

ON THE NORMAL EXPONENTIAL MAP IN SINGULAR CONFORMAL METRICS

ROBERTO GIAMBÒ, FABIO GIANNONI AND PAOLO PICCIONE

ABSTRACT. Brake orbits and homoclinics of autonomous dynamical systems correspond, via Maupertuis principle, to geodesics in Riemannian manifolds endowed with a metric which is singular on the boundary (Jacobi metric). Motivated by the classical, yet still intriguing in many aspects, problem of establishing multiplicity results for brake orbits and homoclinics, as done in [6, 7, 10], and by the development of a Morse theory in [8] for geodesics in such kind of metric, in this paper we study the related normal exponential map from a global perspective.

1. INTRODUCTION

The purpose of this paper is to prove global regularity results for the distance-to-the-boundary function in Riemannian manifolds with singular metrics on the boundary. This kind of study is motivated by the use of the degenerate Jacobi metric (via Maupertuis' principle) for the problem of brake orbits and homoclinics in the autonomous case, as done in [6, 7, 9, 10]. This approach was suggested for the first time by Seifert in [19], where a famous conjecture concerning a multiplicity results for brake orbits was formulated. The metric singularity on the boundary is of a very special type, being produced by the first order vanishing of a conformal factor which multiplies a fixed background metric. Following the local theory developed in [8], in this paper we will introduce a suitable notion of normal exponential map adapted to this type of degenerate boundaries, and we will determine its regularity properties.

There exists a huge amount of literature concerning the study of brake orbits – see e.g., [12, 15, 21, 22] – and more generally on the study of periodic solutions of autonomous Hamiltonian systems with prescribed energy [13, 14, 16, 17, 18]. We also observe here that manifolds with singular boundary of the type investigated in the present paper arise naturally in the study of certain compactifications of incomplete Riemannian manifolds. They constitute an important class of the so-called *singular manifolds*, see [1] and the references therein, where the singularity is described by the vanishing (or the diverging) of some conformal factor, called the *singularity function*.

In order to describe the results of the present paper, let us consider a Riemannian manifold (M, g) of class C^3 , representing the configuration space of some conservative dynamical system, and let $V : M \rightarrow \mathbb{R}$ be a map of class C^2 on M , which represents the potential function of the system. Fix an energy level $E \in \mathbb{R}$,

Date: July 27th, 2014.

2010 *Mathematics Subject Classification.* 30F45, 58E10.

$E > \inf_M V$, and consider the Jacobi metric:

$$(1.1) \quad g_* = \frac{1}{2}(E - V)g,$$

defined in the open sublevel $V^{-1}(]-\infty, E[)$, the so called *potential well*. Note that g_* is singular on the boundary $V^{-1}(E)$.

For any $Q \in V^{-1}(]-\infty, E[)$, denote by $d_V(Q)$ the distance of Q from $V^{-1}(E)$ with respect to the Jacobi metric (1.1). In the recent work [8], the following assumptions:

- V is of class C^2 in a neighborhood of $V^{-1}(]-\infty, E[)$;
- E is a regular value for V ;
- the sublevel $V^{-1}(]-\infty, E[)$ is compact;

were used to prove that if the minimizer that realizes $d_V(Q)$ is unique, then d_V is differentiable at Q , and its gradient with respect to the Riemann metric g is given by

$$\nabla^g d_V(Q) = \frac{E - V(Q)}{2d_V(Q)} \dot{\gamma}_Q(1),$$

where γ_Q is the minimizer (affinely parametrized in the interval $[0, 1]$) joining $V^{-1}(E)$ with Q . Uniqueness of the minimizer is guaranteed for all points Q sufficiently close to the boundary $V^{-1}(E)$. Moreover, in [8, Section 4] a definition of Jacobi fields along Jacobi geodesic starting from $V^{-1}(E)$ is given and a Morse Index Theorem was proved.

Following along this path, in this paper we introduce a normal exponential map \exp^\perp , defined in terms of g_* -geodesics $\gamma :]0, a] \rightarrow V^{-1}(]-\infty, E[)$ satisfying $\lim_{s \downarrow 0} \gamma = P \in V^{-1}(E)$, see Section 2. Such geodesic is necessarily “orthogonal” to $V^{-1}(E)$, in the sense that a suitable normalization of $\dot{\gamma}(s)$, when s goes to 0, admits as limit as a vector $v \in T_P(V^{-1}(E))$ which is g -orthogonal to $V^{-1}(E)$ at P . We prove the regularity of \exp^\perp , and we establish the equivalence between conjugate points to $V^{-1}(E)$ and critical values of the exponential map (Proposition 2.3 and Theorem 2.9).

In section 3 we apply the above result to prove that if the minimizer between $V^{-1}(E)$ and Q_0 is unique and if Q_0 is not conjugate to $V^{-1}(E)$, then $d_V(Q)$ is of class C^2 in a neighborhood of Q_0 (cf. Theorem 3.1). This extends the result proved in [6], where the C^2 -regularity is proved only for points Q_0 sufficiently close to $V^{-1}(E)$.

