D$ is a co-meager set in $\mathcal{M}^N$ such that each of its N-tuples $(p_1, \ldots, p_N)$ verify the assumptions of Lemma 15. □

We point out what seems to us as the most interesting aspect of Theorem A, that is that, although the sets $\mathfrak{T}^2, \mathfrak{T}^3, \mathfrak{T}$ depend in general on the sequences $\{\rho_n^i\}$, the theorem holds independently of these sequences, provided that they are decreasing and converging to zero.

4 Examples

In this section, we provide some examples of applications of Theorem A, as well as a counterexample showing that, firstly, by weakening the transitivity assumption, we can have the co-meager set of indecisive points collapsing to the empty set; secondly, the converse of Lemma 14 is not true, meaning that $\mathfrak{T}^2(i)$ can be topologically large even if p_i does not have a dense backward orbit.

4.1 Irrational rotations

Let $\mathcal{M} := \mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n}$ be the n-dimensional torus ($n \in \mathbb{N}$) with the product topology, and $f : \mathcal{M} \to \mathcal{M}$ be an irrational rotation, defined, in the standard angular coordinates, by

$$(x_1 + \alpha_1)_{\mathrm{mod}1}, \ldots, (x_n + \alpha_n)_{\mathrm{mod}1},$$

where $\alpha_1, \ldots, \alpha_n$ are rationally independent real numbers.
Since the system $(\mathcal{M}, f)$ is minimal, and hence transitive, Theorem A applies. In fact, in this case, for every choice of points $p_1, \ldots, p_N \in \mathcal{M}$ with pairwise disjoint orbits and every set of positive, decreasing, infinitesimal sequences $\{\rho_n^i\}_{n\in\mathbb{N}}$ ($i = 1, \ldots, N$), we have that $\mathfrak{T}^1 = \emptyset$ and $\mathfrak{T}$ is co-meager by Lemma 15.
In this case f is an isometry, so ball preimages preserve radii. We remark that, independently of the rate of convergence to zero of the radii ρ_n^i, all the sets $\mathfrak{T}(i)$ are co-meager, while $\mathfrak{T}^2$ is a meager, nonempty subset of $\mathbb{T}^n$. For instance, since the orbits of the points $\{p_i\}_{i=1,\ldots,N}$ are pairwise disjoint by assumption, for every $i \in [[1, N]]$ the set $\mathcal{O}^-(p_i)$ belongs to $\mathfrak{T}^2(i)$ and therefore to $\mathfrak{T}^2$.

4.2 A one-dimensional counterexample to the converse of Lemma 14

Set $\mathcal{M} := [0, 1]$ and let the map $f : \mathcal{M} \to \mathcal{M}$ be defined by $f(x) = x^\alpha$, $\alpha > 1$. Moreover, let $N = 2$, $p_1 = 0$ and $p_2 = y \in (0, 1]$. This elementary example illustrates that a converse of Lemma

14 does not hold in general, in the sense that $\mathfrak{T}(i)$ can be co-meager without the backward orbit of p_i being dense. Indeed, in this case p_1 is a fixed point, but nonetheless we have:

$$\begin{aligned}
&\mathfrak{T}^1 = \{1\} \text{ (if } y \neq 1) \text{ or } \emptyset \text{ (if } y = 1), \quad \mathfrak{T}^2(1) = [0,1] \setminus \mathcal{O}^-(y), \\
&\mathfrak{T}^2(2) = \mathcal{O}^-(y), \ \mathfrak{T}^3 = \emptyset, \quad \mathfrak{T}(1) = [0,1] \setminus \mathcal{O}^-(y),
\end{aligned} \tag{7}$$

where in particular the last equality shows our point.
More generally, if one of the points p_i is an attractive fixed point with basin of attraction $\mathfrak{X} \subseteq \mathcal{M}$, then

$$\mathfrak{T}^2(i) = \mathfrak{X} \setminus \cup_{j \neq i} \mathcal{O}^-(p_j).$$

This example also shows that the transitivity hypothesis cannot be eliminated from the assumptions of Theorem A. Indeed, for every $p_1, p_2 \in (0,1]$, the sets $\mathcal{O}^-(p_1)$ and $\mathcal{O}^-(p_2)$ are in $\mathfrak{T}^2(1)$ and $\mathfrak{T}^2(2)$ respectively, while every $x \in [0,1] \setminus (\mathcal{O}^-(p_1) \cup \mathcal{O}^-(p_2))$ is in $\mathfrak{T}^1$. Therefore, for every choice of holes' centers, all points are decisive.

4.3 A general construction for systems in which all points are decisive

Let $\mathcal{M}$ be compact metric (with metric d) and $f : \mathcal{M} \circlearrowleft$ a homeomorphism. Consider finitely many points $p_1, \dots, p_N$ such that

$$\overline{\mathcal{O}(p_i)} \cap \overline{\mathcal{O}(p_j)} = \emptyset$$

whenever $i \neq j$. Set $\Sigma := \cup_{i=1}^N \overline{\mathcal{O}(p_i)}$. Take also a family $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ of non-empty, closed, invariant subsystems such that, for every $\alpha \in \mathcal{A}$,

$$S_\alpha \cap \Sigma = \emptyset.$$

Notice that, if the system is minimal, it must be $N = 1$ and, defining $S := \cup_{\alpha \in \mathcal{A}} S_\alpha$, it is $S = \emptyset$.
If $S \cup \Sigma$ is compact, the open dynamical system

$$(S \cup \Sigma, d, f|_{S \cup \Sigma}, \mathfrak{S})$$

is such that every point is decisive, that is $S \cup \Sigma = \mathfrak{T}^1 \cup \mathfrak{T}^2$.
Indeed, if $x \in S$, then $d(\overline{\mathcal{O}^+(x)}, p_i) > 0$ for every $i \in [[1, N]]$, so that $x \in \mathfrak{T}^1$. On the other hand, if $x \in \overline{\mathcal{O}(p_i)}$, then $d(\overline{\mathcal{O}^+(x)}, p_j) > 0$ for every $j \neq i$, so that $x \in \mathfrak{T}^1 \cup \mathfrak{T}^2(i)$.
Notice that this example does not verify the assumptions of Theorem A, since the system is not transitive (recall that $N \geq 2$). Moreover $\mathcal{M} = \Sigma \cup S_\alpha$ could have isolated points, as for instance if, for a certain $\bar{\alpha}$, $S_{\bar{\alpha}}$ is a periodic orbit $\{x_1, \dots, x_m\}$. In the latter case, $\{(x_1, \dots, x_m)\}$ would be a non-meager subset of $(\Sigma \cup S_{\bar{\alpha}})^m$ such that the corresponding open dynamical system

$$(\Sigma \cup S_{\bar{\alpha}}, d, f_{\Sigma \cup S_{\bar{\alpha}}}, \mathfrak{S}'),$$

with $\mathfrak{S}' = \{x_i, \{\rho_n^i\}, 1 \leq i \leq m, n \in \mathbb{N}\}$, has an empty set of indecisive points.

4.4 Decisive points in symbolic spaces

Let $\mathcal{M} := \{0,1\}^{\mathbb{Z}}$ be the set of bi-infinite binary words and $\sigma \colon \{0,1\}^{\mathbb{Z}} \circlearrowleft$ the full shift defined by

$$(\sigma(u))_n = u_{n+1}, \quad \text{for all } n \in \mathbb{Z},$$

where by w_k we denote the digit indexed by the integer k of the word w. Set moreover $d(u,v) = 2^{-n}$, where $n := \min\{n \in \mathbb{N} : u_{-n} \neq v_{-n} \vee u_n \neq v_n\}$. Let us first notice that, as the system

is transitive, Theorem A applies also to this example. The study of topological, dimensional- and measure-theoretic characterization of decisive/indecisive points in the full shift "with holes" seems to be linked to delicate problems in combinatorics of words. Here we provide a density result for the set of decisive points.
Let us introduce the following notation:

$$\text{for } a, b \in \mathbb{Z},\ u_{a,b} := u_a u_{a+1} \dots u_{b-1} u_b \in \{0,1\}^{b-a+1},$$

where $\{0,1\}^n$ is the set of binary words having length n. Fix $p_1, p_2 \in \{0,1\}^{\mathbb{Z}}$ with $p_1 \neq p_2$. For every $n \in \mathbb{N}$, set

$$B_n^1 := (p_1)_{-n,n} \quad \text{and} \quad B_n^2 := (p_2)_{-n,n}.$$

Proposition 16. *The set $\mathfrak{T}^2(i)$ is dense in $\mathcal{M}$ for $i = 1, 2$.*