2. EXPONENTIAL MAP AND FOCAL POINTS

A geodesic $x : I \subset \mathbb{R} \rightarrow V^{-1}(]-\infty, E[)$ relative to the metric g_* (1.1) will be called a *Jacobi geodesic*; such a curve satisfies the second order differential equation:

$$(2.1) \quad (E - V(x(s))) \frac{D}{ds} \dot{x}(s) - g(\nabla V(x(s)), \dot{x}(s)) \dot{x}(s) + \frac{1}{2} g(\dot{x}(s), \dot{x}(s)) \nabla V(x(s)) = 0.$$

Moreover, a non constant Jacobi geodesic γ satisfies the conservation law

$$(2.2) \quad \frac{1}{2}(E - V(\gamma))g(\dot{\gamma}, \dot{\gamma}) \equiv \lambda_\gamma \in \mathbb{R}^+ \setminus \{0\}.$$

Given a Jacobi geodesic $\gamma :]0, a] \rightarrow V^{-1}(]-\infty, E[)$ satisfying $\lim_{s \downarrow 0} \gamma(s) = P \in V^{-1}(E)$, then the g -normalized tangent vector $v_s := \dot{\gamma}(s)/g(\dot{\gamma}(s), \dot{\gamma}(s))^{\frac{1}{2}}$ admits limit $\lim_{s \downarrow 0} v_s = v_0 \in T_P(V^{-1}(E))^{\perp}$. Indeed, using [8, Lemma 2.2], it can be seen that $v_s + \nabla V(\gamma(s))/g(\nabla V(\gamma(s)), \nabla V(\gamma(s)))^{1/2}$ is bounded in norm by an function that is infinitesimal for $s \rightarrow 0$. In this situation, we will say that γ is a Jacobi geodesic *starting orthogonally* to $V^{-1}(E)$. These geodesics will be used later in the definition of the normal exponential map of $V^{-1}(E)$ using these geodesics.

Remark 2.1. Let us now recall a basic result that will be repeatedly used throughout the paper. We refer the reader to [8, eq (3.16) and following discussion] for the details of this construction, that we will now briefly sketch. For any $P \in V^{-1}(E)$ consider the trajectories $t \mapsto q(t, P)$ that are solutions of the Cauchy problem:

$$(2.3) \quad \begin{cases} \frac{D}{dt} \dot{q} + \text{grad } V(q) = 0 \\ q(0) = P \\ \dot{q}(0) = 0, \end{cases}$$

where $\frac{D}{dt}$ is the covariant derivative of vector fields along q , and $\text{grad } V$ is the gradient of V with respect to the Riemannian metric g .

Moreover, called $\gamma(P, \lambda)$ the unique Jacobi geodesic starting from $P \in V^{-1}(E)$ and satisfying $\lambda_{\gamma} = \lambda$ (note that the uniqueness of $\gamma(P, \lambda)$ follows again from the Maupertuis Principle), we have that

$$q(t, P) = \gamma(P, \lambda)(s), \quad \text{where } t = t(s) = \int_0^s \frac{\sqrt{\lambda} dr}{E - V(\gamma(P, \lambda)(r))},$$

and then $t \mapsto q(t, P)$ is a reparameterization of $s \mapsto \gamma(P, \lambda)(s)$.

Using Maupertuis–Jacobi principle [8, Proposition 2.1], one can show that, setting $\tau = \sqrt{t}$, the map $q(\tau, P)$, defined in $[0, \tau_0[\times V^{-1}(E)$, for a suitable τ_0 sufficiently small, is a C^1 -diffeomorphism. Moreover, $q(\tau, P)$ is a coordinate system in a neighborhood of $V^{-1}(E)$ and $\frac{\partial q}{\partial \tau}(0, P) = -\frac{1}{2} \text{grad } V(P)$, while $\frac{\partial q}{\partial P}(0, P)$ is the identity map. In particular, it is worth remarking that a C^1 -diffeomorphism, say Φ , between $V^{-1}(E)$ and $d_V^{-1}(\delta)$ is obtained by setting $\Phi(P) = q(\delta, P)$.

Definition 2.2. For any $s > 0, \lambda > 0$ and $P \in V^{-1}(E)$ we denote by *exponential map* (starting from $V^{-1}(E)$) the map:

$$(2.4) \quad \exp^{\perp}(P, \lambda)(s) = \gamma(P, \lambda)(s),$$

First of all, let us prove the following

Proposition 2.3. Fix any $\delta > 0$ such that $d_V^{-1}(\delta)$ is a C^2 -hypersurface (cf [6, 8]). Let Φ be the C^1 -diffeomorphism between $V^{-1}(E)$ and $d_V^{-1}(\delta)$ (see Remark 2.1). Let γ_{δ} be the solutions of the Cauchy problem for Jacobi geodesics with initial position, $\gamma(\frac{\delta}{\sqrt{\lambda}})$ and initial speed $\dot{\gamma}(\frac{\delta}{\sqrt{\lambda}})$. Then for any $s > 0$

$$(2.5) \quad \gamma(P, \lambda)(s) = \gamma_{\delta}(P, \lambda)(s - \frac{\delta}{\sqrt{\lambda}}),$$

and $\exp^{\perp}(\cdot, \cdot)(s)$ is of class C^1 .