Proof. We prove the case $i = 1$. We adress at the end the cases $p_2 = (\dots 000 \dots)$ and $p_2 = (\dots 111 \dots)$, so for now we assume that $(p_2)_a \neq (p_2)_{a+1}$ for some $a \in \mathbb{Z}$. It will be enough to show that for any given word w of length $2k+1$, we can find an element $x \in \mathfrak{T}^2(1)$ such that $x_{-k,k} = w$. Note first that, as $p_2 \neq p_1$, there is some c such that, for all $n \geq c$, $B_n^1 \neq B_n^2$. For the construction of x we will suppose $c > \max(2|a|, k, 1000)$.
Let us now construct x. Set $x_i = 0$ for $i < -k$, and $x_{-k,k} = w$. The interesting part is the construction of $x_{k+1,\infty}$. We set

$$x_{k+1} x_{k+2} \cdots = \prod_{n=c}^{\infty} w_n B_n^1, \tag{8}$$

where the words w_n, $n \geq c$, are words of length n constructed to satisfy the following two properties:

1. B_n^2 is not a subword of $B_{n-1}^1 w_n B_n^1$.
2. w_n is not a subword of $B_{n^3}^2$.

We first we prove that properties 1 and 2 are enough to prove proposition 1, leaving the construction of w_n to the end of the proof. Let k_n be defined by

$$k_n = k + \sum_{i=c}^{n} (3i+1),$$

so that $x_{k_n - n, k_n + n}$ is the word B_n^1 from Eq. (8).
We will prove that, for any $n > c^3$, we have $B_n^2 \neq x_{m-n,m+n}$ for any $m < k_n$, which, since arbitrarily large central windows of p_1 appear in the positive digits of x, proves $x \in \mathfrak{T}^2(1)$, as we wanted. Let m be the least positive integer such that $B_n^2 = x_{m-n,m+n}$, and suppose $m \leq k_n$ (notice that $n \geq c$ in fact excludes the equality). Then we have two cases, both of which lead to a contradiction.

Case 1: If the digit x_m is inside the words w or w_c, then $x_{m-n,m+n}$ contains some subword w_l for $l > \sqrt[3]{n}$, which contradicts property 2 above (see Fig. 1, top).

Case 2: If not, then we have $k_{l-1} < m \leq k_l$ for some $l \leq n$, thus $B_l^2 = x_{m-l,m+l}$ is contained in $B_{l-1}^1 w_l B_l^1$, contradicting property 1 above (see Fig. 1, bottom).