Proof. First of all, recall from Remark 2.1 that $q(\sqrt{t}, P)$ is a coordinate system in a neighborhood of $V^{-1}(E)$. Therefore, for any $\delta > 0$ such that $\gamma(P, \lambda)$ is defined in a neighborhood of $\frac{\delta}{\sqrt{\lambda}}$ (the value of the parameter at which γ reaches $d_V^{-1}(\delta)$), there exists one and only one $t_\delta(P, \lambda)$ such that $q(t_\delta(P, \lambda), P)$ intersects $d_V^{-1}(\delta)$.

Using the Implicit Function Theorem, we deduce the existence of δ sufficiently small such that the map $t_\delta(P, \lambda)$ is of class C^1 .

Now, since

$$\dot{\gamma}(P, \lambda)(s) = \frac{dt}{ds} \dot{q}(t, P) \frac{dt}{ds} = \frac{\sqrt{\lambda}}{E - V(q(t, P))} \dot{q}(t, P)$$

we have that (2.5) holds for any $s > 0$. Now recall that γ_δ is the Jacobi geodesic satisfying the initial conditions

$$(2.6) \quad \begin{cases} \gamma_\delta(P, \lambda)(0) = q(t_\delta(P, \lambda), P), \\ \dot{\gamma}_\delta(P, \lambda)(0) = \frac{\sqrt{\lambda}}{E - V(q(t_\delta(P, \lambda)), P)} \dot{q}(t_\delta(P, \lambda), P), \end{cases}$$

Since $(t, P) \mapsto q(t, P)$ and $(t, P) \mapsto \dot{q}(t, P)$ are of class C^1 (Remark 2.1), standard regularity properties of ODE's give the C^1 -regularity of the map $(P, \lambda) \mapsto \dot{\gamma}_\delta(P, \lambda)(s - \frac{\delta}{\sqrt{\lambda}})$ concluding the proof. \square

Remark 2.4. Note that classical regularization methods show that the map $(P, \lambda) \mapsto \dot{\gamma}_\delta(P, \lambda)(s - \frac{\delta}{\sqrt{\lambda}})$ is also of class C^1 .

Remark 2.5. It is important to note that γ_δ starts from the point:

$$P_\delta := q(t_\delta(P), P) \in d_V^{-1}(\delta) =: N_\delta,$$

with velocity orthogonal to the hypersurface N_δ .

Remark 2.6. Consider a Riemannian metric conformal to g , say ϕg with ϕ positive and smooth real map, and the corresponding action integral

$$h(x) = \frac{1}{2} \int_0^1 \phi(x) g(\dot{x}, \dot{x}) ds$$

on the space X of the $H^{1,2}$ -curves from $[0, 1]$ to M such that $x(0) \in N$, $x(1) = Q$, where N is a smooth hypersurface of M , and Q is a fixed point in M . We recall that the critical points γ on $h|_X$ satisfy the ordinary differential equation:

$$\frac{D}{ds} [\varphi(\gamma(s)) \dot{\gamma}(s)] = \frac{1}{2} g(\dot{\gamma}(s), \dot{\gamma}(s)) \nabla \varphi(\gamma(s)),$$

and the boundary conditions

$$\gamma(0) \in N, \quad \gamma(1) = Q.$$

(Here $\nabla \varphi$ denotes the gradient of φ with respect to the metric g). Moreover, the tangent space of X at γ is given by the $H^{1,2}$ vector fields ξ along γ such that $\xi(0) \in T_{\gamma(0)} N$, the tangent space of N at $\gamma(0)$ and $\xi(1) = 0$, while the Hessian of h at a critical point γ is given by

$$\begin{aligned} H^h(\gamma) = & \int_0^1 \frac{1}{2} g(\dot{\gamma}, \dot{\gamma}) g(H^\varphi(\gamma)[\xi], \xi) + 2g(\nabla \varphi(\gamma), \xi) g\left(\frac{D}{ds} \xi, \dot{\gamma}\right) \\ & + \varphi(\gamma) \left[g(R(\xi, \dot{\gamma})\xi, \dot{\gamma}) + g\left(\frac{D}{ds} \xi, \frac{D}{ds} \xi\right) \right] ds - \varphi(\gamma) g(\nabla_\xi \xi, \dot{\gamma})|_{s=0} \end{aligned}$$

(where ∇ is the covariant derivative with respect to the metric g). By polarization, we obtain the related bilinear form:

$$(2.7) \quad I(\xi, \eta) = \int_0^1 \left[\frac{1}{2} g(\dot{\gamma}, \dot{\gamma}) g(H^\varphi(\gamma)[\xi], \eta) \right. \\ \left. + g(\nabla\varphi(\gamma), \xi) g\left(\frac{D}{ds}\eta, \dot{\gamma}\right) + g(\nabla\varphi(\gamma), \eta) g\left(\frac{D}{ds}\xi, \dot{\gamma}\right) \right. \\ \left. + \varphi(\gamma) \left[g(R(\dot{\gamma}, \xi)\dot{\gamma}, \eta) + g\left(\frac{D}{ds}\xi, \frac{D}{ds}\eta\right) \right] \right] ds + \\ - \frac{1}{2} \varphi(\gamma) \left[g(\nabla_\xi \eta, \dot{\gamma}) + g(\nabla_\eta \xi, \dot{\gamma}) \right] \Big|_{s=0}.$$

The Jacobi field equation is:

$$\frac{1}{2} g(\dot{\gamma}, \dot{\gamma}) H^\varphi(\gamma)[\xi] - \frac{D}{ds} \left[g(\nabla\varphi(\gamma), \xi) \dot{\gamma} + \varphi(\gamma) \frac{D}{ds} \xi \right] \\ + g\left(\frac{D}{ds}\xi, \dot{\gamma}\right) \nabla\varphi(\gamma) + \varphi(\gamma) R(\dot{\gamma}, \xi)\dot{\gamma} = 0,$$

while the vector fields in the kernel of I are Jacobi fields ξ such that $\xi(1) = 0$ and

$$(2.8) \quad \varphi(\gamma) \frac{D}{ds} \xi + g(\nabla\varphi(\gamma), \xi) \dot{\gamma} - \varphi(\gamma) \nabla_\xi \dot{\gamma}$$

is orthogonal to $T_{\gamma(0)}N$ at the instant $s = 0$.

As pointed out in [8, Definition 4.6], for the Jacobi metric the Jacobi fields differential equation in the interval $[0, a]$ is

$$(2.9) \quad - \frac{D}{ds} \left((E - V(\gamma)) \frac{D}{ds} \xi \right) + (E - V(\gamma)) R(\dot{\gamma}, \xi)\dot{\gamma} + \\ + \frac{D}{ds} (g(\nabla V(\gamma), \xi)\dot{\gamma}) - g(\dot{\gamma}, \frac{D}{ds}\xi) \nabla V(\gamma) + \\ - \frac{1}{2} g(\dot{\gamma}, \dot{\gamma}) H^V(\gamma)[\xi] = 0 \text{ for all } s \in]0, a].$$

We recall now the definition (given in [8]) of point which is conjugate to $V^{-1}(E)$.

Definition 2.7. A point $\gamma(s_0)$, on a Jacobi geodesic γ starting from $V^{-1}(E)$ is said to be *conjugate* to $V^{-1}(E)$ if there exists a non identically zero vector field ξ along γ defined in $[0, a]$, with $\xi(a) = 0$, such that

- (a) $\xi \in C^0([0, s_0]) \cap C^2(]0, a])$;
- (b) $\int_0^a (E - V(\gamma)) g\left(\frac{D}{ds}\xi, \frac{D}{ds}\xi\right) ds < +\infty$;
- (c) ξ satisfies equation (2.9) in $]0, a]$;
- (d) $\xi(0) \in T_{\gamma(0)}V^{-1}(E)$;
- (e) the continuous extension at $s = 0$ of the vector field:

$$(2.10) \quad (E - V(\gamma)) \frac{D}{ds} \xi - g(\nabla V(\gamma), \xi) \dot{\gamma}$$

is a multiple of $\nabla V(\gamma(0))$.

Recall that $\nabla V(\gamma(0))$ is parallel to the limit unit vector of $\dot{\gamma}$. A vector field ξ along γ satisfying (a)—(e) above will be called an *E-Jacobi field*.

Let $\gamma : [0, a] \rightarrow M$ be a Jacobi geodesic starting orthogonally to $V^{-1}(E)$. For $s > 0$ small enough, let \mathcal{N}_s denote the set of points having d_V -distance from $V^{-1}(E)$ equal to s , so that $p_s = \gamma(\sigma_s) \in \mathcal{N}_s$, where $\sigma_s = s/\sqrt{\lambda_\gamma}$. Observe that, if $s > 0$ is sufficiently small, \mathcal{N}_s is a C^2 embedded hypersurface of M , as shown in

[6]. Denote by Σ_s^* the shape operator of \mathcal{N}_s at p_s relatively to the metric g_* . From the relation

$$(2.11) \quad \nabla_X^* Y = \nabla_X Y - \frac{1}{2(E-V)} [g(\nabla V, X)Y + g(\nabla V, Y)X - g(X, Y)\nabla V]$$

the following expression for the covariant differentiation $\frac{D^*}{ds}$ along γ relative to the Levi–Civita connection of g_* holds:

$$(2.12) \quad \frac{D^*}{ds}\eta = \frac{D}{ds}\eta - \frac{1}{2(E-V(\gamma))} [g(\nabla V(\gamma), \dot{\gamma})\eta + g(\nabla V(\gamma), \eta)\dot{\gamma} - g(\dot{\gamma}, \eta)\nabla V(\gamma)].$$

Using (2.11) again, given $\eta, \xi \in T_{p_s}\mathcal{N}_s$, the shape operator with respect to the conformal metric Σ_s^* satisfies the identity

$$(2.13) \quad g_*(\Sigma_s^*(\xi), \eta) = \frac{1}{2}(E-V(\gamma))g(\Sigma_s(\xi), \eta) + \frac{1}{4}g(\xi, \eta)g(\nabla V(\gamma), \dot{\gamma}),$$

where we have exploited the fact that $\dot{\gamma} \in (T_{p_s}\mathcal{N}_s)^\perp$.