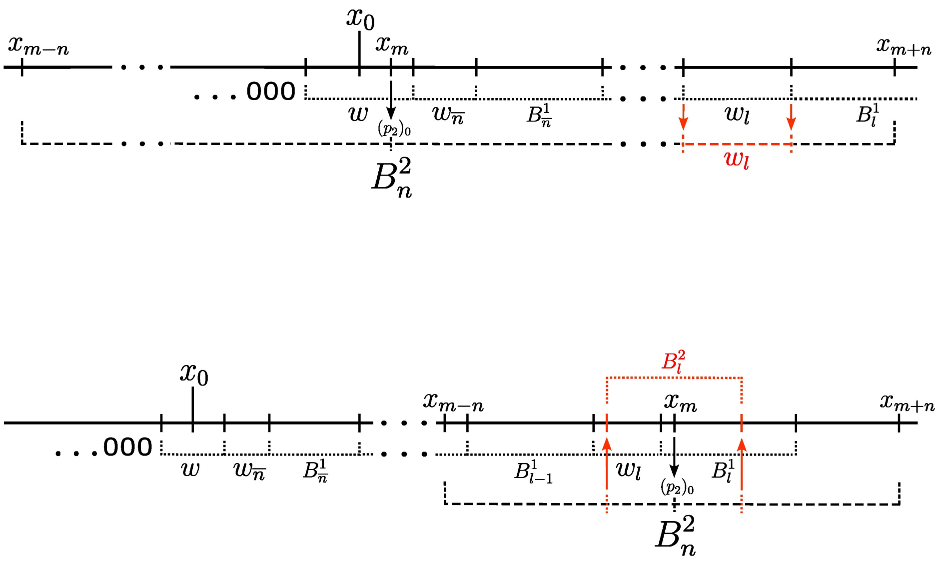

Figure 1: A graphic representation of cases 1. (top) and 2. (bottom) in the proof.

Finally, we show the existence of the words w_n of length n that satisfy the properties above. Set

$$y = y_1 y_2 \ldots y_{5n} := B^1_{n-1} w_n B^1_n.$$

For property 1 to fail, we need to have for some $l \in \{1, 2, \ldots, 3n-1\}$ that

$$B^2_n = y_l y_{l+1} \ldots y_{l+2n}. \tag{9}$$

The digits y_i are already chosen for $i = 1, 2, \ldots, 2n-1, 3n, 3n+1, \ldots, 5n$, so we just have to choose the n remaining digits adequately. We have four possible choices for y_{2n} and y_{2n+1}: we will choose them so that the word (y_{2n}, y_{2n+1}) is not equal to $((p_2)_{2n}, (p_2)_{2n+1})$, $((p_2)_{2n-1}, (p_2)_{2n})$ or $((p_2)_{2n-2}, (p_2)_{2n-1})$. This ensures that l cannot be $1, 2$ or 3. On the other end of the word w, there are two cases.

1. Suppose that $(p_2)_{-n+1,n} \neq (p_1)_{-n,n-1}$. Then $l = 3n-1$ is not possible. In this case, we have 2^{n-2} options for the digits $y_{2n+2}, y_{2n+3}, \ldots, y_{3n-1}$. Of them, at most $2n^3$ do not satisfy property 2, because there are $\leq 2n^3$ length-n subwords of $(p_2)_{-n^3,n^3}$. Moreover, for each l, the number of options that lead to Eq.(9) is:

   - 2^{n-l-1} for $l = 4, 5, \ldots, n-2$.
   - $2^{l-(2n+2)}$ for $l = 2n+2, 2n+3, \ldots, 3n-2$
   - 1 for $l = n-1, \ldots, 2n+1$.

   So the total number of options that either do not satisfy property 1 or do not satisfy property 2 is at most $2n^3 + (2^{n-4} + 2^{n-3} + n + 3) < 2^{n-2}$, which implies that we can find w_n satisfying properties 1 and 2.

2. Suppose that $(p_2)_{-n+1,n} = (p_1)_{-n,n-1}$. Then, we have a unique choice of y_{3n-1} that invalidates the case $l = 3n-1$. But, as $(p_2)_a \neq (p_2)_{a+1}$ for some a such that $|a| < \frac{n}{2}$, we cannot have $l = 3n-2$ in this case. We have 2^{n-3} options for choosing $y_{2n+2}, \ldots, y_{3n-2}$. Of them, as before, at most $2n^3$ do not satisfy property 2. Moreover, for each l, the number of options that lead to Eq.(9) is:

   - 2^{n-l-2} for $l = 4, 5, \ldots, n-3$.
   - $2^{l-(2n+2)}$ for $l = 2n+2, 2n+3, \ldots, 3n-3$
   - 1 for $l = n-2, \ldots, 2n+1$.

   So the total number of options that either do not satisfy property 1 or do not satisfy property 2 is at most $2n^3 + (2^{n-5} + 2^{n-4} + n + 4) < 2^{n-3}$, so also in this case we can find w_n satisfying properties 1 and 2.

Only the cases $p_2 = (\ldots 000 \ldots)$ and $p_2 = (\ldots 111 \ldots)$ are left. If for example $p_2 = (\ldots 000 \ldots)$, we can set $w_n = 0$ for all n in Eq. 8, and then properties 1 and 2 are obviously satisfied, and similarly for $p_2 = (\ldots 111 \ldots)$. $\square$

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