Moreover, introduced the Riemann curvature tensor of g_* , i.e.

$$R^*(X, Y) = [\nabla_X^*, \nabla_Y^*] - \nabla_{[X, Y]}^*,$$

then it can be seen that (2.9) is equivalent to the equation

$$(2.14) \quad \left(\frac{D^*}{ds}\right)^2 \xi(s) = R^*(\dot{\gamma}(s), \xi(s))\dot{\gamma}(s).$$

This equation and the skew–symmetry of the Riemann tensor R^* also imply that, for every E -Jacobi field along γ , the quantity $\lambda_\xi = 2g_*(\frac{D^*}{ds}\xi(s), \dot{\gamma}(s))$ is constant on $]0, a[$. Also observe that (2.12) implies that:

$$(2.15) \quad \lambda_\xi = 2g_*(\frac{D^*}{ds}\xi, \dot{\gamma}) = -\frac{1}{2}g(\nabla V(\gamma), \xi)g(\dot{\gamma}, \dot{\gamma}) + [E - V(\gamma)]g(\frac{D}{ds}\xi, \dot{\gamma}).$$

Remark 2.8. Observe that, if ξ is an E -Jacobi field along γ with $\xi(a) = 0$, then $g_*(\xi, \gamma)$ identically vanishes on $[0, a]$. Indeed, we have just proved that the map $g_*(\frac{D^*}{ds}\xi(s), \dot{\gamma}(s))$ is constant, from which it follows that $g_*(\xi, \gamma)$ is an affine function. Moreover, $\xi(a) = 0$ implies that $g_*(\xi, \gamma)$ vanishes at $s = a$. Let us now prove that $g_*(\xi, \gamma)$ can be extended continuously by setting it equal to 0 at $s = 0$.

First, observe that ξ is continuous at $s = 0$ (by condition (a) of Definition 2.7), and it can be seen that $(E - V(\gamma))$ behaves like $s^{2/3}$ near $s = 0$, see [8, Remark 2.4]. The conservation law (2.2) then implies that $\dot{\gamma}$ behaves like $s^{-1/3}$ near $s = 0$, and therefore $g_*(\xi, \gamma)$ vanishes at $s = 0$. Thus, $g_*(\xi, \gamma)$ vanishes identically on $[0, a]$ and, in particular, $\xi(\sigma_s) \in T_{\gamma(\sigma_s)}\mathcal{N}_s$.

Finally, we can give the main result of this section:

Theorem 2.9. *The point $\gamma(a)$ is conjugate to $V^{-1}(E)$ if and only if $\gamma(a)$ is a critical value of $\exp^\perp(\cdot, \cdot)(a)$.*

The proof is divided into two Propositions.

Proposition 2.10. *Let $\gamma(a)$ be conjugate to $V^{-1}(E)$. Then it is a critical value of $\exp^\perp(\cdot, \cdot)(a)$.*

Proof. Let ξ be an E -Jacobi field in the sense of Definition 2.7 such that $\xi(a) = 0$. Fix $\delta > 0$ sufficiently small and consider $\mathcal{N}_{\frac{\delta}{\sqrt{\lambda}}}$ and $\xi(\frac{\delta}{\sqrt{\lambda}})$. By Remarks 2.5 and

2.8, we have that $\xi(\frac{\delta}{\sqrt{\lambda}}) \in T_{\gamma(\frac{\delta}{\sqrt{\lambda}})}\mathcal{N}_{\frac{\delta}{\sqrt{\lambda}}}$. Now fix $Q(r)$ a smooth curve in $\mathcal{N}_{\frac{\delta}{\sqrt{\lambda}}}$ such that $Q'(0) = \xi(\frac{\delta}{\sqrt{\lambda}})$. If we fix δ and λ , the map

$$P \in V^{-1}(E) \rightarrow q(P, t_\delta(P, \lambda))$$

is a diffeomorphism $\Phi(P)$ of class C^1 . Choose $P(r) = \Phi^{-1}(Q(r))$ and keep λ fixed. Note that by (2.4) with $s = a$

$$d \exp^\perp(P, \lambda)(a)[P'(0)] = d\gamma(P, \lambda)(a)[P'(0)] = \lim_{r \rightarrow 0} \frac{d}{dr} \gamma(P(r), \lambda).$$

Then, by (2.5) and standard argument in the classical theory of ODE's, the values of $d\gamma(P, \lambda)(a)[P'(0)]$ is given by the solution z of the linearized equation of geodesics, evaluated at $s = a$ with initial position $\xi(\frac{\delta}{\sqrt{\lambda}})$ and initial velocity $\frac{D}{ds}\xi(\frac{\delta}{\sqrt{\lambda}})$. Then by the uniqueness of the solutions of the Cauchy problem we have $z(s) = \xi(s)$ for any s , so

$$d \exp^\perp(P, \lambda)(a)[P'(0)] = \xi(a) = 0,$$

concluding the proof. \square

Before proving the converse, we need the following:

Lemma 2.11. *Let ξ be an \mathcal{N}_δ -Jacobi field along γ . Then for all $s \in]0, \delta]$ we have $\xi(\sigma_s) \in T_{\gamma(\sigma_s)}\mathcal{N}_s$, and the vector*

$$(2.16) \quad \frac{D^*}{d\sigma} \Big|_{\sigma=\sigma_s} \xi(\sigma_s) + \Sigma_s^*(\xi(\sigma_s))$$

is parallel to $\dot{\gamma}(\sigma_s)$, where $\sigma_s = s/\sqrt{\lambda_\gamma}$.

Proof. Let us fix $s_0 \in]0, \delta]$ such that $\gamma(\sigma_{s_0}) \in \mathcal{N}_{s_0}$, and prove that (2.16) evaluated when $s = s_0$ is parallel to $\dot{\gamma}(\sigma_{s_0})$. To this aim, choose an arbitrary $\bar{s} \in]0, s_0[$ and let $r \mapsto \gamma_r$ be a 1-parameter family of g_* -geodesics $\gamma_r : [\sigma_{\bar{s}}, \sigma_{s_0}] \rightarrow M$, $r \in]-\varepsilon, \varepsilon[$, such that for all $s \in [\bar{s}, s_0]$,

- $\frac{d}{dr} \Big|_{r=0} \gamma_r(\sigma_s) = \xi(\sigma_s)$,
- $\gamma_r(\sigma_s) \in \mathcal{N}_s$.

Now let us fix $v \in \dot{\gamma}(\sigma_{s_0})^\perp$, consider the two-parameter map $z(r, \sigma) = \gamma_r(\sigma)$, and let $\nu(r, \sigma)$ be a smooth vector field along z such that $\nu(0, \sigma_{s_0}) = v$ and

$$(2.17) \quad \frac{D^*}{d\sigma} \nu(r, \sigma) = 0$$

for all r . Then using the properties of the family γ_r we get

$$(2.18) \quad g_* \left(\Sigma_{s_0}^*(\xi(\sigma_{s_0})), v \right) = g_* \left(\Sigma_{s_0}^*(\xi(\sigma_{s_0})), \nu(0, \sigma_{s_0}) \right) = g_* \left(\frac{D^*}{dr} \Big|_{r=0} \nu(r, \sigma_{s_0}), \dot{\gamma}(\sigma_{s_0}) \right),$$

where last equality is obtained easily using the properties of the shape operator Σ_s^* .

Now let us prove that the function $\sigma \mapsto g_* \left(\frac{D^*}{d\sigma} \xi(\sigma) + \Sigma_s^*(\xi(\sigma)), \nu(0, \sigma) \right)$ is constant (we drop the subscript s from σ and recall that s depends on σ , i.e. $s = \sigma\sqrt{\lambda_\gamma}$). From (2.14) and (2.17) we obtain:

$$(2.19) \quad \begin{aligned} \frac{d}{d\sigma} g_* \left(\frac{D^*}{d\sigma} \xi(\sigma), \nu(0, \sigma) \right) \\ = g_* \left(\frac{(D^*)^2}{d\sigma^2} \xi(\sigma), \nu(0, \sigma) \right) = g_* \left(R^*(\dot{\gamma}(\sigma), \xi(\sigma)) \dot{\gamma}(\sigma), \nu(0, \sigma) \right). \end{aligned}$$

Using (2.17) and (2.18), we get:

$$\begin{aligned}
(2.20) \quad \frac{d}{d\sigma} g_*(\Sigma_s^*(\xi(\sigma)), \nu(0, \sigma)) &= g_*\left(\frac{D^*}{d\sigma} \frac{D^*}{dr} \Big|_{r=0} \nu(r, \sigma), \dot{\gamma}(\sigma)\right) \\
&= g_*\left(\frac{D^*}{dr} \Big|_{r=0} \frac{D^*}{d\sigma} \nu(r, \sigma), \dot{\gamma}(\sigma)\right) + g_*(R^*(\dot{\gamma}(\sigma), \xi(\sigma)) \nu(0, \sigma), \dot{\gamma}(\sigma)) \\
&= -g_*(R^*(\dot{\gamma}(\sigma), \xi(\sigma)) \dot{\gamma}(\sigma), \nu(0, \sigma)).
\end{aligned}$$

Finally, using (2.19) and (2.20) we obtain:

$$\frac{d}{d\sigma} g_*\left(\frac{D^*}{d\sigma} \xi(\sigma) + \Sigma_s^*(\xi(\sigma)), \nu(0, \sigma)\right) = 0,$$

i.e., $g_*\left(\frac{D^*}{d\sigma} \xi(\sigma) + \Sigma_s^*(\xi(\sigma)), \nu(0, \sigma)\right)$ is constant for $\sigma \in [\sigma_{\bar{s}}, \sigma_{s_0}]$. But if we consider $s_0 = \delta$, by assumptions of ξ we have

$$g_*\left(\frac{D^*}{d\sigma} \xi(\sigma) + \Sigma_s^*(\xi(\sigma)), \nu(0, \sigma)\right) \Big|_{\sigma=\sigma_{s_0}} = 0$$

and the proof is complete. \square

Proposition 2.12. *Suppose that $\gamma(a)$ is a critical value of the exponential map $\exp^\perp(\cdot, \cdot)(a)$. Then $\gamma(a)$ is conjugate to $V^{-1}(E)$.*

Proof. Let $\gamma(a)$ be a critical value for the exponential map. Then by (2.5) it is a critical value for the exponential map defined by geodesics starting orthogonally from \mathcal{N}_δ .

Then by classical results (cf [5], Proposition 4.4, cap.10), there exists a C^2 Jacobi field along γ defined in the interval $[\sigma_\delta \equiv \frac{\delta}{\sqrt{\lambda}}, a]$ such that

$$(2.21) \quad \xi(\sigma_\delta) \in T_{\gamma(\sigma_\delta)} \mathcal{N}_\delta, \quad \xi(a) = 0$$

and

$$(2.22) \quad \frac{D}{ds} \xi(\sigma_\delta) + \Sigma_\delta(\xi(\sigma_\delta)) \text{ is parallel to } \dot{\gamma}(\sigma_\delta).$$

Note that ξ can be extended as Jacobi field along γ to the whole interval $]0, a]$. Moreover, since $g_*(\xi(s), \dot{\gamma}(s))$ has second derivative identically zero, and it is null at σ_δ and a , we deduce that

$$(2.23) \quad g_*(\xi(s), \dot{\gamma}(s)) = 0 \text{ for any } s \in]0, a].$$

Moreover, by Lemma 2.11:

$$(2.24) \quad \frac{D^*}{ds} \xi(\sigma) + \Sigma_\delta(\xi(\sigma)) \text{ is parallel to } \dot{\gamma}(\sigma) \text{ for any } \sigma \in]0, \sigma_\delta].$$

Now denote by $I_s^a(\xi, \eta)$ the index form (2.7) with the interval $[0, 1]$ replaced by the interval $[s, a]$ ($s > 0$) and ϕ replaced by $E - V$. By (2.23) and (2.24) we deduce that

$$(2.25) \quad I_s^a(\xi, \eta) = 0 \text{ for any smooth vector field } \eta \text{ along } \gamma \text{ satisfying}$$

$$\eta(s) \in T_{\gamma(s)} \mathcal{N}_{s\sqrt{\lambda}}, \text{ and } \eta(a) = 0.$$

Now, arguing as in the proof of [8, Propositions 4.15], we show that there exists $s_{**} < s_* \in]0, \sigma_\delta]$ such that for any $s \in]0, s_{**}[$ the quadratic form $I_s^{s_*}(\eta, \eta)$ has a minimizer in the affine space $Y_s^{s_*}$ of the absolutely continuous vector fields η along γ such that $\eta(s) \in T_{\gamma(s)} \mathcal{N}_{s\sqrt{\lambda}}$ and $\eta(s_*) = \xi(s_*)$. Such a minimizer is a Jacobi field. Moreover, as the proof of [8, Proposition 4.16], we see also that $I_s^{s_*}$ is strictly positive definite and there is only one unique Jacobi field along γ in $Y_s^{s_*}$: therefore it coincides with the Jacobi field ξ .

The same estimate in the proof of [8, Proposition 4.15] gives also the existence of a constant C independent on s such that

$$\int_s^{s^*} ((E - V(\gamma))g(\frac{D}{ds}\xi, \frac{D}{ds}\xi)) ds \leq C,$$

from which we deduce (d) of Definition 2.7. Note that property (b) implies the continuity of ξ in a $s = 0$, and taking the limit as $s \rightarrow 0$ in (2.23) (using the unit vector of $\dot{\gamma}$) gives also property (d).

Finally, integrating by parts in (2.25) (with $s = 0$) and using standard regularization methods as in [8] we obtain the continuity at $s = 0$ of the map

$$(E - V(\gamma))\frac{D}{ds}\xi - g(\nabla V(\gamma), \xi)\dot{\gamma}$$

and taking the limit as $s \rightarrow 0$ in (2.24) allows to obtain also property (e). \square

3. ON THE C^2 -REGULARITY OF THE DISTANCE FROM THE BOUNDARY OF THE POTENTIAL WELL

In this last section we prove the C^2 -regularity for the Jacobi distance from the boundary of the potential well in a neighborhood of a point Q_0 with a unique minimizer and such that Q_0 is not conjugate to $V^{-1}(E)$. Indeed we prove the following

Theorem 3.1. *Let d_V be the Jacobi distance from $V^{-1}(E)$, Assume that $Q_0 \in V^{-1}(]-\infty, E[)$ is such that there is a unique minimizer that realizes $d_V(Q_0)$. Assume also that Q_0 is not conjugate to $V^{-1}(E)$. Then d_V is of class C^2 in a neighborhood of Q_0 .*

Proof. By Theorem 2.9, it follows that Q_0 is a regular value of the exponential map. Since it is of class C^2 we see that any Q sufficiently close to Q_0 is a regular value of the exponential map. Then, always by Theorem 2.9 we have that Q is not conjugate to $V^{-1}(E)$.

Then, for any Q sufficiently close to Q_0 there is a unique minimizer. Indeed suppose by contradiction there exists a sequence Q_n of points with at least two minimizers γ_n^1 and γ_n^2 . By Lemma 3.4 in [8], they converge (with respect to the H^1 -norm) to the unique minimizer that realizes $d_V(Q_0)$. But this would be in contradiction with the fact that Q_0 is a regular value of the exponential map.

Finally, by Proposition 3.5 of [8], for any Q nearby Q_0 the gradient of d_V at Q is given by

$$\nabla d_V(Q) = \frac{(E - V(Q))}{2d_V(Q)}\dot{\gamma}_Q(1),$$

where γ_Q is the unique minimizer between $V^{-1}(E)$ and Q , parameterized in the interval $[0, 1]$. Then to prove the C^2 regularity of d_V it suffices to prove the C^1 -regularity of $\dot{\gamma}_Q(1)$. Since $\gamma_Q(1) = Q$, thanks to the invertibility of \exp^\perp and its C^1 -regularity, formula (2.5) and Remark 2.4 allows to obtain the conclusion of the proof. \square

REFERENCES

- [1] H. AMANN, *Function spaces on singular manifolds*, Math. Nachr. **286** (2013), no. 5–6, 436–475.

- [2] S. BOLOTIN, *Libration motions of natural dynamical systems*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **6** (1978) 72–77 (in Russian)
- [3] S. BOLOTIN, V.V. KOZLOV, *Librations with many degrees of freedom*, J. Appl. Math. Mech. **42** (1978) 245–250 (in Russian).
- [4] N-C. CHEN, *Periodic brake orbits in the planar isosceles three-body problem*, Nonlinearity **26** (2013) 2875–2898
- [5] M. P. DO CARMO, *Riemannian Geometry*, Birkhäuser, Boston, 1992.
- [6] R. GIAMBÒ, F. GIANNONI, P. PICCIONE, *Orthogonal Geodesic Chords, Brake Orbits and Homoclinic Orbits in Riemannian Manifolds*, Adv. Diff. Eq. **10**, 2005, pp 931–960.
- [7] R. GIAMBÒ, F. GIANNONI, P. PICCIONE, *Existence of orthogonal geodesic chords on Riemannian manifolds with concave boundary and homeomorphic to the N -dimensional disk*, Nonlinear Analysis Series A: Theory, Methods, Applications, 73 (2010) 290–337.
- [8] R. GIAMBÒ, F. GIANNONI, P. PICCIONE, *Morse Theory for geodesics in singular conformal metrics*, to appear on Communication in Analysis and Geometry (preprint arXiv:1406.0644).
- [9] R. GIAMBÒ, F. GIANNONI, P. PICCIONE, *Examples with minimal number of brake orbits and homoclinics in annular potential regions*, J. Differential Equations **256** (2014) 2677–2690
- [10] R. GIAMBÒ, F. GIANNONI, P. PICCIONE, *Multiple Brake Orbits and Homoclinics in Riemannian Manifolds*, Archive for Rational Mechanics and Analysis: 200, (2011), 691–724.
- [11] H. GLUCK, W. ZILLER, *Existence of Periodic Motions of Conservative Systems*, in “Seminar on Minimal Surfaces” (E. Bombieri Ed.), Princeton University Press, 65–98, 1983.
- [12] H. LIU, Y. LONG, *Resonance identity for symmetric closed characteristics on symmetric convex Hamiltonian energy hypersurfaces and its applications*, J. Differential Equations **255** (2013) 2952–2980
- [13] H. LIU, Y. LONG, W. WANG, *Resonance identities for closed characteristics on compact star-shaped hypersurfaces in \mathbb{R}^{2n}* , J. Funct. Anal. **266** (2014) 5598–5638
- [14] C. LIU, Y. LONG, C. ZHU, *Multiplicity of closed characteristics on symmetric convex hypersurfaces in \mathbb{R}^{2n}* , Math. Ann. **323** (2) (2002) 201–215.
- [15] Y. LONG, D. ZHANG, C. ZHU, *Multiple brake orbits in bounded convex symmetric domains*, Adv. Math. **203** (2006), no. 2, 568–635.
- [16] Y. LONG, C. ZHU, *Closed characteristics on compact convex hypersurfaces in \mathbb{R}^{2n}* Ann. of Math. (2) **155** (2002), no. 2, 317–368.
- [17] P.H. RABINOWITZ, *On the existence of periodic solutions for a class of symmetric Hamiltonian systems*, Nonlinear Anal. **11** (1987) 599–611.
- [18] P. H. RABINOWITZ, *Critical point theory and applications to differential equations: a survey*. Topological nonlinear analysis, 464–513, Progr. Nonlinear Differential Equations Appl., 15, Birkhäuser Boston, Boston, MA, 1995.
- [19] H. SEIFERT, *Periodische Bewegungen Mechanischer Systeme*, Math. Z. **51** (1948), 197–216.
- [20] A. SZULKIN, *An Index Theory and Existence of Multiple Brake Orbits for star-shaped Hamiltonian Systems*, Math. Ann. **283** (1989), 241–255.
- [21] D. ZHANG, *Brake type closed characteristics on reversible compact convex hypersurfaces in \mathbb{R}^{2n}* , Nonlinear Analysis **74** (2011) 3149–3158
- [22] D. ZHANG, C. LIU, *Multiple brake orbits on compact convex symmetric reversible hypersurfaces in \mathbb{R}^{2n}* , Ann. I. H. Poincaré **31**(3) (2014) 531–554

SCUOLA DI SCIENZE E TECNOLOGIE

UNIVERSITÀ DI CAMERINO

ITALY

E-mail address: roberto.giambo@unicam.it, fabio.giannoni@unicam.it

DEPARTAMENTO DE MATEMÁTICA

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

UNIVERSIDADE DE SÃO PAULO

BRAZIL

E-mail address: piccione@ime.usp.br